The number of "magic squares" [1]

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Abstract: We define a magic square to be a square matrix whose entries are nonnegative integers and whose rows, columns, and main diagonals sum up to the same number. We prove structural results for the number of such squares as a function of the size of the matrix and the line sum. We give examples for small sizes and show similar results for symmetric, pandiagonal magic squares, and magic hypercubes.

1 Introduction

We define a *semi-magic square* to be a square matrix whose entries are nonnegative integers and whose rows and columns (that is, lines) sum up to the same number. A *magic square* is a semi-magic square whose main diagonals also add up to the line sum. A *symmetric magic square* is a magic square which is symmetric as a matrix. A *pandiagonal magic square* is a semi-magic square whose pandiagonals from the upper left to the lower right add up to the line sum. These definitions clash somewhat with that of the *classical magic square*, a magic square of order n whose entries are the integers $1, \ldots, n^2$.

Our goal is to count these various magic squares. In the classical case, this is in some sense not very interesting: for each order there is a fixed number of classic magic squares. For example, there are 8 classic magic squares of order 3. This situation becomes more interesting if we drop the condition of classicality. That is, we study the number of magic squares as a function of the row-column-(pan)diagonal sum.

Definition 1 *We denote the counting function for semi-magic, magic, symmetric magic, and pandiagonal magic squares of order n and row-column-(pan)diagonal sum t by $H_n(t), M_n(t), S_n(t)$, and $P_n(t)$, respectively.*

The case $n = 2$ is not very complicated: here a semi-magic square is determined once we know one entry; a magic square has to have identical entries in each coefficient. Hence

$$H_2(t) = t + 1 \qquad \text{and} \qquad M_2(t) = S_2(t) = P_2(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

These easy results already hint at something: the counting function H_n is of a different nature from the functions M_n, S_n, and P_n.

The oldest nontrivial results go back to 1915: Macmahon [6] proved that

$$H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2} \qquad \text{and} \qquad M_3(t) = \begin{cases} \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise.} \end{cases}$$

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The first structural result was proved in 1970:

Theorem 1 (Stein-Stein [11]) *$H_n(t)$ is a polynomial in t of degree $(n-1)^2$.*

This result was conjectured earlier by Anand, Dumir, and Gupta [1]. An elementary proof can be found in [8]. Ehrhart [4] and Stanley [9] rediscovered Theorem 1 and proved that $H_n(t)$ satisfies a *reciprocity law* and has a number of special roots:

Theorem 2 (Ehrhart, Stanley)

$$H_n(-n-t) = (-1)^{(n-1)^2} H_n(t) \ , \qquad H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0 \ .$$

Stanley also proved analogous results for the counting function for *symmetric* semi-magic squares.

Theorem 1 allows us to derive $H_n(t)$ in a simple way: Compute the first $(n-1)^2+1$ values of $H_n(t)$ and interpolate. Theorem 2 makes this computation easier by roughly halving the number of values we have to compute.

In this paper we give analogous results for the counting functions for magic squares and symmetric and pandiagonal magic squares. As hinted earlier, these counting functions are of a different nature than the one for semi-magic squares.

Theorem 3 *$M_n(t)$, $S_n(t)$, and $P_n(t)$ are quasipolynomials in t of degree n^2-2n-1, $\frac{1}{2}n^2-\frac{1}{2}n-2$, and n^2-3n+2, respectively. They satisfy*

$$\begin{array}{ll} M_n(-n-t) = (-1)^{n-1}\, M_n(t) \ , & M_n(-1) = M_n(-2) = \cdots = M_n(-n+1) = 0 \ , \\ S_n(-n-t) = (-1)^{n(n-1)/2}\, S_n(t) \ , & S_n(-1) = S_n(-2) = \cdots = S_n(-n+1) = 0 \ , \\ P_n(-n-t) = P_n(t) \ , & P_n(-1) = P_n(-2) = \cdots = P_n(-n+1) = 0 \ . \end{array} \tag{1}$$

A *quasipolynomial* Q is an expression of the form

$$Q(t) = c_d(t)\ t^d + \cdots + c_1(t)\ t + c_0(t) \ ,$$

where $c_0, \ldots, c_d$ are periodic functions in t. The least common multiple of the periods of $c_0, \ldots, c_d$ is called the *period* of Q.

We prove Theorem 3 in Sections 2 and 3. To be able to compute the counting functions M_n, S_n, and P_n we need the periods of these quasipolynomials. We describe in Section 4 methods for finding these periods and hence for the actual computation of M_n, S_n, and P_n. Finally, in Section 5 we extend the function H_n to the enumeration of magic hypercubes. All of this is based on the idea that one can interpret the various counting functions as enumerating integer points ("lattice points") in certain polytopes. We give the necessary background information next.

2 Some geometric combinatorics

Let $\mathcal{P}$ be a rational polytope, that is, a polytope with rational vertices. For a positive integer t, let $L_{\mathcal{P}}(t)$ denote the number of lattice points in the dilated polytope $t\mathcal{P} = \{tx : x \in \mathcal{P}\}$. Similarly, we define $L_{\mathcal{P}}^*(t)$ as the number of lattice points in the (relative) interior of $t\mathcal{P}$. Ehrhart, who initiated the study of the lattice point count in dilated polytopes, proved

Theorem 4 (Ehrhart [3]) *$L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}}^*(t)$ are quasipolynomials in t whose degree is the dimension of $\mathcal{P}$ and whose period is the least common multiple of the denominators of the vertices of $\mathcal{P}$.*

In particular, if $\mathcal{P}$ has integer vertices then $L_{\mathcal{P}}$ and $L_{\mathcal{P}}^*$ are polynomials. Ehrhart [3] conjectured the following reciprocity law, which was proved by Macdonald [5] (for the case that $\mathcal{P}$ has integer vertices) and McMullen [7] (for the case that $\mathcal{P}$ has rational vertices). The version we state is due to Stanley [10]:

Theorem 5 (Ehrhart-Macdonald reciprocity law) *If $\mathcal{P}$ is a rational polytope which is homeomorphic to a d-sphere then*

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}}^*(t) \ .$$

Note that the conditions of the theorem hold in particular if $\mathcal{P}$ is convex.

The connection to our counting functions M_n, S_n, and P_n is the following: all the various magic square conditions are linear inequalities (the entries are nonnegative) and linear equalities (the entries in each line sum up to t) in the n^2 variables forming the entries of the square. In other words, what we are counting is nonnegative integer solutions to the linear system

$$\mathbf{A}\ \mathbf{x} = \mathbf{b} \ ,$$

where $\mathbf{x} \in \mathbb{R}^{n^2}$, $\mathbf{A}$ denotes the matrix determining the respective magic-sum conditions, and $\mathbf{b}$ is the column vector whose entries are all t. For example, if we study magic 3×3 squares,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} .$$

Furthermore, by writing $\mathbf{b} = t\ \mathbf{1}$, where $\mathbf{1}$ denotes a column vector whose entries are all 1, we can see that our counting function enumerates nonnegative integer solutions to the linear system $\mathbf{A}\ \mathbf{x} = t\ \mathbf{1}$, in other words, the lattice points in the t-dilate of the polytope

$$\mathcal{P} = \left\{ \mathbf{x} = (x_1, \ldots, x_{n^2}) \in \mathbb{R}^{n^2} \ : \ x_k \geq 0, \ \mathbf{A}\ \mathbf{x} = \mathbf{1} \right\} \ ,$$

if we choose the matrix $\mathbf{A}$ according to the magic-sum conditions. Note that $\mathcal{P}$ is an intersection of half-spaces and hyperplanes, and is therefore convex. No matter which counting function the matrix $\mathbf{A}$ corresponds to, the entries of $\mathbf{A}$ are 0's and 1's. One obtains the vertices of $\mathcal{P}$ by converting some of the inequalities $x_k \geq 0$ to equalities. It is easy to conclude from this that the vertices of $\mathcal{P}$ are rational. Hence by Ehrhart's Theorem 4, $M_n(t), S_n(t)$, and $P_n(t)$ are quasipolynomials in t whose degrees are the dimensions of the corresponding polytopes. Geometrically, the "magic-sum variable" t is the dilation factor of these polytopes.

The Ehrhart-Macdonald reciprocity law (Theorem 5) connects the lattice-point count in $\mathcal{P}$ to that of the interior of $\mathcal{P}$. In our case, this interior (again, using any matrix $\mathbf{A}$ suitable for one of the counting functions) is

$$\mathcal{P}^{\text{int}} = \left\{ \mathbf{x} = (x_1, \ldots, x_{n^2}) \in \mathbb{R}^{n^2} \,:\, x_k > 0,\ \mathbf{A}\ \mathbf{x} = \mathbf{1} \right\} .$$

The lattice-point count is the same as before with the difference that we now only allow *positive* integers as solutions to the linear system. This motivates the following

Definition 2 *Denote by $M_n^*(t), S_n^*(t)$, and $P_n^*(t)$ the counting functions for magic squares, symmetric, and pandiagonal magic squares as before, but now with the restriction that the entries are positive integers.*

The Ehrhart-Macdonald reciprocity law (Theorem 5) now says that, for example,

$$M_n(-t) = (-1)^{\deg(M_n)} M_n^*(t) . \tag{2}$$

On the other hand, by its very definition, $M_n^*(t) = 0$ for $t = 1, 2, \ldots, n-1$. Hence we obtain $M_n(-t) = 0$ for those t. Also, since each row of the matrix $\mathbf{A}$ defined by some magic-sum conditions has exactly n 1's and all other entries 0, it is not too hard to conclude that $M_n^*(t) = M_n(t-n)$. Combining this with the reciprocity law (2), we obtain

$$M_n(t) = (-1)^{\deg(M_n)} M_n(-t-n) .$$

There are analogous statements for S_n and P_n, and (1) follows, once we prove the degree formulas, with

$$\begin{aligned} &(-1)^{n^2-2n-1} = (-1)^{n-1} , \\ &(-1)^{\frac{1}{2}n^2-\frac{1}{2}n-2} = (-1)^{n(n-1)/2} , \text{ and} \\ &(-1)^{n^2-3n+2} = (-1)^{n(n-3)} = 1 . \end{aligned}$$

This proves Theorem 3 except for the formulas for the degrees of M_n, S_n, and P_n.

3 Proof of the degree formulas

Let's start with magic squares: the degree of M_n equals, by Ehrhart's Theorem 4, n^2 (the number of places we have to fill) minus the number of linearly independent constraints. In other words, we have to find the rank of

$$\begin{pmatrix}
1 & \cdots & 1 & & & & & & & \\
& & & 1 & \cdots & 1 & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & 1 & \cdots & 1 \\
1 & & & 1 & & & & 1 & & \\
& \ddots & & & \ddots & & \cdots & & \ddots & \\
& & 1 & & & 1 & & & & 1 \\
1 & & & & 1 & & \cdots & & & 1 \\
& & 1 & & & & \cdots & 1 & &
\end{pmatrix}.$$

Here we only showed the entries with 1's, every other entry is 0. The sum of the first n rows equals the sum of the next n rows, so we can eliminate one of the first $2n$ rows; we choose the first. Furthermore, we can add the difference of the second and $(n+1)$st row to the $(2n+1)$st and subtract the $(2n)$th row from the $(2n+2)$nd. These operations yield, after a row exchange,

$$\begin{pmatrix}
1 & & & 1 & & & & 1 & & \\
& \ddots & & & \ddots & & \cdots & & \ddots & \\
& & 1 & & & 1 & & & & 1 \\
& & & 1 & \cdots & 1 & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & 1 & \cdots & 1 \\
& & & & 2 & \cdots & & & & \\
& & & & & -1 & \cdots & & &
\end{pmatrix}.$$

These represent $2n+1$ linearly independent restrictions on our magic square. The polytope corresponding to M_n has therefore dimension n^2-2n-1: it lies in an affine subspace of that dimension, and the point $\left(\frac{1}{n}, \dots, \frac{1}{n}\right)$ is an interior point of the polytope. Hence, by Ehrhart's Theorem 4, the degree of M_n is n^2-2n-1.

The case of symmetric magic squares is simpler: here we have $\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$ places to fill, and the row-sum condition is equivalent to the column-sum condition. Hence there are $n+2$ constraints, which are easily seen to be linear independent. The dimension of the corresponding polytope, and hence the degree of S_n, is therefore equal to $\frac{1}{2}n^2 - \frac{1}{2}n - 2$.

Finally, we discuss pandiagonal magic squares. Again we have n^2 places to fill; the constraints are this time represented by

$$\left(\begin{array}{ccccccccccccc}
1 & 1 & \cdots & 1 & & & & & & & & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1 \\
1 & & & & 1 & & & & & 1 & & & \\
 & 1 & & & & 1 & & & & & 1 & & \\
 & & \ddots & & & & \ddots & & \cdots & & & \ddots & \\
 & & & 1 & & & & 1 & & & & & 1 \\
1 & & & & & 1 & & & & & & & 1 \\
 & 1 & & & & & \ddots & & \cdots & 1 & & & \\
 & & \ddots & & & & & 1 & & & \ddots & & \\
 & & & 1 & 1 & & & & & & & 1 &
\end{array}\right).$$

Similar to the first case, the sum of the first n rows equals the sum of the next n rows and the sum of the last n rows. We can therefore eliminate the first and the last rows, and get after rearranging the rows

$$\left(\begin{array}{ccccccccccccc}
1 & & & & 1 & & & & & 1 & & & \\
 & 1 & & & & 1 & & & & & 1 & & \\
 & & \ddots & & & & \ddots & & \cdots & & & \ddots & \\
 & & & 1 & & & & 1 & & & & & 1 \\
1 & & & & & 1 & & & & & & & 1 \\
 & \ddots & & & & & \ddots & & \cdots & 1 & & & \\
 & & 1 & & & & & 1 & & & \ddots 1 & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1
\end{array}\right).$$

In this new matrix, we can now replace the $(n+1)$st row by the difference of the first and $(n+1)$st rows, the $(n+2)$nd row by the difference of the second and $(n+2)$nd rows, and so on:

$$\left(\begin{array}{ccccccccccccc}
1 & & & & 1 & & & & & 1 & & & \\
 & 1 & & & & 1 & & & & & 1 & & \\
 & & \ddots & & & & \ddots & & \cdots & & & \ddots & \\
 & & & 1 & & & & 1 & & & & & 1 \\
 & & & & 1 & -1 & & & & & & & \\
 & & & & & \ddots & & & \cdots & & & & \\
 & & & & & & 1 & -1 & & & & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1
\end{array}\right).$$

Finally, we can now replace the $(n+2)$nd row by the sum of the $(n+1)$st and $(n+2)$nd rows, the $(n+3)$nd row by the sum of the $(n+2)$st and $(n+3)$nd rows, etc. Subtracting the $(2n)$th row from the $(n+1)$st row gives a matrix of full rank, that is, rank $3n-2$. The dimension of the corresponding polytope, which equals the degree of P_n, is therefore n^2-3n+2.

This finishes the proof of Theorem 3.

4 Computations

To interpolate a quasipolynomial of degree d and period p, we need to compute $p(d+1)$ values. The periods of M_n, S_n, and P_n are not as simple to derive as their degrees. What we can do, however, is compute, for fixed n, the vertices of the respective polytopes, whose denominators give the periods of the quasipolynomials by Ehrhart's Theorem. This is easy for very small n but gets involved very quickly. For example, one can practically find by hand that the vertices of the polytope corresponding to S_3 are

$$\left(\frac{2}{3},0,\frac{1}{3},\frac{1}{3},\frac{2}{3},0\right) \qquad \text{and} \qquad \left(0,\frac{2}{3},\frac{1}{3},\frac{1}{3},0,\frac{2}{3}\right) ,$$

whereas the polytope corresponding to S_5 has 74 vertices. To make computational matters worse, the least common multiple of the denominators of these vertices is 60; one would have to do a lot of interpolation to obtain S_5.

The reciprocity laws in Theorem 3 essentially halve the number of computations for each quasipolynomial. Nevertheless, the task of computing our quasipolynomials becomes impractical without further tricks or large computing power. The following table contains data about the vertices of the polytopes corresponding to M_n, S_n, and P_n for $n=3$ and 4. It was produced using `Maple`.

Polytope corresponding to	Number of vertices	lcm of denominators
M_3	4	3
S_3	2	3
P_3	3	1
M_4	20	2
S_4	12	4
P_4	28	2

With this information, it is now easy (that is, using a computer) to interpolate each quasipolynomial. Here are the results:

$$M_3(t) = \begin{cases} \frac{2}{9}t^2 + \frac{2}{3}t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise,} \end{cases}$$

$$S_3(t) = \begin{cases} \frac{2}{3}t + 1 & \text{if } 3|t, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_3(t) = \frac{1}{2}t^2 + \frac{2}{3} + 1,$$

$$M_4(t) = \begin{cases} \frac{1}{480}t^7 + \frac{7}{240}t^6 + \frac{89}{480}t^5 + \frac{11}{16}t^4 + \frac{49}{30}t^3 + \frac{38}{15}t^2 + \frac{71}{30}t + 1 & \text{if } t \text{ is even,} \\ \frac{1}{480}t^7 + \frac{7}{240}t^6 + \frac{89}{480}t^5 + \frac{11}{16}t^4 + \frac{779}{480}t^3 + \frac{593}{240}t^2 + \frac{1051}{480}t - \frac{3}{16} & \text{if } t \text{ is odd,} \end{cases}$$

$$S_4(t) = \begin{cases} \frac{5}{128}t^4 + \frac{5}{16}t^3 + t^2 + \frac{3}{2}t + 1 & \text{if } t \equiv 0 \mod 4, \\ \frac{5}{128}t^4 + \frac{5}{16}t^3 + t^2 + \frac{3}{2}t + \frac{7}{8} & \text{if } t \equiv 2 \mod 4, \\ 0 & \text{if } t \text{ is odd,} \end{cases}$$

$$P_4(t) = \begin{cases} \frac{7}{1440}t^6 + \frac{7}{120}t^5 + \frac{23}{72}t^4 + t^3 + \frac{341}{180}t^2 + \frac{31}{15}t + 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

5 Leaving the plane: magic hypercubes

We can extend the counting function H_n for semi-magic squares in the following way: Define a *magic hypercube* as a d-dimensional $n \times \cdots \times n$ array of n^d nonnegative integers, which sum up to the same number t parallel to any axis. Again we will count all such cubes in terms of d, n and t; we denote the corresponding enumerating function $H_n^d(t)$. Theorems 1 and 2 discuss the case $d = 2$; the case $d = n = 3$ seems to have appeared first in [2]. We have the following result for general d.

Theorem 6 *$H_n^d(t)$ is a quasipolynomial in t of degree $(n-1)^d$. It satisfies*

$$H_n^d(-n-t) = (-1)^{n-1}\, H_n^d(t) \ , \qquad\qquad H_n^d(-1) = H_n^d(-2) = \cdots = H_n^d(-n+1) = 0 \ .$$

The fact that $H_n^d(t)$ is a quasipolynomial, the reciprocity law, and the special zeros for $H_n^d(t)$ follow in exactly the same way as the respective statements in Theorem 3. As there, the remaining task is finding the degree of $H_n^d(t)$, that is, the dimension of the corresponding polytope. Again we only have to find the dimension of the affine subspace of $\mathbb{R}^{n^d}$ in which our polytope lives. We do this by counting linearly independent constraint equations.

To this end, we denote the entries of our hypercube by $c(a_1, \ldots, a_d)$, $1 \le a_j \le n$, viewed as a point in $[0,1]^{n^d}$. Once the $(n-1)^d$ entries $c(a_1, \ldots, a_d)$, $1 \le a_j \le n-1$, are chosen, the other entries are clearly determined. Therefore, there are at least $n^d - (n-1)^d$ linearly independent constraint equations.

On the other hand, every constraint involves at least one of the $n^d - (n-1)^d$ entries $c(a_1, \ldots, a_d)$ for which at least one a_j equals n. Consider, say, the entry $c(a_1, \ldots, a_k, n, \ldots, n)$ with $0 \le k < d$ and $a_1, \ldots, a_k < n$:

$$
\begin{aligned}
c(a_1, \ldots, a_k, n, \ldots, n) &= 1 - \sum_{j_d=1}^{n-1} c(a_1, \ldots, a_k, n, \ldots, n, j_d) \\
&= 1 - \sum_{j_d=1}^{n-1} \left(1 - \sum_{j_{d-1}=1}^{n-1} c(a_1, \ldots, a_k, n, \ldots, n, j_{d-1}, j_d) \right) \\
&= 1 - (n-1) + \sum_{1 \le j_{d-1}, j_d \le n-1} c(a_1, \ldots, a_k, n, \ldots, n, j_{d-1}, j_d) \\
&= \ldots = \chi + (-1)^{d-k} \sum_{1 \le j_{k+1}, \ldots, j_d \le n-1} c(a_1, \ldots, a_k, j_{k+1}, \ldots, j_d) \ ,
\end{aligned}
$$

where $\chi = 2 - n$ if $d - k$ is even, $\chi = 1$ if $d - k$ is odd. The order of summation is irrelevant, which implies that *all* constraints involving $c(a_1, \ldots, a_k, n, \ldots, n)$ with $0 \le k < d$ and $a_1, \ldots, a_k < n$, and not involving any entry with one of $a_1, \ldots, a_k = n$, are equivalent to the *one* constraint

$$
c(a_1, \ldots, a_k, n, \ldots, n) = \chi + (-1)^{d-k} \sum_{1 \le j_{k+1}, \ldots, j_d \le n-1} c(a_1, \ldots, a_k, j_{k+1}, \ldots, j_d) \ .
$$

Therefore there are at most $n^d - (n-1)^d$ linearly independent constraint equations.

Hence the dimension of the polytope and the degree of $H_n^d(t)$ is $(n-1)^d$.

6 Closing remarks

One big remaining open question is that of the periods of our counting functions. The only evidence gained from our data seems the fact that the period is increasing in some fashion with n. We believe the following is true.

Conjecture 7 *M_n, S_n, and P_n are not polynomials for $n \ge 5$.*

As for magic hypercubes, [2] showed that H_3^3 really is a quasipolynomial, not a polynomial. We challenge the reader to (dis)prove

Conjecture 8 *H_n^d is not a polynomial for $n, d \ge 3$.*

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