ON A CONJECTURE OF BAUES IN THE THEORY OF LOOP SPACES

L.J. Billera, M.M. Kapranov and B. Sturmfels

Introduction

An important problem in the theory of loop spaces is to find explicit CW-models for iterated loop spaces $\Omega^i X$ of a given CW-space X. It was shown by J.F. Adams [1] that the Pontryagin algebra $H_*(\Omega X)$ can be obtained as the cohomology algebra of the cobar-construction of the chain coalgebra $C_* X$. H.J. Baues [2] demonstrated that Adams' theorem can be strengthened to an explicit model for ΩX. More precisely, if X is a simplicial space [6] with one vertex and no edges, then Baues' model $\underline{\Omega} X$ is a space glued from cubes. Here the cube I^{n-1} serves as an approximation for the space of paths in the simplex Δ^n joining the 0-th and the n-th vertex. Combinatorially, faces of I^{n-1} correspond to *cellular strings* between the 0-th and n-th vertex in Δ_n, and so I^{n-1} is embedded into the space of all such paths.

R. Milgram [8] noted that a similar approximation to the path space of the cube I^n is provided by the $(n-1)$-dimensional *permutohedron* P_n, which is the convex hull of a generic orbit of the symmetric group S_n in $\mathbf{R}^n$. In fact, P_n can be recovered combinatorially as the complex of cellular strings in I^n. This permitted Baues to iterate his geometric cobar-construction and apply it as well to cubical spaces X with trivial 1−skeleton. He constructed a CW-space $\underline{\Omega} X$, glued from products of permutohedra, which is a model for ΩX. In particular, if Y is a simplicial space with trivial 2−skeleton and $X = \underline{\Omega} Y$, then $\underline{\Omega} X$ will be a model for the iterated loop space $\Omega^2 Y$.

One is tempted to continue this process, introducing "hyperpermutohedra" as complexes of cellular strings in P_n and so on. However, these complexes are no longer isomorphic to face complexes of convex polytopes (and even fail to be combinatorial spheres). Nevertheless, Baues conjectured that they will be homotopy equivalent to spheres and that one can choose a subcomplex homeomorphic to a sphere whose embedding is a homotopy equivalence [2; Conjecture 7.4]. This would yield a model for $\Omega^3 Y$, and so on.

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In this note we prove Baues' conjecture in the more general setting of a convex polytope $P \subset \mathbf{R}^n$ whose vertices are ordered by means of a generic linear functional f. For the desired subcomplex homeomorphic to a sphere one can take the boundary complex of the *monotone path polytope* $\Sigma_f(P)$ introduced in [3]. The polytopal nature of this subcomplex opens up the possibility of iterating Baues' construction to get CW-models for iterated loop spaces by gluing products of iterated monotone path polytopes. We are grateful to David Stone for bringing the work of Adams and Baues to our attention and to Anders Björner and Günter Ziegler for many helpful discussions.

1. Sphericity of the cellular string complex

Let $P \subset \mathbf{R}^n$ be an n-dimensional convex polytope. A *stippling* on P (see [2, Definition III.2.3]) is a rule which assigns to each face Γ of P a vertex $\eta(\Gamma)$ of Γ such that whenever Γ_1 is a face of Γ_2 and $\eta(\Gamma_1) \in \Gamma_2$, then $\eta(\Gamma_1) = \eta(\Gamma_2)$.

An important class of stipplings is provided by convex geometry. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a linear functional. We say that f is *generic* with respect to P if it is non-constant on each edge of P. In this case on each face $\Gamma \subset P$ the function f attains its minimum (resp. maximum) in exactly one point, which is a vertex denoted $s_f(\Gamma)$ (resp. $t_f(\Gamma)$).

Proposition 1.1. *If f is a generic linear functional on P, then s_f and t_f are stipplings.*

Let P be a polytope with a fixed pair of stipplings (s,t). A *cellular string* on P is a sequence of proper faces $(\Gamma_0, \Gamma_1, \ldots, \Gamma_m)$ such that $s(\Gamma_0) = s(P)$, $t(\Gamma_m) = t(P)$, and $s(\Gamma_i) = t(\Gamma_{i-1})$ for each $i = 1, 2, \ldots, m$. The set $\omega = \omega(P,s,t)$ of all cellular strings is partially ordered by the relation: $(\Gamma_0, \Gamma_1, \ldots, \Gamma_m) \le (\Delta_0, \Delta_1, \ldots, \Delta_k)$ if there exist $0 = i_0 < i_1 < i_2 < \ldots < i_m < i_{m+1} = k$ such that $(\Gamma_{i_\nu}, \Gamma_{i_\nu+1}, \ldots \Gamma_{i_{\nu+1}})$ is a cellular string on the face Δ_ν, for each $\nu = 0, 1, \ldots, k$.

The geometric realization $|\omega|$ of the simplicial complex of chains in ω can be viewed as a model for the space of all paths on the boundary ∂P joining $s(P)$ and $t(P)$. So it is a natural question whether $|\omega|$ has the homotopy type of the $(n-2)$-sphere S^{n-2}. The following affirmative answer is the main result of this paper.

Theorem 1.2. *Let f be a generic linear functional on an n-dimensional convex polytope $P \subset \mathbf{R}^n$. Then the poset $\omega(P, s_f, t_f)$ is homotopy equivalent to S^{n-2}.*

Proof: Let $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ be the values of f at the vertices of P. So, the segment $Q := [\lambda_1, \lambda_N] \subset \mathbf{R}$ is the image of P under f. Choose an intermediate point μ_i in each open interval $(\lambda_i, \lambda_{i+1})$. Denote by L_i (resp. M_i) the poset of proper faces of the polytope $f^{-1}(\lambda_i)$ (resp. $f^{-1}(\mu_i)$). When a value λ moves from μ_i to λ_i, each face of $f^{-1}(\lambda)$ contracts to some face of $f^{-1}(\lambda_i)$ of possibly smaller dimension. This defines a surjective morphism of posets $\alpha_i : M_i \to L_i$. Similarly, we have the degeneration in the other direction, defining a morphism $\beta_i : M_i \to L_{i+1}$ (see also Lemma 2.2).

Lemma 1.3. *The poset $\omega(P, s_f, t_f)$ is isomorphic to the inverse limit of the diagram*

$$M_1 \xrightarrow{\beta_1} L_2 \xleftarrow{\alpha_2} M_2 \xrightarrow{\beta_2} L_3 \xleftarrow{\alpha_3} M_3 \xrightarrow{\beta_4} \quad \cdots\cdots \quad \xrightarrow{\beta_{N-2}} L_{N-1} \xleftarrow{\alpha_{N-1}} M_{N-1}.$$

Proof of Lemma 1.3: By definition, an element of this inverse limit is a sequence $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{N-1})$ of polytopes, where $\Gamma_i \in M_i$ is a face of $f^{-1}(\mu_i)$, such that $\beta_i(\Gamma_i) = \alpha_i(\Gamma_{i+1})$ for $i = 1, 2, \ldots, N-2$. Since f is generic, we have either $dim\, \Gamma_i = dim\, \beta_i(\Gamma_i) = dim\, \alpha_i(\Gamma_{i+1}) = dim\, \Gamma_{i+1}$ (which means that Γ_i and Γ_{i+1} are hyperplane sections of the same face of P), or $\beta_i(\Gamma_i) = \alpha_i(\Gamma_{i+1})$ is a point. In either case we get a unique cellular string in P, and each cellular string in P is obtained in this way. $\triangleleft$

Lemma 1.4. *The inverse image under α_i of any upper interval $\{\Phi \in L_i : \Phi \geq \Gamma\}$ in L_i is a contractible subposet of M_i, and similarly for the map β_i.*

Proof of Lemma 1.4: We view $f^{-1}(\lambda_i)$, $f^{-1}(\mu_i)$ and their normal cones as polyhedra in the kernel of f. Let $\mathcal{N}_\Gamma$ be the normal cone of $f^{-1}(\lambda_i)$ at Γ. A face Ξ of $f^{-1}(\mu_i)$ lies in $\alpha_i^{-1}(\{\Phi \in L_i : \Phi \geq \Gamma\})$ if and only if the normal cone of $f^{-1}(\mu_i)$ at Ξ is contained in $\mathcal{N}_\Gamma$. Moreover, these cones form a polyhedral subdivision of $\mathcal{N}_\Gamma$. Hence $\alpha_i^{-1}(\{\Phi \in L_i : \Phi \geq \Gamma\})$ is contractible. $\triangleleft$

It follows from Lemma 1.4 and Quillen's Theorem A [9] that the maps of geometric realizations $|\alpha_i| : |M_i| \to |L_i|$ and $|\beta_i| : |L_i| \to |M_{i+1}|$ are homotopy equivalences and that all their fibers are contractible.

Proof of Theorem 1.2: Let X_i denote the poset inverse limit of the left segment of our diagram up to M_i. Since the operations "inverse limit" and "geometric realization" commute [6; Chapter 3, §3], the geometric realization $|X_i|$ is the inverse limit of the corresponding diagram of topological spaces. Consider the natural projection $p_i : |X_i| \to$

$|X_{i-1}|$. Each fiber of p_i is contractible since it coincides with some fiber of $|\alpha_i|$. Hence p_i is a homotopy equivalence by Quillen's Theorem A [9]. Using induction on i, we conclude that $|X_{N-1}| = |\omega|$ is homotopy equivalent to $|X_1|$. However, $|X_1|$ is the boundary of the $(n-1)$-polytope $f^{-1}(\mu_1)$ and hence homeomorphic to S^{n-2}. $\triangleleft$

We note here that Björner [5] has given another proof of Theorem 1.2 in the case where P is a zonotope (i.e., a Minkowski sum of line segments).

2. Coherent strings and the monotone path polytope

In this section we will sharpen Theorem 1.2 by identifying a subcomplex ω_{coh} of $\omega = \omega(P, s_f, t_f)$ which is homeomorphic to S^{n-2}. To this end we introduce a larger space Ω homotopy equivalent to ω, and we describe an explicit retraction from Ω onto ω_{coh}.

Let Γ denote the subdivision of the open segment $int(Q) = (\lambda_1, \lambda_N)$ into open segments $(\lambda_i, \lambda_{i+1})$ and singletons $\{\lambda_j\}$, $1 < j < N$. We call Γ the *chamber complex* of (P, Q). A *face bundle* is a map Δ which assigns to each $\sigma \in \Gamma$ a proper face $\Delta(\sigma)$ of $f^{-1}(\sigma)$. We usually identify Δ with a subcomplex of the boundary complex ∂P, whence the set $\mathcal{F}(P)$ of all face bundles is finite and partially ordered by inclusion. Every point $\psi = (\psi_\sigma)_{\sigma\in\Gamma} \in (S^{n-2})^\Gamma$ defines a face bundle Δ_ψ via $\Delta_\psi(\sigma) := f^{-1}(\sigma)^{\psi_\sigma}$ (this denotes the extreme face of $f^{-1}(\sigma)$ in direction ψ_σ). Thus $\mathcal{F}(P)$ is the face poset of a piecewise-linear subdivision of the product of spheres $(S^{n-2})^\Gamma$.

We define the *monotone path complex* $\mathcal{M} = \mathcal{M}_f(P)$ to be the subcomplex of $\mathcal{F}(P)$ consisting of all cells which are contained in

$$\Omega \quad := \quad \{\ \psi \in (S^{n-2})^\Gamma \ \mid \ f^{-1}(\tau)^{\psi_\tau} = f^{-1}(\tau)^{\psi_\sigma} \text{ whenever } \tau \subset \overline{\sigma}\ \}. \tag{1}$$

Proposition 2.1. *The monotone path complex $\mathcal{M}$ equals the cellular string complex ω.*

Proposition 2.1 is a consequence of Lemma 1.3 and the following Lemma 2.2 whose elementary proof we omit.

Lemma 2.2. *For any $\psi \in S^{n-2}$, we have $\beta_{i-1}(f^{-1}(\mu_{i-1})^\psi) = f^{-1}(\lambda_i)^\psi = \alpha_i(f^{-1}(\mu_i)^\psi)$.*

Using Proposition 2.1 and a covering argument, it can be shown that the cellular string complex ω is homotopy equivalent to the space Ω. We now consider its diagonal

$$\Omega_{coh} \quad := \quad \{\psi \in (S^{n-2})^\Gamma \ \mid \ \psi_\sigma = \psi_\tau \text{ for all cells } \tau, \sigma \text{ in } \Gamma\ \} \tag{2}$$

which is clearly homeomorphic to the $(n-2)$-sphere. The following result yields an alternative proof of Theorem 1.2.

Theorem 2.3. *The inclusion of topological spaces $\Omega_{coh} \subset \Omega$ is a homotopy equivalence.*

Proof: Let $\sigma_1 < \sigma_2 < \ldots < \sigma_{2N-1}$ be the natural ordering of all cells of the chamber complex Γ (thus σ_{2j-1} is a 1-cell and σ_{2j} is a 0-cell). We abbreviate $\psi_i := \psi_{\sigma_i} \in S^{n-2}$ and consider the intermediate spaces

$$\Omega_i \quad := \quad \{\psi \in \Omega \mid \psi_i = \psi_{i+1} = \ldots = \psi_{2N-1}\} \qquad \text{for} \quad i = 1, 2, \ldots, 2N-1. \tag{3}$$

Since $\Omega_{coh} = \Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \Omega_{2N-1} = \Omega$, it suffices to show that the inclusion $\Omega_i \subseteq \Omega_{i+1}$ is a homotopy equivalence. We will prove this for the case $i = 2j-1$ (so σ_i is a 1-cell); the case $i = 2j$ is analogous.

Let $\psi \in \Omega_{i+1}$. The local coherence condition (1) states that both ψ_{i+1} and ψ_i are support vectors for the same face of the fiber over the point σ_{i+1}. This implies $\psi_i \neq -\psi_{i+1}$, and thus the convex combinations

$$\psi_i(\lambda) \quad = \quad \frac{\lambda\,\psi_i + (1-\lambda)\,\psi_{i+1}}{||\,\lambda\,\psi_i + (1-\lambda)\,\psi_{i+1}\,||} \qquad \text{for } \lambda \in [0,1] \tag{4}$$

are well-defined vectors on the unit sphere S^{n-2}. Now check that the point

$$\psi(\lambda) \quad := \quad \big(\psi_1, \psi_2, \ldots, \psi_i,\, \psi_i(\lambda), \ldots, \psi_i(\lambda)\big) \quad \in \quad (S^{n-2})^{\Gamma} \tag{5}$$

lies in Ω for all $\lambda \in [0,1]$. This is clearly the case for $\lambda = 0$, since $\psi(0) = \psi \in \Omega_{i+1}$. On the other hand, the local coherence condition (1) is preserved at each vertex σ_{2l} as the parameter λ increases from 0 to 1. Therefore the map $\psi \mapsto \psi(1)$ provides the desired explicit retraction from Ω_{i+1} onto Ω_i. ◁

We define ω_{coh} to be the subposet of ω consisting of all face bundles (or cellular strings) Δ_ψ with $\psi \in \Omega_{coh}$. Such face bundles (or cellular strings) Δ_ψ are called *coherent*. It was shown in [3] that the poset ω_{coh} of coherent face bundles is isomorphic to the face poset of the $(n-1)$-dimensional *monotone path polytope* $\Sigma_f(P)$, which is defined as the Minkowski sum $\Sigma_f(P) = \sum_{\sigma \in \Gamma} f^{-1}(\sigma)$. Thus ω_{coh} is a natural subcomplex of ω which is homeomorphic to the $(n-2)$-sphere.

3. Examples and the generalized Baues problem

We first discuss the three classes of examples mentioned in the introduction.

Example 3.1. The standard double stippling on the simplex Δ^n is defined as follows. Faces of Δ^n are identified with ordered subsets $\sigma = \{\sigma_0 < \ldots < \sigma_k\} \subset \{0,1,2,\ldots,n\}$, and we set $s(\sigma) = \sigma_0$, $t(\sigma) = \sigma_k$. It was noticed in [1],[2] that the cellular string complex $\omega(\Delta^n, s, t)$ is isomorphic to the boundary complex of the cube I^{n-1}. This double stippling is realized geometrically by the linear functional $f : \Delta^n \to \mathbf{R}$ which takes the value i at the i-th vertex. The monotone path polytope $\Sigma_f(\Delta^n)$ is combinatorially (but not affinely) isomorphic to I^{n-1}. In fact, $\Sigma_f(\Delta^n)$ is the Newton polytope of the discriminant of a univariate polynomial of degree n [7]. Here all cellular strings are coherent [3].

Example 3.2. The vertices of the n-cube $I^n = \{(x_1,\ldots,x_n) \in \mathbf{R}^n \,|\, 0 \le x_i \le 1\}$ correspond to subsets of $\{1,2,\ldots,n\}$, and hence are partially ordered by inclusion. The standard combinatorial double stippling (s,t) associates to each face its unique minimal (resp. maximal) vertex with respect to this order. The complex of cellular strings $\omega(I^n, s, t)$ is isomorphic to the face lattice of the $(n-1)$-dimensional permutohedron P_n (see [3],[10] and below). This double stippling is realized geometrically by the linear functional $f(x_1,\ldots,x_n) = x_1 + \ldots + x_n$. It was shown in [3] that all the cellular strings are coherent and the monotone path polytope $\Sigma_f(I^n)$ is linearly isomorphic to P_n.

Example 3.3. The permutohedron P_n is realized as the convex hull in $\mathbf{R}^n$ of the points $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ for all permutations $\sigma \in S_n$. The face lattice of P_n is well-known (cf. [2],[10]); in particular, two permutations $\sigma, \tau \in S_n$ are connected by an edge of P_n if and only if they differ by an adjacent transposition (i.e. $\sigma = \tau \circ (i, i+1)$). The standard combinatorial double stippling of P_n associates to each face $\Gamma \subset P_n$ its unique minimal vertex $s(\Gamma)$ (resp. maximal vertex $t(\Gamma)$) with respect to the weak Bruhat order. Recall that the *weak Bruhat order* on the symmetric group S_n is defined as the transitive closure of the relation $\preceq$, where $\sigma \preceq \tau$ if $\sigma = \tau$, or $\sigma = \tau \circ (i, i+1)$ for some i and the number of inversions in τ is larger than the number of inversions of σ.

The double stippling (s,t) may be realized geometrically by the linear functional $f(x_1,\ldots,x_n) = \sum_{i=1}^{n} t_i x_i$, where $t_1 < \ldots < t_n$ is any increasing sequence of real numbers. It follows from the results in [2, Section III.7] that many of the cellular chains in $\omega(P_n, s, t)$ are non-coherent.

We now discuss a natural generalization of Baues' conjecture. Consider a convex polytope $P := conv(\mathcal{A}_P)$ where $\mathcal{A}_P = \{p_1, p_2, \dots, p_m\} \subset \mathbf{R}^n$, and let $\pi : \mathbf{R}^n \to \mathbf{R}^d$ be an affine map with $\pi(p_1) = q_1, \dots, \pi(p_m) = q_m$. We consider $\mathcal{A}_Q = \{q_1, q_2, \dots, q_m\} \subset \mathbf{R}^d$ as an m-element multiset, and we define $Q := conv(\mathcal{A}_Q)$.

A *polyhedral complex* is a collection of convex polytopes having the property that the intersection of any two is a face of each and is itself in the collection. A *polyhedral subdivision of* Q is a collection Π of subsets of $\mathcal{A}_Q$ whose convex hulls form a polyhedral complex whose union equals Q. A polyhedral subdivision Π of Q is *induced by* π *from* P if each cell $\sigma \in \Pi$ is of the form $\pi(F_\sigma)$ for some face F_σ of P. Since both cells of Π and faces of P are regarded as labeled, $\sigma = \pi(F_\sigma)$ uniquely specifies the face F_σ, and $\pi^{-1}(F_\sigma) \cap \pi^{-1}(F_{\sigma'}) = \pi^{-1}(F_\sigma \cap F_{\sigma'})$. Note also $dim\, \sigma \leq dim\, F_\sigma$. An induced subdivision Π is called *tight* if $dim\, \sigma = dim\, F_\sigma$ for each $\sigma \in \Pi$. If P is a simplex, then the tight induced subdivisions of Q are precisely the triangulations of Q with vertices in $\mathcal{A}_Q$.

The set $\mathcal{S}(P, Q)$ of proper (i.e. $\Pi \neq \{\mathcal{A}_Q\}$) polyhedral subdivisions of P is partially ordered by refinement. Thus we view $\mathcal{S}(P, Q)$ as a subposet of the partition lattice on A_Q. In the case where $dim\ Q = 1$, the poset $\mathcal{S}(P, Q)$ is the complex ω of cellular strings. In view of Baues' conjecture, it is natural to ask the following question.

Generalized Baues Problem. *Is the poset $\mathcal{S}(P, Q)$ homotopy equivalent to a sphere of dimension $dim\ P - dim\ Q - 1$?*

As in Section 2, there is a natural candidate for a spherical subcomplex of $\mathcal{S}(P, Q)$. This is the boundary complex of the *fiber polytope* $\Sigma(P, Q)$ which was introduced in [3]. When $dim\ Q = 1$, then this is the monotone path polytope $\Sigma_f(P)$ where f denotes the linear map from P onto Q. As was shown in [3], the faces of $\Sigma(P, Q)$ correspond to *P-coherent polyhedral subdivisions* of Q. These can be defined analogously to Section 2. Thus the set of coherent polyhedral subdivisions forms a spherical subposet in $\mathcal{S}(P, Q)$.

In general, we do not even know whether the complex $\mathcal{S}(P, Q)$ is connected. A partial positive answer to the Generalized Baues Problem, establishing just the connectedness of $\mathcal{S}(P, Q)$, would have the following interesting implications:

(a) Any two triangulations of a simplicial polytope Q can be joined by a sequence of *bistellar moves* [4] involving its vertices (consider the case when P is a simplex).

(b) Any two one-element extensions of a given realizable oriented matroid can be joined by a sequence of *mutations* (i.e., "inverting" of simplicial regions). This corresponds to the case when $P = I^n$ is a cube and Q is a zonotope. See [11] for a detailed discussion of the generalized Baues problem from the oriented matroid perspective.

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Department of Mathematics, Cornell University, Ithaca, New York 14853

Department of Mathematics, Northwestern University, Evanston, Illinois 60208

Department of Mathematics, Cornell University, Ithaca, New York 14853