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Polyhedral Subdivisions and the Newton Polytope of the Product of all Minors of an $m \times n$ Matrix

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Abstract

We consider the Newton polytope $\Sigma(m,n)$ of the product of all minors of an $m \times n$ matrix of indeterminates. Using the fact that this polytope is the secondary polytope of the product $\Delta_{m-1} \times \Delta_{n-1}$ of simplices, and thus has faces corresponding to regular polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$, we study facets of $\Sigma(m,n)$, which correspond to the coarsest, nontrivial such subdivisions. We make use of the relation between secondary and fiber polytopes, which in this case gives a representation of $\Sigma(m,n)$ as the Minkowski average of all $m \times n$ transportation polytopes.

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1. Newton, secondary and fiber polytopes.

Given an $m \times n$ matrix $A = (a_{ij})$ of indeterminates, we consider the polytope $\Sigma(m,n) \subset \mathbb{R}^{mn}$ defined as the *Newton polytope* of the product of *all* minors of A, that is, the convex hull of all exponent vectors obtained when one expands this product as a polynomial in the a_{ij}. For example, when $m = n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and $\Sigma(2,2)$ is the line segment joining $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ in $\mathbb{R}^4$.

We address here the problem of determining the polytope $\Sigma(m,n)$ for general m and n. Determining a polytope can be accomplished in several different ways, for example, by giving a description of it set of vertices, or by determining a minimal set of linear inequalities which define it. The latter amounts to describing all its faces of codimension 1, that is, its *facets*. This is the point of view we adopt in this paper. We begin a description of $\Sigma(m,n)$ by describing some classes of facets and giving a means to determine the linear inequalities to which they correspond.

Our approach to $\Sigma(m,n)$ depends on the fact that it is equal to the *secondary polytope* $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$ of the product $\Delta_{m-1} \times \Delta_{n-1}$ [GZK;§3E.3]. (Here Δ_k denotes the standard k-simplex, defined as the convex hull of the $k+1$ unit vectors in $\mathbb{R}^k$.) Thus, the lattice of faces of $\Sigma(m,n)$ is isomorphic to the poset of all regular polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$ that add no new vertices. (A subdivision is *regular* if it supports a strictly convex piecewise linear function, where *strictly* convex means there is a different linear function on each maximal cell of the subdivision. See [GZK] or [BFS] for a discussion of secondary polytopes and regular subdivisions.) Vertices of $\Sigma(m,n)$ correspond to the regular triangulations of $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, while facets correspond to the coarsest possible regular subdivisions.

So to study faces of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, it is sufficient to study the structure of the associated regular subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. To do this, we make use of the fact that secondary polytopes are (up to scalar multiple) equal to fiber polytopes of projections of simplices, in this case, the projection of the $(mn-1)$-simplex onto $\Delta_{m-1} \times \Delta_{n-1}$.

We begin with some definitions. If $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^d$ are convex polytopes, and $\pi : \mathbb{R}^n \to \mathbb{R}^d$ is a linear map with $\pi(P) = Q$, then we define the *fiber polytope* $\Sigma(P,Q)$ to be the Minkowski average of all fibers $\pi^{-1}(q)$ over $q \in Q$, i.e.,

$$\Sigma(P,Q) = \frac{1}{vol\, Q} \int_Q \pi^{-1}(q)\; dq.$$

Here the set-valued integral can be defined as the set of integrals of all sections $\gamma : Q \to P$ of π (i.e., $\pi \circ \gamma$ is the identity on Q). Alternatively, it can be defined as a Riemann-type limit of Minkowski sums of sets $\pi^{-1}(q)$ or as the convex set whose support function is pointwise the integral, over q, of the support functions of the sets $\pi^{-1}(q)$. See [FP], where it is shown that $\Sigma(P,Q) \subset \mathbf{R}^n$ is a polytope of dimension $dim\, P - dim\, Q$, whose face lattice is isomorphic to a poset of polyhedral subdivisions of Q (the *P-coherent* subdivisions). In particular, when P is a simplex, say $P = \Delta_{n-1}$, the fiber polytope $\Sigma(P,Q)$ is (up to scaling) the secondary polytope $\Sigma(Q)$ (or, more precisely, the secondary polytope $\Sigma(\mathcal{A})$, where $\mathcal{A}$ is the set of images under π of the vertices of P).

In the case of $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, the corresponding map $\pi : \Delta_{mn-1} \to \Delta_{m-1} \times \Delta_{n-1}$ has as fibers all $m \times n$ ***transportation polytopes***, i.e., polytopes of nonnegative $m \times n$ matrices with prescribed row and column sums. Thus the study of the Newton polytope $\Sigma(m,n)$ is equivalent to the study of the average transportation polytope. From this we conclude, for example, that $\Sigma(m,n)$ has dimension $(m-1)(n-1)$.

In general, the fiber polytope is a subset of the fiber $\pi^{-1}(x_Q)$ over the centroid of Q, and the exact relationship between secondary and fiber polytopes is given by

$$\Sigma(Q) = (dim\, Q + 1)\; vol(Q)\; \Sigma(\Delta_{n-1}, Q).$$

Thus $\Sigma(\Delta_{n-1}, \Delta_{m-1} \times \Delta_{n-1})$ consists of $m \times n$ matrices with row sums $\frac{1}{m}$ and column sums $\frac{1}{n}$. In order to make the identification between the Newton polytope $\Sigma(m,n)$ and the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, we normalize volume so that $vol(\Delta_{m-1} \times \Delta_{n-1}) = \binom{m+n-2}{m-1}$. (All simplices in $\Delta_{m-1} \times \Delta_{n-1}$ have the same volume, which we take to be 1; see §5). Thus, $\Sigma(m,n)$ consists of matrices with row sums $\binom{m+n-1}{m}$ and column sums $\binom{m+n-1}{n}$. Thus when $m = n = 2$, the row and column sums must be 3, as seen above.

In §2, we consider the general case of fiber polytopes and give a concrete description of the isomorphism between faces and coherent polyhedral subdivisions. This is specialized in §3 to the case of faces of $\Sigma(m,n)$ and regular polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. In particular, for each $\Theta \in \mathbb{R}^{mn}$, we give a description of the subdivision Π_Θ corresponding to the face of $\Sigma(m,n)$ having outward normal Θ. If Θ is a 0–1-matrix, the maximal cells of Π_Θ can be read directly from the minimal line covers of the $1's$ of Θ (sets of rows and columns including all the $1's$). In some cases the resulting subdivision can be shown to be coarsest possible, and so the corresponding Θ will be normal to a facet of $\Sigma(m,n)$.

The notion of an indecomposable weighting of a bipartite graph is defined in §4, where it is used to characterize facet normals, as well as to give a lower bound on the codimension of certain faces of $\Sigma(m,n)$. In §5, formulas for the support function are derived for fiber and secondary polytopes. In the case of $\Sigma(m,n)$, the value of the support function h_{mn} at a 0–1 matrix Θ can be expressed in terms of the minimal line covers of Θ. This calculation is carried out in §6 for some special Θ and for small values of m and n.

Finally, some notational conventions. For P a convex polyhedron in $\mathbb{R}^n$ and $\theta \in \mathbb{R}^n$, we denote by P^θ and P_θ the faces of P where the linear form $\langle \theta, \cdot \rangle$ achieves its maximum and minimum, respectively. Note that $P^{-\theta} = P_\theta$. For a set $X \subset \mathbb{R}^n$, we denote by *cone* X (resp. *pos* X) the set of all nonnegative (resp. positive) linear combinations of the points in X. We will denote the ith row and the jth column of a matrix B by B_i and B^j, respectively. The set $\{1, \ldots, n\}$ will be denoted $[n]$.

2. Coherent subdivisions and faces of $\Sigma(P, Q)$.

We consider first the general fiber polytope $\Sigma(P, Q)$, where $P = conv\{p_1, \ldots, p_m\} \subset \mathbb{R}^n$, $Q = conv\{q_1, \ldots, q_m\} \subset \mathbb{R}^d$ and $\pi : P \to Q$ a linear map such that $\pi(p_i) = q_i$. By Theorem 2.1 of [FP], each face $\Sigma(P, Q)^\theta$ of $\Sigma(P, Q)$ corresponds to a certain (coherent) polyhedral subdivision Π_θ. We describe this correspondence in this section and specialize it to obtain the correspondence between faces of the secondary polytope $\Sigma(Q)$ and regular subdivisions of Q. As in [BFS], [FP] and [GZK], by a *polyhedral subdivision of* Q we mean a collection Π of subsets of $\{q_1, \ldots, q_m\}$ whose convex hulls form a polyhedral complex that covers Q.

The fiber $\pi^{-1}(q)$ over a point $q \in Q$ can be written as

$$\pi^{-1}(q) = \{ \, p \in \mathbb{R}^n \mid p = \sum_{i=1}^{m} \lambda_i p_i, \; \sum \lambda_i q_i = q, \; \sum \lambda_i = 1, \; \lambda_i \geq 0 \, \}. \tag{2.1}$$

Let $(y_0, y) = (y_0, y_1, \ldots, y_d)$ denote a point in $\mathbb{R}^{d+1}$. For $\theta \in \mathbb{R}^n$ define the polyhedron

$$S(\theta) = \{ \, (y_0, y) \in \mathbb{R}^{d+1} \mid y_0 + \langle y, q_i \rangle \geq \langle \theta, p_i \rangle, \; i = 1, \ldots, m \, \}. \tag{2.2}$$

For $y \in S(\theta)$, define

$$\sigma_y = \{ \, q_i \mid y_0 + \langle y, q_i \rangle = \langle \theta, p_i \rangle \, \}. \tag{2.3}$$

The following gives a concrete description of the lattice isomorphism given by Theorem 2.1 of [FP].

Theorem 2.1. *In the correspondence between faces of the fiber polytope $\Sigma(P, Q)$ and polyhedral subdivisions of Q, the face $\Sigma(P, Q)^\theta$ corresponds to the coherent subdivision*

$$\Pi_\theta = \{ \, \sigma_y \mid y \in S(\theta) \, \}.$$

Proof: By general linear programming duality we can write

$$max\{ \, \langle \theta, p \rangle \mid p \in \pi^{-1}(q) \, \} = min\{ \, y_0 + \langle y, q \rangle \mid (y_0, y) \in S(\theta) \, \}. \tag{2.4}$$

Since $max\{ \langle\theta, p\rangle \mid p \in \pi^{-1}(q) \}$ is finite precisely when $q \in Q$, we have that $(1, q)$ is an element of the relatively open inner normal cone $N(S(\theta), y)$ to the polyhedron $S(\theta)$ at some point y if and only if $q \in Q$. In fact, for $y \in relint\, S(\theta)_{(1,q)}$, we have

$$(1, q) \in N(S(\theta), y) = pos\{ (1, z) \mid z \in \sigma_y \}.$$

Thus $\Pi := \{ \sigma_y \mid y \in S(\theta) \}$ is a polyhedral subdivision of Q; it is given by the intersection of the (sets of generators of the cones in the) normal fan of $S(\theta)$ with the hyperplane $y_0 = 1$ in $\mathbb{R}^{d+1}$.

By Theorem 2.1 of [FP], $\sigma \in \Pi_\theta$ if and only if for $q \in conv\, \sigma$, the maximum value of $\langle\theta, p\rangle$ over the fiber $\pi^{-1}(q)$ is attained on the face

$$\pi^{-1}(\sigma) := conv\{ p_i \mid q_i = \pi(p_i) \in \sigma \} \tag{2.5}$$

of P. That is,

$$\pi^{-1}(q)^\theta = \pi^{-1}(\sigma) \cap \pi^{-1}(q). \tag{2.6}$$

For $q \in relint\, conv\, \sigma$, the face $\pi^{-1}(q)^\theta$ contains a point $p \in relint\, \pi^{-1}(\sigma)$, and so

$$max\{ \langle\theta, p\rangle \mid p \in \pi^{-1}(q) \}$$

is attained at a point $p = \sum \lambda_i p_i$ for which $\lambda_i > 0$ whenever $q_i \in \sigma$. Thus for any (y_0, y) solving

$$min\{ y_0 + \langle y, q\rangle \mid (y_0, y) \in S(\theta) \},$$

$q_i \in \sigma$ implies $y_0 + \langle y, q_i\rangle = \langle\theta, p_i\rangle$, and so $\sigma \subset \sigma_y$. Thus Π_θ refines Π.

On the other hand, any $p = \sum \lambda_i p_i$ solving $max\{ \langle\theta, p\rangle \mid p \in \pi^{-1}(q) \}$ must satisfy $\lambda_i = 0$ when $q_i \notin \sigma$ and so, by strict complementarity, there is a $(y_0, y) \in S(\theta)_{(1,q)}$ with $y_0 + \langle y, q_i\rangle > \langle\theta, p_i\rangle$ whenever $q_i \notin \sigma$. Thus $\sigma_y \subset \sigma$, and we conclude $\Pi_\theta = \Pi$. ◁

One way to understand this association $\theta \mapsto \Pi_\theta$ is to observe that, for fixed θ, the maximum in (2.4) varies piecewise-linearly in q. The subdivision Π_θ gives the associated regions of linearity. A consequence of the proof of Theorem 2.1 is that the lattice of faces of the subdivision Π_θ is the lattice of faces of the normal fan to the (necessarily unbounded) polyhedron $S(\theta)$.

Corollary 2.2. *The lattice of faces of the subdivision Π_θ is anti-isomorphic to the lattice of faces of the polyhedron $S(\theta)$. In particular, the maximal cells of the subdivision Π_θ correspond to the minimal faces of $S(\theta)$.* ◁

In the case of the secondary polytope of Q, we have $m = n$ and $P = \Delta_{n-1}$, and so (2.1), (2.2) and (2.3) simplify, respectively, to $\pi^{-1}(q) = \{ \lambda \in \mathbb{R}^n \mid \sum \lambda_i q_i = q \}$, the set of all convex representations of q, $S(\theta) = \{ (y_0, y) \in \mathbb{R}^{d+1} \mid y_0 + \langle y, q_i \rangle \geq \theta_i, \ i = 1, \dots, m \}$, the set of all downward pointing normals to the convex hull of the points $(1, q_i, \theta_i)$, and $\sigma_y = \{ q_i \mid y_0 + \langle y, q_i \rangle = \theta_i \}$, the set of points on some bottom face of this convex hull.

3. Regular subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$ and faces of $\Sigma(m,n)$.

We consider here the case of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, which corresponds to the special case of the situation described in §2 in which P is the $(mn-1)$-simplex Δ_{mn-1} and $Q = \Delta_{m-1} \times \Delta_{n-1}$. As in the general case, we make more explicit the association between faces and subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. For faces defined by 0–1 normals, this association leads to consideration of the classical combinatorial notion of a line cover of a 0–1 matrix.

We define the map π in this case as follows. We consider elements of $\mathbb{R}^{mn}$ to be $m \times n$ matrices; unit vectors are then the 0–1 matrices E_{ij} having a single 1 in the i^{th} row and j^{th} column. Then

$$\begin{aligned} \Delta_{mn-1} &= \{X \in \mathbb{R}^{mn} \mid X = (x_{ij}) \geq 0, \ \sum_{ij} x_{ij} = 1 \ \} \\ &= conv\{ \ E_{ij} \mid i \in [m], \ j \in [n] \ \}. \end{aligned}$$

For $X \in \mathbb{R}^{mn}$, let $r(X) \in \mathbb{R}^m$ and $c(X) \in \mathbb{R}^n$ denote the vectors of row sums and column sums of X. The map π is then defined by $\pi(X) = (r(X), c(X))$. We define $v_{ij} := \pi(E_{ij}) = (e_i, e_j)$, where e_i and e_j are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Thus

$$\Delta_{m-1} \times \Delta_{n-1} = conv\{ \ v_{ij} \mid i \in [m], \ j \in [n] \ \}.$$

For $(a,b) \in \Delta_{m-1} \times \Delta_{n-1}$, the fiber

$$\pi^{-1}(a,b) = \{ \ X \in \mathbb{R}^{mn} \mid X \geq 0, \ r(X) = a, \ c(X) = b \ \}$$

has been studied in the optimization literature under the name *transportation polytope*; see [KW] and [YKK]. Note that in this case, the constraint $\sum_{ij} x_{ij} = 1$ in (2.1) is implied by $r(X) = a$. As in §2, we observe that for $\Theta = (\theta_{ij}) \in \mathbb{R}^{mn}$,

$$\begin{aligned} max\{ \ \langle \Theta, X \rangle \mid X \in \pi^{-1}(a,b) \ \} &= max\{ \ \langle \Theta, X \rangle \mid X \geq 0, \ r(X) = a, \ c(X) = b \ \} \\ &= min\{ \ \langle u, a \rangle + \langle v, b \rangle \mid u_i + v_j \geq \theta_{ij} \ \}. \end{aligned} \tag{3.1}$$

If we let

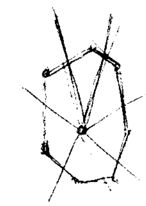
$$S(\Theta) = \{ \ (u,v) \in \mathbb{R}^m \times \mathbb{R}^n \mid u_i + v_j \geq \theta_{ij} \ \} \tag{3.2}$$

and define, for $(u,v) \in S(\Theta)$,

$$\sigma_{uv} = \{ \ v_{ij} \mid u_i + v_j = \theta_{ij} \ \}, \tag{3.3}$$

interpreting Theorem 2.1 and Corollary 2.2 in this case leads to the following

Theorem 3.1. *For any $\Theta \in \mathbb{R}^{mn}$, the face $\Sigma(m,n)^\Theta$ of the secondary polytope $\Sigma(m,n) = \Sigma(\Delta_{m-1} \times \Delta_{n-1})$ corresponds to the regular subdivision $\Pi_\Theta = \{\ \sigma_{uv} \mid (u,v) \in S(\Theta)\ \}$. The lattice of faces of Π_Θ is anti-isomorphic to the lattice of faces of the polyhedron $S(\Theta)$, and so Π_Θ is a triangulation precisely when $S(\Theta)$ is simple.* $\triangleleft$



Note that in this case, $S(\Theta)$ is simple if every 1-face is on precisely $m+n-1$ facets. $S(\Theta)$ has no vertices so a maximal cell of Π_Θ is of the form σ_{uv}, where (u,v) lies on a 1-face.

We consider next subdivisions Π_Θ where $\Theta = (\theta_{ij})$ is a 0–1 matrix. In this case, we define a *line cover* of Θ to be a pair (I,J), with $I \subset [m], J \subset [n]$ such that for all (i,j) with $\theta_{ij} = 1$, either $i \in I$ or $j \in J$. A line cover (I,J) is said to be *minimal* if both I and J are minimal. Note that line covers of Θ correspond directly to 0–1 vectors in the polyhedron $S(\Theta)$.

Given 0–1 matrix Θ and line cover (I,J), we say θ_{ij} is *exactly covered* by (I,J) if $\theta_{ij} = 1$ and $(i,j) \notin I \times J$, or $\theta_{ij} = 0$ and $(i,j) \in ([m] \setminus I) \times ([n] \setminus J)$. Define

$$\sigma_{IJ} = \{\ v_{ij} \mid \theta_{ij} \text{ exactly covered by } (I,J)\ \}. \tag{3.4}$$

Corollary 3.2. *For 0–1 matrices Θ, the maximal cells of the subdivision Π_Θ are precisely the cells σ_{IJ} where (I,J) is a minimal line cover of Θ with $J \neq [n]$.*

Proof: We first show for every $(u,v) \in S(\Theta)$ there is a 0–1 vector $(\bar{u},\bar{v}) \in S(\Theta)$ with $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$. We can assume that both $u \geq 0$ and $v \geq 0$ since, for example, if $u_1 = min\{u_i, v_j\} < 0$, then $u' = u - u_1 e \geq 0$, $v' = v + u_1 e \geq 0$ and $\sigma_{u'v'} = \sigma_{uv}$. In this case, set $\bar{u}_i = [u_i \geq \frac{1}{2}]$ and $\bar{v}_j = [v_j > \frac{1}{2}]$ for all i and j, where $[R]$ is 1 or 0 depending on whether the relation R is true or false. Then $u_i + v_j = 0$ implies $\bar{u}_i + \bar{v}_j = 0$, and $u_i + v_j = 1$ implies $\bar{u}_i + \bar{v}_j = 1$, and so $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$.

Since always $\sigma_{\emptyset\,[n]} \subset \sigma_{[m]\,\emptyset}$, there is no loss in eliminating the former from consideration as a maximal cell. Since the lineality space of the polyhedron $S(\Theta)$ is generated by $(u,v) = (1,\ldots,1,-1,\ldots,-1)$, there can be at most one 0–1 point (u,v) on a 1-face of $S(\Theta)$ unless either $u = (1,\ldots,1)$ or $v = (1,\ldots,1)$. Thus in cases with $I \neq [m]$ and $J \neq [n]$, I and J can be recovered from σ_{IJ} (given Θ), and so the correspondence between maximal cells and minimal covers is bijective. $\triangleleft$

Example 3.3. $\Theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $m = n = 2$. *Here Π_Θ is a triangulation of a square into triangles $\sigma_1 = \{v_{11}, v_{12}, v_{21}\}$ and $\sigma_2 = \{v_{12}, v_{21}, v_{22}\}$. The cell σ_1 corresponds to the minimal line cover $(\{1,2\},\emptyset)$ (as well as to $(\emptyset,\{1,2\})$), while σ_2 corresponds to $(\{1\},\{1\})$.*

Note that the cell $\sigma_1 \cap \sigma_2$ does not correspond to a line cover; however $\sigma_1 \cap \sigma_2 = \sigma_{uv}$ for $u = (1, \frac{1}{2})$ and $v = (\frac{1}{2}, 0)$. Finally, note that if $\Theta' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\Pi_\Theta = \Pi_{\Theta'}$.

Corollary 3.2 gives a simple means to verify that certain 0–1 matrices Θ are facet normals of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$. We consider the case in which the $1's$ in Θ form the union of two rectangles. We call a 0–1 matrix Θ a *generalized hook* if there are proper nonempty subsets $I' \subseteq I \subset [m]$ and $J' \subseteq J \subset [n]$ such that $\Theta_{ij} = 1$ if and only if $(i, j) \in (I' \times J) \cup (I \times J')$. Included in this class are all 0–1 matrices having all $1's$ only in one row or one column.

Proposition 3.4. *Suppose Θ is a 0–1 matrix which is a generalized hook. Then the face $\Sigma(m, n)^\Theta$ is a facet.*

Proof: We consider first the case in which $I' = I$ (and, without loss of generality $J' = J$). Then the $1's$ of Θ form a rectangle with rows in I and columns in J. There are only two minimal line covers of Θ, $(I, \emptyset)$ and $(\emptyset, J)$, and so by Corollary 3.2 the subdivision Π_Θ has only two maximal cells. Thus Π_Θ must be maximal in the refinement order, and the conclusion follows.

In the case $I' \neq I$ and $J' \neq J$, Θ has three minimal covers, $(I, \emptyset)$, $(\emptyset, J)$ and (I', J'), and so Π_Θ has three maximal cells. We show Π_Θ to be maximal in the refinement order in this case by showing that the union of no two of these cells is convex.

Let $\sigma_1 = \sigma_{I\,\emptyset}$, $\sigma_2 = \sigma_{\emptyset\,J}$ and $\sigma_3 = \sigma_{I'J'}$, and choose $i_1 \in I'$, $i_2 \in I \setminus I'$, $i_3 \in [m] \setminus I$, $j_1 \in J'$, $j_2 \in J \setminus J'$ and $j_3 \in [n] \setminus J$. It is straightforward to verify that $\frac{1}{2}v_{i_2 j_3} + \frac{1}{2}v_{i_3 j_2} \in conv(\sigma_1 \cup \sigma_2) \setminus (\sigma_1 \cup \sigma_2)$, $\frac{1}{2}v_{i_1 j_1} + \frac{1}{2}v_{i_3 j_2} \in conv(\sigma_2 \cup \sigma_3) \setminus (\sigma_2 \cup \sigma_3)$ and $\frac{1}{2}v_{i_1 j_1} + \frac{1}{2}v_{i_2 j_3} \in conv(\sigma_1 \cup \sigma_3) \setminus (\sigma_1 \cup \sigma_3)$. $\triangleleft$

When $min\{m, n\} \leq 2$ or $max\{m, n\} \leq 3$, all facets have 0–1 normals. That this fails to hold in general is shown by the following example, constructed with E. Babson from a larger example suggested by A. Schrijver. See [BB] for further details.

Example 3.5. *The matrix*

$$\Theta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

gives a facet of $\Sigma(\Delta_3 \times \Delta_3)$ that does not have a 0–1 normal.

4. Facet normals and weightings of bipartite graphs.

We give a characterization of normal vectors to facets of the general fiber polytope $\Sigma(P,Q)$, where $\pi : P \to Q$ as in §2, in terms of a decomposition property for the polyhedra $S(\theta)$. When specialized to the polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, we get a correspondence between facet normals and indecomposable weighted vertex covers in bipartite graphs.

Note that since $\Sigma(P,Q) \subset \mathbb{R}^n$ and generally $dim\,\Sigma(P,Q) = dim\,P - dim\,Q < n$, facet normals are only determined up to the addition of an element of $\mathcal{K}(\pi, P) := (ker\,\pi)^\perp + (aff\,P)^\perp$. For $\theta \in \mathbb{R}^n$, recall the polyhedron $S(\theta) = \{\ (y_0, y) \in \mathbb{R}^{d+1} \mid y_0 + \langle y, q_i\rangle \geq \langle \theta, p_i\rangle,\ i = 1, \ldots, m\ \}$.

We say a direction $\theta \in \mathbb{R}^n$ is *decomposable* if $\theta \in \mathcal{K}(\pi, P)$ or if $\theta = \theta_1 + \theta_2$ and $S(\theta) = S(\theta_1) + S(\theta_2)$ (Minkowski sum), with not both θ_1 and θ_2 in $\mathcal{K}(\pi, P) + cone(\theta)$. Otherwise we say θ is *indecomposable.* Note that we always have $S(\theta_1+\theta_2) \supset S(\theta_1)+S(\theta_2)$.

Theorem 4.1. *For $\theta \in \mathbb{R}^n$, the face $\Sigma(P,Q)^\theta$ is a facet of $\Sigma(P,Q)$ if and only if θ is indecomposable .*

Proof: Assume $\theta \notin \mathcal{K}(\pi, P)$. If $\Sigma(P,Q)^\theta$ is not a facet, then $\theta = \theta_1 + \theta_2$ where $\Sigma(P,Q)^{\theta_1}$ and $\Sigma(P,Q)^{\theta_2}$ are faces properly containing $\Sigma(P,Q)^\theta$ and

$$\Sigma(P,Q)^\theta = \Sigma(P,Q)^{\theta_1} \cap \Sigma(P,Q)^{\theta_2}.$$

Then for almost every $q \in Q$

$$\begin{aligned} max\{\ \langle \theta, p\rangle \mid p \in \pi^{-1}(q)\ \} &= max\{\ \langle \theta_1, p\rangle \mid p \in \pi^{-1}(q)\ \} \\ &\qquad + max\{\ \langle \theta_2, p\rangle \mid p \in \pi^{-1}(q)\ \} \\ &= min\{\ y_0 + \langle y, q\rangle \mid (y_0, y) \in S(\theta_1)\ \} \\ &\qquad + min\{\ y_0 + \langle y, q\rangle \mid (y_0, y) \in S(\theta_2)\ \} \\ &= min\{\ y_0 + \langle y, q\rangle \mid (y_0, y) \in S(\theta_1) + S(\theta_2)\ \} \\ &\geq min\{\ y_0 + \langle y, q\rangle \mid y \in S(\theta)\ \}. \end{aligned} \tag{4.1}$$

From the equality of the first and the last expressions above, we conclude that $S(\theta) = S(\theta_1) + S(\theta_2)$. Since both $\Sigma(P,Q)^{\theta_1}$ and $\Sigma(P,Q)^{\theta_2}$ properly contain $\Sigma(P,Q)^\theta$, we have *neither* θ_1 nor θ_2 in $\mathcal{K}(\pi, P) + cone(\theta)$, so θ must be decomposable.

Conversely, suppose $\Sigma(P,Q)^\theta$ is a facet of $\Sigma(P,Q)$. If $\theta = \theta_1 + \theta_2$ and $S(\theta) = S(\theta_1) + S(\theta_2)$, then a similar argument allows us to conclude that

$$\Sigma(P,Q)^\theta \subset \Sigma(P,Q)^{\theta_1} \cap \Sigma(P,Q)^{\theta_2}. \tag{4.2}$$

Since $\Sigma(P,Q)^\theta$ is a facet, both $\Sigma(P,Q)^{\theta_1}$ and $\Sigma(P,Q)^{\theta_2}$ must be either $\Sigma(P,Q)^\theta$ or $\Sigma(P,Q)$, so both θ_1 and θ_2 must lie in $\mathcal{K}(\pi,P) + cone(\theta)$. Thus θ is indecomposable. $\triangleleft$

In the case of $\Sigma(m,n) = \Sigma(\Delta_{m-1} \times \Delta_{n-1})$, $\mathcal{K}(\pi,P)$ is the linear span of the $m+n$ 0–1 matrices consisting of a row or column of $1's$. This is the same as the space of all *additive* matrices, that is, matrices of the form $X = (x_{ij})$ with $x_{ij} = \alpha_i + \beta_j$ for $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$. We call matrices Θ and Θ' *equivalent* if $\Theta - \Theta'$ is additive. In this case $\Sigma(m,n)^\Theta = \Sigma(m,n)^{\Theta'}$; that is, equivalent matrices yield the same face (see Example 3.3.).

We can interpret coordinates of $\Theta \in \mathbb{R}^{mn}$ as weights on the edges of the complete bipartite graph $K_{m,n}$ and $S(\Theta) = \{\ (u,v) \in \mathbb{R}^m \times \mathbb{R}^n \mid u_i + v_j \geq \theta_{ij}\ \}$ as the polyhedron of weighted vertex covers of the weighted graph Θ. So a weighting Θ of $K_{m,n}$ is *indecomposable* if it is not additive and whenever $\Theta = \Theta_1 + \Theta_2$ and $S(\Theta) = S(\Theta_1) + S(\Theta_2)$, then both Θ_1 and Θ_2 are nonnegative multiples of Θ plus additive matrices.

Corollary 4.2. *Facets of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$ correspond to indecomposable weightings of $K_{m,n}$.* $\triangleleft$

The matrix given in Example 3.5 gives an indecomposable weighting of $K_{4,4}$ that is not equivalent to a 0–1 matrix.

Remark 4.3. It is shown in [BB] that a 0–1 weighting Θ of $K_{m,n}$ corresponding to a subgraph of $K_{m-1,n-1}$ (i.e., Θ has a zero row and a zero column) yields a facet (and, hence, is indecomposable) if this subgraph is connected. In fact, if Θ corresponds to a subgraph of $K_{m-1,n-1}$ having k components, then $codim\, \Sigma(m,n)^\Theta \leq k$. We show equality here.

Suppose $\Theta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are nonnegative nonzero matrices. Let $\Theta_1 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $\Theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. It is straightforward to show that $S(\Theta) = S(\Theta_1) + S(\Theta_2)$ and so Θ is decomposable. By (4.2), $\Sigma(m,n)^\Theta \subset \Sigma(m,n)^{\Theta_1} \cap \Sigma(m,n)^{\Theta_2}$, and so $codim\, \Sigma(m,n)^\Theta \geq 2$. (Alternately, one can use Corollary 3.2 to show that the subdivision Π_Θ is a strict refinement of Π_{Θ_1}.) Repeating this argument leads to the following.

Corollary 4.4. *If*

$$\Theta = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix},$$

where $A_1, A_2, \dots, A_k$ are nonnegative nonzero matrices, then $codim\, \Sigma(m,n)^\Theta \geq k$. $\triangleleft$

5. The support function of $\Sigma(m, n)$.

The *support function* $h_P : \mathbb{R}^n \to \mathbb{R}$ of a polytope $P \subset \mathbb{R}^n$ is defined by

$$h_P(\theta) := max\{ \langle x, \theta \rangle \mid x \in P \}$$

for $\theta \in \mathbb{R}^n$; thus

$$P^\theta = \{ x \in P \mid \langle x, \theta \rangle = h_P(\theta) \}. \tag{5.1}$$

Support functions are always positively homogeneous ($h_P(t\theta) = th_P(\theta)$, for $t \geq 0$) and subadditive ($h_P(\theta + \theta') \leq h_P(\theta) + h_P(\theta')$), and we have

$$P = \{ x \in \mathbb{R}^n \mid \langle x, \theta \rangle \leq h_P(\theta), \quad \text{for all} \quad \theta \in \mathbb{R}^n \}.$$

See, for example, [R] for a general discussion of support functions. We note that if $P = \{ x \mid Ax \leq b \}$, then $h_P(\theta) = min\{ \langle A_1, \theta \rangle, \ldots, \langle A_m, \theta \rangle \}$.

We first consider the support function of a general $\Sigma(P, Q)$. Let $\theta \in \mathbb{R}^n$ and for each $\sigma \in \Pi_\theta$, let $\Delta(\sigma)$ be any fixed triangulation of the polytope σ. For $\tau \in \Delta(\sigma)$, denote by x_τ the centroid of the subset $\pi^{-1}(\tau)$ of P (see (2.5)).

Proposition 5.1. *For $\theta \in \mathbb{R}^n$,*

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\mathrm{vol}\, \tau) \ \langle \theta, x_\tau \rangle. \tag{5.2}$$

Proof: By definition of fiber polytopes and properties of the Minkowski integral, we have for each $\theta \in \mathbb{R}^n$

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \int_Q h_{\pi^{-1}(q)}(\theta) \ dq = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \int_\sigma h_{\pi^{-1}(q)}(\theta) \ dq. \tag{5.3}$$

The function $h_{\pi^{-1}(q)}(\theta)$ is given by (2.4) and, for fixed θ, is a linear function of $q \in \sigma \in \Pi_\theta$. By (2.6) and (5.1), for a simplex $\tau \in \Delta(\sigma)$ and $q \in \tau$ we have $h_{\pi^{-1}(q)}(\theta) = \langle \theta, p \rangle$, where p is the unique point of $\pi^{-1}(\tau)$ such that $\pi(p) = q$. Thus by (5.3) we get

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} \int_\tau h_{\pi^{-1}(q)}(\theta) \ dq = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\mathrm{vol}\, \tau) \ \langle \theta, x_\tau \rangle. \triangleleft$$

Note that $vol\, \tau = 0$ if τ is not a maximal simplex of $\Delta(\sigma)$. For a triangulation Δ refining Π_θ, Proposition 5.1 follows directly from [FP; Cor. 2.6]. There is a slight advantage to the more general formulation here in that it allows one to use arbitrary triangulations of each cell of Π_θ.

To give the support function for secondary polytopes, we must be a bit more careful about the scaling involved in passing from fiber polytopes. Recall that if Q is a polytope with n vertices then the secondary polytope $\Sigma(Q)$ is homothetic to the fiber polytope $\Sigma(\Delta_{n-1}, Q)$, i.e.,

$$\Sigma(Q) = (dim\, Q + 1)\; vol(Q)\; \Sigma(\Delta_{n-1}, Q)$$

[FP; Thm. 2.5]. Again, letting Π_θ be any regular subdivision of Q and for each $\sigma \in \Pi_\theta$, $\Delta(\sigma)$ a triangulation of σ, define $e_\tau := \sum_{q_i \in \tau} e_i = (dim\, Q + 1)x_\tau$ for each $\tau \in \Delta(\sigma)$.

Corollary 5.2. *For* $\theta \in \mathbb{R}^n$,

$$h_{\Sigma(Q)}(\theta) = \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\mathrm{vol}\, \tau)\; \langle \theta, e_\tau \rangle. \tag{5.4}$$

Denote by h_{mn} the support function of the Newton polytope $\Sigma(m, n)$. This is a subadditive function on the space of $m \times n$ matrices Θ. Determining $h_{mn}(\Theta)$ for all integral matrices Θ is equivalent to determining a complete set of inequalities determining $\Sigma(m, n)$. Restricted to 0–1 matrices, h_{mn} can be viewed as giving a monotone, subadditive function on subsets of $[m] \times [n]$ (or *shapes* fitting in an $m \times n$ array); i.e., for $S, T \subset [m] \times [n]$, if $S \subseteq T$, then $h_{mn}(S) \leq h_{mn}(T)$, and if $S \cap T = \emptyset$, then $h_{mn}(S \cup T) \leq h_{mn}(S) + h_{mn}(T)$. It would be of interest to determine $h_{mn}(S)$ for all $S \subset [m] \times [n]$. We give below a formula for $h_{mn}(S)$ in terms of line covers of S and describe how to evaluate it for certain simple shapes.

By (5.4), we can relate the evaluation of $h_{mn}(\Theta)$ to the subdivision Π_Θ. Since each simplex in $\Delta_{m-1} \times \Delta_{n-1}$ has the same volume by the unimodularity of its coordinates (see, e.g., [Hai; Lemma 2]), the support function h_{mn} is obtained from (5.4) by setting $vol\, \tau = 1$ for each maximal simplex τ. In particular, when Θ is a 0–1 matrix, the indicator function of some shape S, we can view this shape as a subset of the vertex set of $\Delta_{m-1} \times \Delta_{n-1}$. In this case, the inner product in (5.4) counts the number of vertices in the set $S \cap \tau$.

Corollary 5.3. *For any shape* $S \subset [m] \times [n]$,

$$h_{mn}(S) = \sum_{I,J} \sum_{\tau \in \Delta(I,J)} |S \cap \tau|, \tag{5.5}$$

where the first sum is over minimal line covers (I, J) *of* S *(with* $J \neq [n]$*) and the second sum is over maximal simplices of any triangulation* $\Delta(I, J)$ *of the cell* σ_{IJ} *in (3.4).*

The primary difficulty in evaluating (5.5) for any particular 0–1 matrix Θ is determining a triangulation of the cell σ_{IJ} for minimal line covers of S. In certain cases this presents no problem. Recall the *standard triangulation* of $\Delta_{m-1} \times \Delta_{n-1}$ whose maximal simplices have vertex sets determined by all *monotone paths* in an $m \times n$ array, i.e., paths from the upper left entry to the lower right entry that only move down or to the right. In [BCS], it is shown that lexicographically ordering these paths gives a shelling of this triangulation. From this it follows that if one takes an initial segment of this triangulation, in the lexicographic order favoring moves down (so the initial path in this order moves down the first column and across the last row), stopping just prior to the addition of a new vertex, then the simplices so defined form a triangulation of the convex hull of the vertices involved. Thus if, after reordering rows and columns, the vertices of a cell σ_{IJ} are arranged in a block triangular form

$$\begin{pmatrix} E & O & \dots & O \\ E & E & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ E & E & \dots & E \end{pmatrix}, \tag{5.6}$$

where the $E's$ and $O's$ represent rectangular arrays of $1's$ and $0's$, then a triangulation of σ_{IJ} is formed by taking all monotone paths among the $1's$ in this array.

For example, if S is a generalized hook, then it is easy to check that the two or three cells σ_{IJ} in Π_S can be put in the form (5.6).

There is a determinantal formula giving the number of monotone paths in the array (5.6). Let λ_i be the number of $1's$ in row $m+1-i$ of (5.6), for $i = 1, \dots, m$, so $n = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$, and define the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of the integer $\lambda_1 + \lambda_2 + \dots + \lambda_m$. (The array (5.6) is essentially the inverted Ferrers diagram of λ.) If $\mu = (n, 1, \dots, 1)$ partitions $m+n-1$, then the number of monotone paths in (5.6) is the number of partitions in the interval $[\mu, \lambda]$ in Young's lattice (i.e., partitions with coordinatewise ordering) and is given by the determinant of the $m \times m$ matrix

$$\left(\binom{\lambda_i - \mu_j + 1}{i - j + 1} \right) = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ \binom{\lambda_2 - n+1}{2} & \lambda_2 & 1 & 0 & \dots & 0 \\ \binom{\lambda_3 - n+1}{3} & \binom{\lambda_3}{2} & \lambda_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \binom{\lambda_m - n+1}{m} & \binom{\lambda_m}{m-1} & \binom{\lambda_m}{m-2} & \binom{\lambda_m}{m-3} & \dots & \lambda_m \end{pmatrix}. \tag{5.7}$$

(Here we understand $\binom{a}{b} = 0$ if $a < 0$, $b < 0$ or $a < b$. See [St; Exer. 3.63].) Note that if (5.6) consists only of $1's$, then the number of monotone paths is easily shown to be $\binom{m+n-2}{m-1}$ (which is the number of simplices in any triangulation of $\Delta_{m-1} \times \Delta_{n-1}$).

It appears to be a considerably more difficult problem to count the number of meetings of these paths with a prescribed fixed set of cells, which is necessary for a complete evaluation of (5.5). In certain cases, this can be done, as is illustrated by the examples in the next section.

6. Evaluation of h_{mn} for some simple examples.

We compute $h_{mn}(\Theta)$ for a 0–1 Θ having all it's $1's$ in a single line, as well as certain other values of $h_{2,n}$, $h_{3,3}$ and $h_{4,4}$.

Proposition 6.1. *If $\Theta_{(k)}$ is a 0–1 $m \times n$ matrix with k $1's$, all contained in a single line (row or column), then*

$$h_{mn}(\Theta_{(k)}) = \sum_{j=1}^{k} \binom{m+n-j-1}{m-1}. \tag{6.1}$$

Proof: By Proposition 3.4, such matrices correspond to facets of $\Sigma(m,n)$. We can assume that $\Theta_{(k)}$ has $1's$ as the first k entries in the first row. In this case the subdivision $\Pi_{\Theta_{(k)}}$ consists of two cells σ_1 and σ_2, corresponding, respectively, to the minimal line covers $(\{1\},\emptyset)$ and $(\emptyset,\{1,\ldots,k\})$. By (3.4), these cells have vertices indicated by the $1's$ in the $m \times n$ arrays

$$\begin{pmatrix} 1\ldots 1 0 \ldots 0 \\ \\ E_1 \end{pmatrix} \text{and} \begin{pmatrix} 1\ldots 1 & E_2 \\ O & \end{pmatrix},$$

where E_1 is an $(m-1)\times n$ array of $1's$ and E_2 is an $m \times (n-k)$ array of $1's$. By the above discussion, one sees that $vol\,\sigma_1 = \binom{m+n-2}{m-1} - \binom{m+n-k-2}{m-1}$ (which is the number of paths in an $m \times n$ array minus the number in an $m \times (n-k)$ array) and $vol\,\sigma_2 = \binom{m+n-k-2}{m-1}$. Counting the incidence of these paths with the first k entries in row 1, we get

$$\begin{aligned} h_{mn}(\Theta_{(k)}) &= \sum_{j=1}^{k}\left[\binom{m+n-j-1}{m-1} - \binom{m+n-k-2}{m-1}\right] + k\binom{m+n-k-2}{m-1} \\ &= \sum_{j=1}^{k}\binom{m+n-j-1}{m-1}. \quad \triangleleft \end{aligned} \tag{6.2}$$

Remark 6.2. When $m = 2$, $\Sigma(2,n)$ is congruent to the permutohedron P_n. (See [SZ; Prop. 1.12] where this is proved for the Newton polytope of the product of *maximal* minors of a $2 \times n$ matrix, from which one obtains $\Sigma(2,n)$ by translation by the $2 \times n$ matrix of $1's$.) In this case, the $2^n - 2$ facets of $\Sigma(2,n)$ have normals given by all matrices $\Theta_{(k)}$ consisting of k $1's$ in the first row, $1 < k < n$, and (6.1) reduces to $h_{2,n}(\Theta_{(k)}) = \binom{n+1}{2} - \binom{k+1}{2}$ (c.f., [YKK; §5.3.1]).

Example 6.3. *The case $m = n = 3$.*

From (6.1) we get $h_{3,3}(\Theta_{(1)}) = 6$ and $h_{3,3}(\Theta_{(2)}) = 9$. The only other 0–1 matrix corresponding to a connected subgraph of $K_{2,2}$, and thus to a facet of $\Sigma(3,3)$, which is not equivalent to one of these two, is (up to row and column permutation)

$$\Theta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We show $h_{3,3}(\Theta) = 14$.

Corresponding to line covers $(\{1,2\},\emptyset)$, $(\emptyset,\{1,2\})$ and $(\{1\},\{1\})$, we have cells σ_1, σ_2 and σ_3, having vertices indicated by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

respectively. These make contributions 5, 5 and 4 to the sum (5.5).

One can check by direct calculation that all row and column permutations of $\Theta_{(1)}$, $\Theta_{(2)}$ and Θ give all the facet normals of $\Sigma(3,3)$. Thus $\Sigma(3,3)$ consists of all 3×3 matrices having row and column sums 10, such that each entry is at most 6, any two entries in the same row or column sum to at most 9, and any 3 entries, two in the same row and the third in one of their columns, sum to at most 14. This polytope has 108 vertices and so $\Delta_2 \times \Delta_2$ has 108 regular triangulations (see [BFS], [FP] or [GZK] for the definition of regular triangulation). It is not known whether $\Delta_2 \times \Delta_2$ (or, in general, $\Delta_{m-1} \times \Delta_{n-1}$) admits triangulations that are not regular. ◁

Example 6.4. *The case $m = n = 4$. We tabulate in Table 1 the values of $h_{4,4}(\Theta)$ for those 0–1 Θ corresponding to connected subgraphs of $K_{3,3}$, and so known to correspond to facets of $\Sigma(4,4)$ by Remark 4.3, as well as the Θ of Example 3.5.*

We have omitted row and column permutations of $\Theta's$ in the table (which have the same values of $h_{4,4}$). We also omit $\Theta's$ which are equivalent to those on the table, and which thus yield the same facet, although usually different, but easily derivable, values of $h_{4,4}$ (using the fact that elements of $\Sigma(4,4)$ have row and column sums equal to 35).

Note, for example that

$$\Theta_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \Theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Θ	$h_{4,4}(\Theta)$	Θ	$h_{4,4}(\Theta)$
$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	20	$\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	62
$\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	30	$\begin{pmatrix}1&1&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$	71
$\begin{pmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	34	$\begin{pmatrix}0&1&1&0\\1&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	74
$\begin{pmatrix}1&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	46	$\begin{pmatrix}1&1&1&0\\1&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	76
$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	52	$\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$	84
$\begin{pmatrix}1&1&1&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	53	$\begin{pmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$	86
$\begin{pmatrix}0&1&1&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	56	$\begin{pmatrix}2&1&0&0\\1&0&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$	95

Table 1

and

$$\Theta_3 = \begin{pmatrix}1&1&1&0\\1&1&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \sim \Theta_2 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&1\end{pmatrix},$$

where $\sim$ denotes equivalence (see §4). Here, $h_{4,4}(\Theta_1) = h_{4,4}(\Theta_2) = 62$, while $h_{4,4}(\Theta_3) = h_{4,4}(\Theta_4) + 35 = 65$, since the row and column sums equal 35 in this case. Note also the monotonicity and subadditivity of $h_{4,4}$, as illustrated, for example, by the first, second and fourth values in the left table.

The values in the table were all obtained by application of (5.5). In some cases, the resulting cells σ_{IJ} could only be partially triangulated by initial segments of the standard triangulation. In these cases, a full triangulation was obtained by "placing" the remaining vertices outside the partial triangulation, forming further simplices by joining to exposed facets on the boundary. In this regard, the following observations are useful. Full-dimensional simplices on the vertices of $\Delta_{m-1} \times \Delta_{n-1}$ (which thus involve $m+n-1$ vertices) correspond to spanning trees in the graph $K_{m,n}$. A simplex τ of codimension 1 corresponds to an acyclic subgraph having 2 components. To form a full simplex containing τ, one must add an edge which joins these components. Each such edge is oriented according to whether it meets the left or right side of the first component, say, and the corresponding vertices of $\Delta_{m-1} \times \Delta_{n-1}$ are on the same or opposite side of the hyperplane generated by τ depending on whether they have the same or opposite orientation. This makes it fairly easy to determine, when placing a new vertex over a partial triangulation, to which of the current facets it is to be joined.

A complete list of the normals of facets of $\Sigma(4,4)$ has not been conjectured. The table accounts for all the 0–1 facet normals having a zero row and column. There appear to be many others; for example,

$$\Theta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

can be shown to be normal to a facet (see [BB]). $\triangleleft$

Looking at the Table 1, one is led to ask whether, on 0–1 matrices, the function h_{mn} can be viewed as giving the energy, in some unspecified sense, of the corresponding configuration of $1's$.

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