MAXIMAL PERIODS OF (EHRHART) QUASI-POLYNOMIALS

MATTHIAS BECK, STEVEN V. SAM, AND KEVIN M. WOODS

Abstract. A *quasi-polynomial* is a function defined of the form $q(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$, where $c_0, c_1, \ldots, c_d$ are periodic functions in $k \in \mathbb{Z}$. Prominent examples of quasi-polynomials appear in Ehrhart's theory as integer-point counting functions for rational polytopes, and McMullen gives upper bounds for the periods of the $c_j(k)$ for Ehrhart quasi-polynomials. For generic polytopes, McMullen's bounds seem to be sharp, but sometimes smaller periods exist. We prove that the second leading coefficient of an Ehrhart quasi-polynomial always has maximal expected period and present a general theorem that yields maximal periods for the coefficients of certain quasi-polynomials. We present a construction for (Ehrhart) quasi-polynomials that exhibit maximal period behavior and use it to answer a question of Zaslavsky on convolutions of quasi-polynomials.

1. Introduction

A *quasi-polynomial* is a function defined on $\mathbb{Z}$ of the form

$$q(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)\,, \tag{1}$$

where $c_0, c_1, \ldots, c_d$ are periodic functions in k, called the *coefficient functions* of q. Assuming c_d is not identically zero, we call d the *degree* of q. Quasi-polynomials play a prominent role in enumerative combinatorics [8, Chapter 4]. Arguably their best known appearance is in Ehrhart's fundamental work on integer-point enumeration in rational polytopes [1, 3]. For more applications, we refer to the recent article [4].

A *rational polytope* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Q}^n$. The *dimension* of a polytope $\mathcal{P}$ is the dimension d of the smallest affine space containing $\mathcal{P}$, in which case we call $\mathcal{P}$ a d-polytope. A *face* of $\mathcal{P}$ is a subset of the form $\mathcal{P} \cap H$, where H is a hyperplane such that $\mathcal{P}$ is entirely contained in one of the two closed half-spaces of $\mathbb{R}^n$ that H naturally defines. A $(d-1)$-face of a d-polytope is a *facet*, and a 0-face is a *vertex*. The smallest $k \in \mathbb{Z}_{>0}$ for which the vertices of $k\mathcal{P}$ are in $\mathbb{Z}^n$ is the *denominator* of $\mathcal{P}$. Ehrhart's theorem states that the integer-point counting function $L_{\mathcal{P}}(k) := \#\left(k\mathcal{P} \cap \mathbb{Z}^n\right)$ is a quasi-polynomial of degree d in $k \in \mathbb{Z}_{>0}$, and the denominator of $\mathcal{P}$ is a period of each of the coefficient functions.

In general, many of the coefficient functions will have smaller periods. Suppose q is given by (1). The *minimum period* of c_j is the smallest $p \in \mathbb{Z}_{>0}$ such that $c_j(k+p) = c_j(k)$ for all $k \in \mathbb{Z}$ (any multiple of p is, of course, also a period of c_j). The *minimum period* of q is the least common multiple of the minimum periods of $c_0, c_1, \ldots, c_d$. In this paper, we study the minimum periods of the c_j. All of our illustrating examples can be realized as Ehrhart quasi-polynomials. Ehrhart's theorem tells us that the minimum period of each c_j divides the denominator of $\mathcal{P}$.

Date: February 6, 2007.

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 52C07.

Key words and phrases. Ehrhart quasi-polynomial, period, lattice points, rational polytope, quasi-polynomial convolution.

The following theorem due to McMullen [7, Theorem 6] gives a more precise upper bound for these periods. For $0 \le j \le d$, define the *j-index* of $\mathcal{P}$ to be the minimal positive integer p_j such that the j-dimensional faces of $p_j\mathcal{P}$ all span affine subspaces that contain integer lattice points.

Theorem 1 (McMullen)**.** *Given a rational d-polytope $\mathcal{P}$, let p_j be the j-index of $\mathcal{P}$. If $L_{\mathcal{P}}(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$ is the Ehrhart quasi-polynomial of $\mathcal{P}$, then the minimum period of c_j divides p_j.*

Note that $p_d | p_{d-1} | \cdots | p_0$. Since p_0 is the denominator of $\mathcal{P}$, this is a stronger version of Ehrhart's theorem. If we further assume that $\mathcal{P}$ is full-dimensional, then $p_d = 1$, and so $c_d(k)$ is a constant function. In this case, it is well known that $c_d(k)$ is the Euclidean volume of $\mathcal{P}$ [1, 3].

These bounds on the periods seem tight for generic rational polytopes, that is, p_j is the minimum period of c_j, but this statement is ill-formed (we make no claim what notion of the term *generic* should be used here) and conjectural. One of the contributions of this paper is a step in the right direction: for any $p_d | p_{d-1} | \cdots | p_0$, there does indeed exist a polytope such that c_j has minimum period p_j.

Theorem 2. *Given distinct positive integers $p_d | p_{d-1} | \cdots | p_0$, the simplex*

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{p_0}, 0, \dots, 0\right), \left(0, \tfrac{1}{p_1}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{p_d}\right) \right\} \subset \mathbb{R}^{d+1}$$

has an Ehrhart quasi-polynomial $L_\Delta(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$, where c_j has minimum period p_j for $j = 0, 1, \dots, d$ (and p_j is the j-index of Δ).

Note that Δ is actually not a full-dimensional polytope; it is a d-dimensional polytope in $\mathbb{R}^{d+1}$. This allows us to state the theorem in slightly greater generality (we don't have to constrain $p_d = 1$, which is necessary for a full-dimensional polytope).

Theorem 2 complements recent literature [2, 6] that contains several special classes of polytopes that defy the expectation that c_j has minimum period p_j. De Loera–McAllister [2] constructed a family of polytopes stemming from representation theory that exhibit *period collapse*, i.e., the Ehrhart quasi-polynomials of these polytopes (which have arbitrarily large denominator) have minimum period 1—they are polynomials. McAllister–Woods [6] gave a class of polytopes whose Ehrhart quasi-polynomials have arbitrary period collapse (though not for the periods of the individual coefficient functions), as well as an example of non-monotonic minimum periods of the coefficient functions.

First, we will prove (in Section 2) that no period collapse is possible in the second leading coefficient $c_{d-1}(k)$:

Theorem 3. *Given a rational d-polytope $\mathcal{P}$, let p_{d-1} be the $(d-1)$-index of $\mathcal{P}$. Let $L_{\mathcal{P}}(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$. Then c_{d-1} has minimum period p_{d-1}.*

In Section 3, we give some general results on quasi-polynomials with maximal period behavior. Namely, we will prove:

Theorem 4. *Suppose $c(k)$ is a periodic function with minimum period n, and m is some nonnegative integer. Then the rational generating function $\sum_{k \ge 0} c(k) k^m x^k$ has as poles only n^{th} roots of unity, and each of these poles has order $m + 1$.*

A direct consequence of this statement is the following:

Corollary 5. *Suppose $r(x)$ is a rational function all of whose poles are primitive n^{th} roots of unity. Then r is the generating function of a quasi-polynomial*

$$r(x) = \sum_{k \geq 0} \Big(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \Big)\, x^k,$$

where each c_j is either identically zero or has minimum period n.

As an application to Theorem 2 (proved in Section 4), we turn to a question that stems from a recent theorem of Zaslavsky [9]. Suppose $A(k) = a_d(k)\, k^d + a_{d-1}(k)\, k^{d-1} + \cdots + a_0(k)$ and $B(k) = b_e(k)\, k^e + b_{e-1}(k)\, k^{e-1} + \cdots + b_0(k)$ are quasi-polynomials, where the minimum period of a_j is α_j and the minimum period of b_j is β_j. Then the *convolution*

$$C(k) := \sum_{m=0}^{k} A(k-m)\, B(m)$$

is another quasi-polynomial. If we write $C(k) = c_{d+e+1}(k)\, k^{d+e+1} + c_{d+e}(k)\, k^{d+e} + \cdots + c_0(k)$, and let c_j have minimum period γ_j, Zaslavsky proved the following result.

Theorem 6 (Zaslavsky)**.** *Define $g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : \ 0 \leq i \leq d, \ 0 \leq j-i \leq e\}$ for $j \geq 0$, and let $g_{-1} = 1$. Then*

$$\gamma_{j+1} \,|\, \operatorname{lcm} \{\alpha_{j+1}, \ldots, \alpha_d, \beta_{j+1}, \ldots, \beta_e, g_j\}\,. \tag{2}$$

We will reprove this result in Section 5 using the generating-function tools we develop. A natural problem, raised by Zaslavsky, is to construct two quasi-polynomials whose convolution satisfies (2) with equality. The answer is given by another application of Theorem 2 (Section 5).

Theorem 7. *Given $d \geq e$ and distinct positive integers $\alpha_d | \alpha_{d-1} | \cdots | \alpha_e | \beta_e | \alpha_{e-1} | \beta_{e-1} | \cdots | \alpha_0 | \beta_0$, let*

$$\Delta_1 = \operatorname{conv} \left\{ \left(\tfrac{1}{\alpha_0}, 0, \ldots, 0 \right), \left(0, \tfrac{1}{\alpha_1}, 0, \ldots, 0 \right), \ldots, \left(0, \ldots, 0, \tfrac{1}{\alpha_d} \right) \right\}$$

and

$$\Delta_2 = \operatorname{conv} \left\{ \left(\tfrac{1}{\beta_0}, 0, \ldots, 0 \right), \left(0, \tfrac{1}{\beta_1}, 0, \ldots, 0 \right), \ldots, \left(0, \ldots, 0, \tfrac{1}{\beta_e} \right) \right\}.$$

Then the convolution of L_{Δ_1} and L_{Δ_2} satisfies (2) *with equality.*

2. The Second Leading Coefficient of an Ehrhart Quasi-Polynomial

In this section we prove Theorem 3, namely the minimum period of the second leading coefficient of the Ehrhart quasi-polynomial of a rational d-polytope $\mathcal{P}$ equals the $(d-1)$-index of $\mathcal{P}$. Most of the work towards Theorem 3 is contained in the proof of the following result.

Proposition 8. *If $\mathcal{P}$ is a rational d-polytope with Ehrhart quasi-polynomial $L_\mathcal{P}(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$, then c_{d-1} is constant if and only if the $(d-1)$-index of $\mathcal{P}$ is* 1*.*

Proof. If the $(d-1)$-index of $\mathcal{P}$ is 1, then c_{d-1} is constant by McMullen's Theorem 1.

For the converse implication, we use the *Ehrhart-Macdonald Reciprocity Theorem* [1, 5]. It says that for a rational d-polytope $\mathcal{P}$, the evaluation of $L_\mathcal{P}$ at negative integers yields the lattice-point enumerator of the interior $\mathcal{P}^\circ$, namely,

$$L_\mathcal{P}(-k) = (-1)^d L_{\mathcal{P}^\circ}(k)\,.$$

This identity implies that the lattice-point enumerator for the boundary of $\mathcal{P}$ is the quasi-polynomial $L_{\partial\mathcal{P}}(k) = L_\mathcal{P}(k) - (-1)^d L_\mathcal{P}(-k)$. Since $L_{\partial\mathcal{P}}(k)$ counts integer points in a $(d-1)$-dimensional object, it is a degree $d-1$ quasi-polynomial, and we see that its leading coefficient is $c_{d-1}(k) + c_{d-1}(-k)$.

Now suppose the $(d-1)$-index of $\mathcal{P}$ is $m > 1$. Then there is a facet of $\mathcal{P}$ whose affine span has no lattice points when dilated by $1, m+1, 2m+1, \dots$ On the other hand, the affine span of *any* facet contains lattice points when dilated by multiples of m. The leading coefficient of the quasi-polynomial $L_{\partial\mathcal{P}}(mk)$ is a constant measuring the volume of the facets relative to the sublattices in their affine spans (for the same reason that the leading coefficient of $L_{\mathcal{P}}(k)$ measures the volume of $\mathcal{P}$: asymptotically, the volume is a good approximation for the number of integer points). The same can be said for the leading coefficient of the polynomial $L_{\partial\mathcal{P}}(1+mk)$; however, now some of these sublattices are empty, and thus the leading coefficient of $L_{\partial\mathcal{P}}(1+mk)$ is smaller than the leading coefficient of $L_{\partial\mathcal{P}}(mk)$, i.e., $c_{d-1}(1+mk)+c_{d-1}(-1-mk) < c_{d-1}(mk)+c_{d-1}(-mk)$. Hence c_{d-1} is not constant. □

Proof of Theorem 3. Let p be the minimal period of c_{d-1} and q be the $(d-1)$-index of $\mathcal{P}$. By McMullen's Theorem 1, $p|q$. On the other hand, the second-leading coefficient of $L_{p\mathcal{P}}$ is constant, and by Proposition 8, the $(d-1)$-index of $p\mathcal{P}$ is 1, which implies $q|p$. □

3. Some General Results on Quasi-Polynomial Periods

A key ingredient to proving Theorem 4 is a basic result (see, e.g., [1, Chapter 3] or [8, Chapter 4]) about a quasi-polynomial $q(k)$ and its generating function $r(x) = \sum_{k \ge 0} q(k) x^k$, which is easily seen to be a rational function.

Lemma 9. *Suppose q is a quasi-polynomial with rational generating function $r(x) = \sum_{k\ge 0} q(k)\, x^k$. Then n is a period of q and q has degree d if and only if all poles of r are n^{th} roots of unity of order $\le d+1$ and there is a pole of order $d+1$.*

The above result will be useful again in the proof of Theorem 2. Recall that the statement of Theorem 4 is that given a periodic function $c(k)$ with minimum period n and a nonnegative integer m, the only poles of the rational generating function $\sum_{k\ge 0} c(k) k^m x^k$ are n^{th} roots of unity, and each pole has order $m+1$.

Proof of Theorem 4. We use induction on m. The case $m = 0$ follows directly from Lemma 9, as

$$\sum_{k\ge 0} c(k) k^0 x^k = \frac{c(0) + c(1)x + \cdots + c(n-1)x^{n-1}}{1 - x^n} \,.$$

The induction step is a consequence of the identity

$$\sum_{k\ge 0} c(k) k^m x^k = x \, \frac{d}{dx} \sum_{k \ge 0} c(k) k^{m-1} x^k$$

and the fact that a pole of order m turns into a pole of order $m+1$ under differentiation. □

Corollary 5 now follows like a breeze. Recall its statement: If $r(x)$ is a rational function all of whose poles are primitive n^{th} roots of unity, then r is the generating function of a quasi-polynomial

$$r(x) = \sum_{k\ge 0} \Big(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \Big)\, x^k,$$

where each $c_j \not\equiv 0$ has minimum period n.

Proof of Corollary 5. Consider the rational generating functions

$$r_j(x) := \sum_{k \ge 0} c_j(k) k^j x^k, \qquad \text{so that} \qquad r(x) = r_d(x) + r_{d-1}(x) + \cdots + r_0(x) \,.$$

We claim that the poles of each (not identically zero) $r_j(x)$ are all primitive n^{th} roots of unity. Indeed, suppose not, and consider the largest j such that $r_j(x)$ has a pole ω which is not a primitive n^{th} root of unity. Theorem 4 says that ω is a pole of $r_j(x)$ of order $j+1$. Since ω is not a pole of $r_d(x), r_{d-1}(x), \ldots, r_{j+1}(x)$ (we chose j as large as possible), ω is a pole of

$$r_d(x) + r_{d-1}(x) + \cdots + r_{j+1}(x) + r_j(x)$$

of order $j+1$. On the other hand, Theorem 4 also implies that $r_{j-1}(x), r_{j-2}(x), \ldots, r_0(x)$ have no poles of order greater than j. Summing over all the r_i, ω must be a pole of $r(x)$ of order $j+1$, contradicting that fact that $r(x)$ has only poles that are primitive n^{th} roots of unity.

Therefore the poles of each (not identically zero) $r_j(x)$ are all primitive roots of unity. Lemma 9 implies that n is a period of each nonzero $r_j(x)$, and Theorem 4 implies that n is the minimum period, proving the corollary. □

4. Ehrhart Quasi-Polynomials with Maximal Periods

Recall that Theorem 2 says that for given distinct positive integers $p_d | p_{d-1} | \cdots | p_0$, the simplex

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{p_0}, 0, \ldots, 0\right), \left(0, \tfrac{1}{p_1}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \tfrac{1}{p_d}\right) \right\} \subset \mathbb{R}^{d+1}$$

has an Ehrhart polynomial $L_\Delta(k) = c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$, where c_j has minimum period p_j for $j = 0, 1, \ldots, d$. Note that p_j is the j-index of Δ.

Proof of Theorem 2. The Ehrhart series of

$$\Delta = \left\{ (x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1} : \, p_0 x_0 + p_1 x_1 + \cdots + p_d x_d = 1 \right\}$$

is, by construction,

$$\operatorname{Ehr}_\Delta(x) := \sum_{k \geq 0} L_\Delta(k)\, x^k = \frac{1}{(1 - x^{p_0})(1 - x^{p_1}) \cdots (1 - x^{p_d})} \, .$$

Given j, let ω be a primitive p_j^{th} root of unity. Then ω is a pole of $\operatorname{Ehr}_\Delta(x)$ of order $j+1$. We expand $\operatorname{Ehr}_\Delta(x)$ to yield the Ehrhart quasi-polynomial:

$$\operatorname{Ehr}_\Delta(x) = \sum_{k \geq 0} L_\Delta(k)\, x^k = \sum_{k \geq 0} \left(c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k) \right) x^k .$$

Let n be the minimum period of $c_j(k)$. By McMullen's Theorem 1, $n | p_j$. Therefore, we need to show that $p_j | n$. As before, let $r_j(x) = \sum_{k \geq 0} c_j(k) k^j x^k$, so that $\operatorname{Ehr}_\Delta(x) = r_d(x) + r_{d-1}(x) + \cdots + r_0(x)$. Since ω is a pole of $\operatorname{Ehr}_\Delta(x)$, it must be a pole of (at least) one of $r_d, \ldots, r_0$. Let J be the largest index such that ω is a pole of $r_J(x)$. By Theorem 4, ω is a pole of $r_J(x)$ of order $J+1$. Since ω is not a pole of $r_d(x), r_{d-1}(x), \ldots, r_{J+1}(x)$, ω is a pole of

$$r_d(x) + r_{d-1}(x) + \cdots + r_{J+1}(x) + r_J(x)$$

of order $J+1$. On the other hand, Theorem 4 also implies that $r_{J-1}(x), r_{J-2}(x), \ldots, r_0(x)$ have no poles of order greater than J. Summing over all the r_i, ω must be a pole of $\operatorname{Ehr}_\Delta(x)$ of order $J+1$. Since we saw that ω is a pole of $\operatorname{Ehr}_\Delta(x)$ of order $j+1$, we have that $J = j$, that is, ω is a pole of $r_j(x)$. Since ω is a primitive p_j^{th} root of unity, Theorem 4 says that p_j must divide the minimum period n, and so $n = p_j$, as desired. □

5. Quasi-Polynomial Convolution with Maximal Periods

We start our last section with a generating-function proof of Zaslavsky's Theorem 6. It uses the following generalization of Lemma 9:

Lemma 10. *Suppose $q(k) = c_d(k)\,k^d + c_{d-1}(k)\,k^{d-1} + \cdots + c_0(k)$ is a quasi-polynomial with rational generating function $r(x) = \sum_{k \ge 0} q(k)\,x^k$.*

(a) *If n is a period of c_j, then there is an n^{th} root of unity that is a pole of r of order at least $j+1$.*
(b) *If all poles of r of order $\ge j+1$ are n^{th} roots of unity, then n is a period of c_j.*

Proof. Part (a) follows from Theorem 4.

For part (b), expand r (crudely) into partial fractions as $r(x) = s(x) + t(x)$, such that s has as poles the poles of r of order $\ge j+1$ and t has as poles those of order $\le j$. Now apply Lemma 9 to s and note that t does not contribute to c_j. □

Proof of Theorem 6. Let $f_A(x) = \sum_{k\ge 0} A(k)\,x^k$ and define f_B and f_C analogously. To determine γ_{j+1}, the period of c_{j+1}, Lemma 10(b) tells us that we need to consider the poles of $f_C(x) = f_A(x) f_B(x)$ of order $\ge j+2$. These poles come in three types:

(1) poles of f_A of order $\ge j+2$;
(2) poles of f_B of order $\ge j+2$;
(3) common poles of f_A and f_B whose orders add up to at least $j+2$.

Lemma 10(a) gives the statement of Theorem 6 instantly; the periods $\alpha_{j+1}, \dots, \alpha_d$ give rise to poles of type (1), $\beta_{j+1}, \dots, \beta_e$ give rise to poles of type (2), and $g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : 0 \le i \le d,\ 0 \le j-i \le e\}$ stems from poles of the type (3). □

Proof of Theorem 7. The convolution of L_{Δ_1} and L_{Δ_2} equals L_Δ, where Δ is the $(d+e+1)$-simplex

$$\Delta = \operatorname{conv}\left\{ \left(\tfrac{1}{\alpha_0}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{\alpha_d}, 0, \dots, 0\right), \left(0, \dots, 0, \tfrac{1}{\beta_0}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \tfrac{1}{\beta_e}\right) \right\},$$

which follows directly from the fact that the generating function of the convolution of two quasipolynomials is the product of their generating functions. Let $L_\Delta(k) = c_{d+e+1}(k)\,k^{d+e+1} + c_{d+e}(k)\,k^{d+e} + \cdots + c_0(k)$ and suppose $c_j(k)$ has minimum period γ_j. By construction and Theorem 2, we have

$$\gamma_{2j} = \beta_j \qquad \text{and} \qquad \gamma_{2j+1} = \alpha_j \qquad \text{for } 0 \le j \le e\,,$$

and $\gamma_{e+j+1} = \alpha_j$ for $j > e$. We will show that these values agree with the upper bounds given by Zaslavsky's Theorem 6. We distinguish three cases.

Case 1: $j \le 2e$ and $j+1 = 2m$ for some integer m. We need to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \dots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \dots, \beta_e, g_j\} = \beta_m\,. \tag{3}$$

Consider

$$g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : 0 \le i \le d,\, 0 \le j-i \le e\}\,.$$

If $2i \ge j$, i.e., $i \ge m$, then $\gcd(\alpha_i, \beta_{j-i}) = \beta_{j-i}$. Thus

$$g_j = \operatorname{lcm}\{\alpha_j, \alpha_{j-1}, \dots, \alpha_{m+1}, \beta_m, \beta_{m+1}, \dots, \beta_j\} = \beta_m\,,$$

which proves (3), since $j+1 > m$.

Case 2: $j \le 2e$ and $j = 2m$ for some integer m. We need to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \dots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \dots, \beta_e, g_j\} = \alpha_m\,. \tag{4}$$

Now

$$g_j = \operatorname{lcm}\{\alpha_j, \alpha_{j-1}, \dots, \alpha_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_j\} = \alpha_m\,,$$

which proves (4), since $j+1>m$.

Case 3: $j>2e$. We would like to show that

$$\gamma_{j+1} = \operatorname{lcm}\{\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_d, \beta_{j+1}, \beta_{j+2}, \ldots, \beta_e, g_j\} = \alpha_{j-e}\,. \tag{5}$$

Here

$$g_j = \operatorname{lcm}\{\gcd(\alpha_i, \beta_{j-i}) : j-e \le i \le j\}\,.$$

However, for $j-e \le i \le j$, we have $\gcd(a_i, \beta_{j-i}) = \alpha_i$, whence $g_j = \alpha_{j-e}$, which proves (5). □

6. Open Problems

(1) For an Ehrhart quasi-polynomial, period collapse cannot happen in relation to the j-index for the first two coefficients. On the other side, McAllister–Woods [6] showed that period collapse can happen for any other coefficient, however, it is still a mystery to what extent. For example, can one construct polygons whose Ehrhart periods are $(1,s,t)$ (the minimum periods of $c_2(x)$, $c_1(x)$, and $c_0(x)$, respectively), for given s and t? Even simple examples are not easy to come by, e.g., we could construct polygons with period sequence $(1,s,1)$ for $1 \le s \le 5$ but have not been successful for $s=6$.

(2) In constructing the simplex with maximal period behavior, we required that the integers $p_0, \ldots, p_d$ be distinct, but perhaps this restriction is not necessary. Does the statement still hold true if we weaken the conditions, or do there exist counterexamples?

(3) In the example of periods of quasi-polynomial convolution, Theorem 7, our methods require that we assume that $\alpha_d|\alpha_{d-1}|\cdots|\alpha_e|\beta_e|\alpha_{e-1}|\beta_{e-1}|\cdots|\alpha_0|\beta_0$, rather than the more natural $\alpha_d|\alpha_{d-1}|\cdots|\alpha_0$ and $\beta_e|\beta_{e-1}|\cdots|\beta_0$. We conjecture that the theorem is still true in this case.

(4) More generally, this would follow from a conjecture about a special class of generating functions:

Conjecture 11. *Let $a_1, a_2, \ldots, a_n$ be given positive integers. Let $q(k) = c_d(k)\,k^d + \cdots + c_0(k)$ be the quasi-polynomial whose generating function $r(x) = \sum_{k\ge 0} q(k)x^k$ is given by*

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_n})}\,.$$

For a positive integer m, define $b_m = \#\{i : m \mid a_i\}$. For $0 \le j \le d$, let $p_j = \operatorname{lcm}\{m : b_m > j\}$. Then the minimum period of $c_j(k)$ is p_j.

(5) There are several multi-parameter versions of Ehrhart polynomials to which a generalization of McMullen's Theorem 1 applies. Beyond McMullen's theorem, not much is know about periods and minimum periods (which are now lattices in some $\mathbb{Z}^m$) of these multivariate quasi-polynomials and coefficient functions.

References

1. Matthias Beck and Sinai Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2007.
2. Jesús A. De Loera and Tyrrell B. McAllister, *Vertices of Gelfand-Tsetlin polytopes*, Discrete Comput. Geom. **32** (2004), no. 4, 459–470.
3. Eugène Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618.
4. Petr Lisoněk, *Combinatorial families enumerated by quasi-polynomials*, Preprint (2006), to appear in *J. Combin. Theory Ser. A*.
5. Ian G. Macdonald, *Polynomials associated with finite cell-complexes*, J. London Math. Soc. (2) **4** (1971), 181–192.
6. Tyrrell B. McAllister and Kevin M. Woods, *The minimum period of the Ehrhart quasi-polynomial of a rational polytope*, J. Combin. Theory Ser. A **109** (2005), no. 2, 345–352, `arXiv:math.CO/0310255`.

7. Peter McMullen, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel) **31** (1978/79), no. 5, 509–516.
8. Richard P. Stanley, *Enumerative Combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.
9. Thomas Zaslavsky, *Periodicity in quasipolynomial convolution*, Electron. J. Combin. **11** (2004), no. 2, Research Paper 11, 6 pp. (electronic).

Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA
E-mail address: beck@math.sfsu.edu
URL: http://math.sfsu.edu/beck

Department of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: ssam@berkeley.edu

Department of Mathematics, Oberlin College, Oberlin, OH 44074, USA
E-mail address: kevin.woods@oberlin.edu
URL: http://www.oberlin.edu/math/faculty/woods.html