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Duality and Minors of Secondary Polyhedra

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Using Minkowski integration, we define the secondary polyhedron of a vector configuration $\mathscr{A}$ and study its behavior under the matroidal operations of duality, deletion, and contraction. A main tool is the identification of the regular polyhedral subdivisions of $\mathscr{A}$ with the cells in the dual chamber complex. As an application we construct a non-regular triangulation of a cyclic polytope. © 1993 Academic Press, Inc.

1. INTRODUCTION

Given a spanning set $\mathscr{A} = \{a_1, ..., a_n\}$ of non-zero vectors in $\mathbf{R}^d$, we are interested in the $(n-d)$-dimensional *secondary polyhedron* $\Sigma(A)$ whose faces correspond to the regular polyhedral subdivisions of the $(d-1)$-dimensional spherical polytope $P(\mathscr{A}) := \text{pos}(\mathscr{A}) \cap S^{d-1}$. The spherical polytope is thought of as a $(d-1)$-polytope in the usual affine sense whenever $\text{pos}(\mathscr{A})$ is a pointed cone, and in this case $\Sigma(\mathscr{A})$ is bounded and is normally equivalent to the secondary polytope defined in [5] (see also [2, 3, 6, 8]). On the other hand, if $\text{pos}(\mathscr{A}) = \mathbf{R}^d$, then $\Sigma(\mathscr{A})$ is unbounded and its vertices correspond to the regular triangulations of the $(d-1)$-sphere with vertices on the rays of $\mathscr{A}$. Our approach extends the work of Oda and Park [10], who have constructed the normal fan of $\Sigma(\mathscr{A})$ by means of linear transforms.

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Each $a_i \in \mathscr{A}$ gives rise to two *minors*. The minor by *deletion* of a_i is the configuration $\mathscr{A} \setminus a_i = \{a_1, ..., a_{i-1}, a_{i+1}, ..., a_n\}$ in $\mathbf{R}^d$. The minor by *contraction* of a_i is the configuration $\mathscr{A}/a_i = \{\pi_{a_i}(a_1), ..., \pi_{a_i}(a_{i-1}), \pi_{a_i}(a_{i+1}), ..., \pi_{a_i}(a_n)\}$ in $\mathbf{R}^{d-1}$, where $\pi_{a_i}: \mathbf{R}^d \to \mathbf{R}^{d-1}$ is any epimorphism with kernel span(a_i). It is our objective to relate the secondary polyhedron $\Sigma(\mathscr{A})$ of $\mathscr{A}$ to the secondary polyhedra $\Sigma(\mathscr{A} \setminus a_i)$ and $\Sigma(\mathscr{A}/a_i)$ obtained by deletion and contraction of any $a_i \in \mathscr{A}$.

In Section 2 we use an integral representation as in [3] to define $\Sigma(\mathscr{A})$, we discuss its combinatorial interpretation in terms of polyhedral subdivisions, and we give formulas for its vertices and the extreme rays of its recession cone. We use this integral representation to give a description of $\Sigma(\mathscr{A} \setminus a_i)$ as a facet of $\Sigma(\mathscr{A})$. Section 3 is concerned with the behavior of the secondary polyhedron $\Sigma(\mathscr{A})$ under duality and under minors by contraction. We show that the boundary complex of $\Sigma(\mathscr{A})$ is antiisomorphic to the *chamber complex* of a linear transform $\mathscr{B}$ of $\mathscr{A}$, and we show that $\Sigma(\mathscr{A}/a_i)$ either is a Minkowski summand of $\Sigma(\mathscr{A})$ or can be obtained from one by removing a single facet. In Section 4 we answer a question raised by Kapranov and Voevodsky [8] by presenting an example of a non-regular triangulation of a cyclic polytope.

2. The Secondary Polyhedron and Deletions

Let $\mathscr{A} = \{a_1, ..., a_n\}$ be a set of n non-zero vectors spanning $\mathbf{R}^d$. The polyhedral cone pos$(\mathscr{A})$ is the image of the non-negative orthant $\mathbf{R}^n_+ = \text{pos}(\{e_1, ..., e_n\})$ in n-space under the linear map $\pi: e_i \mapsto a_i$. The fiber of a point $x \in \text{pos}(\mathscr{A})$ is the polyhedron

$$\pi^{-1}(x) = \{(\lambda_1, ..., \lambda_n) \in \mathbf{R}^n_+ : \lambda_1 a_1 + \cdots + \lambda_n a_n = x\} \tag{2.1}$$

consisting of all positive representations of x with respect to $\mathscr{A}$. Note that each k-face of the spherical polytope $P(\mathscr{A}) = \text{pos}(\mathscr{A}) \cap S^{d-1}$ is of the form $F \cap S^{d-1}$, where F is a $(k+1)$-face of pos$(\mathscr{A})$. We define the *secondary polyhedron* of $\mathscr{A}$ to be the Minkowski integral

$$\Sigma(\mathscr{A}) := \int_{P(\mathscr{A})} \pi^{-1}(x)\, dx \tag{2.2}$$

with respect to the rotation invariant probability measure on the unit sphere S^{d-1}. This means that $\Sigma(\mathscr{A})$ is the set of all points $\int_{P(\mathscr{A})} \gamma(x)\, dx$ in $\mathbf{R}^n$, where $\gamma: P(\mathscr{A}) \to \mathbf{R}^n_+$ is a measurable function such that $\pi \circ \gamma$ is the identity (see [3]).

We will now derive a description of the recession cone and the face lattice of $\Sigma(\mathscr{A})$. A *circuit* of $\mathscr{A}$ is any non-zero vector of the form

$$C_v := \sum_{i=1}^{d+1} (-1)^i \det(a_{v_1}, ..., a_{v_{i-1}}, a_{v_{i+1}}, ..., a_{v_{d+1}})\, e_{v_i}, \tag{2.3}$$

where v is a $(d+1)$-subset of $\{1, 2, ..., n\}$. We call C_v a *positive circuit* if $C_v \in \mathbf{R}^n_+$.

Given any basis $\tau = \{a_{\tau_1}, ..., a_{\tau_d}\}$ of $\mathscr{A}$, we define $L_{\tau,i}$ to be the unique linear functional on $\mathbf{R}^d$ with $L_{\tau,i}(a_{\tau_j}) = \delta_{ij}$ (Kronecker delta). For $x \in \operatorname{pos}(\mathscr{A})$ let Ω_x denote the set of all bases τ of $\mathscr{A}$ with $x \in \operatorname{pos}(\tau)$. The following straightforward lemma shows that all fibers have the same recession cone.

LEMMA 2.1. (a) *The zero fiber $\pi^{-1}(0)$ equals the positive hull of all positive circuits C_v of $\mathscr{A}$.*

(b) *For all $x \in \operatorname{pos}(\mathscr{A})$ we have $\pi^{-1}(x) = \pi^{-1}(0) + \operatorname{conv}\{\sum_{i=1}^{d} L_{\tau,i}(x) \cdot e_{\tau_i} \mid \tau \in \Omega_x\}$.*

Note that $\operatorname{pos}(\mathscr{A})$ is pointed if and only if $\pi^{-1}(0) = \{0\}$. Since the integral in (2.2) is additive with respect to the Minkowski sum in Lemma 2.1(b), we get the following result.

COROLLARY 2.2. *The secondary polyhedron $\Sigma(A)$ has the recession cone $\pi^{-1}(0)$. Thus $\Sigma(A)$ is a polytope if and only if* $\operatorname{pos}(\mathscr{A})$ *is pointed.*

A *subdivision* of $\mathscr{A}$ is a collection Π of subsets of $\mathscr{A}$ such that the polyhedral cones $\{\operatorname{pos}(\sigma) \mid \sigma \in \Pi\}$ form a fan (i.e., a complex of cones) which covers $\operatorname{pos}(\mathscr{A})$. Equivalently, Π can be viewed as a subdivision of $P(\mathscr{A})$ into spherical polytopes $\operatorname{pos}(\sigma) \cap S^{d-1}$. A *triangulation* of $\mathscr{A}$ is a subdivision into simplicial cones (respectively, spherical simplices). Given polyhedral subdivisions Π_1 and Π_2, we say Π_1 *refines* Π_2, written $\Pi_2 \prec \Pi_1$, if every face of Π_1 is a subset of some face of Π_2.

For a polyhedron $Q \subset \mathbf{R}^n$ and a vector $\psi \in \mathbf{R}^n$, we say that Q is *bounded in direction* ψ if the linear functional $\langle \psi, \cdot \rangle$ attains a finite minimum over Q. In this case ψ defines a proper face $Q^\psi := \{y \in Q \mid \langle \psi, y \rangle \leqslant \langle \psi, Q \rangle\}$, having an *inward* pointing normal ψ. We call $\psi \in \mathbf{R}^n$ *feasible* if $\Sigma(\mathscr{A})$ is bounded in direction ψ. By Corollary 2.2, the cone of feasible vectors is the polar of $\pi^{-1}(0)$. Note also that $\psi = (\psi_1, ..., \psi_n)$ is feasible if and only if $(0, .., 0, -1)$ does not lie in

$$\operatorname{pos}\{(a_1, \psi_1), (a_2, \psi_2), ..., (a_n, \psi_n)\} \subseteq \mathbf{R}^{d+1}. \tag{2.4}$$

In this case the projection of the "bottom" faces of the polyhedron in (2.4) onto the first d coordinates defines a subdivision $\Pi(\psi)$ of $\mathscr{A}$. This method of obtaining subdivisions goes back to Walkup and Wets [12]. We call a subdivision Π *regular* if it arises in this way. If Π is a regular subdivision, then the set

$$\mathscr{F}_{\mathscr{A}}(\Pi) := \{\psi \in \mathbf{R}^n \mid \Pi(\psi) = \Pi\} \tag{2.5}$$

is a non-empty relatively open convex polyhedral cone. These cones define a polyhedral fan $\mathscr{F}_{\mathscr{A}}$, which we call the *secondary fan*. The face lattice of the secondary fan is isomorphic via (2.5) to the poset of regular subdivisions, ordered by refinement. Here the maximal cells of $\mathscr{F}_{\mathscr{A}}$ correspond to regular triangulations of $\mathscr{A}$.

We remark that if $\operatorname{pos}(\mathscr{A})$ fails to be pointed, then already four vectors in the plane can have non-regular subdivisions. For example, $\Pi = \{\{1, 2\}, \{1, 4\}, \{2, 3, 4\}\}$ is a non-regular subdivision of $\mathscr{A} = \{a_1, a_2, a_3, a_4\} = \{(1, 1), (1, 0), (1, -1), (-1, 0)\} \subset \mathbf{R}^2$.

Every triangulation Δ of $\mathscr{A}$ gives rise to a piecewise linear section $\gamma_\Delta : \operatorname{pos}(\mathscr{A}) \to \mathbf{R}^n_+$ via $\gamma_\Delta(x) := \sum_{i=1}^{d} L_{\tau,i}(x)\, e_{\tau_i}$ whenever $x \in \tau \in \Delta$. If Δ is a regular triangulation of $\mathscr{A}$, then any vector $\psi \in \mathscr{F}_{\mathscr{A}}(\Delta)$ satisfies $\pi^{-1}(x)^{\psi} = \{\gamma_\Delta(x)\}$ for all points x in the interior of $P(\mathscr{A})$. By [3, Proposition 1.2], integration over $P(\mathscr{A})$ yields the following result.

PROPOSITION 2.3. *For any regular triangulation Δ the vector $\phi_\Delta := \int_{P(\mathscr{A})} \gamma_\Delta(x)\, dx$ is a vertex of $\Sigma(\mathscr{A})$, and all vertices are of this form. The inner normal cone of $\Sigma(\mathscr{A})$ at ϕ_Δ equals $\mathscr{F}_{\mathscr{A}}(\Delta)$.*

Proposition 2.3 states in other words that the maximal cells of the secondary fan are precisely the maximal cells of the normal fan $\mathscr{N}(\Sigma(\mathscr{A}))$ of the secondary polyhedron. Since all cells of a polyhedral cell complex are obtained by intersecting closures of maximal cells, this implies that $\mathscr{N}(\Sigma(\mathscr{A})) = \mathscr{F}_{\mathscr{A}}$.

THEOREM 2.4. *The face lattice of $\Sigma(\mathscr{A})$ is antiisomorphic to the poset of regular subdivisions of $\mathscr{A}$, ordered by refinement.*

We now prove the following direct geometric description for the secondary polyhedron by deletion $\Sigma(\mathscr{A} \setminus a_i)$.

THEOREM 2.5. *The face $\Sigma(\mathscr{A})^{e_i}$ of the secondary polyhedron in the direction of the ith unit vector is a facet, which is a translate of the secondary polyhedron $\Sigma(\mathscr{A} \setminus a_i)$ by deletion.*

Proof. The $(d-1)$-polyhedron $\Sigma(\mathcal{A}\backslash a_i)$ is defined as $\int_{P(\mathcal{A}\backslash a_i)} \theta_i^{-1}(x)\, dx$, where

$$\begin{aligned}\theta_i\colon (\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_n) \mapsto \lambda_1 a_1 + \cdots \\ + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_n a_n.\end{aligned} \tag{2.6}$$

Each fiber $\theta_i^{-1}(x)$ in $\mathbf{R}_+^{n-1}$ of a point $x \in \operatorname{pos}(\mathcal{A}\backslash a_i)$ is a subset of $\pi^{-1}(x)$ via the ith coordinate inclusion of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$. More precisely, $\theta_i^{-1}(x) = \pi^{-1}(x)^{e_i}$ is the face of $\pi^{-1}(x)$ on which the i-th coordinate function is zero and hence minimal. This face is a facet if x lies in the interior of $\operatorname{pos}(\mathcal{A}\backslash a_i)$. This implies

$$\int_{P(\mathcal{A}\backslash a_i)} \theta_i^{-1}(x)\, dx = \int_{P(\mathcal{A}\backslash a_i)} \pi^{-1}(x)^{e_i}\, dx = \left(\int_{P(\mathcal{A}\backslash a_i)} \pi^{-1}(x)\, dx\right)^{e_i}. \tag{2.7}$$

If a_i is contained in $\operatorname{pos}(\mathcal{A}\backslash a_i)$, then the right-hand integral equals $\Sigma(\mathcal{A})$ and we are done. Let us now assume that $a_i \notin \operatorname{pos}(\mathcal{A}\backslash a_i)$. Pick a sufficiently generic point $x \in \operatorname{pos}(\mathcal{A})\backslash\operatorname{pos}(\mathcal{A}\backslash a_i)$. Then there exists a unique simplicial $(d-1)$-cone $\operatorname{pos}(a_{j_1}, ..., a_{j_{d-1}})$ which spans a facet of $\operatorname{pos}(\mathcal{A}\backslash a_i)$ and such that $x \in \operatorname{pos}(a_i, a_{j_1}, ..., a_{j_{d-1}})$; say, $x = \mu_i a_i + \mu_{j_1} a_{j_1} + \cdots + \mu_{j_{d-1}} a_{j_{d-1}}$, where $\mu \in \mathbf{R}_+^n$ and $\mu_k = 0$ for $k \notin \{i, j_1, ..., j_{d-1}\}$. The point μ gives the unique minimum of the ith coordinate function e_i over $\pi^{-1}(x)$, i.e., $\pi^{-1}(x)^{e_i} = \{\mu\}$. We have shown that the face $\pi^{-1}(x)^{e_i}$ is a vertex of $\pi^{-1}(x)$ for almost all $x \in P(\mathcal{A})\backslash P(\mathcal{A}\backslash a_i)$. Thus $\int_{P(\mathcal{A})\backslash P(\mathcal{A}\backslash a_i)} \pi^{-1}(x)^{e_i}\, dx$ is a point, which completes the proof of Theorem 2.5. ∎

3. Duality and Contractions

Let $\mathcal{A} = \{a_1, ..., a_n\} \subset \mathbf{R}^d$ as before. The *circuit space* of $\mathcal{A}$ is the $(n-d)$-dimensional subspace $\mathscr{C}(\mathcal{A})$ of $\mathbf{R}^n$ spanned by all circuits C_ν as in (2.3). Equivalently,

$$\mathscr{C}(\mathcal{A}) = \left\{\beta = (\beta_1, ..., \beta_n) \in \mathbf{R}^n \,\middle|\, \sum_{i=1}^{n} \beta_i a_i = 0\right\}. \tag{3.1}$$

The *cocircuit space* of $\mathcal{A}$ is the d-dimensional subspace $\mathscr{D}(\mathcal{A})$ of $\mathbf{R}^n$ spanned by the vectors

$$D_\mu := \sum_{i=1}^{n} \det(a_{\mu_1}, ..., a_{\mu_{d-1}}, a_i)\, e_i, \tag{3.2}$$

called *cocircuits* of $\mathscr{A}$, where μ ranges over all $(d-1)$-subsets of $\{1, ..., n\}$. Equivalently,

$$\mathscr{D}(\mathscr{A}) = \{(\phi(a_1), ..., \phi(a_n)) \in \mathbf{R}^n \mid \phi\colon \mathbf{R}^d \to \mathbf{R} \text{ any linear functional}\}. \quad (3.3)$$

Note that the vector spaces $\mathscr{C}(\mathscr{A})$ and $\mathscr{D}(\mathscr{A})$ are orthogonal complements in $\mathbf{R}^n$.

A spanning subset $\mathscr{B} = \{b_1, ..., b_n\}$ of $\mathbf{R}^{n-d}$ is called a *linear transform* of $\mathscr{A}$ provided $\mathscr{C}(\mathscr{A}) = \mathscr{D}(\mathscr{B})$. This is equivalent to $\mathscr{D}(\mathscr{A}) = \mathscr{C}(\mathscr{B})$ and thus to $\mathscr{A}$ being a linear transform of $\mathscr{B}$. In this case the oriented matroids associated with $\mathscr{A}$ and $\mathscr{B}$ are dual. If we define $I_{\mathscr{A}} = \{i \mid -a_i \in \text{pos}(\mathscr{A})\}$, then it is a consequence of oriented matroid duality that the sets $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$ partition $\{1, ..., n\}$. Thus, for example, pos$(\mathscr{A})$ is pointed (i.e., $I_{\mathscr{A}} = \emptyset$) if and only if pos$(\mathscr{B}) = \mathbf{R}^{n-d}$ ($I_{\mathscr{B}} = \{1, ..., n\}$). See [4, 9, 11].

The *chamber complex* $\Gamma(\mathscr{A})$ of $\mathscr{A}$ is defined to be the coarsest polyhedral complex that covers pos$(\mathscr{A})$ and that refines all triangulations of $\mathscr{A}$. Given $x_0 \in \text{pos}(\mathscr{A})$, the unique (relatively open) cell of $\Gamma(\mathscr{A})$ containing x_0 is

$$\Gamma(\mathscr{A}, x_0) = \bigcap \{\text{rel int pos } \mathscr{A}' \mid \mathscr{A}' \subseteq \mathscr{A},\ x_0 \in \text{rel int pos } \mathscr{A}'\}. \quad (3.4)$$

For a combinatorial study of chamber complexes we refer to [1]. We now relate the secondary polyhedra and the chamber complexes of $\mathscr{A}$ and $\mathscr{B}$. For the special case pos$(\mathscr{A}) = \mathbf{R}^n$, Theorem 3.1 leads to an association between polytopes with normal vectors in $\mathscr{A}$ and cells in $\mathscr{B}$. This statement can also be inferred from [9, Theorem 5A2].

THEOREM 3.1. *Let $\mathscr{A} \subset \mathbf{R}^d$ and $\mathscr{B} \subset \mathbf{R}^{n-d}$ be linear transforms of each other. Then the boundary complex of the secondary polyhedron $\Sigma(\mathscr{A})$ is antiisomorphic to the chamber complex $\Gamma(\mathscr{B})$, and the boundary complex of $\Sigma(\mathscr{B})$ is antiisomorphic to $\Gamma(\mathscr{A})$.*

For the proof of Theorem 3.1 we will need the following lemma.

LEMMA 3.2. *Given any subset $\sigma \subset \{1, ..., n\}$, then $\sigma \in \Pi(\psi)$ if and only if $\sum_{i=1}^{n} \psi_i b_i \in \text{rel int pos}\{b_k \mid k \notin \sigma\}$.*

Proof. The cone pos$\{(a_i, \psi_i) \mid i \in \sigma\}$ is a bottom face of pos$\{(a_1, \psi_1), ..., (a_n, \psi_n)\} \subset \mathbf{R}^{d+1}$ if and only if there is a linear functional $\phi\colon \mathbf{R}^d \to \mathbf{R}$ such that $\phi(a_i) + \psi_i = 0$ for $i \in \sigma$ and $\phi(a_i) + \psi_i > 0$ for $i \notin \sigma$. This is equivalent to the existence of a vector $v \in \mathscr{D}(\mathscr{A}) = \mathscr{C}(\mathscr{B})$ with $v_i + \psi_i = 0$ for $i \in \sigma$ and $v_i + \psi_i > 0$ for $i \notin \sigma$. In this case, $\sum_{i=1}^{n} \psi_i b_i = \sum_{k \notin \sigma} (v_k + \psi_k) b_k$, which completes the proof. ∎

Proof of Theorem 3.1. Consider the linear map $B\colon \mathbf{R}^n \to \mathbf{R}^{n-d}$, $\psi \mapsto \sum_{i=1}^{n} \psi_i b_i$. Fix a feasible vector $\psi \in \mathbf{R}^n$. It lies in $\mathscr{F}_{\mathscr{A}}(\Pi)$ for some regular subdivision $\Pi = \Pi(\psi)$ of $\mathscr{A}$. Applying Lemma 3.2 to any $\sigma \in \Pi$, we see that $B(\psi)$ lies in pos($\mathscr{B}$) and thus lies in a unique cell $\Gamma(\mathscr{B}, B(\psi))$ of the chamber complex $\Gamma(\mathscr{B})$. Lemma 3.2 and (3.4) imply the relations

$$\Gamma(\mathscr{B}, B(\psi)) = \bigcap_{\sigma \in \Pi(\psi)} \operatorname{rel\,int} \operatorname{pos}\{b_k \mid k \notin \sigma\}. \tag{3.5}$$

We now define a map from the boundary complex of the secondary polyhedron $\Sigma(\mathscr{A})$ to the chamber complex $\Gamma(\mathscr{B})$ by

$$\Sigma(\mathscr{A})^{\psi} \mapsto \Gamma(\mathscr{B}, B(\psi)). \tag{3.6}$$

This map is well-defined and order-reversing because if $\Sigma(\mathscr{A})^{\psi} \subseteq \Sigma(\mathscr{A})^{\psi'}$ then $\Pi(\psi)$ refines $\Pi(\psi')$, by Theorem 2.4, and in this case $\Gamma(\mathscr{B}, B(\psi')) \subseteq \Gamma(\mathscr{B}, B(\psi))$ by (3.5). By Lemma 3.2, the assignment $\Gamma(\mathscr{B}, x) \mapsto \Sigma(\mathscr{A})^{\psi}$, for any ψ such that $B(\psi) = x$, defines the inverse to (3.6). ∎

The secondary polyhedron $\Sigma(\mathscr{A})$ as defined in (2.2) has codimension d in $\mathbf{R}^n$. For the following discussion we will consider $\Sigma(\mathscr{A})$ and all fibers $\pi^{-1}(x)$ to be embedded in $\mathbf{R}^{n-d}$ via the map B, which is an isomorphism when restricted to translates of the kernel of π. In fact, the bijection (3.6) shows that under this embedding the normal fan of $\Sigma(\mathscr{A})$ equals $\Gamma(\mathscr{B})$. In order to describe minors by contraction we will now construct a polyhedron in $\mathbf{R}^{n-d}$ which is *normally equivalent* to (i.e., has the same normal fan as) $\Sigma(\mathscr{A})$. The support function of any such polyhedron is a strictly convex, piecewise linear function over the chamber complex $\Gamma(\mathscr{B})$.

First note that $\Gamma(\mathscr{B})$ is the coarsest polyhedral complex that refines all *regular* triangulations of $\mathscr{B}$, since every basis of $\mathscr{B}$ appears in some regular triangulation. A triangulation Δ of $\mathscr{B}$ is regular if and only if there exists a vector $\psi \in \mathbf{R}^n$ that induces a *convex* piecewise linear function $g_\Delta = g_{\psi,\Delta}$ over the fan defining Δ (see [2] for the affine case). The function g_Δ is the *support function* of a polyhedron Q_Δ with normal fan Δ. Recalling that Minkowski addition of convex polyhedra corresponds both to addition of support functions and to intersection of normal fans (see [7, p. 309] or [2, Proposition 1.2.2]), we get the following.

PROPOSITION 3.3. *The function $\sum_\Delta g_\Delta$, the sum taken over all regular triangulations of $\mathscr{B}$, is the support function of a polyhedron Q normally equivalent to the secondary polyhedron $\Sigma(\mathscr{A})$. In fact, Q is the Minkowski sum $\sum_\Delta Q_\Delta$.*

The summands of $\Sigma(\mathscr{A})$ in Proposition 3.3 may be taken to be $Q_\Delta = \int_\sigma \pi^{-1}(x)\, dx$, where σ is the maximal cell of $\Gamma(\mathscr{A}) \cap S^{d-1}$ corresponding to the regular triangulation Δ of $\mathscr{B}$. In fact, the regular polyhedral subdivisions of $\mathscr{B}$ are precisely the normal fans of the fibers

$\pi^{-1}(x)$, $x \in \operatorname{pos}(\mathscr{A})$ (under the embedding B). Thus each face of $\Sigma(\mathscr{B})$ gives rise to a Minkowski summand of $\Sigma(\mathscr{A})$.

Again, let $\mathscr{A} = \{a_1, ..., a_n\}$ be a spanning subset of $\mathbf{R}^d$, and let $\mathscr{B} = \{b_1, ..., b_n\} \subset \mathbf{R}^{n-d}$ be a linear transform of $\mathscr{A}$. We assume that a_i is neither a loop nor a coloop of $\mathscr{A}$, which means that none of the a_i and none of the b_i are zero. The following lemma is a straightforward analogue to the matroidal duality of deletion and contraction.

LEMMA 3.4. *Deletion and contraction are dual in the sense that*

(i) *$\mathscr{A}/a_i$ is a linear transform of $\mathscr{B}\backslash b_i$ and*

(ii) *$\mathscr{A}\backslash a_i$ is a linear transform of $\mathscr{B}/b_i$.*

We have identified the normal fan of $\Sigma(\mathscr{A})$ with the chamber complex $\Gamma(\mathscr{B})$. Using Lemma 3.4, we can therefore identify the normal fan of $\Sigma(\mathscr{A}/a_i)$ with the chamber complex $\Gamma(\mathscr{B}\backslash b_i)$, and similarly the normal fan of $\Sigma(\mathscr{A}\backslash a_i)$ with $\Gamma(\mathscr{B}/b_i)$. Thus our problem is reduced to describing the behavior of the chamber complex under deletion and contraction. Using this point of view we now describe the relationship of the secondary polyhedron by contraction $\Sigma(\mathscr{A}/a_i)$ to $\Sigma(\mathscr{A})$.

We say a point a_i is *extreme* in $\mathscr{A}$ if $a_i \notin \operatorname{pos}(\mathscr{A}\backslash a_i)$. We note that a_i is extreme in $\mathscr{A}$ if and only if b_i is *not* extreme in $\mathscr{B}$. Every convex polyhedron Q can be written uniquely as a minimal intersection of halfspaces in its affine hull, each of which then defines a facet. If Q is a convex polyhedron and F is a facet of Q, then we say *Q' results from Q by removing F* if Q' is the intersection of all halfspaces in the minimal representation of Q except the one corresponding to F.

THEOREM 3.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be as above. If a_i is extreme in $\mathscr{A}$, then $\Sigma(\mathscr{A}/a_i)$ is a Minkowski summand of $\Sigma(\mathscr{A})$. If a_i is not extreme in $\mathscr{A}$, then $\Sigma(\mathscr{A}/a_i)$ is unbounded and is obtained from a Minkowski summand of $\Sigma(\mathscr{A})$ by removing the facet with inner normal e_i.*

Proof. We consider the chamber complex $\Gamma(\mathscr{B}\backslash b_i)$. If a_i is extreme in $\mathscr{A}$, then since b_i is not extreme in $\mathscr{B}$, regular triangulations of $\mathscr{B}\backslash b_i$ are just those regular triangulations of $\mathscr{B}$ that do not involve b_i as a vertex. Thus by Proposition 3.3

$$\Sigma(\mathscr{A}) = \Sigma(\mathscr{A}/a_i) + \sum_{\Delta} Q_\Delta, \tag{3.7}$$

the summation over those regular triangulations of $\mathscr{B}$ that do contain b_i.

On the other hand, if a_i is not extreme in $\mathscr{A}$, then since b_i is extreme in $\mathscr{B}$, all regular triangulations of $\mathscr{B}$ must contain b_i. In this case, let $R = \sum_\Delta Q_\Delta$, where the sum here is over all regular triangulations Δ of $\mathscr{B}$

such that the complex $\Delta \backslash b_i$ triangulates $\mathscr{B} \backslash b_i$. (By $\Delta \backslash b_i$ is meant the complex consisting of all simplices that do not contain b_i.) By Proposition 3.3 again, R is a Minkowski summand of $\Sigma(\mathscr{A})$.

Each polytope Q_Δ in the definition of R has a facet with inner normal b_i. Removing this facet corresponds to restricting the support function g_Δ of Q_Δ to $\text{pos}(\mathscr{B} \backslash b_i)$, or equivalently, to replacing g_Δ by the new support function $g_{\Delta \backslash b_i}$ gotten by setting $\psi_i = +\infty$. The sum over these new support functions is the support function of the polyhedron R' which results from R by removing the facet with inner normal b_i. Thus the support function of R' is strictly convex and piecewise linear over $\Gamma(\mathscr{B} \backslash b_i)$, and therefore R' is normally equivalent to $\Sigma(\mathscr{A}/a_i)$. Now Theorem 3.5 follows because the facet of $\Sigma(\mathscr{A})$ (or of its summand R) with inner normal b_i becomes the facet with inner normal e_i when these polyhedra are considered in the original embedding in $\mathbf{R}^n$. ∎

4. A Non-Regular Triangulation of a Cyclic Polytope

In this section we apply our duality results to triangulations of the cyclic 8-polytope $C(8, 12)$ with 12 vertices. In particular, we show that $C(8, 12)$ admits a non-regular triangulation; this proves a conjecture of Kapranov and Voevodsky [8, Remark 3.5].

Let $\mathscr{A} := \{a_1, a_2, ..., a_{12}\} \subset \mathbf{R}^9$, where $a_i := (1, i, i^2, ..., i^8)$ for $i = 1, 2, ..., 12$. The cyclic polytope $C(8, 12)$ is defined as $\text{conv}(\mathscr{A})$. It will here be identified with the spherical 8-polytope $P(\mathscr{A})$ or with its positive hull $\text{pos}(\mathscr{A})$. A linear transform of $\mathscr{A}$ is given by $\mathscr{B} = \{b_1, b_2, ..., b_{12}\} \subset \mathbf{R}^3$, where

$$(b_1, b_2, b_3, ..., b_{12}) = \begin{pmatrix} 1 & 0 & 0 & -165 & 990 & -2772 & 4620 & -4950 & 3465 & -1540 & 396 & -45 \\ 0 & -1 & 0 & 45 & -240 & 630 & -1008 & 1050 & -720 & 315 & -80 & 9 \\ 0 & 0 & 1 & -9 & 36 & -84 & 126 & -126 & 84 & -36 & 9 & -1 \end{pmatrix}.$$

If we replace the vectors b_2, b_4, b_6, b_8, b_{10}, and b_{12} by their negatives, then we obtain a pointed cone in 3-space, which can be represented by the 2-dimensional affine configuration depicted in Fig. 1. This diagram is an *affine Gale diagram* [11] of $C(8, 12)$. (See also [2, Fig. 2]).

By Theorem 3.1, the maximal cells of chamber complex $\Gamma(\mathscr{B})$ are in one-to-one correspondence with the regular triangulations of $C(8, 12)$. We now consider the specific maximal cell $\Gamma(\mathscr{B}, x_0)$ which contains the vector $x_0 = (770, -159, 20)$. As is illustrated in Fig. 1, this cell is the intersection

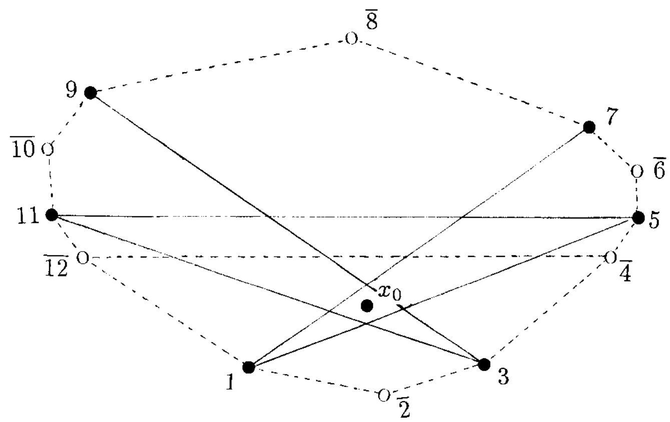

FIGURE 1

of the simplicial cones $\text{pos}(\{b_i, b_j, b_k\})$, where $i\,j\,k$ ranges over the triples in the following list:

$$\begin{array}{ccccccccc} 1\,2\,3 & 1\,2\,5 & 1\,3\,7 & 1\,3\,9 & 1\,4\,5 & 1\,5\,7 & 1\,5\,9 & \underline{1\,5\,11} & 1\,7\,8 \\ 1\,7\,10 & 1\,7\,12 & 1\,9\,10 & 1\,9\,12 & 1\,11\,12 & 2\,3\,11 & 3\,4\,5 & 3\,4\,7 & 3\,4\,9 \\ \underline{3\,5\,11} & 3\,6\,7 & 3\,6\,9 & 3\,7\,11 & 3\,8\,9 & 3\,9\,11 & 3\,11\,12 & 5\,6\,7 & 5\,6\,9 \\ \underline{5\,6\,11} & 5\,8\,9 & \underline{5\,8\,11} & \underline{5\,10\,11} & 7\,8\,9 & 7\,8\,11 & 7\,10\,11 & 9\,10\,11 & \end{array} \tag{4.1}$$

The regular triangulation Δ_0 of $C(8, 12)$ corresponding to the region $\Gamma(\mathscr{B}, x_0)$ has as its maximal simplices the complements of all triples in (4.1), that is, the 35 maximal simplices of Δ_0 are 4 5 6 7 8 9 10 11 12, 3 4 6 7 8 9 10 11 12, 2 4 5 6 8 9 10 11 12, etc. We now replace the five underlined triples in (4.1) by

$$2\,5\,11 \qquad 4\,5\,11 \qquad 5\,7\,11 \qquad 5\,9\,11 \qquad 5\,11\,12. \tag{4.2}$$

This corresponds to bending the line $\overline{5\ 11}$ until it curves under the point x_0 in Fig. 1.

The new collection of complementary sets defines a simplicial 8-ball Δ_1 which is obtained from Δ_0 by a bistellar operation (see [2]) on $\{5, 11\}^c = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 12\}$. This bistellar operation can be carried out geometrically for the triangulation Δ_0 of $C(8, 12)$ because $(\{1, 3, 6, 8, 10\}, \{2, 4, 7, 9, 12\})$ is a Radon partition of $C(8, 12)$. For, we can see in Fig. 1 that $(-, +, -, +, 0, -, +, -, +, -, 0, +)$ is a signed cocircuit of $\mathscr{B}$ and hence is a signed circuit of $\mathscr{A}$.

PROPOSITION 4.1. *The triangulation Δ_1 of $C(8, 12)$ is not regular.*

Proof. If the triangulation Δ_1 were regular, then there would exist a vector x_1 in the interior of all simplicial cones $\text{pos}(\{b_i, b_j, b_k\})$, where ijk ranges over all triples in (4.2) and all non-underlined triples in (4.1). However, as can be seen in Fig. 1, the intersection of these simplicial cones (or spherical triangles) is empty. This shows that Δ_1 is a non-regular triangulation of the cyclic polytope $C(8, 12)$. ∎

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