PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 2, October 1994

CELLULAR STRINGS ON POLYTOPES

L. J. BILLERA, M. M. KAPRANOV, AND B. STURMFELS

(Communicated by Jeffry N. Kahn)

ABSTRACT. The complex of cellular strings with respect to a generic linear functional on a d-dimensional convex polytope has the homotopy type of the $(d-2)$-sphere. This result was conjectured in a special case by H.-J. Baues.

INTRODUCTION

While from a topological point of view all convex polytopes have the same boundary, the rich combinatorial structure of these objects has been studied since the earliest days of mathematics. This paper deals with a new combinatorial invariant, the poset of cellular strings on a polytope. We determine the homotopy type of this poset, and as a corollary we prove a conjecture of Baues in the theory of loop spaces.

Baues's conjecture arose from the problem of finding explicit CW-models for iterated loop spaces $\Omega^i X$ of a given CW-space X. Using cellular strings on cubes, Baues [2] demonstrated that the algebraic cobar construction of Adams [1] can be strengthened to a geometric model for ΩX. More precisely, if X is a simplicial space [6] with one vertex and no edges, then Baues's model $\underline{\Omega} X$ is a space glued from cubes, where each cube I^{n-1} approximates the space of paths in the simplex Δ_n joining the 0th and the nth vertex. Here faces of I^{n-1} correspond to cellular strings on Δ_n.

Milgram [8] noted that the $(n-1)$-dimensional *permutohedron* P_n (the convex hull of a generic orbit of the symmetric group S_n in $\mathbf{R}^n$) similarly approximates the path space of the cube I^n. More precisely, the face poset of P_n is isomorphic to the poset of cellular strings in I^n. This permitted Baues to iterate his geometric cobar-construction and apply it to cubical spaces X with trivial 1-skeleton. He constructed a CW-space $\underline{\Omega} X$, glued from products of permutohedra, which is a model for ΩX. In particular, if Y is a simplicial space with trivial 2-skeleton and $X = \underline{\Omega} Y$, then $\underline{\Omega} X$ is a model for the iterated loop space $\Omega^2 Y$.

One is tempted to continue this process, introducing "hyperpermutohedra" as complexes of cellular strings in P_n and so on. However, these complexes are

Received by the editors February 1, 1993.
1991 *Mathematics Subject Classification.* Primary 52B11; Secondary 55P35.
Partially supported by ARO (DAAG29-85-C-0018) and NSF (DMS-9004123, DMS-9002056) at Cornell University. This work was done while the second author was visiting the Mathematical Sciences Institute at Cornell.

no longer isomorphic to face complexes of convex polytopes (and even fail to be spheres). Nevertheless, Baues conjectured that they are *homotopy equivalent* to spheres and that one can choose a subcomplex homeomorphic to a sphere whose embedding is a homotopy equivalence [2, Conjecture 7.4].

In §1 we prove Baues's conjecture for an arbitrary convex polytope $P \subset \mathbf{R}^n$ whose vertices are ordered by means of a generic linear functional f. In §2 we show that the desired subcomplex homeomorphic to a sphere can be taken to be the boundary complex of the *monotone path polytope* $\Sigma_f(P)$. These results open up the possibility of iterating Baues's construction to get CW-models for iterated loop spaces by gluing products of iterated monotone path polytopes. In §3 we discuss a natural generalization of Baues's conjecture arising from the theory of fiber polytopes [3].

We are grateful to David Stone for bringing the work of Adams and Baues to our attention and to Anders Björner and Günter Ziegler for helpful discussions.

1. Sphericity of the cellular string complex

Let $P \subset \mathbf{R}^n$ be an n-dimensional convex polytope. A *stippling* on P (see [2, Definition III.2.3]) is a rule which assigns to each face Γ of P a vertex $\eta(\Gamma)$ of Γ such that whenever Γ_2 is a face of Γ_1 and $\eta(\Gamma_1) \in \Gamma_2$, then $\eta(\Gamma_1) = \eta(\Gamma_2)$.

An important class of stipplings is provided by convex geometry. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a linear functional. We say that f is *generic* with respect to P if it is nonconstant on each edge of P. In this case on each face $\Gamma \subset P$ the function f attains its minimum (resp. maximum) in exactly one point, which is a vertex denoted $s_f(\Gamma)$ (resp. $t_f(\Gamma)$).

Proposition 1.1. *If f is a generic linear functional on P, then s_f and t_f are stipplings.*

Let P be a polytope with a fixed pair of stipplings (s, t). A *cellular string* on P is a sequence of proper faces $(\Gamma_0, \Gamma_1, \dots, \Gamma_m)$ such that $s(\Gamma_0) = s(P)$, $t(\Gamma_m) = t(P)$, and $s(\Gamma_i) = t(\Gamma_{i-1})$ for each $i = 1, 2, \dots, m$. The set $\omega = \omega(P, s, t)$ of all cellular strings is partially ordered by the relation $(\Gamma_0, \Gamma_1, \dots, \Gamma_m) \le (\Delta_0, \Delta_1, \dots, \Delta_k)$ if there exist $0 = i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1} = m$ such that $(\Gamma_{i_\nu}, \Gamma_{i_\nu+1}, \dots, \Gamma_{i_{\nu+1}})$ is a cellular string on the face Δ_ν, for each $\nu = 0, 1, \dots, k$.

The geometric realization $|\omega|$ of the simplicial complex of chains in ω can be viewed as a model for the space of all monotone paths on the boundary ∂P joining $s(P)$ and $t(P)$. So it is a natural question whether $|\omega|$ has the homotopy type of the $(n-2)$-sphere S^{n-2}. The following affirmative answer is the main result of this paper.

Theorem 1.2. *Let f be a generic linear functional on an n-dimensional convex polytope $P \subset \mathbf{R}^n$. Then the poset $\omega(P, s_f, t_f)$ is homotopy equivalent to S^{n-2}.*

Proof. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be the values of f at the vertices of P, so the segment $Q := [\lambda_1, \lambda_N] \subset \mathbf{R}$ is the image of P under f. Choose an intermediate point μ_i in each open interval $(\lambda_i, \lambda_{i+1})$. Denote by L_i (resp. M_i) the poset of proper faces of the polytope $f^{-1}(\lambda_i)$ (resp. $f^{-1}(\mu_i)$). When a value λ moves from μ_i to λ_i, each face of $f^{-1}(\lambda)$ contracts to some face of $f^{-1}(\lambda_i)$ of possibly smaller dimension. This defines a surjective morphism of

posets $\alpha_i : M_i \to L_i$. Similarly, we have the degeneration in the other direction, defining a morphism $\beta_i : M_i \to L_{i+1}$ (see also Lemma 2.2).

Lemma 1.3. *The poset $\omega(P, s_f, t_f)$ is isomorphic to the inverse limit of the diagram*

$$M_1 \xrightarrow{\beta_1} L_2 \xleftarrow{\alpha_2} M_2 \xrightarrow{\beta_2} L_3 \xleftarrow{\alpha_3} M_3 \xrightarrow{\beta_4} \cdots \xrightarrow{\beta_{N-2}} L_{N-1} \xleftarrow{\alpha_{N-1}} M_{N-1}.$$

Proof of Lemma 1.3. By definition, an element of this inverse limit is a sequence $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1})$ of polytopes, where $\Gamma_i \in M_i$ is a face of $f^{-1}(\mu_i)$, such that $\beta_i(\Gamma_i) = \alpha_{i+1}(\Gamma_{i+1})$ for $i = 1, 2, \ldots, N-2$. Since f is generic, we have either $\dim \Gamma_i = \dim \beta_i(\Gamma_i) = \dim \alpha_{i+1}(\Gamma_{i+1}) = \dim \Gamma_{i+1}$ (which means that Γ_i and Γ_{i+1} are hyperplane sections of the same face of P) or $\beta_i(\Gamma_i) = \alpha_{i+1}(\Gamma_{i+1})$ is a point. In either case we get a unique cellular string in P, and each cellular string in P is obtained in this way. □

Lemma 1.4. *The inverse image under α_i of any upper interval $\{\Phi \in L_i : \Phi \geq \Gamma\}$ in L_i is a contractible subposet of M_i, and similarly for the map β_i.*

Proof of Lemma 1.4. We view $f^{-1}(\lambda_i)$, $f^{-1}(\mu_i)$, and their normal cones as polyhedra in the kernel of f. Let $\mathcal{N}_\Gamma$ be the normal cone of $f^{-1}(\lambda_i)$ at Γ. A face Ξ of $f^{-1}(\mu_i)$ lies in $\alpha_i^{-1}(\{\Phi \in L_i : \Phi \geq \Gamma\})$ if and only if the normal cone of $f^{-1}(\mu_i)$ at Ξ is contained in $\mathcal{N}_\Gamma$. Moreover, these cones form a polyhedral subdivision of $\mathcal{N}_\Gamma$. Hence $\alpha_i^{-1}(\{\Phi \in L_i : \Phi \geq \Gamma\})$ is contractible. □

It follows from Lemma 1.4 and Quillen's Theorem A [9] that the maps of geometric realizations $|\alpha_i| : |M_i| \to |L_i|$ and $|\beta_i| : |M_i| \to |L_{i+1}|$ are homotopy equivalences and that all their fibers are contractible.

Proof of Theorem 1.2. Let X_i denote the poset inverse limit of the left segment of our diagram up to M_i. Since the operations "inverse limit" and "geometric realization" commute [6, Chapter 3, §3], the geometric realization $|X_i|$ is the inverse limit of the corresponding diagram of topological spaces. Consider the natural projection $p_i : |X_i| \to |X_{i-1}|$. Each fiber of p_i is contractible since it coincides with some fiber of $|\alpha_i|$. Hence p_i is a homotopy equivalence by Quillen's Theorem A [9]. Using induction on i, we conclude that $|X_{N-1}| = |\omega|$ is homotopy equivalent to $|X_1|$. However, $|X_1|$ is the boundary of the $(n-1)$-polytope $f^{-1}(\mu_1)$ and hence homeomorphic to S^{n-2}. □

We note that Björner [5] has given another proof of Theorem 1.2 in the case where P is a zonotope (i.e., a Minkowski sum of line segments).

2. Coherent strings and the monotone path polytope

We now sharpen Theorem 1.2 by identifying a subcomplex ω_{coh} of $\omega = \omega(P, s_f, t_f)$ which is homeomorphic to S^{n-2}. To this end we introduce a larger space Ω homotopy equivalent to ω, and we describe an explicit retraction from Ω onto ω_{coh}.

Let Γ denote the subdivision of the open segment $\text{int}(Q) = (\lambda_1, \lambda_N)$ into open segments $(\lambda_i, \lambda_{i+1})$ and singletons $\{\lambda_j\}$, $1 < j < N$. We call Γ the *chamber complex* of (P, Q). A *face bundle* is a map Δ which assigns to each $\sigma \in \Gamma$ a proper face $\Delta(\sigma)$ of $f^{-1}(\sigma)$. We usually identify Δ with a subcomplex of the boundary complex ∂P, whence the set $\mathcal{F}(P)$ of all face bundles is finite

and partially ordered by inclusion. Every point $\psi = (\psi_\sigma)_{\sigma\in\Gamma} \in (S^{n-2})^\Gamma$ defines a face bundle Δ_ψ via $\Delta_\psi(\sigma) := f^{-1}(\sigma)^{\psi_\sigma}$ (this denotes the extreme face of $f^{-1}(\sigma)$ in direction ψ_σ). Thus $\mathscr{F}(P)$ is the face poset of a piecewise-linear subdivision of the product of spheres $(S^{n-2})^\Gamma$.

We define the *monotone path complex* $\mathscr{M} = \mathscr{M}_f(P)$ to be the subcomplex of $\mathscr{F}(P)$ consisting of all cells which are contained in

$$\Omega := \{\psi \in (S^{n-2})^\Gamma \mid f^{-1}(\tau)^{\psi_\tau} = f^{-1}(\tau)^{\psi_\sigma} \text{ whenever } \tau \subset \overline{\sigma}\}. \tag{1}$$

Proposition 2.1. *The monotone path complex $\mathscr{M}$ equals the cellular string complex ω.*

Proposition 2.1 is a consequence of Lemma 1.3 and the following Lemma 2.2 whose elementary proof we omit.

Lemma 2.2. *For any $\psi \in S^{n-2}$, we have $\beta_{i-1}(f^{-1}(\mu_{i-1})^\psi) = f^{-1}(\lambda_i)^\psi = \alpha_i(f^{-1}(\mu_i)^\psi)$.*

Using Proposition 2.1 and a covering argument, it can be shown that the cellular string complex ω is homotopy equivalent to the space Ω. We now consider its diagonal

$$\Omega_{\text{coh}} := \{\psi \in (S^{n-2})^\Gamma \mid \psi_\sigma = \psi_\tau \text{ for all cells } \tau, \sigma \text{ in } \Gamma\} \tag{2}$$

which is clearly homeomorphic to the $(n-2)$-sphere. The following result yields an alternative proof of Theorem 1.2.

Theorem 2.3. *The inclusion of topological spaces $\Omega_{\text{coh}} \subset \Omega$ is a homotopy equivalence.*

Proof. Let $\sigma_1 < \sigma_2 < \cdots < \sigma_{2N-1}$ be the natural ordering of all cells of the chamber complex Γ (thus σ_{2j-1} is a 1-cell and σ_{2j} is a 0-cell). We abbreviate $\psi_i := \psi_{\sigma_i} \in S^{n-2}$ and consider the intermediate spaces

$$\Omega_i := \{\psi \in \Omega \mid \psi_i = \psi_{i+1} = \cdots = \psi_{2N-1}\} \quad \text{for} \quad i = 1, 2, \ldots, 2N-1. \tag{3}$$

Since $\Omega_{\text{coh}} = \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_{2N-1} = \Omega$, it suffices to show that the inclusion $\Omega_i \subseteq \Omega_{i+1}$ is a homotopy equivalence. We will prove this for the case $i = 2j-1$ (so σ_i is a 1-cell); the case $i = 2j$ is analogous.

Let $\psi \in \Omega_{i+1}$. The local coherence condition in (1) states that both ψ_{i+1} and ψ_i are support vectors for the same face of the fiber over the point σ_{i+1}. This implies $\psi_i \neq -\psi_{i+1}$, and thus the convex combinations

$$\psi_i(\lambda) = \frac{\lambda\psi_i + (1-\lambda)\psi_{i+1}}{\|\lambda\psi_i + (1-\lambda)\psi_{i+1}\|} \qquad \text{for } \lambda \in [0, 1] \tag{4}$$

are well-defined vectors on the unit sphere S^{n-2}. Now check that the point

$$\psi(\lambda) := (\psi_1, \psi_2, \ldots, \psi_i, \psi_i(\lambda), \ldots, \psi_i(\lambda)) \in (S^{n-2})^\Gamma \tag{5}$$

lies in Ω for all $\lambda \in [0, 1]$. This is clearly the case for $\lambda = 0$, since $\psi(0) = \psi \in \Omega_{i+1}$. On the other hand, the local coherence condition in (1) is preserved at each vertex σ_{2l} as the parameter λ increases from 0 to 1. Therefore, the map $\psi \mapsto \psi(1)$ provides the desired explicit retraction from Ω_{i+1} onto Ω_i. □

We define ω_{coh} to be the subposet of ω consisting of all face bundles (or cellular strings) Δ_ψ with $\psi \in \Omega_{\text{coh}}$. Such face bundles (or cellular strings) Δ_ψ

are called *coherent*. It was shown in [3] that the poset ω_{coh} of coherent face bundles is isomorphic to the face poset of the $(n-1)$-dimensional *monotone path polytope* $\Sigma_f(P)$, which is defined as the Minkowski sum $\Sigma_f(P) = \sum_{\sigma \in \Gamma} f^{-1}(\sigma)$. Thus ω_{coh} is a natural subcomplex of ω which is homeomorphic to the $(n-2)$-sphere.

3. Examples and the generalized Baues problem

We first discuss the three classes of examples mentioned in the introduction.

Example 3.1. The standard double stippling on the simplex Δ^n is defined as follows. Faces of Δ^n are identified with ordered subsets $\sigma = \{\sigma_0 < \cdots < \sigma_k\} \subset \{0, 1, 2, \ldots, n\}$, and we set $s(\sigma) = \sigma_0$, $t(\sigma) = \sigma_k$. It was noticed in [1, 2] that the cellular string complex $\omega(\Delta^n, s, t)$ is isomorphic to the boundary complex of the cube I^{n-1}. This double stippling is realized geometrically by the linear functional $f : \Delta^n \to \mathbf{R}$ which takes the value i at the ith vertex. The monotone path polytope $\Sigma_f(\Delta^n)$ is combinatorially (but not affinely) isomorphic to I^{n-1}. In fact, $\Sigma_f(\Delta^n)$ is the Newton polytope of the discriminant of a univariate polynomial of degree n [7]. Here all cellular strings are coherent [3].

Example 3.2. The vertices of the n-cube $I^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | 0 \le x_i \le 1\}$ correspond to subsets of $\{1, 2, \ldots, n\}$, which are partially ordered by inclusion. The standard combinatorial double stippling (s, t) associates to each face its unique minimal (resp. maximal) vertex with respect to this order. The complex of cellular strings $\omega(I^n, s, t)$ is isomorphic to the face lattice of the $(n-1)$-dimensional permutohedron P_n (see [3, 10], and below). This double stippling is realized geometrically by the linear functional $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. It was shown in [3] that all the cellular strings are coherent and the monotone path polytope $\Sigma_f(I^n)$ is linearly isomorphic to P_n.

Example 3.3. The permutohedron P_n is defined as the convex hull in $\mathbf{R}^n$ of the points $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ for all permutations $\sigma \in S_n$. The face lattice of P_n is well known (cf. [2, 10]); in particular, two permutations $\sigma, \tau \in S_n$ are connected by an edge of P_n if and only if they differ by an adjacent transposition (i.e., $\sigma = \tau \circ (i, i+1)$). The standard combinatorial double stippling of P_n associates to each face $\Gamma \subset P_n$ its unique minimal vertex $s(\Gamma)$ (resp. maximal vertex $t(\Gamma)$) with respect to the weak Bruhat order. Recall that the *weak Bruhat order* on the symmetric group S_n is defined as the transitive closure of the relation $\preceq$, where $\sigma \preceq \tau$ if $\sigma = \tau$ or $\sigma = \tau \circ (i, i+1)$ for some i and the number of inversions in τ is larger than the number of inversions of σ.

The double stippling (s, t) may be realized geometrically by the linear functional $f(x_1, \ldots, x_n) = \sum_{i=1}^n t_i x_i$, where $t_1 < \cdots < t_n$ is any increasing sequence of real numbers. It follows from examples in [2, §III.7] that many of the cellular chains in $\omega(P_n, s, t)$ are noncoherent. The sphericity of the cellular string complex for P_n was the original Baues's conjecture. Note that Baues's description in [2] is purely combinatorial, while our proof of the conjecture makes use of convex geometry.

We now discuss a natural generalization of the posets $\omega(P, f)$ for the case where the linear functional f is replaced by a linear operator into $\mathbf{R}^k$. In the topological setting discussed in the introduction this corresponds to the k-fold

loop space $\Omega^k(X)$ regarded as the space of pointed maps of the k-sphere to X rather than iteratively as $\Omega(\Omega^{k-1}(X))$.

Consider a convex polytope $P := \text{conv}(\mathscr{A}_P)$, where $\mathscr{A}_P = \{p_1, p_2, \dots, p_m\} \subset \mathbf{R}^n$, and let $\pi : \mathbf{R}^n \to \mathbf{R}^d$ be an affine map with $\pi(p_1) = q_1, \dots, \pi(p_m) = q_m$. We consider $\mathscr{A}_Q = \{q_1, q_2, \dots, q_m\} \subset \mathbf{R}^d$ as an m-element multiset, and we define $Q := \text{conv}(\mathscr{A}_Q)$.

A *polyhedral complex* is a collection of convex polytopes having the property that the intersection of any two is a face of each and is itself in the collection. A *polyhedral subdivision of* Q is a collection Π of subsets of $\mathscr{A}_Q$ whose convex hulls form a polyhedral complex whose union equals Q. A polyhedral subdivision Π of Q is said to be *induced by* π *from* P if

(1) each cell $\sigma \in \Pi$ is of the form $\pi(F_\sigma)$, where F_σ is the intersection of $\mathscr{A}_P$ with a supporting hyperplane of P, and
(2) for each $\sigma, \sigma' \in \Pi$, $F_{\sigma \cap \sigma'} = F_\sigma \cap F_{\sigma'}$.

Note that $\sigma = \pi(F_\sigma)$ uniquely specifies the set F_σ. Also note that $\dim \sigma \leq \dim F_\sigma$, where by the dimension of a set we mean the dimension of its affine span. An induced subdivision Π is called *tight* if $\dim \sigma = \dim F_\sigma$ for each $\sigma \in \Pi$. Such subdivisions are just polyhedral sections of the projections $\pi : P \to Q$, i.e., they are subcomplexes in P which project onto Q bijectively. If P is a simplex, then the tight induced subdivisions of Q are precisely the triangulations of Q with vertices in $\mathscr{A}_Q$.

A polyhedral subdivision Π of Q is said to be *proper* if $\Pi \neq \{\mathscr{A}_Q\}$. Let $\mathscr{S}(P, Q)$ denote the set of all proper induced polyhedral subdivisions of Q. This set is partially ordered by inclusion, so we view $\mathscr{S}(P, Q)$ as a subposet of the partition lattice on A_Q.

It can be viewed as a combinatorial model of the space of all continuous sections of the projection $\pi : P \to Q$ lying in the boundary of Q. In the case where $\dim Q = 1$, the poset $\mathscr{S}(P, Q)$ is the poset ω of cellular strings. In view of Theorem 1.2, it is natural to ask the following question.

Generalized Baues Problem. *Is the poset* $\mathscr{S}(P, Q)$ *homotopy equivalent to a sphere of dimension* $\dim P - \dim Q - 1$?

As in §2, there is a natural candidate for a spherical subcomplex of $\mathscr{S}(P, Q)$. This is the boundary complex of the *fiber polytope* $\Sigma(P, Q)$ which was introduced in [3] as the Minkowski integral of all fibers of the projection π. When $\dim Q = 1$, this is the monotone path polytope $\Sigma_f(P)$, where f denotes the linear map from P onto Q. As was shown in [3], the faces of $\Sigma(P, Q)$ correspond to *P-coherent polyhedral subdivisions* of Q. These can be defined analogously to §2. Thus the set of coherent polyhedral subdivisions forms a spherical subposet in $\mathscr{S}(P, Q)$.

Thus the essence of the problem is to compare two ways of averaging the fibers: one combinatorial, by means of the inverse limit (cf. Lemma 1.3) and the other geometric, by means of Minkowski summation of polytopes.

In general, it is already unknown whether the complex $\mathscr{S}(P, Q)$ is connected. A partial positive answer to the Generalized Baues Problem, this connectedness, would imply that any two triangulations of a simplicial polytope Q can be joined by a sequence of *bistellar moves* [4] involving its vertices (consider the case where P is a simplex). Positive results have been obtained in [11] for

the case where P is a regular n-cube for specific choices of Q, including those where Q has dimension at most 2 or at least $n-3$. We can also obtain a positive answer in the case where P is a simplex and $\dim Q = 2$.

References

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412.
2. H. J. Baues, *Geometry of loop spaces and the cobar construction*, Mem. Amer. Math. Soc., vol. 25, Amer. Math. Soc., Providence, RI, 1980.
3. L. J. Billera and B. Sturmfels, *Fiber polytopes*, Ann. of Math. (2) **135** (1992), 527–549.
4. L. J. Billera, P. Filliman, and B. Sturmfels, *Constructions and complexity of secondary polytopes*, Adv. Math. **83** (1990), 155–179.
5. A. Björner, *Essential chains and homotopy type of posets*, Proc. Amer. Math. Soc. **116** (1992), 1179–1181.
6. P. Gabriel and M. Zisman, *Calculus of functions and homotopy theory*, Ergebnisse der Math., bd. 35, Springer, Berlin, 1967.
7. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Newton polytopes of the classical discriminant and resultant*, Adv. Math. **84** (1990), 237–254.
8. R. J. Milgram, *Iterated loop spaces*, Ann. of Math. (2) **84** (1966), 386–403.
9. D. Quillen, *Higher algebraic K-theory*: I, Higher K-Theories (H. Bass, ed.), Lecture Notes in Math., vol. 341, Springer, New York, 1973, pp. 77–139.
10. R. Rado, *An inequality*, J. London Math. Soc. (2) **27** (1952), 1–6.
11. B. Sturmfels and G. Ziegler, *Extension spaces of oriented matroids*, Discrete Comput. Geom. **10** (1993), 23–45.

(L. J. Billera and B. Sturmfels) Department of Mathematics, Cornell University, Ithaca, New York 14853

E-mail address: billera@math.cornell.edu

E-mail address: bernd@math.cornell.edu

(M. M. Kapranov) Department of Mathematics, Northwestern University, Evanston, Illinois 60208

E-mail address: kapranov@chow.math.nwu.edu