BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

SUFFICIENCY OF McMULLEN'S CONDITIONS FOR f-VECTORS OF SIMPLICIAL POLYTOPES

BY LOUIS J. BILLERA[1] AND CARL W. LEE[2]

For convex d-polytope P let $f_i(P)$ equal the number of faces of P of dimension i, $0 \leqslant i \leqslant d-1$. $f(P) = (f_0(P), \ldots, f_{d-1}(P))$ is called the *f-vector of P*. An important combinatorial problem is the characterization of the class of all f-vectors of polytopes, and in particular of simplicial polytopes (i.e. those for which each facet is a simplex). McMullen in [5] conjectures a set of necessary and sufficient conditions for $(f_0, \ldots, f_{d-1})$ to be the f-vector of a simplicial d-polytope and proves this conjecture in the case of polytopes with few vertices. We sketch here a proof of the sufficiency[3] of these conditions, and derive in a related way a general solution to an upper bound problem posed by Klee.

The f-vectors of simplicial d-polytopes satisfy the *Dehn-Sommerville equations*

$$\sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i(P) = (-1)^{d-1} f_j(P), \qquad -1 \leqslant j \leqslant d-1,$$

where we put $f_{-1}(P) = 1$. As in [6, p. 170], for d-vector $f = (f_0, \ldots, f_{d-1})$ and integer $e \geqslant d$ let

$$g_j^{(e)}(f) = h_{j+1}^{(e)}(f) = \sum_{i=-1}^{j} (-1)^{j-i} \binom{e-i-1}{e-j-1} f_i, \qquad -1 \leqslant j \leqslant e-1,$$

with the convention that $f_{-1} = 1$ and $f_i = 0$ for $i < -1$ or $i > d-1$. We note here that these relations are invertible, allowing us to express the f_i as nonnegative linear combinations of the $h_j^{(e)}(f)$. The Dehn-Sommerville equations for f are, for any $e \geqslant d$, equivalent to $g_i^{(e)}(f) = (-1)^{e-d} g_{e-i-2}^{(e)}(f)$, $-1 \leqslant i \leqslant [e/2] - 1$. Let h and i be positive integers. Then h can be written uniquely as

Received by the editors July 18, 1979.

1980 *Mathematics Subject Classification*. Primary 52A25; Secondary 05A15, 05A19, 05A20, 90C05, 13H10.

Key words and phrases. Convex polytope, f-vector, 0-sequence, shelling, simplicial complex.

[1]Supported in part by NSF grant MCS77-28392 and ONR contract N00014-75-C-0678.

[2]Supported, in addition, by an NSF Graduate Fellowship.

[3]ADDED IN PROOF. R. Stanley has proved necessity since this was written.

$$h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geqslant j \geqslant 1$. Following McMullen put

$$h^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}$$

and define $0^{\langle i \rangle} = 0$. McMullen conjectured ([5], [6, p. 179]) that $(f_0, \ldots, f_{d-1})$ is the f-vector of a simplicial d-polytope if and only if the following three conditions hold:

$$g_i^{(d+1)}(f) = -g_{d-i-1}^{(d+1)}(f), \qquad -1 \leqslant i \leqslant [\tfrac{1}{2}(d+1)] - 1, \tag{1}$$

$$g_i^{(d+1)}(f) \geqslant 0, \qquad 0 \leqslant i \leqslant n - 1, \tag{2}$$

$$g_i^{(d+1)}(f) \leqslant (g_{i-1}^{(d+1)}(f))^{\langle i \rangle}, \qquad 1 \leqslant i \leqslant n - 1, \tag{3}$$

where $n = [d/2]$. Condition (1) is just the set of Dehn-Sommerville equations, the conjectured necessity of (2) is known as the *Generalized Lower Bound Conjecture* ([7], [6, p. 178]).

We will sketch a proof of the following

THEOREM 1. *If* $f = (f_0, \ldots, f_{d-1})$ *satisfies* (1), (2), *and* (3) *above, then* $f = f(P)$ *for some simplicial d-polytope P.*

The case $d < 2$ is easily dispensed with, so assume $d \geqslant 2$. For finite $(d-1)$-dimensional simplicial complex Σ let $|\Sigma|$ denote the underlying topological space of Σ. $f(\Sigma) = (f_0(\Sigma), \ldots, f_{d-1}(\Sigma))$ is the *f-vector of* Σ, where $f_i(\Sigma)$ is the number of i-dimensional simplices in Σ. For $e \geqslant d$ write $h^{(e)}(\Sigma)$ for $h^{(e)}(f(\Sigma))$. We call $h^{(d)}(\Sigma)$ the *h-vector of* Σ. This is equivalent to Stanley's h-vector of [9]. If $|\Sigma|$ is a $(d-1)$-sphere then the Dehn-Sommerville equations hold (see, for example, Grünbaum [1, p. 152]). If $|\Sigma|$ is a d-ball then $\partial|\Sigma|$ is a $(d-1)$-sphere with associated complex $\partial\Sigma$. $|\Delta|$ is then a d-sphere, where $\Delta = \Sigma \cup v \cdot \partial\Sigma$. It can be shown that $h_i^{(d+1)}(\Delta) = h_i^{(d+1)}(\Sigma) + h_{i-1}^{(d)}(\partial\Sigma)$, $0 \leqslant i \leqslant d+1$ (where we take $h_{-1}^{(d)}(\partial\Sigma) = 0$). The Dehn-Sommerville equations for Δ and $\partial\Sigma$ allow us to solve for $h_i^{(e)}(\partial\Sigma)$ in terms of $h_j^{(d+1)}(\Sigma)$. In particular [7]

$$h_i^{(d+1)}(\partial\Sigma) = h_i^{(d+1)}(\Sigma) - h_{d+1-i}^{(d+1)}(\Sigma), \qquad 0 \leqslant i \leqslant [\tfrac{1}{2}(d+1)].$$

A nonvoid set M of monomials $Y_1^{a_1} \cdots Y_s^{a_s}$ is said to be an *order ideal of monomials* if whenever $m_1 \in M$ and $m_2 | m_1$ then $m_2 \in M$. Let Φ be the set of all monomials in the variables $Y_1, \ldots, Y_s$. Give the elements of Φ the lexicographic linear order $<$ induced by $Y_1 < \cdots < Y_s$. A finite or infinite sequence $(H(0), H(1), \cdots)$ of nonnegative integers is said to be an *0-sequence* if there exists an order ideal M of monomials in the variables $Y_1, \ldots, Y_s$ with each $\deg Y_i = 1$ such that $H(i) = \operatorname{card}\{m \in M: \deg m = i\}$. Stanley in **[10]** gives the following

THEOREM. *Let* $H: \mathbf{N} \longrightarrow \mathbf{N}$. *The following statements are equivalent*:

(i) $(H(0), H(1), \cdots)$ *is an* 0*-sequence.*

(ii) $H(0) = 1$ *and for all* $i \geqslant 1$, $H(i+1) \leqslant H(i)^{\langle i \rangle}$.

(iii) *Let* $s = H(1)$ *and for each* $i \geqslant 0$ *let* M_i *be the first* (*in the ordering above*) $H(i)$ *monomials of degree* i *in the variables* $Y_1, \ldots, Y_s$. *Define* $M = \bigcup_{i \geqslant 0} M_i$. *Then* M *is an order ideal of monomials. Call* M *the lexicographic order ideal of monomials associated with* $(H(0), H(1), \cdots)$.

IDEA OF PROOF OF THEOREM 1. If $(f_0, \ldots, f_{d-1})$ satisfies (1), (2), and (3), then by the above theorem $(H(0), \ldots, H(d+1))$ is an 0-sequence, where $H(i) = h_i^{(d+1)}(f)$ for $0 \leqslant i \leqslant n$ and $H(i) = 0$ for $n+1 \leqslant i \leqslant d+1$. A simplicial complex Σ is constructed by choosing as its maximal simplices certain $(d+1)$-sets from a ν-set, where $\nu = H(1) + d + 1$, such that Σ is shellable in the sense of [9] and such that $(H(0), \ldots, H(d+1))$ is its h-vector. It is then shown that Σ is the complex associated with a shellable proper collection $\mathcal{B}$ of facets of the cyclic polytope $C(\nu, d+1)$, implying that $|\Sigma|$ is a d-ball. $\partial|\Sigma|$ is then a $(d-1)$-sphere with associated complex $\partial\Sigma$. From $H(i) = 0$, $n+1 \leqslant i \leqslant d+1$, it can be concluded that $h_i^{(d+1)}(\partial\Sigma) = h_i^{(d+1)}(f)$, $0 \leqslant i \leqslant n$. This and the Dehn-Sommerville equations for $\partial\Sigma$ yield $h^{(d+1)}(\partial\Sigma) = h^{(d+1)}(f)$, whence we conclude $f = f(\Sigma)$. Next, with an appropriate realization of $C(\nu, d+1)$ in $\mathbf{R}^{d+1}$, a point $z \in \mathbf{R}^{d+1}$ can be found such that z is beyond those facets of $C(\nu, d+1)$ that are in $\mathcal{B}$ and beneath the rest. Then the vertex figure P of z in $\operatorname{conv}(C(\nu, d+1) \cup \{z\})$ is a d-polytope whose boundary complex is isomorphic to $\partial\Sigma$, demonstrating sufficiency. (In fact, P is n-stacked in the sense of [7].)

A sketch of the construction of Σ follows. The case $H(1) = 0$ is easily dealt with. For $H(1) \geqslant 1$, let $U = \{u_1, \ldots, u_{\nu'}\}$ where $\nu' = H(1) + 2n$. Let Ψ' be the set of all $2n$-subsets W' of U of the form $\{u_{i_1}, u_{i_1+1}\} \cup \cdots \cup \{u_{i_n}, u_{i_n+1}\}$ where $1 \leqslant i_1, i_n + 1 \leqslant \nu'$, and $i_{j+1} > i_j + 1$, $1 \leqslant j \leqslant n-1$. Let $V' = \{v_1, \ldots, v_{d+1-2n}\}$, $V = V' \cup U$ and Ψ be the set of all $(d+1)$-subsets W of V of the form $V' \cup W'$ for $W' \in \Psi'$. Give the elements of Ψ the lexicographic linear order $<$ induced by $u_1 < \cdots < u_{\nu'}$. Let Φ_n be the set of all monomials in the variables $Y_1, \ldots, Y_s$ of degree at most n, where $s = H(1)$. A one-to-one order preserving correspondence $\beta: \Phi_n \longrightarrow \Psi$ can be defined. From Φ_n choose the lexicographic order ideal of monomials M associated with $(H(0), \ldots, H(n))$. List the elements of M in order $m_1 < \cdots < m_\mu$. Consider the corresponding elements of Ψ, $F_i = \beta(m_i)$. Let Σ be the d-dimensional simplicial complex whose maximal simplices are $F_1, \ldots, F_\mu$. It can be shown that Σ is shellable with shelling order $F_1, \ldots, F_\mu$ and that $h^{(d+1)}(\Sigma) = (H(1), \ldots, H(d+1))$.

Relabel the elements of $V = \{v_1, \ldots, v_{d+1-2n}, u_1, \ldots, u_{\nu'}\}$ as $\{v_1, \ldots, v_\nu\}$ where $\nu = H(1) + d + 1$. Consider the cyclic polytope $C(\nu, d+1) = \text{conv}\{v_1, \ldots, v_\nu\}$ where $v_i = (t_i, t_i^2, \ldots, t_i^{d+1}) \in \mathbf{R}^{d+1}$, $t_1 < \cdots < t_\nu$. This notation implicitly defines a one-to-one correspondence between V and the vertex set of $C(\nu, d+1)$. Then $\{F_1, \ldots, F_\mu\}$ is a representation of a shellable proper collection $\mathcal{B}$ of facets of $C(\nu, d+1)$. The existence of a realization of $C(\nu, d+1) \subseteq \mathbf{R}^{d+1}$ and of a point $z \in \mathbf{R}^{d+1}$ beyond precisely the facets in $\mathcal{B}$ reduces to finding rational numbers $t_1 < \cdots < t_\nu$ satisfying a finite number of polynomial inequalities. This can be accomplished by an application of a version of Tarski's Principle (see e.g. [2, Theorem 13, p. 290]). Once this is done, the desired simplicial polytope P can be obtained as previously described.

A PROBLEM OF KLEE ON UPPER BOUNDS. For $3 \leqslant d \leqslant r < \nu$, a polytope (resp. spherical complex) P is of type (d, ν, r) if P is a d-polytope (resp. $(d-1)$-spherical complex) with ν vertices, one of which is incident to precisely r edges. The problem, stated by Klee in a dual fashion, is to determine max $f_{d-1}(P)$ over all simplicial polytopes P of type (d, ν, r). Klee places bounds on this number and determines it in some particular cases [3], [4]. We offer the complete solution with the following

THEOREM 2. *Let S be a simplicial sphere of type (d, ν, r). Then $f_i(S) \leqslant f_i(C(\nu-1, d)) + f_i(C(r+1, d)) - f_i(C(r, d))$, $0 \leqslant i \leqslant d-1$. Further, there exists a simplicial d-polytope P^* that satisfies all of the above expressions with equality. (Here $f_i(C(d, d))$ is 2 if $i = d-1$, and is $f_i(C(d, d-1))$ otherwise.)*

The bounds are established in the same manner that Stanley uses in [8], relying on the fact that h-vectors of simplicial spheres are 0-sequences. P^* is obtained from a construction similar to that used in the proof of Theorem 1. Here, however, the desired polytope is $\text{conv}(C(\nu-1, d) \cup \{z\})$ for an appropriate z. By a triangulation argument similar to that of pulling vertices of polytopes it can in fact be shown that P^* achieves the maximum number of i-dimensional faces over the class of all (not necessarily simplicial) spherical complexes of type (d, ν, r). (Spherical complexes are defined in [6].)

REFERENCES

1. B. Grünbaum, *Convex polytopes*, Wiley, New York, 1967.

2. N. Jacobson, *Lectures in abstract algebra*, Vol. III, Van Nostrand, Princeton, N. J., 1964.

3. V. Klee, *Polytope pairs and their relationship to linear programming*, Acta Math. 133 (1974), 1–25.

4. ———, *Convex polyhedra and mathematical programming*, Proc. Internat. Congr. Mathematicians, Vol. I, Vancouver, 1974, pp. 485–490.

5. P. McMullen, *The numbers of faces of simplicial polytopes*, Israel J. Math. 9 (1971), 559–570.

6. P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, London Math. Soc. Lecture Note Ser. 3, Cambridge, 1971.

7. P. McMullen and D. W. Walkup, *A generalized lower-bound conjecture for simplicial polytopes*, Mathematika **18** (1971), 264–273.

8. R. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Studies in Appl. Math. **54** (1975), 135–142.

9. ———, *Cohen-Macaulay complexes*, Higher combinatorics, M. Aigner (ed.), D. Reidel, Dordrecht, 1977, pp. 51–62.

10. ———, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978), 57–83.

SCHOOL OF OPERATIONS RESEARCH AND CENTER FOR APPLIED MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853