The cd-index of zonotopes and arrangements

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To Gian-Carlo Rota, for years of inspiration.

Abstract

We investigate a special class of polytopes, the zonotopes, and show that their flag f-vectors satisfy only the affine relations fulfilled by flag f-vectors of *all* polytopes. In addition, we determine the lattice spanned by flag f-vectors of zonotopes. By duality, these results apply as well to the flag f-vectors of central arrangements of hyperplanes.

1 Introduction

The flag f-vector of a convex polytope is an enumerative invariant of its lattice of faces, containing more information than the usual f-vector. While the latter counts the numbers of faces in each dimension, the former counts the numbers of chains (flags) having any possible set of dimensions.

The Euler relation is the only affine relation satisfied by f-vectors of all polytopes. For simplicial (or simple) d-polytopes, there are $\lfloor \frac{d-1}{2} \rfloor$ additional relations, called the Dehn-Sommerville equations, which provide a complete description of the affine space generated by all such f-vectors [8]. The information contained in the f-vector of a simplicial polytope is nicely summarized in the form of the h-vector [16].

In the case of the flag f-vector, there is a large set of equations that are satisfied for all polytopes. The corresponding affine space has dimension given by the Fibonacci sequence [1]. The **cd**-index provides an efficient way to summarize this information [2].

In the case of simplicial, simple and cubical polytopes, the flag f-vector reduces directly to the f-vector. In this paper we investigate another special class of polytopes, the zonotopes, and show for these that there is no reduction whatsoever; that is, we show that the flag f-vectors of zonotopes satisfy only the affine relations satisfied by flag f-vectors of *all* polytopes. This strengthens a result of Liu [12, Theorem 4.7.1]. Zonotopes are of particular interest in the study of hyperplane arrangements (see [18]), to which they are dual. A direct consequence of our result is that the **cd**-index of a central hyperplane arrangement is the most efficient encoding of the affine information of its flag f-vector.

We define the basic terminology used throughout this paper. For a convex d-dimensional polytope Q, and for a subset $S \subseteq \{0, \ldots, d-1\}$, we denote by f_S the number of chains of faces (*flags*) in Q, $F_1 \subset \cdots \subset F_k$, with $S = \{\dim F_1, \ldots, \dim F_k\}$. The vector consisting of all the numbers f_S, $S \subseteq \{0, \ldots, d-1\}$, is called the *flag f-vector* of Q. The affine span of the flag f-vectors of all polytopes (more generally, of all Eulerian posets) is described by a system of linear equations, known as the generalized Dehn-Sommerville equations [1].

*Supported in part by NSF grant DMS-9500581.

For any $S \subseteq \{0, \ldots, d-1\}$, we set $h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T$. Define a polynomial in the non-commuting variables $\mathbf{a}$ and $\mathbf{b}$, called the **ab**-*index*, by

$$\Psi(Q) = \sum_S h_S \cdot u_S,$$

where $u_S = z_0 \cdots z_{d-1}$, $z_i = \mathbf{b}$ if $i \in S$ and $z_i = \mathbf{a}$ if $i \notin S$. An implicit encoding of the generalized Dehn-Sommerville equations is given by the fact that $\Psi(Q)$ is always a polynomial in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. We call the polynomial the **cd**-*index* of Q.

As an example, the **cd**-index of a polygon Q is given by

$$\Psi(Q) = \mathbf{c}^2 + (f_0 - 2) \cdot \mathbf{d} \tag{1.1}$$

and the **cd**-index of a 3-dimensional polytope Q is given by

$$\Psi(Q) = \mathbf{c}^3 + (f_0 - 2) \cdot \mathbf{dc} + (f_2 - 2) \cdot \mathbf{cd}. \tag{1.2}$$

In Section 2, we discuss the operations of taking pyramids and prisms, and we use them to give a direct proof that the flag f-vectors of all polytopes span the linear space determined by the generalized Dehn-Sommerville equations. We next discuss zonotopes and three operations on them – projection, Minkowski sum with a line segment and prism. We use the coalgebra techniques of [7] to determine their effect on the **cd**-index. In Section 4, we show the the **cd**-index of an n-fold iterated Minkowski sum is a polynomial function of n, and we use this in Section 5 to show that the flag f-vectors of zonotopes also span the space of all flag f-vectors. This result is extended in Section 6 by determining the lattice spanned by the **cd**-indices of all zonotopes. It is the ring of all integral polynomials in $\mathbf{c}$ and $2\mathbf{d}$. In terms of flag f-vectors, this is equivalent to saying that f_S is divisible by $2^{|S|}$. Some observations and concluding remarks are indicated in the final section.

The authors thank Gàbor Hetyei for reading and making useful comments on an earlier version of this paper.

2 Polytopes span

For a field $\mathbf{k}$ of characteristic 0, let $\mathcal{F}$ be the polynomial algebra in non-commuting variables $\mathbf{c}$ and $\mathbf{d}$ over the field $\mathbf{k}$, that is, $\mathcal{F} = \mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$. (In fact, everything we do here works in any characteristic other than 2.) If we set the degree of $\mathbf{c}$ to 1 and the degree of $\mathbf{d}$ to 2, we define $\mathcal{F}_d$ to be all polynomials in $\mathcal{F}$ that are homogeneous of degree d.

Recall that a derivation f on an algebra A is a linear map satisfying the product rule $f(x \cdot y) = f(x) \cdot y + x \cdot f(y)$. Observe that it is enough to determine how the derivation acts on a set of generators, and hence we may describe a derivation on $\mathcal{F}$ by giving its value on the elements $\mathbf{c}$ and $\mathbf{d}$. Define two derivations D and G on $\mathcal{F}$ by $D(\mathbf{c}) = 2 \cdot \mathbf{d}$, $D(\mathbf{d}) = \mathbf{c} \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{c}$, $G(\mathbf{c}) = \mathbf{d}$, and $G(\mathbf{d}) = \mathbf{c} \cdot \mathbf{d}$. Observe that both these derivations increase the degree by 1, that is, they are maps from $\mathcal{F}_d$ to $\mathcal{F}_{d+1}$.

For a polytope Q we denote the pyramid over Q by $\text{Pyr}(Q)$. Likewise, denote the prism over Q by $\text{Pri}(Q)$. We similarly denote two linear maps $\text{Pyr}, \text{Pri} : \mathcal{F} \longrightarrow \mathcal{F}$, by

$$\text{Pyr}(w) = w \cdot \mathbf{c} + G(w)$$

and

$$\text{Pri}(w) = w \cdot \mathbf{c} + D(w).$$

The following results are proved by using coalgebra techniques in [7] (see Theorem 4.4 and Theorem 5.2).

Proposition 2.1 *For a polytope Q we have that*

$$\begin{aligned} \Psi(\text{Pyr}(Q)) &= \text{Pyr}(\Psi(Q)), \\ \Psi(\text{Pri}(Q)) &= \text{Pri}(\Psi(Q)). \end{aligned}$$

Lemma 2.2 *The linear span of the two sets* $\text{Pyr}(\mathcal{F}_d)$ *and* $\text{Pri}(\mathcal{F}_d)$ *is the linear space* $\mathcal{F}_{d+1}$.

Proof: Define a third derivation G' on the algebra $\mathcal{F}$ by $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{d} \cdot \mathbf{c}$. It follows that $w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$ for all $w \in \mathcal{F}$ (see [7, Lemma 5.1]).

Observe that $\text{Pri}(w) - \text{Pyr}(w) = D(w) - G(w) = G'(w)$. Thus the statement of the lemma is equivalent to that $\text{Pyr}(\mathcal{F}_d)$ and $G'(\mathcal{F}_d)$ span the space $\mathcal{F}_{d+1}$. Let $V = \text{Pyr}(\mathcal{F}_d) + G'(\mathcal{F}_d)$.

Let w be an element in $\mathcal{F}_d$. Then we have that $\mathbf{c} \cdot w = w \cdot \mathbf{c} + G(w) - G'(w) = \text{Pyr}(w) - G'(w)$. Hence $\mathbf{c} \cdot w$ belongs to V.

Let v be in $\mathcal{F}_{d-1}$. Then we have that $G'(\mathbf{c} \cdot v) = \mathbf{d} \cdot v + \mathbf{c} \cdot G'(v)$. Since $\mathbf{c} \cdot G'(v)$ belongs to V by the previous paragraph and $G'(\mathbf{c} \cdot v)$ also belongs to V, we have $\mathbf{d} \cdot v \in V$.

Since a monomial in $\mathcal{F}_{d+1}$ begins either with a $\mathbf{c}$ or a $\mathbf{d}$, we conclude $V = \mathcal{F}_{d+1}$. ■

From Lemma 2.2, we conclude directly the basic result that the linear span of all flag f-vectors has dimension given by the Fibonacci numbers [1].

Theorem 2.3 *Beginning with a point, one can produce, by repeated use of the operations* Pyr *and* Pri*, a set of polytopes whose* cd*-indices span* $\mathcal{F}$.

We note that this approach does not identify a specific basis, as was done in [1] and [10]. We end this section with a useful fact.

Lemma 2.4 *The four linear maps D, G,* Pri, *and* Pyr *are injective.*

Proof: Let $\mathcal{F}_d^{(i)}$ be the linear span of all monomials of degree d containing i $\mathbf{d}$'s. Define two derivations D_0 and D_1 by: $D_0(\mathbf{c}) = 0$, $D_0(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$, $D_1(\mathbf{c}) = 2 \cdot \mathbf{d}$, and $D_1(\mathbf{d}) = 0$. Observe that $D = D_0 + D_1$ and D_j is a linear map from $\mathcal{F}_d^{(i)}$ to $\mathcal{F}_{d+1}^{(i+j)}$.

Define a linear map $\phi : \mathcal{F}_d^{(i)} \longrightarrow \mathbf{k}[x_0, \ldots, x_i]$ by

$$\phi(\mathbf{c}^{n_0}\mathbf{d}\mathbf{c}^{n_1}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{n_i}) = x_0^{n_0} x_1^{n_1} \cdots x_i^{n_i}.$$

This map takes the linear space $\mathcal{F}_d^{(i)}$ isomorphically onto the linear space of homogeneous polynomials of degree $d - 2 \cdot i$ in the variables $x_0, \ldots, x_i$. Moreover, we have

$$\phi(D_0(w)) = (x_0 + 2 \cdot x_1 + \cdots + 2 \cdot x_{i-1} + x_i) \cdot \phi(w).$$

Since the ring of polynomials is an integral domain, we have $(x_0 + 2 \cdot x_1 + \cdots + 2 \cdot x_{i-1} + x_i)$ is not a zero divisor. Hence $D_0 : \mathcal{F}_d^{(i)} \longrightarrow \mathcal{F}_{d+1}^{(i)}$ is an injective map. The linear map D corresponds to a block matrix, where the blocks on the diagonal are described by D_0 and the blocks below the diagonal are equal to zero. We thus conclude D is also injective.

For the other three linear maps the proof is similar. The only difference is that we obtain another polynomial of degree 1 for each linear map. For Pri, G and Pyr we get, respectively, $x_0 + 2x_1 + \cdots + 2x_i$, $x_0 + \cdots + x_{i-1}$ and $x_0 + \cdots + x_i$. ■

3 Zonotopes

The *Minkowski sum* of two subsets X and Y of $\mathbb{R}^d$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^d \ : \ \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

Notably, the Minkowski sum of two convex polytopes is another convex polytope. For a vector $\mathbf{x}$ we denote the set $\{\lambda \cdot \mathbf{x} \ : \ 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathbf{x}]$. We denote by aff(X) the *affine span* of X, that is, the intersection of all affine subspaces containing the set X.

We say that the nonzero vector $\mathbf{x} \in \text{aff}(Q)$ lies in *general position* with respect to the convex polytope Q if the line $\{\lambda \cdot \mathbf{x} + \mathbf{u} \in \mathbb{R}^d \ : \ \lambda \in \mathbb{R}\}$ intersects the boundary of the polytope Q in at most two points for all $\mathbf{u} \in \mathbb{R}^d$. Alternatively, $\mathbf{x} \in \text{aff}(Q)$ is in general position if $\mathbf{x}$ is parallel to no proper face of Q.

From [7, Prop. 6.3] we have the following result. Let Q be a d-dimensional convex polytope and $\mathbf{x}$ a nonzero vector that lies in general position with respect to the polytope Q. Let H be a hyperplane orthogonal to the vector $\mathbf{x}$, and let Proj(Q) be the orthogonal projection of Q onto the hyperplane H. Observe that Proj(Q) is a $(d-1)$-dimensional convex polytope.

Proposition 3.1 *The* **cd**-*index of the Minkowski sum of* Q *and* $[\mathbf{0}, \mathbf{x}]$ *is given by*

$$\Psi(Q + [\mathbf{0}, \mathbf{x}]) = \Psi(Q) + D(\Psi(\text{Proj}(Q))).$$

A *zonotope* is the Minkowski sum of line segments. That is, if $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$, then the zonotope they generate is the Minkowski sum

$$Z = [\mathbf{0}, \mathbf{x}_1] + \cdots + [\mathbf{0}, \mathbf{x}_n].$$

A (central) *hyperplane arrangement* is a finite collection $\mathcal{H}$ of linear hyperplanes in $\mathbb{R}^d$. An arrangement is called *essential* if the intersection of all its hyperplanes is the origin. An arrangement $\mathcal{H}$ induces a subdivision of $\mathbb{R}^d$ into relatively open cells, whose closures are ordered by inclusion. The resulting poset is a lattice, called the *face lattice* of $\mathcal{H}$. An arrangement $\mathcal{H} \subset \mathbb{R}^d$ has a natural flag f-vector with components $f_S(\mathcal{H})$, where $S \subseteq \{1, \ldots, d\}$. The face lattice of Z is anti-isomorphic to that of the central arrangement $\mathcal{H}$ of the n hyperplanes with normals $\mathbf{x}_1, \ldots, \mathbf{x}_n$ [5, Prop. 2.2.2]. If Z is d-dimensional, then its flag f-vector and that of its dual hyperplane arrangement are related by $f_S(Z) = f_{d-S}(\mathcal{H})$, where $S = \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, d-1\}$ and $d - S = \{d - i_k, \ldots, d - i_1\}$.

Two important and useful facts about the combinatorial behavior of zonotopes are the following:

1. The face lattice of Z is determined by the *oriented matroid* of the point configuration $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ [5, Prop. 2.2.2], and
2. The flag f-vector of Z is determined by the *matroid* of the configuration $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ [5, Cor. 4.6.3].

For a zonotope Z we note that, up to combinatorial type (that is, up to face lattice), the prism over Z can be realized as the zonotope $\text{Pri}(Z) = Z + [\mathbf{0}, \mathbf{x}]$ for any $\mathbf{x} \notin \text{aff}(Z)$.

We define a zonotope $M(Z)$ by

$$M(Z) = Z + [\mathbf{0}, \mathbf{x}],$$

where $\mathbf{x}$ lies in general position with respect to Z. While the combinatorial type of $M(Z)$ depends on the choice of $\mathbf{x}$, the flag f-vector is invariant. This follows since the underlying matroid of $M(Z)$ is always a free extension (of the same rank) of the matroid of Z, that is, an extension such that $\mathbf{x}$ lies on no proper subspace spanned by the generators $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Finally, we define the zonotope $\pi(Z)$ to be the projection of $M(Z)$ along the direction $\mathbf{x}$, that is, onto the hyperplane orthogonal to $\mathbf{x}$. Observe that $\pi(Z)$ is the projection $\text{Proj}(Z)$ in a general direction. The zonotope $\pi(Z)$ is well-defined up to flag f-vector since its underlying matroid will be the one obtained by contracting $\mathbf{x}$ in the matroid of $M(Z)$.

Directly as a corollary of Proposition 3.1 we have

Corollary 3.2 *For a zonotope Z we have*

$$\Psi(M(Z)) - \Psi(Z) = D(\Psi(\pi(Z))).$$

The operations Pri, M, and π were used by Liu [12] to give a lower bound on the dimension of span of the flag f-vectors of zonotopes. The relationship between these operations is given by the following lemma. The second relation was first observed by Liu in [12, Theorem 4.2.7]. We say d-zonotopes Z and W are *equal up to flag f-vector* if, for every $S \subseteq \{0, \ldots, d-1\}$, $f_S(Z) = f_S(W)$.

Lemma 3.3 *For a zonotope Z we have, up to flag f-vector,*

$$\pi(M(Z)) = M(\pi(Z))$$

and

$$\pi(\text{Pri}(Z)) = M(Z).$$

Proof: In each pair it is enough to check that the underlying matroids are the same.

For $\pi(M(Z))$, one makes a free extension of Z by $\mathbf{x}$ and again by $\mathbf{y}$ (both in aff(Z)), then contracting $\mathbf{y}$. The image of $\mathbf{x}$ under this contraction is still free with respect to the images of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, so the resulting matroid is the same as that of $M(\pi(Z))$.

For $\pi(\text{Pri}(Z))$, the description is the same, except now neither $\mathbf{x}$ nor $\mathbf{y}$ is in aff(Z). In this case, the images of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ will have the same matroid as $M(Z)$. ∎

4 Polynomial functions

In this section we define polynomial functions and derive some of their properties. These functions play a role in the proof of our main theorem. Let V and W be vector spaces over the field $\mathbf{k}$.

Definition 4.1 *A function $f : \mathbb{N} \longrightarrow V$ is called a* polynomial function of degree d *if it can be written in the form*

$$f(n) = \mathbf{v}_d \cdot \binom{n}{d} + \mathbf{v}_{d-1} \cdot \binom{n}{d-1} + \cdots + \mathbf{v}_0 \cdot \binom{n}{0},$$

where $\mathbf{v}_0, \ldots, \mathbf{v}_d \in V$ and $\mathbf{v}_d \neq \mathbf{0}$. We call $\mathbf{v}_d$ the leading coefficient.

Observe that $\binom{n}{d}$ is defined by the Pascal relations in any characteristic. We define the *difference operator* Δ by $\Delta f(n) = f(n+1) - f(n)$. The following proposition contains the essential results we will need about polynomial functions.

Proposition 4.2 *Let $f : \mathbb{N} \longrightarrow V$ be a polynomial function of degree d.*

(i) If $\phi : V \longrightarrow W$ is a linear map then the composition $\phi \circ f : \mathbb{N} \longrightarrow W$ is a polynomial function of degree at most d. If ϕ is injective, then the degree is d.

(ii) The function $\Delta f(n)$ is a polynomial function of degree $d-1$.

(iii) If g is a function from $\mathbb{N}$ to V such that $\Delta g(n) = f(n)$ then g is a polynomial function of degree $d+1$ with the same leading coefficient as f.

(iv) The vector $f(0)$ is in the linear span of $f(1), \ldots, f(d+1)$.

Proof: Let $f(n)$ be the polynomial function of degree d

$$f(n) = \mathbf{v}_d \cdot \binom{n}{d} + \mathbf{v}_{d-1} \cdot \binom{n}{d-1} + \cdots + \mathbf{v}_0 \cdot \binom{n}{0}.$$

(i) Observe that

$$(\phi \circ f)(n) = \phi(\mathbf{v}_d) \cdot \binom{n}{d} + \phi(\mathbf{v}_{d-1}) \cdot \binom{n}{d-1} + \cdots + \phi(\mathbf{v}_0) \cdot \binom{n}{0},$$

which is a polynomial function of degree at most d. When ϕ is injective, we have $\phi(\mathbf{v}_d) \neq \mathbf{0}$ and hence $\phi \circ f$ is of degree d.

(ii) It is straightforward to obtain

$$\Delta f(n) = \mathbf{v}_d \cdot \binom{n}{d-1} + \mathbf{v}_{d-1} \cdot \binom{n}{d-2} + \cdots + \mathbf{v}_1 \cdot \binom{n}{0},$$

which proves (ii).

(iii) By induction on n we have

$$g(n) = \mathbf{v}_d \cdot \binom{n}{d+1} + \mathbf{v}_{d-1} \cdot \binom{n}{d} + \cdots + \mathbf{v}_0 \cdot \binom{n}{1} + g(0),$$

which is a polynomial function of degree $d+1$. The leading coefficient is $\mathbf{v}_d$, which is the leading coefficient of f.

(iv) By property (ii) we know that $\Delta^d f(n)$ is a polynomial function of degree 0, hence it is a constant. Thus $\Delta^d f(0) = \Delta^d f(1)$. But $\Delta^d f(0)$ is a linear combination of $f(0), \ldots, f(d)$ and $\Delta^d f(1)$ is a linear combination of $f(1), \ldots, f(d+1)$. The coefficient of $f(0)$ in $\Delta^d f(0)$ is $(-1)^d$, which is nonzero, and hence the relation $\Delta^d f(0) = \Delta^d f(1)$ gives the desired result. ■

Observe that Proposition 4.2 and its proof is valid in any characteristic for the field $\mathbf{k}$, since $(-1)^d$ is never zero. Moreover, it applies to Abelian groups ($\mathbb{Z}$-modules) as well. This last fact will be used in Section 6.

The main result of this section shows that the **cd**-index of iterates of the operation M is a polynomial function.

Theorem 4.3 *Let Z be a d-dimensional zonotope. Then the mapping $n \mapsto \Psi(M^n(Z))$ is a polynomial function of degree $d-1$ into $\mathcal{F}_d$, with leading coefficient $D^{d-1}(\mathbf{c})$.*

Proof: The proof is by induction on d. The base case is $d = 2$. Assume that Z is a 2-dimensional zonotope, that is, Z is a $2k$-gon. Then $M(Z)$ is a $(2k+2)$-gon, and $M^n(Z)$ is a $(2k+2n)$-gon. By equation (1.1) we have the **cd**-index of $M^n(Z)$ is given by $\Psi(M^n(Z)) = \mathbf{c}^2 + (2k+2n-2)\cdot\mathbf{d} = 2\cdot n\cdot\mathbf{d} + \mathbf{c}^2 + (2k-2)\cdot\mathbf{d}$. This is a polynomial function of degree 1 in n with leading coefficient $2\cdot\mathbf{d} = D(\mathbf{c})$.

Assume that $d \geq 3$ and let $W = \pi(Z)$. Observe that W is a $(d-1)$-dimensional zonotope. Now by Corollary 3.2 and Lemma 3.3 we have

$$\begin{aligned} \Delta(\Psi(M^n(Z))) &= \Psi(M^{n+1}(Z)) - \Psi(M^n(Z)) \\ &= D(\Psi(M^n(\pi(Z)))) \\ &= D(\Psi(M^n(W))). \end{aligned}$$

By the induction hypothesis we know that $n \mapsto \Psi(M^n(W))$ is a polynomial function of degree $d-2$ with leading coefficient $D^{d-2}(\mathbf{c})$. Since D is an injective linear map (see Lemma 2.4), by property (i) in Proposition 4.2 we have $n \mapsto D(\Psi(M^n(W)))$ is polynomial function of degree $d-2$. Its leading term is $D^{d-1}(\mathbf{c})$. Finally, by property (iii) in the same proposition we complete the induction. ■

5 Zonotopes span

Let $\mathcal{G}_d$ be the linear span of the **cd**-indices of zonotopes of dimension d. Liu proved that $\dim \mathcal{G}_d \geq \dim \mathcal{G}_{d-1} + \dim \mathcal{G}_{d-3}$ [12, Theorem 4.7.1]. In this section we prove that $\dim \mathcal{G}_d = \dim \mathcal{G}_{d-1} + \dim \mathcal{G}_{d-2}$, that is, $\mathcal{G}_d$ equals $\mathcal{F}_d$.

Since zonotopes are polytopes, we know that $\mathcal{G}_d \subseteq \mathcal{F}_d$. We first prove a variation of Lemma 2.2 that substitutes D for Pyr in order to be able to operate solely with zonotopes.

Lemma 5.1 *The linear span of the two sets $D(\mathcal{F}_d)$ and $\mathrm{Pri}(\mathcal{F}_d)$ is the whole space $\mathcal{F}_{d+1}$.*

Proof: Let V be the subspace of $\mathcal{F}_d$ which is spanned by $D(\mathcal{F}_d)$ and $\mathrm{Pri}(\mathcal{F}_d)$, that is, $V = D(\mathcal{F}_d) + \mathrm{Pri}(\mathcal{F}_d)$.

Let $w \in \mathcal{F}_d$. Since $w \cdot \mathbf{c} = \mathrm{Pri}(w) - D(w) \in V$, we know that every **cd**-monomial which ends with a $\mathbf{c}$ belongs to $\mathrm{Pri}(\mathcal{F}_d) + D(\mathcal{F}_d)$.

Consider $v \in \mathcal{F}_{d-1}$. We have that $D(v\cdot\mathbf{c}) = D(v)\cdot\mathbf{c} + 2\cdot v\cdot\mathbf{d}$, and hence $v\cdot\mathbf{d} = \frac{1}{2}\cdot(D(v\cdot\mathbf{c}) - D(v)\cdot\mathbf{c})$. We have $D(v\cdot\mathbf{c}) \in V$. Moreover $D(v)\cdot\mathbf{c} \in V$ by the previous paragraph. Hence $v\cdot\mathbf{d} \in V$, and we conclude that every **cd**-monomial belongs to V. ■

The following result shows that the flag f-vectors of zonotopes made by the successive application of the operators Pri and M, beginning with $Z = \mathbf{0}$, span the space of all flag f-vectors of polytopes.

Theorem 5.2 *The* **cd**-*indices of d-dimensional zonotopes linearly span the space of* **cd**-*polynomials of degree d, that is, $\mathcal{G}_d = \mathcal{F}_d$.*

Proof: The proof is by induction on the dimension d; the case $d \leq 2$ is clear. We assume that the theorem holds for $d \geq 2$, hence $\mathcal{G}_d = \mathcal{F}_d$, and prove it for $d+1$. Assume that $\{Z_1, \ldots, Z_N\}$ form a spanning set of zonotopes of dimension d. Since $\Psi(\mathrm{Pri}(Z_i)) = \mathrm{Pri}(\Psi(Z_i))$ we have that $\mathrm{Pri}(\mathcal{F}_d) \subseteq \mathcal{G}_{d+1}$.

By combining Theorem 4.3 and property (iv) in Proposition 4.2, we know that $\Psi(Z_i)$ lies in the linear span of $\Psi(M(Z_i)), \ldots, \Psi(M^d(Z_i))$. Hence, we know that $\{M^j(Z_i) \mid 1 \leq i \leq N, 1 \leq j \leq d\}$ is a spanning set of zonotopes. Observe that every zonotope in this spanning set is the Minkowski sum of a line segment with a d-dimensional zonotope. Hence we can describe this spanning set as $\{M(W_1), \ldots, M(W_{N \cdot d})\}$.

By Lemma 3.3 and Corollary 3.2 we have

$$\begin{aligned} \Psi(M(\text{Pri}(W_i))) - \Psi(\text{Pri}(W_i)) &= D(\Psi(\pi(\text{Pri}(W_i)))) \\ &= D(\Psi(M(W_i))). \end{aligned}$$

Since both $M(\text{Pri}(W_i))$ and $\text{Pri}(W_i)$ are $(d+1)$-dimensional zonotopes, we have $D(\Psi(M(W_i))) \in \mathcal{G}_{d+1}$. But since $\{M(W_i)\}$ forms a spanning set for $\mathcal{F}_d$, we obtain that $D(\mathcal{F}_d) \subseteq \mathcal{G}_{d+1}$. By Lemma 5.1 we obtain that $\mathcal{G}_{d+1} = \mathcal{F}_{d+1}$, which completes the induction. ∎

Since the face lattice of a central hyperplane arrangement is an Eulerian poset, it has a **cd**-index, obtainable from that of its dual zonotope by reversing each **cd**-monomial.

Corollary 5.3 *The* **cd***-indices of essential hyperplane arrangements in* $\mathbb{R}^d$ *linearly span the space of* **cd***-polynomials of degree* d.

6 The integral span

We turn now to the problem of finding the *integral* span of flag f-vectors of zonotopes. This leads to an integral **c**-2**d**-index for zonotopes and central arrangements.

Let $\mathcal{R}$ be the ring in the non-commuting variables **c** and **d** over the integers $\mathbb{Z}$, that is, $\mathcal{R} = \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. As before let the degree of **c** be 1 and the degree of **d** be 2. Let $\mathcal{R}_d$ be all polynomials in $\mathcal{R}$ that are homogeneous of degree d. We view $\mathcal{R}_d$ as an Abelian group. Similarly, let $\mathcal{T} = \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ and let $\mathcal{T}_d = \mathcal{T} \cap \mathcal{R}_d$. For a **cd**-monomial w, let $p(w)$ be the number of **d**'s that occur in w. A generating set of $\mathcal{T}_d$ is $2^{p(w)} \cdot w$, where w ranges over all **cd**-monomials of degree d.

Observe that Lemma 2.2 and Theorem 2.3 have the following integer analogues.

Lemma 6.1 *The Abelian group* $\mathcal{R}_{d+1}$ *is generated by* $\text{Pyr}(\mathcal{R}_d)$ *and* $\text{Pri}(\mathcal{R}_d)$.

Theorem 6.2 *The Abelian group* $\mathcal{R}_d$ *is generated by the* **cd***-index of* d*-dimensional polytopes.*

The goal of this section is to prove the analogous result of Theorem 6.2 for zonotopes. Let $\mathcal{S}_d$ be the subgroup of $\mathcal{R}_d$ generated by the elements $\Psi(Z)$, where Z ranges over all d-dimensional zonotopes. We begin by showing that $\mathcal{T}_d \subseteq \mathcal{S}_d$. This proof is essentially the same as the proof of Theorem 5.2. We need the following lemma.

Lemma 6.3 *The Abelian group* $\mathcal{T}_{d+1}$ *is generated by* $\text{Pri}(\mathcal{T}_d)$ *and* $D(\mathcal{T}_d)$.

The proof differs from the proof of Lemma 5.1 in only one point. We do not divide by 2; we use the relation $2 \cdot v \cdot \mathbf{d} = D(v \cdot \mathbf{c}) - D(v) \cdot \mathbf{c}$ and the fact that the monomial $v \cdot \mathbf{d}$ contains one more **d** than v, that is, $p(v \cdot d) = p(v) + 1$. We thus have that the generating set of $\mathcal{T}_{d+1}$ lies in the integral span of $\text{Pri}(\mathcal{T}_d)$ and $D(\mathcal{T}_d)$.

The results in Section 4 also apply to Abelian groups as well as vector spaces. Hence the proof of Theorem 5.2 generalizes to a proof of the following result.

Proposition 6.4 *The Abelian group $\mathcal{T}_d$ is contained in the the group $\mathcal{S}_d$.*

It remains to show the inclusion in the other direction, that is, $\mathcal{S}_d \subseteq \mathcal{T}_d$. For S a subset of $\{0, 1, \ldots, d-1\}$, we call S *sparse* if for all i, $\{i, i+1\} \not\subseteq S$ and $d-1 \notin S$. Suppose that S has cardinality p. Let w be a **cd**-monomial of degree d containing p **d**'s. We say that w *covers* the sparse set S if u_S appear in the expansion of $w = w(\mathbf{c}, \mathbf{d})$ as an **ab**-polynomial $w = w(\mathbf{a}+\mathbf{b}, \mathbf{ab}+\mathbf{ba})$. More explicitly, we can write $w = \mathbf{c}^{i_0} \cdot \mathbf{d} \cdot \mathbf{c}^{i_1} \cdot \mathbf{d} \cdots \mathbf{d} \cdot \mathbf{c}^{i_p}$, where $i_k \geq 0$. Define $j_0, \ldots, j_{p-1}$ by $j_0 = i_0$ and $j_{h+1} = j_h + 2 + i_{h+1}$. Observe that the hth **d** in w covers the positions j_h and j_{h+1}. Then w covers the sparse set S if and only if S is contained in the set $\{j_0, j_0+1, j_1, j_1+1, \ldots, j_{p-1}, j_{p-1}+1\}$. (Compare this notion with Stanley's definition of $\mathcal{W}_S$ [14].)

For a **cd**-monomial w and a **cd**-polynomial $F(\mathbf{c}, \mathbf{d})$, we denote the coefficient of w in $F(\mathbf{c}, \mathbf{d})$ by $[w]F(\mathbf{c}, \mathbf{d})$.

Definition 6.5 *For a d-dimensional polytope Q and a sparse subset S of $\{0, 1, \ldots, d-1\}$, define k_S by*

$$k_S = \sum_w [w]\Psi(Q),$$

where the sum ranges over all **cd**-*monomials w of degree d that cover S and contain exactly $|S|$* **d**'*s.*

We call the vector (k_S), where S ranges over all sparse subsets, the *flag k-vector*. As an example, let $d = 8$ and $S = \{0, 3, 5\}$. Then we have

$$k_{\{0,3,5\}} = [\mathbf{d}^3\mathbf{c}^2]\Psi(Q) + [\mathbf{d}^2\mathbf{cdc}]\Psi(Q) + [\mathbf{dcd}^2\mathbf{c}]\Psi(Q).$$

As a refinement of Proposition 1.3 in [14] we obtain the following relation.

Proposition 6.6 *The coefficients of the* **cd**-*monomials containing p* **d**'*s can be expressed as an integer linear combination of k_S's where S has cardinality p. That is, for w containing p* **d**'*s we have*

$$[w]\Psi(Q) = \sum_{|S|=p} q_{w,S} \cdot k_S,$$

where the sum ranges over sparse sets S and $q_{w,S}$ are integers.

The proof follows by ordering the sets and the monomials by lexicographic order. It is then easy to see that the defining relation of k_S corresponds to an lower triangular matrix with 1's on the main diagonal. Thus this linear relation is invertible over the integers.

Lemma 6.7 *For T a sparse subset of $\{0, 1, \ldots, d-1\}$ we have that*

$$h_T = \sum_{U \subseteq T} k_U.$$

The proof is by expanding the **cd**-index in terms of **a**'s and **b**'s and collecting terms.

Combining Lemma 6.7 with the relation $f_S = \sum_{T \subseteq S} h_T$ we obtain

$$f_S = \sum_{U \subseteq S} 2^{|S \setminus U|} \cdot k_U. \tag{6.1}$$

By the Principle of Inclusion-Exclusion the inverse of this relation is

$$k_S = \sum_{U \subseteq S} (-1)^{|S \setminus U|} \cdot 2^{|S \setminus U|} \cdot f_U. \tag{6.2}$$

Lemma 6.8 *For a zonotope Z we have that $2^{|S|}$ divides f_S.*

Proof: Observe that a zonotope is centrally symmetric and every face of a zonotope is a zonotope. Hence every face of the zonotope Z is centrally symmetric.

We may count f_S, where $S = \{i_1 < \cdots < i_k\}$, by first choosing a face F_{i_k} of dimension i_k, then choosing an i_{k-1}-dimensional face of F_{i_k}, and so on.

But since at each selection the face F_{i_j} is centrally symmetric (including Z), we know that there is an even number of choices of $F_{i_{j-1}}$. By multiplying together all these factors of 2, we obtain $2^{|S|}$. ■

Lemma 6.9 *For a zonotope Z we have that $k_S \equiv 0 \bmod 2^{|S|}$.*

Proof: It is enough to observe that $2^{|S|}$ divides $2^{|S \setminus U|} \cdot f_U$. ■

By combining Proposition 6.6 and Lemma 6.9 we obtain

Proposition 6.10 *The* **cd**-*index of a zonotope Z of dimension d belongs to $\mathcal{T}_d$. That is, $\mathcal{S}_d \subseteq \mathcal{T}_d$.*

Proof: It is enough to prove for a zonotope Z and a **cd**-monomial w that the coefficient of w in $\Psi(Z)$ is divisible by $2^{p(w)}$ where $p(w) = p$ is the number of **d**'s occurring in w. That is, $[w]\Psi(Z) \equiv 0 \bmod 2^p$.

Indeed, by Proposition 6.6 and Lemma 6.9 we have

$$[w]\Psi(Z) = \sum_{|S|=p} q_{w,S} \cdot k_S \equiv 0 \bmod 2^p,$$

where S ranges over all sparse subsets of $\{0, 1, \ldots, d-1\}$ having cardinality p. ■

Combining Propositions 6.4 and 6.10 gives us the main result of this section.

Theorem 6.11 *The Abelian group generated by the* **cd**-*indices of zonotopes of dimension d is precisely $\mathcal{T}_d$, that is, all integral polynomials of degree d in the variables* **c** *and* 2**d**.

As a direct consequence of this theorem, Proposition 6.6 and equation (6.2), we get the following.

Corollary 6.12 *The lattice spanned by flag f-vectors of all d-zonotopes is the set of all integral vectors $f = \{f_S\}$ where f_S is divisible by $2^{|S|}$.*

Since in the relation $f_S(Z) = f_{d-S}(\mathcal{H})$ between a d-zonotope Z and its dual (essential) hyperplane arrangement $\mathcal{H}$, the sets S and $d - S$ have the same cardinality, we obtain the following.

Corollary 6.13 *The lattice spanned by flag f-vectors of all essential hyperplane arrangements in $\mathbb{R}^d$ is the set of all integral vectors $f = \{f_S\}$ where f_S is divisible by $2^{|S|}$.*

7 Concluding remarks

Our method proves that zonotopes span, but is there a nice basis? We describe one possible basis, suggested in [12]. To do so, we define two operations P and B on a zonotope Z, where $PZ := \text{Pri}(Z)$ and $BZ := M(\text{Pri}(Z))$. Note that both result in a zonotope of one higher dimension. Now if we write a BP-word of length d, that is, a word of length d made with the letters B and P, we may view this as a sequence of d operations performed on the 0-dimensional zonotope $\mathbf{0}$, and so as a d-dimensional zonotope. Liu [12] conjectured that a basis for the flag f-vectors (and hence, the **cd**-indices) of all d-dimensional zonotopes could be constructed by forming all BP-words of length d ending in P and having no two consecutive B's. This should be compared to the basis for all polytopes given in [1], which was made up of similar combinations of pyramid and bipyramid operations.

We have described the lattice spanned by all **cd**-indices of zonotopes. The next natural problem is to determine all linear inequalities they must satisfy. It is known that the **cd**-index of any polytope must be nonnegative [14]. What more can be said about zonotopes? There is a family of linear inequalities known to be satisfied by flag f-vectors of zonotopes.

Theorem 7.1 (Varchenko/Liu) *If Z is a d-dimensional zonotope and $S = \{i_1, \ldots, i_k\}$, then*

$$\frac{f_S(Z)}{f_{i_1}(Z)} < \binom{d - i_1}{i_2 - i_1, \ldots, i_k - i_{k-1}, d - i_k} \cdot 2^{i_k - i_1}.$$

For the case $k = 2$ this was proved in [17] (see also [5, §4.6]), while for this generality a proof is given in [12]. Theorem 7.1 bounds the average number of $S \setminus \{i_1\}$ chains in links of i_1-faces of a d-dimensional zonotope by the number of $S \setminus \{i_1\}$ chains in a $(d-i_1)$-dimensional crosspolytope (all with the dimensions shifted appropriately). It is easy to find polytopes for which the inequalities in Theorem 7.1 fail. For example the cyclic polytope $C_d(n)$ does not satisfy the inequality for $S = \{0, 1\}$ when $n \geq 2d+1$.

For 3-zonotopes it is enough to consider the pairs (f_0, f_2). In this case the convex hull taken over all 3-zonotopes can be completely described as the cone with apex $(8, 6)$ (corresponding to the 3-cube) and extreme rays $(1, 1)$ and $(2, 1)$. Along the ray $(1, 1)$ can be found all zonotopes of the form $M^n(\text{Pri(square)})$, while on the other ray one finds all those of the form $\text{Pri}(M^n(\text{square}))$ (prisms over even polygons). Other than the fact that only even points appear, the problem of determining which lattice points in this cone are actually realized by zonotopes (or by oriented matroids) appears to be a difficult one. See, for example, [8, Chap. 18].

The **cd**-index of a zonotope does depend only on the underlying matroid, and not on the oriented matroid. This suggests that there is a **cd**-index for matroids, in fact, a **c**-2**d**-index, independent of whether they are orientable or not. The authors are currently investigating the **cd**-index without reference to orientation.

For certain classes of polytopes and Eulerian posets the flag h-vector has been given a combinatorial interpretation. We wonder if the flag k-vector can be also given a combinatorial interpretation. Observe that we only define k_S for sparse sets S. We may extend the flag k-vector to all sets by inverting the relation given in Lemma 6.7. It is known that the flag h-vector of a polytope, being the "fine" h-vector of a balanced Cohen-Macaulay complex, must be the fine f-vector of another balanced simplicial complex Δ [16, Thm 4.6]. Thus, in this case, the flag k-vector can be interpreted as the fine h-vector of this Δ.

By the definition of the flag k-vector, $k_S \geq 0$ for all sparse S whenever the **cd**-index is nonnegative. Is there a larger interesting class of posets for which the sparse flag k-vector is always nonnegative? For example, in the case just described this will occur if the complex Δ is itself Cohen-Macaulay; here the full flag k-vector will be nonnegative. That the full flag k-vector is not always nonnegative can be seen by examining the flag k-vector of a tetrahedron, for which $k_{\{0,1\}} = -4$.

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to contain the Adin g-cone, then our work would show that cubical spheres do not satisfy the toric g-theorem, which holds for rational polytopes.

3. Adin's cubical h-vector is a sum of alternating sums of components of the simplicial h-vectors of links of vertices. Is there a convenient interpretation of these alternating sums? Charney and Davis considered such a sum in the context of metric geometry, as did McMullen in his decomposition of the polytope algebra.

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as $m \to \infty$. Fixing $d \geq 2$ and $1 \leq i \leq \frac{d}{2}$, consider the doubly indexed sequence of cubical d-spheres $C^n_{P_{m,i}}$, indexed by m and n. By a diagonalization argument it follows that for every $k \geq 1$, there are cubical d-spheres $C^d_{k,i}$ such that

$$\frac{1}{g^c_i(C^d_{k,i})}\, g^c(C^d_{k,i}, t) \to t^i - t^{d+2-i}$$

as $k \to \infty$. This completes the proof. □

Remark: For any degree d polynomial $q(t)$, with $q_i = -q_{d-i} \geq 0$ for every $i \leq \lfloor \frac{d-1}{2} \rfloor$, and $q_{\lfloor \frac{d}{2} \rfloor} \geq 0$ if d is even, a similar construction yields cubical spheres S_k and numbers r_k with

$$\frac{1}{r_k}\, g^c(S_k, t) \to tq(t)$$

as $k \to \infty$. This is achieved by noting that the g^c-polynomials of two PL cubical spheres are essentially added by removing one maximal cube from each and identifying the resulting boundaries. (The resulting complex is again a PL sphere [14, Corollary 3.13].) To achieve a fixed ratio $a = q_i/q_j$, for example, we choose the sequences $P_{m,i}$ and $P_{m,j}$ as above and form the corresponding sequences $C^n_{P_{m,i}}$ and $C^n_{P_{m,j}}$. We simultaneously diagonalize these sequences by $n_{i,m} = 2^{r_j} g^s_j(P_{m,j}, t) q_i k_m$ and $n_{j,m} = 2^{r_i} g^s_i(P_{m,i}, t) q_j k_m$, attaching $C^{n_{i,m}}_{P_{m,i}}$ and $C^{n_{j,m}}_{P_{m,j}}$ along maximal faces, as above.

6 Further Comments

1. Since we have shown that the Adin g-cone is contained in the closure of the convex hull of f-vectors of PL cubical spheres, it would be especially interesting to determine whether the same is true for cubical polytopes (Conjecture 5.2), and conversely, if the Adin g-cone contains the convex hull of f-vectors of cubical polytopes (Conjecture 5.1). In fact, it would be nice to know if any of the results here remain true if *sphere* is replaced by *polytope*. If the fissuring operation as applied in §3 were to preserve shellability, then all of the spheres constructed there could be asserted to be *shellable* rather than just *PL*.

2. It is natural to compare the Adin h-vector and toric h-vector for cubical complexes. Both are invertible linear transformations of the f-vectors of cubical complexes, symmetric for Eulerian cubical complexes, nonnegative for shellable cubical complexes, and satisfy the reciprocity theorem for relative subcomplexes of balls [16]. A major difference is that the Adin h-vector is a lower-triangular linear transformation of the f-vector but the toric h-vector is not.

It would be of interest to determine if the toric g-cone contains the Adin g-cone. (Hetyei [10] has done this for the corresponding h-cones.) Were the toric g-cone not ratio [11].

In general, it is not at all clear whether there exist k-stacked cubical polytopes with many $(k-1)$-faces relative to smaller faces. However for $k=1$, we have the following, which was noted in [11] for $d=4$.

Corollary 5.6 *For any $n \geq 2^d$ there exists a 1-stacked cubical d-polytope P with at least n vertices, hence $g^c(\partial P)$ lies on the ray e_1, with $g_1^c(\partial P)$ arbitrarily large.*

5.2 PL Cubical Spheres

Though we can go no further with cubical polytopes, we can show that Conjecture 5.2 holds for PL cubical spheres. We prove the following

Theorem 5.7 *For each $1 \leq i \leq \frac{d}{2}$, there exist PL cubical d-spheres with g^c arbitrarily close to the ray e_i.*

Proof: For a simplicial $(d-1)$-polytope P, we let C be the PL cubical d-sphere and K the triangulation of P, as given by Theorem 3.1. For $n > 0$, define the iterated fissuring

$$C_P^n := C(MK, C \setminus MK^\circ)^n \tag{12}$$

defined by (4). Since, as in the proof of Theorem 3.1, ∂MK is collared in MK, C_P^n is a PL d-sphere.

By (1) and (5), we compute

$$\begin{aligned} f(C_P^n, t) &= f(C,t) + n(1+t)f(M\partial P, t) \\ &= f(C,t) + n2^r(1+t)f\Big(\partial P, \frac{t}{2}\Big), \end{aligned}$$

and

$$\begin{aligned} g^c(C_P^n, t) &= \frac{t(1-t)^{d+1}}{1+t} f\Big(C_P^n, \frac{2t}{1-t}\Big) + 2^d \frac{(1-t)(1+\bar{\chi}_{C_P^n}(-t)^{d+2})}{1+t} \\ &= g^c(C,t) + n2^r t(1-t)^d f\Big(\partial P, \frac{t}{1-t}\Big) \\ &= g^c(C,t) + n2^r t g^s(\partial P, t). \end{aligned}$$

Noting that $C = C_P^0$, we rewrite this as

$$g^c(C_P^n, t) = g^c(C_P^0, t) + n2^r t g^s(\partial P, t) \tag{13}$$

to emphasize that the first term is independent of n.

By [13], for fixed d, there are simplicial $(d-1)$-polytopes $P_{m,i}$ such that

$$\frac{1}{g_{i-1}^s(\partial P_{m,i})}\, g^s(\partial P_{m,i}, t) \to t^{i-1} - t^{d+1-i}$$

5.1 Stacked Cubical Polytopes

As a first attempt, we consider *stacked cubical polytopes,* analogous to the stacked simplicial polytopes considered by McMullen and Walkup in their paper introducing the generalized lower bound conjecture for simplicial polytopes [13].

Definition 5.3 *A cubical d-polytope is* k-stacked *if its boundary is the boundary of a cubical complex with no interior $(d-1-k)$-faces. Similarly, a cubical d-ball is* k-stacked *if it has no interior $(d-1-k)$-faces.*

Definition 5.4 *A simplicial complex is called* k-neighborly *if every set of vertices of cardinality k is a face.*

Neighborly stacked polytopes were used in [13, Theorem 3] to produce examples of simplicial polytopes having a simplicial g-vector with a dominant coordinate. The following shows that the cubical g-vector behaves analogously.

Proposition 5.5 *If $k \leq \frac{d}{2}$ and $\{P_n\}$ is a sequence of k-stacked cubical d-polytopes such that $f_{k-1}(\partial P_n)$ dominates $f_i(\partial P_n)$ for all $i < k-1$, that is, for each such* i,

$$\frac{f_i(\partial P_n)}{f_{k-1}(\partial P_n)} \to 0 \quad \text{as} \quad n \to \infty,$$

then $g_k^c(\partial P_n)$ dominates $g_i^c(\partial P_n)$ for all $i \neq k$.

Proof: For each n, if P_n is k-stacked, then $\partial P_n = \partial K_n$, where K_n is a cubical complex with no interior $(d-1-k)$-faces. Thus $f_i(K_n^\circ) = 0$ for all $i \leq d-1-k$. Since $h^c(K_n^\circ)$ is a lower-triangular linear transformation of $f(K_n^\circ)$, this means that $h_i^c(K_n^\circ) = 0$ for all $i \leq d-k$. Thus by Proposition 4.1, $h_i^c(K_n) = 0$ for all $i \geq k$ and so $g_i^c(\partial P_n) = h_i^c(K_n)$ for all $i \leq k$, and $g_i^c(\partial P_n) = 0$ for $k < i \leq \lfloor \frac{d}{2} \rfloor$. Since $f_i(K_n^\circ) = 0$ for $i \leq d-1-k$ and $f_{k-1}(\partial P_n)$ dominates $f_i(\partial P_n)$ for all $i < k-1$, the same is true for $f(K_n)$. Thus $h_k^c(K_n)$ dominates $h_i^c(K_n)$ for all $i < k$. Thus $g_k^c(\partial P_n)$ dominates $g_i^c(\partial P_n)$ for all $i \neq k$. □

For cubical 4-polytopes, we can write

$$g^c = (g_1^c, g_2^c) = (f_0 - 16, 16 - 3f_0 + 4f_3).$$

It is easy to verify that the boundary of any 1-stacked 4-polytope has $g_2^c = 0$, and hence the ray e_1 is in the convex hull of f-vectors of cubical 4-polytopes. The difficulty is in finding cubical 4-polytopes with g^c arbitrarily close to the ray e_2, *i.e.,* cubical 4-polytopes with an arbitrarily high ratio of facets to vertices. Jockusch was able to construct cubical 4-polytopes with a higher ratio of facets to vertices than previously expected possible, but did not determine if there is any bound on this

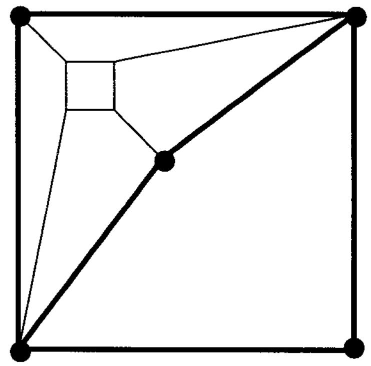

Figure 6: A cubical 2-ball containing $K_{2,3}$.

since C_r has the $\lfloor\frac{d-1}{2}\rfloor$-skeleton of the r-cube. Thus also $f(C) \leq f(C_r)$, since the coefficients f_j are nonnegative linear combinations of the coefficients of h^{sc}. □

For each $d \geq 1$, it is easy to find an example of a cubical d-sphere whose 1-skeleton does not lie in the 1-skeleton of any cube. Note first that the complete bipartite graph $K_{2,3}$ is not a subcomplex of any cube. This is clear because any two vertices in a cube that are joined by a path of length 2 must differ in exactly two coordinates, and so there must be precisely two such paths. Next consider the cubical 2-ball C in Figure 6. Note that $K_{2,3}$ is a subgraph of $(C)_1$, as indicated. For $d \geq 2$, take $S^d = \partial(C \times I^{d-1}) \supset C$. For $d = 1$ simply choose the boundary of a triangle.

5 A Cubical Lower Bound Conjecture

Adin raised the following "generalized lower bound conjecture" as a question [1, Question 2].

Conjecture 5.1 *If K is a cubical d-polytope, then $g_i^c(\partial K) \geq 0$, for all $i \leq \lfloor\frac{d}{2}\rfloor$.*

Here we conjecture that these are the best possible linear inequalities.

Conjecture 5.2 *The closed convex hull of the f-vectors of all cubical d-polytopes is the translated cone given by the inequalities $g_i^c \geq 0$, for all $i \leq \lfloor\frac{d}{2}\rfloor$, and $g_0^c = 2^{d-1}$.*

Let e_i denote the positive ray in the i^{th} coordinate direction. One approach to proving Conjecture 5.2 is to show that, for any $1 \leq i \leq \frac{d}{2}$, there exist cubical d-polytopes with g^c arbitrarily close in direction to the ray e_i. If both conjectures are correct, then the "Adin g-cone" so defined (*i.e.*, the set of f-vectors whose corresponding g^c-vectors are nonnegative) is exactly the closure of the convex hull of all f-vectors of cubical polytopes. The first conjecture is only known to be true in the case $i = 1$ [1, 5]. We address the second conjecture here.

Adin has shown that the cubical h-vector has many properties analogous to those of the simplicial h-vector. For example, if K is Eulerian then $h^c(K)$ is symmetric, and for any K, $h^c(K)$ is a lower-triangular linear transformation of $f(K)$. In particular, for any cubical $(d-1)$-complex K, and for all $i \leq d$, we have

$$h_i^c(K) = (-1)^i 2^{d-1} f_{-1}(K) + \sum_{j=1}^{i} (-1)^{i-j} 2^{j-1} f_{j-1}(K) \sum_{k=0}^{i-j} \binom{d-j}{k}, \tag{11}$$

where $f_{-1}(K) = 1$ [1, Lemma 1]. The relation (11) can be inversed to give the f_j as nonnegative linear combinations of the h_i^c.

We state without proof a few more such properties of h^c. In what follows, we assume that K is a cubical complex homeomorphic to a $(d-1)$-ball. For $K^\circ = K \setminus \partial K$, we have

$$f(K^\circ, t) = f(K, t) - f(\partial K, t),$$

and we define h^{sc} for K° as in (8), with the same rank as K, namely, $d-1$, and h^c as in (11), but with $f_{-1}(K^\circ) = 0$.

Proposition 4.1 *If K is a cubical $(d-1)$-ball, then*

1. *$h_i^c(K) = h_{d-i}^c(K^\circ)$ for all i, and*
2. *$g_i^c(\partial K) = h_i^c(K) - h_{d-i}^c(K)$ for all $i \geq 1$.*

We now consider Kalai's upper bound conjecture for cubical spheres. Let C_r be a $\lfloor \frac{d-1}{2} \rfloor$-neighborly cubical d-sphere with 2^r vertices. By definition, $f_i(C_r)$ is the number of i-faces in an r-cube, for $i \leq \lfloor \frac{d-1}{2} \rfloor$. By the cubical Dehn-Sommerville equations, the remaining $f_i(C_r)$ are determined and thus are independent of the particular C_r chosen.

Conjecture 4.2 *(Kalai): If C is a cubical d-sphere with 2^r vertices, then $f_i(C) \leq f_i(C_r)$, for all i.*

Using the Adin h-vector it is easy to prove the conjecture in the case of odd d for any cubical d-sphere whose 1-skeleton lies in the r-cube.

Theorem 4.3 *If d is odd and C is any cubical d-sphere for which every vertex has degree at most r, then $f_i(C) \leq f_i(C_r)$ for all i.*

Proof: If C is any such cubical d-sphere C , then $lk_C v$ is a simplicial $(d-1)$-sphere with at most r vertices, for each v. Thus by the upper bound theorem for simplicial spheres, $h^s(lk_C v) \leq h^s(S)$, where S is any $\lfloor \frac{d}{2} \rfloor$-neighborly simplicial $(d-1)$-sphere with r vertices. Since d is odd, $\lfloor \frac{d}{2} \rfloor = \lfloor \frac{d-1}{2} \rfloor$. Thus

$$h^{sc}(C) = \sum_v h^s(lk_C v) \leq \sum_v h^s(S) = h^{sc}(C_r),$$

4 Adin's h-vector and Kalai's Upper Bound Conjecture

Recently, Adin defined a "cubical h-vector" for studying the face numbers of cubical complexes [1]. This invariant appears to be a good analog of the usual h-vector for simplicial complexes.

For a ranked poset P, with $rank(P) = d$, we denote by $f_i = f_i(P)$ the number of elements of rank i and define the polynomial

$$f(P,t) := \sum_{i=0}^{d} f_i t^i. \tag{6}$$

From this, we define polynomials $h^s(P,t)$, $h^{sc}(P,t)$ and $h^c(P,t)$ by

$$h^s(P,t) = (1-t)^d f\Big(P, \frac{t}{1-t}\Big), \tag{7}$$

$$h^{sc}(P,t) = (1-t)^d f\Big(P, \frac{2t}{1-t}\Big) \tag{8}$$

and

$$h^c(P,t) = \frac{t(1-t)^d}{1+t} f\Big(P, \frac{2t}{1-t}\Big) + 2^d \frac{1+\tilde{\chi}_P(-t)^{d+2}}{1+t} \tag{9}$$

$$= \frac{1}{1+t}\Big(t\, h^{sc}(P,t) + 2^d(1+\tilde{\chi}_P(-t)^{d+2})\Big), \tag{10}$$

where $\tilde{\chi}_P := f(P,-1) - 1$ is the reduced Euler characteristic of P (*c.f.* [1, (1-3)]).

In general, the coefficient of t^i in a polynomial $q(t)$ will be denoted by q_i. The coefficients of h^s, h^{sc} and h^c will be referred to, respectively, as the ***simplicial, short*** ***cubical*** and ***cubical h-vectors*** of the ranked poset P. (Note that in h^s, one uses the simplicial rank in the f-polynomial, whereas in h^{sc} and h^c one uses the cubical rank.) As observed by Hetyei,

$$h^{sc}(K,t) = \sum_{v \in V} h^s(lk_K\, v, t).$$

It is also useful to observe that $h^{sc}_i = h^c_i + h^c_{i+1}$ for all $0 \le i \le d$ [1].

Each h-polynomial has an associated g-polynomial, defined by multiplying it by $1-t$. The associated "g-vector" is usually taken to be the first half of the coefficients of the g-polynomial (excluding the middle term, when the degree is even, as well as the constant term). Thus, if P is the boundary complex of a cubical d-polytope, the degrees of h^{sc}, h^c and g^c are $d-1$, d and $d+1$, respectively, so the relevant coefficients of g^c are $g^c_1, \ldots, g^c_{\lfloor \frac{d}{2} \rfloor}$.

links in K_i. Thus ∂MK_i is collared in MK_i [14, Corollary 2.26], and hence by (3), it follows that $C_{i+1} \cong_{PL} C_i$.

Now take $C := C_n$ and $K := K_n$. □

Figure 5 illustrates the construction of the 3-sphere C corresponding to a pentagon. C_0 is the 3-sphere consisting of two 3-cubes joined along their boundaries, and MK_0 is the "inside" cube. C_1 is then the boundary of the 4-cube. Since $K_1 = T * v_3$, where T is the complex of two edges, $MK_1 = MT \times I$, where MT is as given in Figure 1. The final complex MK (not pictured) is the product of I with the cubical complex shown in Figure 2.

A first consequence of Theorem 3.1 and its proof is the existence of "neighborly" cubical spheres.

Definition 3.2 *A cubical complex is said to be* n-neighborly *if its* n*-skeleton is that of a cube.*

Corollary 3.3 *There exist* $\lfloor \frac{d-1}{2} \rfloor$*-neighborly cubical* d*-spheres with* 2^k *vertices for every* $k > d$.

Proof: Choose $P = C(k, d-1)$, the cyclic *(d-1)*-polytope with k vertices, ordered arbitrarily. Note that each ball K_i in the proof of Theorem 3.1 has as boundary the cyclic polytope $C(d+i, d-1)$. Hence, each K_i has as its $\lfloor \frac{d-3}{2} \rfloor$-skeleton the $\lfloor \frac{d-3}{2} \rfloor$-skeleton of a $(d+i-1)$-simplex. Thus the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of $M\partial K_i$ is that of a $(d+i)$-cube.

We will show, by induction, that $(C_i)_{\lfloor \frac{d-1}{2} \rfloor} = (M\partial K_i)_{\lfloor \frac{d-1}{2} \rfloor}$. First, recall that C_0 has the $(d-1)$-skeleton of a d-cube, and $(C_0)_{d-1} = (M\partial K_0)_{d-1}$. Assuming the assertion for i, it follows from of the fissure construction that

$$\begin{aligned}
(C_{i+1})_{\lfloor \frac{d-1}{2} \rfloor} &= \Big(C_i(MK_i, C_i \setminus (MK_i)^\circ)\Big)_{\lfloor \frac{d-1}{2} \rfloor} \\
&= \Big(M\partial K_i \times I\Big)_{\lfloor \frac{d-1}{2} \rfloor} \\
&= (I^{d+i+1})_{\lfloor \frac{d-1}{2} \rfloor} \\
&= \Big(M\partial K_{i+1}\Big)_{\lfloor \frac{d-1}{2} \rfloor}.
\end{aligned}$$

The second equality follow from the fact that the fissuring is taking place along $M\partial K_i$, which has the same $\lfloor \frac{d-1}{2} \rfloor$-skeleton as C_i by the induction hypothesis.

Thus, C_{k-d-1} is the desired $\lfloor \frac{d-1}{2} \rfloor$-neighborly cubical d-sphere with 2^k vertices. □

We remark that the spheres constructed in Corollary 3.3 are always PL. It is an open question whether there exist neighborly *polytopal* spheres.

$$\begin{aligned} MK_i &\subset M(\partial K_{i-1} * \{v_{d+i}\}) \\ &= M\partial K_{i-1} \times I \\ &= \partial M K_{i-1} \times I \\ &\subset C_{i-1}(MK_{i-1}, C_{i-1} \setminus (MK_{i-1})^\circ) \\ &= C_i. \end{aligned}$$

Note that beginning with C_1, all the C_i are cubical *complexes* and that both the inclusions above are inclusions of complexes (preserve rank). This shows MK_i to be a subcomplex of C_i.

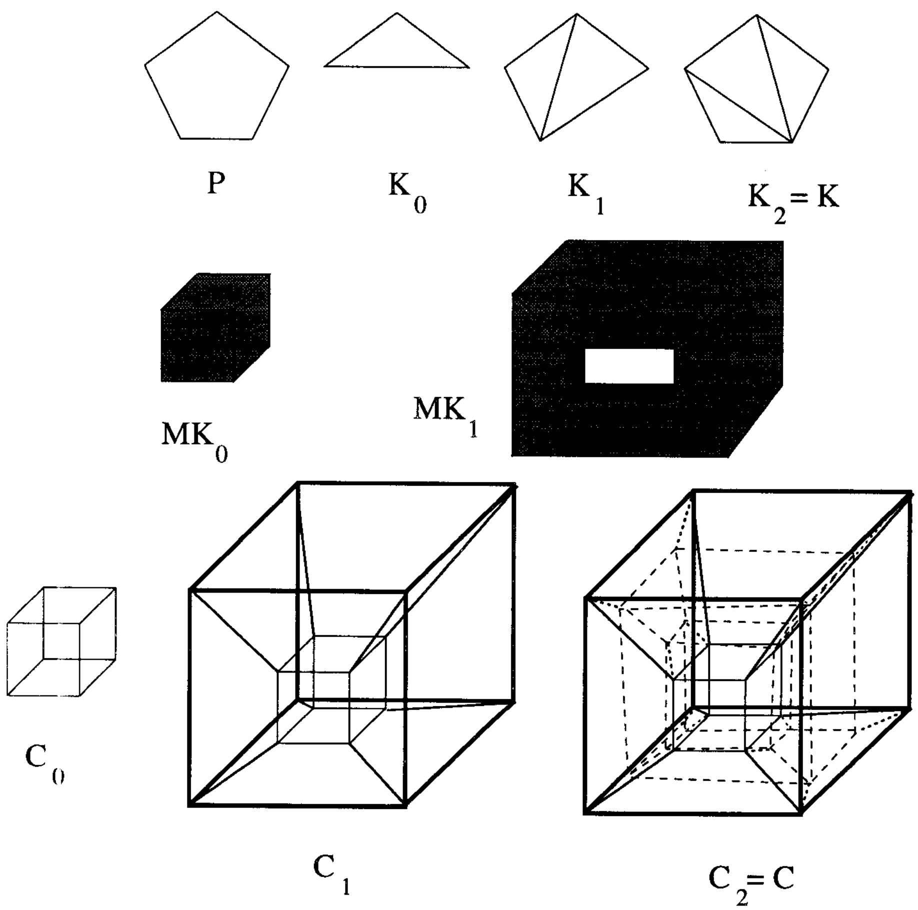

Figure 5: A cubical 3-sphere C from a pentagon P.

To verify that $C_i \setminus (MK_i)^\circ$ is an order ideal in C_i, we note, by induction, that MK_i is a full-dimensional (pure) subcomplex in the the sphere C_i, and so the complement of its interior is a complex. Finally, to check that C_{i+1} is a PL $(d+1)$-sphere, note that MK_i is a PL $(d+1)$-manifold with boundary, since its links are

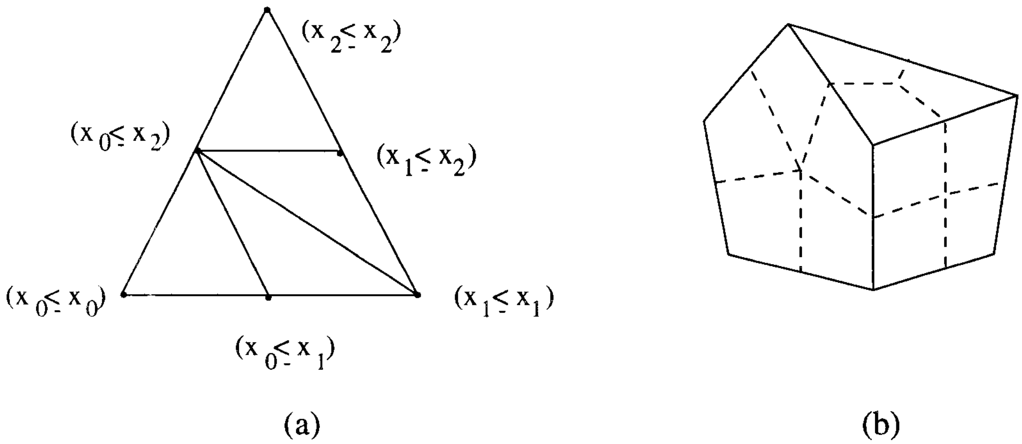

Figure 4: (a) The subdivision of the simplex $x_0 \leq x_1 \leq x_2$. (b) The barycentric cover of a complex with three maximal cells.

3 Neighborly Cubical Spheres

Now we use the mirroring and fissuring operations to produce a "neighborly" cubical sphere for each $n \geq d+1$, *i.e.*, a cubical d-sphere with the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of the n-cube. The existence of such spheres was suggested by Kalai (personal communication). We begin by constructing, for a given simplicial d-polytope P, a cubical $(d+1)$-sphere having the mirror complex of ∂P as a subcomplex.

Theorem 3.1 *If P is a simplicial d-polytope, then there is a triangulation K of P and a PL-cubical (d+1)-sphere C such that the mirror complex MK is a subcomplex of C having $M\partial P$ as its boundary.*

Proof: Given the polytope P, along with an arbitrary ordering on its vertices $v_0, v_1, \ldots, v_{n+d}$ (which we assume are in general position), we form a sequence of simplicial d-balls $K_0, \ldots, K_n$ such that $K_i \subset \partial K_{i-1} * \{v_{d+i}\}$, where K_0 is the d-simplex spanned by $v_0, \ldots, v_d$ and K_i is the join of v_{d+i} with that part of the boundary of K_{i-1} that it does not see. By construction, the K_i are all PL d-balls and $\partial K_n = \partial P$, so $\partial MK_n = M\partial P$.

Let C_0 be the cubical poset made up of two $(d+1)$-cubes sharing a common boundary. We consider the $(d+1)$-cube MK_0 to be one of these cubes, and hence $C_0 \setminus (MK_0)^\circ$ is the other. Finally, $|C_0|$ is a PL $(d+1)$-sphere.

Define

$$C_{i+1} := C_i(MK_i, C_i \setminus (MK_i)^\circ).$$

We assert that, for $1 \leq i < n$, C_{i+1} is well-defined and is a cubical PL $(d+1)$-sphere. To verify that C_{i+1} is well-defined, we must check that both MK_i and $C_i \setminus (MK_i)^\circ$ are subcomplexes (*i.e.*, order ideals) in C_i. Assuming this to be true for C_{i-1} and MK_{i-1}, we observe that for $i \geq 1$

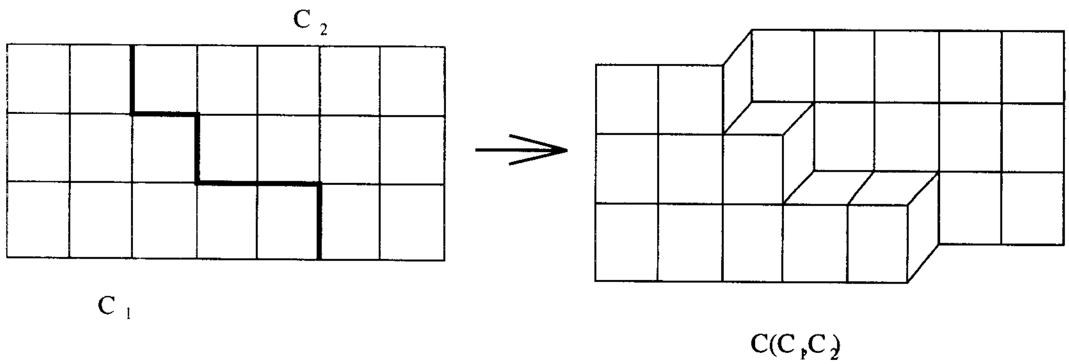

Figure 3: Fissuring along $C_1 \cap C_2$.

2.3 Barycentric Covers

Finally, we define a cubical complex midway between a complex of *simple* polytopes and its barycentric subdivision. We call it the *barycentric cover*. In the case of a simplicial complex, this is the same as the *cubical barycentric subdivision* used by Hetyei [10]. The remainder of this section is not used in what follows.

Let KP be the poset with elements the order relations of an arbitrary poset P, partially ordered by inclusion (*i.e.*, $(u \leq v) \leq (x \leq y)$ if and only if $x \leq u \leq v \leq y$). As shown below, KP is cubical whenever P is a poset having all intervals Boolean algebras. This includes simplicial and cubical posets (more generally, face posets of polyhedral complexes with simple, nonempty cells) as well as their duals. In fact $KP = K(P^{op})$. It is straightforward to check that K distributes over product, *i.e.*, $K(P \times Q) = K(P) \times K(Q)$.

Proposition 2.2 *If P has Boolean intervals then KP is a cubical poset. Further, if $\hat{P}$ is a lattice, then so is $\widehat{KP}$.*

Theorem 2.3 *$|KP|$ gives a polyhedral subdivision of $|P|$ by the map taking the point $(x \leq x)$ to itself and the point $(x < y)$ in $|KP|$ to the midpoint of the edge $x < y$ in $|P|$, and extending linearly over every closed simplex in $|KP|$.*

In the case in which P is itself the face poset of a polyhedral complex, we obtain the following corollary, which justifies calling KP the *barycentric cover* of P.

Corollary 2.4 *If P is the face poset of a polyhedral complex, then KP is the face poset of a polyhedral subdivision of P, lying between P and $|P|$ in the refinement order.*

When the underlying complex of P has simple facets and P does not include the empty set (so P has Boolean intervals), KP will be a cubical complex by Proposition 2.2. See Figure 4(b). Further, the triangulation of each d-cube in KP induced by $|P|$ is the standard triangulation into $d!$ simplices given by all the coordinate permutations.

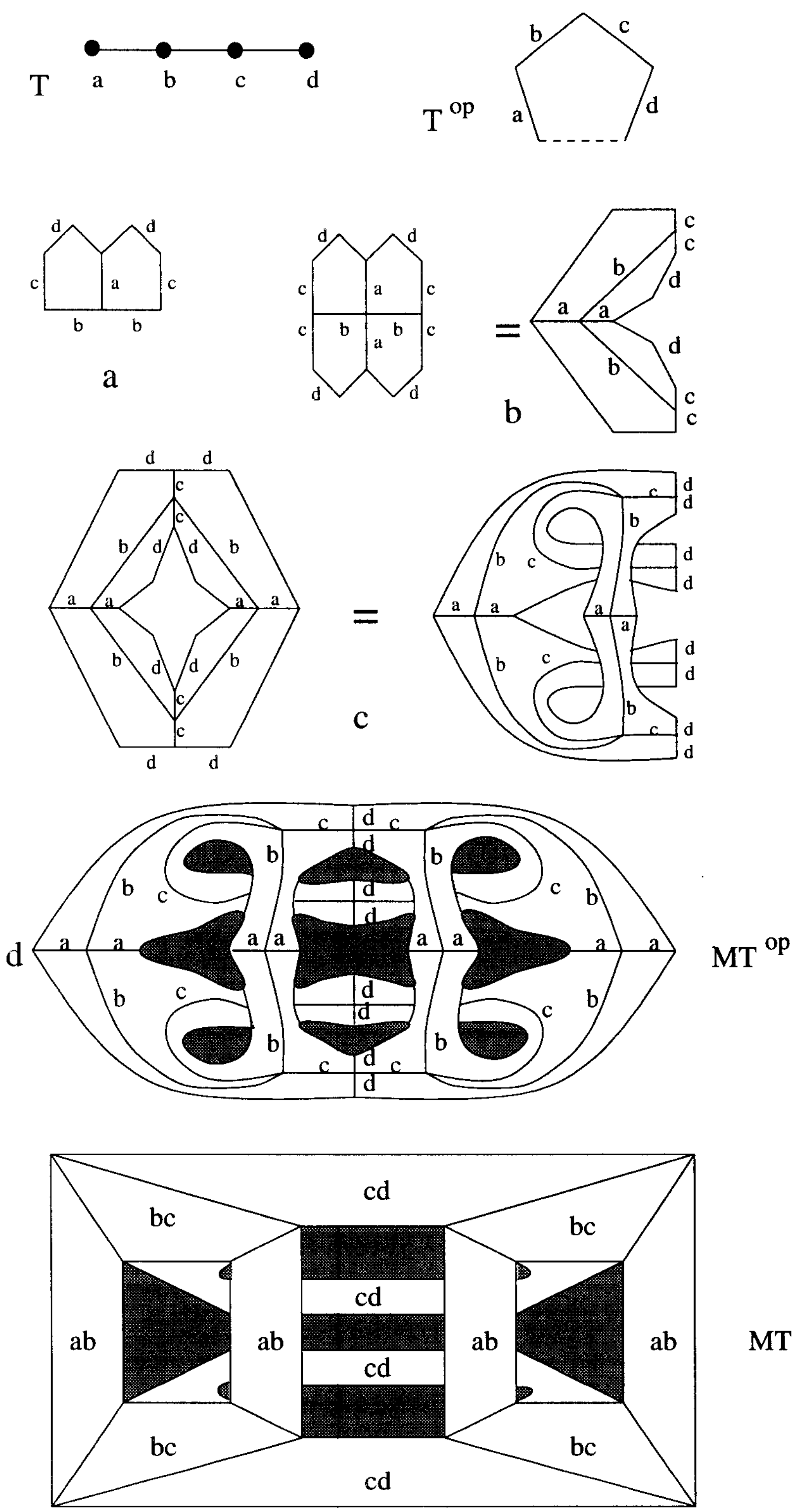

Figure 2: The mirror complex of three edges.

Proof: To see that MT is a cubical complex, note that it can be realized as a subcomplex of the n-cube I^n by associating the point $a = (a_1, a_2, \ldots, a_n)$ with the face of the cube having a as its centroid. The interval lying above any minimal element of MT is, up to signs, the poset T. The statement about skeletons follows directly. □

It follows immediately that if T is a d-simplex, then MT is a $(d+1)$-cube. Further, $\partial(MT) = M(\partial T)$.

We say MT results from a mirroring of T, since if T is the boundary complex of a simplicial polytope P then by taking the dual P^{op} of P, mirroring P^{op} across all its facets, and then taking the dual of the resulting cell complex, we get back MT. This also works for more general T, as is illustrated in Figure 2, where T is a line segment divided into three edges.

We note here that since the operation M commutes with poset product, we get $M(T_1 * T_2) = M(T_1) \times M(T_2)$, where $T_1 * T_2$ is the join of complexes T_1 and T_2. Further note that if T is a subcomplex of σ^{n-1}, the f-polynomials of T and MT are related by

$$f(MT, t) = 2^n f(T, \frac{t}{2}). \tag{1}$$

2.2 Cubical fissures

We define next an operation on a cubical poset C that depends on a pair of order ideals C_1 and C_2 in C. Let $C(C_1, C_2) \subset C \times I$ be the poset defined by

$$C(C_1, C_2) := (C_1 \times \{1\}) \ \cup \ (C_1 \cap C_2 \times \{0\}) \ \cup \ (C_2 \times \{-1\}). \tag{2}$$

We call this the *fissure* of C *between* C_1 and C_2.

That $C(C_1, C_2)$ is cubical follows from the fact that it is an order ideal in the cubical poset $C \times I$. Topologically, we have the relation

$$|C(C_1, C_2)| := |C_1| \ \cup_{C_1 \cap C_2 \times \{1\}} \ \Big(|C_1 \cap C_2| \times [-1, 1]\Big) \ \cup_{C_1 \cap C_2 \times \{-1\}} \ |C_2|. \tag{3}$$

When C is the poset of a cubical complex (also denoted by C), $C(C_1, C_2)$ is the poset of the complex obtained by lifting C_1 by height one, dropping C_2 by one, and filling in the resulting fissure by $(C_1 \cap C_2) \times [-1, 1]$. See Figure 3.

We can iterate the fissuring of C between a pair of complexes as follows. Let $C(C_1, C_2)^0 := C$, $C(C_1, C_2)^1 := C(C_1, C_2)$ and, for $r \geq 1$

$$C(C_1, C_2)^r := \Big(C(C_1, C_2)^{r-1}\Big)\Big(C_1(C_1, C_1 \cap C_2)^{r-1}, C_2\Big). \tag{4}$$

Note that successive fissurings separate C_1 and C_2 by more and more copies of $(C_1 \cap C_2) \times [-1, 1]$. It follows directly from (2) and (4) that

$$f(C(C_1, C_2)^r, t) = f(C_1 \cup C_2, t) + r(1 + t) f(C_1 \cap C_2, t), \tag{5}$$

for $r \geq 1$.

poset) where each face is a 0-1 vector with a 0 in the i^{th} place if and only if the vertex i belongs to the face. Thus T is partially ordered by $0 > 1$ extended componentwise. Then we construct a partially ordered set MT as follows:

$$MT = \{(a_1, a_2, \ldots, a_n) : (|a_1|, |a_2|, \ldots, |a_n|) \in T\} \subseteq I^n,$$

partially ordered by $0 > 1$ and $0 > -1$ extended componentwise. Note that MT depends on the ambient simplex σ^{n-1} as well as the complex T.

This operation has a long history. For the case of the m-gon, it was used by Coxeter [6] to produce regular maps $\{4, m|4^{[\frac{1}{2}m]-1}\}$ on surfaces. It has been used by Davis in the study of reflection groups and toric varieties (see [7, p.108], for the Coxeter system $(\mathbf{Z}_2^{[n]}, [n])$, and [8]). It has also been studied by Schulte [15, §5], where it is denoted 2^T, and in [12], where the topology of 2^T is studied for neighborly T. A dual version, as illustrated in Figure 2, can be found in [4, §3.2]; indeed, using the notation there, $MT = (\mathcal{B}_{T^{op}})^{op}$, where the subspace arrangement is a poset with ordering by inclusion.

A simple example is given in Figure 1. Here T is the simplicial complex consisting of two adjacent edges, and MT is the boundary of the 3-cube minus two opposite (open) facets.

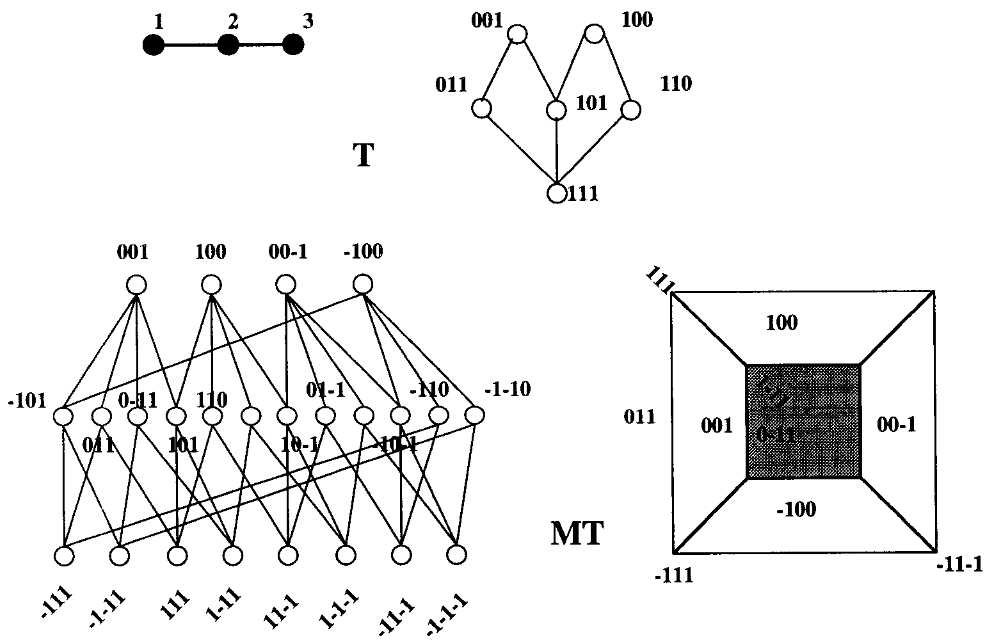

Figure 1: The mirror complex of two edges.

Proposition 2.1 *MT is the face lattice of a cubical complex in which the link of any vertex is isomorphic to the original simplicial complex T. If T has the k-skeleton of the $(n-1)$-simplex, then MT has the $(k+1)$-skeleton of the n-cube.*

Some preliminaries on posets are in order here. For a poset P, we denote by $[x, y]$ or $[x, y]_P$ the interval $\{z \in P : x \leq z \leq y\}$, by $\bigwedge(x)$ or $\bigwedge_P(x)$ the principal (lower) *order ideal* $\{z \in P : z \leq x\}$, and by $\bigvee(x)$ or $\bigvee_P(x)$ the principal *filter* $\{z \in P : z \geq x\}$. We will also refer to $\bigvee_P(x)$ as the link in P of x or $lk_P\, x$. By $\widehat{P}$ we mean P with a 0 and 1 adjoined, and by P^{op} we mean the underlying set of P with the order reversed. We denote by $|P|$ the (simplicial) complex of chains in P.

By a *simplicial poset* we mean one in which every order ideal $\bigwedge(x)$ is a Boolean algebra (*i.e.*, a product of copies of B_1, the Boolean algebra on one element). By a *cubical poset* we mean one in which each order ideal $\bigwedge(x)$ is a product of copies of I, the face poset of an interval, excluding the empty set. We consider I to be the poset $\{0, 1, -1\}$ with ordering $1 < 0$ and $-1 < 0$ (and so the Hasse diagram of I is $\bigwedge$). Thus the face poset of any simplicial complex (including the empty set) is simplicial, while the face poset of any cubical complex (excluding the empty set) is cubical. For this reason, throughout this paper when we consider the face poset of any simplicial complex, it will always *include* the empty set, while that of a cubical complex will always *exclude* the empty set. Two other concepts we will use are the boundary and interior of a poset. If P is a finite poset, denote by $\partial P = \{y : y \leq x, |\bigvee_P(x)| = 2\}$ the boundary of P, and call $P^\circ = P \backslash \partial P$ the interior of P. To keep notation to a minimum, we will usually denote a complex and its face poset by the same symbol.

Both simplicial and cubical posets are ranked, the rank of an element being one less than the cardinality of a maximal chain ending at this element. Thus in a simplicial complex, the rank of a face is one more than its dimension, while in a cubical complex, rank is the same as dimension. We will restrict our attention here to simplicial posets with unique minimal element that are meet semilattices (*i.e.*, simplicial complexes) and to cubical posets P such that $\widehat{P}$ is a lattice (called cubical complexes). A map will be called a complex map if it preserves rank. Thus being a subcomplex of a cubical complex is a stronger property than merely being a subposet. We note that poset product corresponds to complex join in the simplicial case and product in the cubical case, and thus order ideals in cubical posets are posets of cubes. Finally, for a complex X, we denote by $(X)_k$ its k-skeleton (the set of all r-faces of X, $r \leq k$).

2 Mirrors, Fissures and Barycentric Covers

We define and study three constructions leading to cubical complexes. Two of these, mirroring and fissuring, are used in the constructions in later sections.

2.1 Mirroring

We begin with an operation which converts a simplicial complex to a cubical complex. Let T be a subcomplex of the $(n-1)$-simplex σ^{n-1}. We can think of T (including the empty set) as a partially ordered set (the corresponding simplicial

Neighborly Cubical Spheres and a Cubical Lower Bound Conjecture

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April 10, 1996

Abstract

Using mirrors and cyclic polytopes, we construct cubical d-spheres which are the analogs of cyclic polytopes in the sense that they have the $\lfloor \frac{d-1}{2} \rfloor$-skeleta of cubes. The existence of these neighborly cubical spheres leads to a special case of an upper bound conjecture for cubical spheres, suggested by Kalai. We extend the same construction to show that the closed convex hull of f-vectors of cubical spheres contains a cone described by Adin, as an analog to the generalized lower bound theorem for simplicial polytopes.

1 Introduction

In the past few years, there has been much activity regarding the enumeration of faces of cubical polytopes. In this paper we continue efforts to put this subject on more of a parallel track with that of simplicial polytopes.

In §2, we discuss some basic constructions of cubical complexes, mirrors, fissures and barycentric covers. In §3 we use the mirror and fissure operations to produce what we call "neighborly" cubical spheres. In §4, we define Adin's "cubical h-vector", an enumerative invariant for cubical complexes [1], and use it to prove a special case of an upper bound conjecture due to Kalai.

In §5 we consider a cubical analog of the generalized lower bound theorem for simplicial polytopes, formulated in terms of Adin's cubical h-vector. We show that if this conjecture holds for all cubical spheres, it gives the tightest set of linear inequalities possible for their face numbers. We conclude with questions in §6.

We would like to thank Rachel Hastings, Gabor Hetyei and an anonymous referee for reading and making useful comments on earlier versions of this paper.

*Supported in part, respectively, by an NSF Postdoctoral Fellowship; NSF Grants DMS 9207700 and 9500581; and an NSF Planning Grant.