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Polarity and Inner Products in Oriented Matroids

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We show that the usual polarity properties of the face lattices of convex polytopes do not extend to the setting provided by oriented matroids. Thus, the classical theorems of Weyl and Minkowski fail to hold in this setting. We extend the notion of inner product to oriented matroids and use it to construct polars in certain cases.

1. INTRODUCTION

The notion of oriented matroid provides an abstract combinatorial setting for the study of linear dependence in vector spaces over ordered fields, or of directed graphs. In his early work in this area, Bland [3, 4] showed that this setting was rich enough to provide the basic results of linear programming, including an algorithmic proof of the duality theorem (see also [9]). Las Vergnas [11] generalized the notion of the lattice of faces of a convex polytope (or pointed polyhedral cone) to oriented matroids, showing this lattice to have many properties of the usual face lattice. Taking a polar approach, Edmonds, Fukuda and Mandel [8, 10, 14] have extended the study of this lattice and its relationship to linear programming.

In this paper, we examine whether the usual polarity properties of convex polytopes can be extended to oriented matroids. In Section 2, we define the lattices of Las Vergnas and of Edmonds and Mandel, and we describe a construction of Lawrence, which he used to show that not all oriented matroid lattices are face lattices of polytopes. In Section 3 we use Lawrence's construction and an adjointness notion due to Cheung [6] to show that polyhedral polarity fails in general for oriented matroids. (Thus, for example, the family of lattices defined by Las Vergnas differs from that defined by Edmonds and Mandel, and as a result, the classical theorems of Weyl and Minkowski do not extend to the oriented matroid setting.) In Section 4, we extend the notion of inner product to oriented matroids, and in Section 5 relate it to polarity. We show that under very special conditions, one can use an inner product, exactly as one does in the polyhedral case, to recognize an oriented matroid polar.

We note here that our results lead to the following apparent inconsistency. In the case of vector spaces over ordered fields, the Farkas lemma is the key result in proving both the duality theorem of linear programming and the theorems of Weyl and Minkowski on polarity for polytopes and polyhedral cones. For general oriented matroids, however, Bland has proved a version of the Farkas lemma [4, Corollary 3.1.1] and of the duality theorem of linear programming [4, Theorem 3.5], yet we will show that polarity fails in general. One is led to conclude that duality and polarity are, from a combinatorial perspective, essentially different phenomena.

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For definitions, notation, and general results about oriented matroids, see [4], [5]. If $M=(E,\mathcal{O})$ is an oriented matroid, we will often denote the rank of M by $\rho(M)$.

2. Face Lattices of Oriented Matroids

Let $M=(E,\mathcal{O})$ be an acyclic oriented matroid (that is, an oriented matroid such that there does not exist $X\in\mathcal{O}$ with $X^-=\varnothing$). Then every $e\in E$ is in some positive cocircuit of M [4, Theorem 3.1] and hence the set $\mathcal{K}^+(\mathcal{O}^\perp)$ of all positive elements of the signed cocircuit span of M is not empty and contains a signed set X with $X^+=E$. Let $L(M)\equiv\{E\setminus\underline{Y}\mid Y\in\mathcal{K}^+(\mathcal{O}^\perp)\}$ be partially ordered by set inclusion. $L(M)$ is then a lattice with $F_1\wedge F_2=F_1\cap F_2$, for F_1, $F_2\in L(M)$, having many of the properties of polyhedral face lattices [12]. We call $L(M)$ the *Las Vergnas lattice* of M. If $F\in L(M)$, F is said to be a *face* of $L(M)$. Note that the unique maximal element of $L(M)$ is E, being the complement of the empty set which is in $\mathcal{K}^+(\mathcal{O}^\perp)$, and the unique minimal element is $\varnothing$, the complement of the signed set with $X^+=E$.

On the other hand, let the elements of $L^*(M)\equiv\{\underline{Y}\mid Y\in\mathcal{K}^+(\mathcal{O}^\perp)\}$ be ordered by set inclusion. Again $L^*(M)$ is a lattice, with $\underline{Y}_1\vee\underline{Y}_2=\underline{Y_1\circ Y_2}$ [10, 14]. (For signed sets Y_1 and Y_2, $Y_1\circ Y_2$ is defined by $(Y_1\circ Y_2)^+=Y_1^+\cup(Y_2^+\setminus Y_1^-)$, $(Y_1\circ Y_2)^-=Y_1^-\cup(Y_2^-\setminus Y_1^+)$.) We will call this lattice, whose elements are the underlying sets of elements of $\mathcal{K}^+(\mathcal{O}^\perp)$, the *Edmonds–Mandel lattice* of the matroid.

Note the relation between $L(M)$ and $L^*(M)$ for a given M. It is easy to see that the map $\phi\colon L^*(M)\to L(M)$ defined by $\phi(Y)=E\setminus Y$ is a bijective order-inverting function, and hence $L(M)$ and $L^*(M)$ are anti-isomorphic. Therefore they form a polar, or dual, pair of lattices. Clearly the class of all Las Vergnas lattices of acyclic oriented matroids is the class of polars of all Edmonds–Mandel lattices of such matroids. For previous results concerning these classes of lattices, we refer the reader to [7, 8, 10, 12, 14 and 15]. In particular, we note here that each of $L(M)$ and $L^*(M)$ is a *point lattice* and a *copoint lattice* (i.e. each element is a join of points and a meet of copoints, see [15; Prop. 2.1.3]).

It is easy to see that if M is any acyclic oriented matroid and M' is the acyclic oriented matroid obtained from M by deleting all but one of any set of parallel elements, then $L(M')\cong L(M)$. Furthermore, suppose $e\in E$ is such that $\{e\}$ is not a point of $L(M)$. Then $L(M\setminus e)\cong L(M)$ [15, Proposition 2.2.2]. Therefore in what follows we may assume when desired that $M=(E,\mathcal{O})$ is a simple acyclic oriented matroid such that every element of E is a point of $L(M)$.

For some time it was not known whether the lattices arising from oriented matroids were all polytopal (i.e., isomorphic to face lattices of convex polytopes). In the fall of 1980, however, Lawrence announced that he had found a matroid construction which could be used to produce a class of acyclic oriented matroids whose lattices were not polytopal. This construction proved useful to us in showing that the class of Las Vergnas lattices is not the same as the class of Edmonds–Mandel lattices. For this reason we describe here without proof the reformulation of Lawrence's construction which was related to us by Edmonds and its use in producing a matroid with a non-polytopal lattice.

Let $M=(E,\mathcal{O})$ be an oriented matroid, where $E=\{e_1,\ldots,e_n\}$. Lawrence's construction produces from M an oriented matroid $\Lambda(M)$ on the set $E\cup E^*$, where $E^*=\{e_1^*,\ldots,e_n^*\}$. For $A\subseteq E$, define $A^*=\{e^*\mid e\in A\}$. $\Lambda(M)$ is the oriented series extension of **M** such that if $X=(X^+,X^-)$ is an oriented circuit of M, then $(X^+\cup(X^-)^*, X^-\cup(X^+)^*)$ is an oriented circuit of $\Lambda(M)$.

In [15], a different description of $\Lambda(M)$ is given, and the following lemma is proved.

Lemma 2.1. *Let Y be a cocircuit of M, and let $A\subseteq Y$. Then Y^A defined by $(Y^A)^+=(Y^+\setminus A)\cup(Y^-\cap A)^*$ and $(Y^A)^-=(Y^-\setminus A)\cup(Y^+\cap A)^*$ is a cocircuit of $\Lambda(M)$. If $e\in E$*

is not a coloop of M, $\{e, e^*\}$ is the underlying set of a positive cocircuit Y_e of $\Lambda(M)$. Furthermore, $\mathcal{O}^\perp(\Lambda(\mathrm{M})) = \mathscr{A} \cup -\mathscr{A} \cup \mathscr{B}$, where $\mathscr{A} = \{Y_e | e \in E$ is not a coloop of $M\}$, $-\mathscr{A} = \{-Y | Y \in \mathscr{A}\}$, and $\mathscr{B} = \{Y^A | Y \in \mathcal{O}^\perp(M),\ \varnothing \subsetneq A \subsetneq Y\}$.

Note that for M any oriented matroid, $\Lambda(M)$ is acyclic and there exist two positive cocircuits of $\Lambda(M)$ corresponding to each hyperplane of M. Notice also that if $e \in E$ is not a loop of M, then e and e^* are both points of $L(\Lambda(M))$. This follows from Lemma 2.1 and [12, Proposition 1.6].

With this lemma one can easily prove Lawrence's result.

THEOREM 2.2. *$\Lambda(M)$ has a polytopal lattice if and only if M is a representable matroid.*

COROLLARY 2.3. *There exist oriented matroids whose lattices are not polytopal face lattices.*

It is perhaps interesting to compare the lattices of $M = (E, \mathcal{O})$ with the lattices of $\Lambda(M)$. Since $\rho(\Lambda(M)) = \rho(M) + |E|$, the rank of $L(\Lambda(M))$, or of $L^*(\Lambda(M))$, is $|E|$ greater than the rank of $L(M)$. The number of copoints of $L(\Lambda(M))$ increases over the number of copoints of $L(M)$ even more rapidly. If M has k coloops, $|\{Y \in \mathcal{O}^\perp(\Lambda(M))\colon Y^- = \varnothing\}| = |\mathcal{O}^\perp(M)| + |E| - k$ with Y and $-Y$ counted as two distinct elements of $\mathcal{O}^\perp(M)$. For example if M is the Vámos matroid, a non-representable matroid having an acyclic orientation [5], $\rho(\Lambda(M)) = 4 + 8 = 12$, and $L(\Lambda(M))$ has 90 copoints and 16 points. Another non-representable oriented matroid, called the non-Pappus matroid [3], has nine elements and is of rank 3, so applying this construction again results in a lattice of rank 12. At this time, these are the smallest non-polytopal matroid lattices known to exist, i.e. those of lowest rank.

If the original matroid M is representable, so is $\Lambda(M)$ and hence its lattice is polytopal. One question of interest in doing polytopal constructions is the effect of the construction on the diameter of the geometric figure. The diameter of a convex polyhedron is the maximum over all pairs of vertices of the minimum length of an edge path connecting the two vertices, where the length is measured as the number of edges in the path. The diameter of a polyhedron is of interest in linear programming as a measure of the worst possible performance of a best possible edge-following algorithm applied to a linear program with that polyhedron as its feasible region. Conjectured bounds for the diameter of an d-dimensional polyhedron with n facets are on the order of $n - d$ (see [16]). It is of interest, then, to construct polyhedra having large diameter from others of small diameter. It seems that the construction of $\Lambda(M)$ has just the opposite effect. For the proof of the following, see [15].

PROPOSITION 2.4. *Let M be any representable oriented matroid. The diameter of the polytope P whose face lattice is isomorphic to $L^*(\Lambda(M))$ is less than or equal to 4.*

3. POLARS OF ORIENTED MATROIDS

We define a *polar* of an acyclic oriented matroid M to be an acyclic oriented matroid M^* such that $L(M)$ is anti-isomorphic to $L(M^*)$ (and hence $L^*(M) \cong L(M^*)$). The question of whether the class of Las Vergnas lattices of oriented matroids is the class of Edmonds-Mandel lattices is then equivalent to the question of whether every acyclic oriented matroid has a polar. This issue is of interest as an analog of the results of Weyl and Minkowski. Every convex polytope in $\mathbb{R}^n$ can be realized both as the convex hull of a finite set of points and as the bounded intersection of a finite number of closed half-spaces; this is the content of the classical theorems of Weyl and Minkowski [17, pp. 55–57]. In looking at the Las Vergnas lattice of a matroid, where elements correspond to points and complements of positive cocircuits to copoints, we are looking at the matroid analog of the convex hull of the elements. In looking at the Edmonds–Mandel lattice of a matroid, elements correspond to copoints and complements of positive cocircuits to points, so we have the matroid analog of the intersection of a finite number of closed half-spaces. An oriented matroid M has a polar M^* if and only if both $L(M) \cong L^*(M^*)$ and $L^*(M) \cong L(M^*)$. In this case each of the lattices $L(M)$ and $L^*(M)$ arises both as the 'convex hull of points' and as the 'intersection of half-spaces'. We shall show that not every oriented matroid has a polar and so not every lattice of the form $L(M)$ [or of the form $L^*(M)$] has this property.

In order to construct an oriented matroid which has no polar, we need the notion of adjoint, due to Cheung [6]. Let M be any matroid (not necessarily oriented), and let $\mathscr{L}(M)$ denote its full lattice of flats [18]. An adjoint of M is a matroid M^{Δ} of the same rank such that there exists a one-to-one, order-inverting function $\psi: \mathscr{L}(M) \to \mathscr{L}(M^{\Delta})$ which maps the copoints of $\mathscr{L}(M)$ onto the points of $\mathscr{L}(M^{\Delta})$. We will show that for any oriented matroid M, if $\Lambda(M)$ has a polar, then M has an adjoint.

Let $\mathbf{A}$ be an $m \times n$ $\{0, 1\}$-matrix, and consider its rows as indicating the incidence of elements of $E = \{e_1, \ldots, e_n\}$ with the sets $S_1, S_2, \ldots, S_m$, i.e., $a_{ij} = 1$ if and only if $e_j \in S_i$. Let $L(\mathbf{A})$ be the collection of subsets B of E of the form $B = \bigcup_{i \in I} S_i$ for $I \subseteq \{1, \ldots, m\}$, partially ordered by set inclusion. Then $L(\mathbf{A})$ is a lattice, with $B \vee C = B \cup C$. The unique minimal element of $L(\mathbf{A})$ is $\varnothing$, since $\varnothing = \bigcup_{i \in I} S_i$ for $I = \varnothing$. Similarly, the transpose $\mathbf{A}^{\mathrm{T}}$ of $\mathbf{A}$ generates a lattice which we will denote by $L(\mathbf{A}^{\mathrm{T}})$.

LEMMA 3.1. *$L(\mathbf{A})$ is anti-isomorphic to $L(\mathbf{A}^{\mathrm{T}})$.*

PROOF. Let $E' = \{e'_1, \ldots, e'_m\}$ be such that e'_i corresponds to the ith row of $\mathbf{A}$, and define a binary relation $\sim$ between E' and E by $e'_i \sim e_j$ if and only if $a_{ij} = 0$. By [2; p. 123, Theorem 19] there is an anti-isomorphism between the lattice of all sets of the form $\{e_j | e'_i \sim e_j \forall i \in I\}$ and that of all sets of the form $\{e'_i | e'_i \sim e_j \forall j \in J\}$. Now sets of the former type are precisely the complements of elements of $L(\mathbf{A})$; those of the latter type are the complements of elements of $L(\mathbf{A}^{\mathrm{T}})$. Thus by a composition of three anti-isomorphisms, we get one between $L(\mathbf{A})$ and $L(\mathbf{A}^{\mathrm{T}})$.

Note that if the ith column of $\mathbf{A}$ is the zero vector and $\mathbf{A}'$ is the matrix $\mathbf{A}$ with column j removed, $\phi: L(\mathbf{A}) \to L(\mathbf{A}')$ defined by $\phi(S) = S$ is an isomorphism. Similarly, if the ith and jth columns of $\mathbf{A}$ are the same and $\mathbf{A}'$ is as above, $\psi: L(\mathbf{A}) \to L(\mathbf{A}')$ defined by $\psi(S) = S \backslash \{e_j\}$ is an isomorphism.

Suppose $M = (E, \mathscr{C})$ is an acyclic oriented matroid and the rows of the matrix $\mathbf{A}$ are the incidence vectors of the positive cocircuits of M. Then, by definition, $L(\mathbf{A})$ is $L^*(M)$, and hence $L(\mathbf{A}^{\mathrm{T}}) \cong L(M)$. Therefore, if M and M' are acyclic oriented matroids such that $L(M) \cong L(M')$ [and hence $L^*(M) \cong L^*(M')$], and $\mathbf{A}$ and $\mathbf{A}'$ are the matrices whose rows are the incidence vectors of the positive cocircuits of M and M', respectively, $L(\mathbf{A}) \cong L(\mathbf{A}')$ and $L(\mathbf{A}^{\mathrm{T}}) \cong L((\mathbf{A}')^{\mathrm{T}})$. Furthermore, if M is simple and $e \in E$ is not a point of $L(M)$, the column $\mathbf{A}^e$ of $\mathbf{A}$ corresponding to e is not minimal, but is the union of minimal columns (i.e. those corresponding to the points in the smallest face of $L(M)$ containing e). Thus if $\mathbf{A}'$ is the matrix $\mathbf{A}$ with column $\mathbf{A}^e$ removed, $L((\mathbf{A}')^{\mathrm{T}}) \cong L(\mathbf{A}^{\mathrm{T}})$.

LEMMA 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be 0–1 matrices, each with the property that both its rows and its columns are pairwise incomparable. Then if $L(\mathbf{A})$ is isomorphic to $L(\mathbf{B})$, $\mathbf{B}$ can be*

PROOF. Since the rows of **A** (respectively, **B**) are incomparable, it follows that the points of $L(\mathbf{A})$ [respectively, $L(\mathbf{B})$] correspond to the rows. Further, the incomparability of the columns of **A** implies that the copoints of $L(\mathbf{A})$ correspond to sets of the form $\{1, 2, \ldots, m\}\setminus\{j\}$, where **A** has m columns. Similarly for **B**. The isomorphism between $L(\mathbf{A})$ and $L(\mathbf{B})$ gives a bijection between the points of $L(\mathbf{A})$ and those of $L(\mathbf{B})$, and so between the rows of **A** and of **B**. Similarly, we get a bijection on copoints, and so between the columns of **A** and of **B**. Assuming that the rows and columns of **B** have been reordered according to these bijections, note that $a_{ij} = 0$ if and only if point i of $L(\mathbf{A})$ is on copoint $\{1, \ldots, m\}\setminus\{j\}$ of $L(\mathbf{A})$. By the isomorphism, this is equivalent to the same statement for $L(\mathbf{B})$ and so is equivalent to $b_{ij} = 0$.

LEMMA 3.3. *Let $M = (E, \mathcal{O})$ be an acyclic oriented matroid, and let* **A** *be the* $\{0, 1\}$*-matrix whose rows are the incidence vectors of the positive cocircuits of M with the elements of E. If M has a polar oriented matroid, then it has a polar $\tilde{M}$ such that the minimal rows of* $\mathbf{A}^{\mathrm{T}}$ *are the incidence vectors of the positive cocircuits of $\tilde{M}$.*

PROOF. Note first that we may assume that M is simple and every $e \in E$ is a point of $L(M)$. Otherwise, we may delete all but one of each set of parallel elements, yielding a matroid M_1, and then delete any elements which are not points of $L(M_1)$, resulting in a matroid $\mathbf{M}'$ such that $L(M') \cong L(M)$. Noting that any polar of M is also a polar of M', and conversely, we may then prove the lemma for M' and $\mathbf{A}'$, the $\{0, 1\}$-incidence matrix of the positive cocircuits of M'. Since the columns of $\mathbf{A}'$ account for all the minimal columns of **A**, the lemma must hold for M and **A**.

If M has any polar, it has a polar $\tilde{M}$ which is simple and such that every element of $\tilde{M}$ is a point of $L(\tilde{M})$. Let **B** be the 0–1 matrix whose rows are the incidence vectors of the positive cocircuits of $\tilde{M}$. By minimality of circuits, the rows of **B** are pairwise incomparable, and by the properties of $\tilde{M}$ above, the columns of **B** are incomparable as well.

By Lemma 3.1, $L(\mathbf{A}^{\mathrm{T}})$ is anti-isomorphic to $L(\mathbf{A}) = L^*(M)$. Further $L^*(\tilde{M}) = L(\mathbf{B})$ is anti-isomorphic to $L^*(M)$ and so $L(\mathbf{A}^{\mathrm{T}})$ and $L(\mathbf{B})$ are isomorphic. By our assumption above on M, the rows as well as the columns of $\mathbf{A}^{\mathrm{T}}$ are pairwise incomparable, and so by Lemma 3.2, after reordering the elements of $\tilde{M}$ if necessary, we obtain the desired conclusion.

We can now prove the main result of this paper. We first note that if $\hat{M} = (\hat{E}, \hat{\mathcal{O}})$ is any oriented matroid and $M = (E, \mathcal{O})$ is the matroid obtained from $\hat{M}$ by deleting all loops and all but one element of any set of parallel elements, the lattice of flats of $\hat{M}$ is isomorphic to the lattice of flats of M.

THEOREM 3.4. *Let $\hat{M}$ be any oriented matroid, and let $M = (E, \mathcal{O})$ be obtained from $\hat{M}$ as above. Let $\Lambda(M) = (E \cup E^*, \mathcal{O}^*)$ be the acyclic oriented matroid resulting from applying Lawrence's construction to M. If $\Lambda(M)$ has a polar, then it has a simple polar $\tilde{\Lambda}(M) = (E', \mathcal{O}')$ such that there exists $D \subseteq E'$ for which $\tilde{\Lambda}(M)/D$, the contraction of D in $\tilde{\Lambda}(M)$, is an adjoint of $\hat{M}$.*

PROOF. Clearly a matroid is an adjoint of $\hat{M}$ if and only if it is an adjoint of M. Therefore we will prove the theorem by proving that there exists $D \subseteq E'$ such that $\tilde{\Lambda}(M)/D$ is an adjoint of M.

(A somewhat [illegible]

The notation Y and Y^* will be used to denote the pair of positive cocircuits of $\Lambda(M)$ corresponding to the same hyperplane of M. If $v \in E$ is not a coloop of M, let Y_v be the positive cocircuit of $\Lambda(M)$ having underlying set $\{v, v^*\}$; otherwise let Y_v correspond to the coloop v^* of $\Lambda(M)$ (see Lemma 2.1). Assume $E = \{v_1, \ldots, v_m\}$ with $\{v_{k+1}, \ldots, v_m\}$ being the set of coloops of M. Let $\mathbf{A}$ be the matrix whose rows are the $\{0, 1\}$-incidence vectors of the positive cocircuits of $\Lambda(M)$, ordered so that the first $2s$ rows correspond to the s pairs of positive cocircuits corresponding to the s hyperplanes of $\mathbf{M}$ which are not the complements of coloops of M, the next $m-k$ correspond to the coloops $v_{k+1}, \ldots, v_m$, the next k correspond to the cocircuits Y_{v_i} for $i = 1, \ldots, k$, and the last $m-k$ correspond to the coloops $v^*_{k+1}, \ldots, v^*_m$ (see Figure 1).

		e_H: $e_Y e_{Y^*}$: $Y_1 Y_1^* \cdots Y_s Y_s^*$	e_H: e_Z: $Z_{k+1} \cdots Z_m$	e_{v_i}: $Y_{v_1} \cdots Y_{v_k}$	e_{v_i}: $Y_{v_{k+1}} \cdots Y_{v_m}$
X_i	$v_1, v_2, \ldots, v_k$		$\mathbf{0}$	$\mathbf{I}$	$\mathbf{0}$
	$v_{k+1}, \ldots, v_m$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{0}$	$\mathbf{0}$
X_i^*	$v_1^*, \ldots, v_k^*$		$\mathbf{0}$	$\mathbf{I}$	$\mathbf{0}$
	$v_{k+1}^*, \ldots, v_m^*$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{I}$

$\mathbf{A}^T$

FIGURE 1

Let $\tilde{\Lambda}(M) = (E', \mathcal{O}')$ be a polar of $\Lambda(M)$. Since M has no loops, each element of $\Lambda(M)$ is a point of $L(\Lambda(M))$, so each row of $\mathbf{A}^{\mathrm{T}}$ is minimal. By Lemma 3.2, we may assume that the incidence vectors of the positive cocircuits of $\tilde{\Lambda}(M)$ are the rows of $\mathbf{A}^{\mathrm{T}}$ (see Figure 1).

Denote by X_i and X_i^* the positive cocircuits of $\tilde{\Lambda}(\mathrm{M})$ which correspond to $v_i \in E$ and $v_i^* \in E^*$, respectively. Note that if $i \in \{k+1, \ldots, m\}$, so that v_i is a coloop of M, $|X_i| = |X_i^*| = 1$, so X_i and X_i^* correspond to coloops of $\tilde{\Lambda}(M)$. The positive cocircuits Y and Y^* of $\Lambda(M)$ correspond, respectively, to elements e_Y and e_{Y^*} of $\tilde{\Lambda}(M)$. We will denote by e_{v_i} the element of $\tilde{\Lambda}(M)$ corresponding to the positive cocircuit $Y_{v_i} = \{v_i, v_i^*\}$ of $\Lambda(M)$ if $i \in \{1, \ldots, k\}$ and to the coloop v_i^* if $i = k+1, \ldots, m$.

For $i = \{1, \ldots, k\}$, since $Y_{v_i} = \{v_i, v_i^*\}$ is the only cocircuit of $\Lambda(M)$ containing both v_i and v_i^*, $X_i \cap X_i^* = \{e_{v_i}\}$. Thus by applying the strong elimination axiom for the signed cocircuit span to X_i and $-X_i^*$, eliminating e_{v_i}, we know there exists $Y^i \in \mathcal{K}(\mathcal{O}^{\perp}(\tilde{\Lambda}(M)))$ such that $Y^{i+} = X_i \backslash \{e_{v_i}\}$ and $Y^{i-} = X_i^* \backslash \{e_{v_i}\}$. If $i \in \{k+1, \ldots, m\}$, let Y^i denote the coloop $\{e_{Z_i}\}$ of $\tilde{\Lambda}(M)$, where $Z_i = \{v_i\} \in \mathcal{O}^{\perp}(\Lambda(M))$.

Let $\tilde{M} = \tilde{\Lambda}(M)/\{e_{v_i} | i \in \{1, \ldots, m\}\}$. For $i = 1, \ldots, m$, $Y^i \subseteq \tilde{E} \equiv E' \backslash \{e_{v_i} | i \in \{1, \ldots, m\}\}$, so $Y^i \in \mathcal{K}(\mathcal{O}^{\perp}(\tilde{M}))$. Let F be any flat of $\tilde{\Lambda}(M)$ generated by a subset of the set $\{Y^i | i \in \{1, \ldots, m\}\}$, i.e., let $F = \bigcap_{i \in I}(E' \backslash Y^i)$ for some $I \subseteq \{1, \ldots, m\}$. Then $F \supseteq \{e_{v_i} | i \in \{1, \ldots, m\}\}$, so $G = F \backslash \{e_{v_i} | i \in \{1, \ldots, m\}\}$ is a flat of M. From here on we will think of the Y^is as elements of the cocircuit span of $\tilde{M}$, and of the flats generated by the Y^is as flats of $\tilde{M}$.

Let H be any hyperplane of M, and let Y and Y^* be the two positive cocircuits of $\Lambda(M)$ corresponding to the cocircuits of M with underlying set $E\backslash H$. Then $(H\cup H^*)\cap(Y\cup Y^*)=\varnothing$. If $|Y|\neq 1$, let $e_H=\{e_Y, e_{Y^*}\}$ be the pair of elements of $\tilde{\Lambda}(M)$ (and of $\tilde{M}$) which correspond to the cocircuits Y and Y^* of $\Lambda(M)$; otherwise, let $e_H=e_Y$, where e_Y is the element of $\tilde{\Lambda}(M)$ and of $\tilde{M}$ corresponding to the cocircuit Y of M with $|Y|=1$. Note that for each $i\in\{1,\dots,m\}$, either $e_H\subseteq \underline{Y}^i$ or $e_H\cap\underline{Y}^i=\varnothing$.

If $v_i\in E$, H is a hyperplane of M, and Y and Y^* are the cocircuits of $\Lambda(M)$ corresponding to H, then $e_H\subseteq \underline{Y}^i$ if and only if $\underline{Y}\cup\underline{Y}^*\supseteq\{v_i, v_i^*\}$, which holds if and only if $v_i\notin H$. So then $v_i\in H$ if and only if $e_H\cap\underline{Y}^i=\varnothing$.

We show now that there exists an anti-isomorphism ϕ between the flats of M and the flats of $\tilde{M}$ generated by subsets of $\{Y^1, Y^2,\dots, Y^m\}$. To see this, let $\mathscr{H}(M)$ denote the set of hyperplanes of M and define a binary relation $\sim$ between $\mathscr{H}(M)$ and $\{Y^1,\dots, Y^m\}$ by $H\sim Y^i$ if and only if $v_i\in H$. By the discussion above, $H\sim Y^i$ if and only if $e_H\cap\underline{Y}^i=\varnothing$. As in the proof of Lemma 3.1, we get, by [2; p. 123], an anti-isomorphism between sets of the form

$$\{H|H\sim Y^i\forall i\in I\}=\{H|v_i\in H\forall i\in I\}$$

and those of the form

$$\{Y^i|H\sim Y^i\forall H\in J\}=\{Y^i|e_H\subseteq\tilde{E}\backslash \underline{Y}^i\forall H\in J\}.$$

The anti-isomorphism ϕ is now defined for a flat F of M as follows. Write $F=\bigcap\{H|H\in J\}$, where $J=\{H|H\supseteq F\}=\{H|v_i\in H\forall v_i\in F\}$. Then

$$\begin{aligned}\phi(F)&=\bigcap\{\tilde{E}\backslash \underline{Y}^i|e_H\subseteq\tilde{E}\backslash \underline{Y}^i\forall H\in J\}\\&=\bigcap\{\tilde{E}\backslash \underline{Y}^i|v_i\in H,\forall H\in J\}\\&=\bigcap\{\tilde{E}\backslash \underline{Y}^i|v_i\in F\}.\end{aligned}$$

We show that for $H\in\mathscr{H}(M)$,

$$\phi(H)=\bigcap\{\tilde{E}\backslash \underline{Y}^i|e_H\subseteq\tilde{E}\backslash \underline{Y}^i\}=e_H.$$

To see this, suppose that for some H', $e_{H'}\cap\phi(H)\neq\varnothing$. Then $e_{H'}\subseteq\phi(H)$, which implies $e_{H'}\cap\underline{Y}^i=\varnothing$ for all i such that $v_i\in H$. But $e_{H'}\cap\underline{Y}^i=\varnothing$ if and only if $v_i\in H'$, so we get $H\subseteq H'$. Thus $H=H'$. A similar argument shows that ϕ is onto the desired set of flats of $\tilde{M}$.

Since for every $i\in\{1,\dots,m\}$, e_{v_i} is not a loop of $\tilde{\Lambda}(M)/\{e_{v_1},\dots,e_{v_{i-1}}\}$, $\rho(\tilde{M})=\rho(\tilde{\Lambda}(M)/\{e_{v_i}|i\in\{1,\dots,m\}\})=\rho(\tilde{\Lambda}(M))-m=\rho(\Lambda(M))-m=\rho(M)+m-m=\rho(M)$. Thus if $H\in\mathscr{H}(M)$, $\phi(H)=e_H$ is a flat of rank 1 in $\tilde{M}$. If there exists any point P of the lattice of flats of $\tilde{M}$ which is not of the form e_H for some $H\in\mathscr{H}(M)$, P must contain some element, e_Y, say, but not e_{Y^*}, where Y is a cocircuit of $\Lambda(M)$ with $|\underline{Y}|>1$. Then $P\cap\{e_Y, e_{Y^*}\}$ is a flat of $\tilde{M}$ which is properly contained in $\{e_Y, e_{Y^*}\}$, contradicting the fact that the rank of $\{e_Y, e_{Y^*}\}$ is one. Thus ϕ maps the hyperplanes of M onto the flats of $\tilde{M}$ or rank 1, and hence $\tilde{M}$ is an adjoint of M.

COROLLARY 3.5. *There exists an acyclic oriented matroid of rank* 12 *which has no polar.*

PROOF. Let V be the Vámos matroid [5] and apply Lawrence's construction to V. Suppose $\Lambda(V)$ has a polar. Then by the theorem, there exists an oriented matroid $\tilde{M}$ which is an adjoint of V. But by a theorem of Cheung [4], the Vámos matroid has no

We remark here that Bachem and Kern [1] have announced a converse to Theorem 3.4 of the form: $\Lambda(M)$ has a polar whenever M has an 'oriented adjoint'.

4. Inner Products: Definition and Properties

In this section we introduce the notion of an inner product on an oriented matroid as a generalization of the notion of inner product on a real vector space. One way of viewing this additional structure is as follows. Given a collection of points in a real vector space, the linear matroid they generate merely summarizes their interdependencies. Adding the structure of the natural orientation to this matroid captures the convexity properties of the set of points, e.g. which of them give extreme rays of the convex cone that they generate. Giving the structure of a sign-valued inner product for this oriented matroid will further specify the spatial arrangement of the set of points, e.g. which pairs of points are orthogonal, or make an acute or obtuse angle. In this sense, the notion of inner product we define here can be considered as a combinatorial abstraction of the geometric notion of angle.

If X is a signed subset of E and $x \in E$ we define

$$\mathrm{sg}_X\, x \equiv \begin{cases} +, & \text{if } x \in X^+, \\ -, & \text{if } x \in X^-, \\ 0, & \text{if } x \notin \underline{X}. \end{cases}$$

If $E = \{e_1, \ldots, e_n\}$, we will sometimes think of X as the $\{+, -, 0\}$-vector with ith entry $\mathrm{sg}_X\, e_i$. Multiplication of signs is defined as if we were working with +1s, −1s and 0s, i.e., $+\cdot+=-\cdot-=+$, $+\cdot-=-$, $+\cdot 0=-\cdot 0=0\cdot 0=0$. If $\lambda_i \in \{0, +, -\}$ for every $i = 1, \ldots, k$, we define $\sum_{i=1}^k \lambda_i \in \{0, +\}$ if $\lambda_i \in \{0, +\}$ for every i, $\sum_{i=1}^k \lambda_i \in \{0, -\}$ if $\lambda_i \in \{0, -\}$ for every i, and, if $\lambda_i \in \{0, +\}$ for every i (or $\lambda_i \in \{0, -\}$ for every i), $\sum_{i=1}^k \lambda_i = 0$ if and only if $\lambda_i = 0$ for every i. If there exist j and j' in $\{1, \ldots, k\}$ such that $\lambda_j = +$ and $\lambda_{j'} = -$, $\sum_{i=1}^k \lambda_i$ is undefined. We define the sum over the empty set, $\sum_\varnothing \lambda_i$, to be zero. We will denote $\lambda_i \in \{0, +\}$ and $\lambda_j \in \{0, -\}$ by $\lambda_i = \oplus$ and $\lambda_j = \ominus$, respectively.

Definition 4.1. Let $M = (E, \mathcal{O})$ be an oriented matroid. A function $\langle\ ,\ \rangle$, where $E \times E \to \{0, +, -\}$ is an *inner product on M* if it satisfies

(I1) $\langle x, x\rangle = \oplus$, and $\langle x, x\rangle = 0$ if and only if x is a loop of M;

(I2) $\langle x, y\rangle = \langle y, x\rangle$ for every $x, y \in E$; and

(I3) if $X \in \mathcal{O}$, $y \in \underline{X}$, and $z \in E$ are such that $\mathrm{sg}_X\, x\langle x, z\rangle = \oplus$ for every $x \in \underline{X}\backslash y$, then $\mathrm{sg}_X\, y\langle y, z\rangle = -\sum_{x\in \underline{X}\backslash y} \mathrm{sg}_X\, x\langle x, z\rangle$.

The first two requirements for the inner product are obvious analogs of the definiteness and symmetry of inner products in real vector spaces. The third property parallels the linearity property; i.e., if $a = \sum_{i=1}^m \lambda_i a_i$, then $\langle a, b\rangle = \sum_{i=1}^m \lambda_i \langle a_i, b\rangle$.

An immediate consequence of the definition is that $\langle x, z\rangle = 0$ for every $z \in E$ if and only if x is a loop of M.

As an example, consider the all-positive inner product defined for acyclic matroids. If $M = (E, \mathcal{O})$ is any acyclic oriented matroid this inner product is given by $\langle x, y\rangle = +$ for every pair x, y of elements of E. Clearly $\langle\ ,\ \rangle$ satisfies (I1) and (I2). Suppose $X \in \mathcal{O}$, $y \in \underline{X}$ and $z \in E$ are such that $\mathrm{sg}_X\, x\langle x, z\rangle = \oplus$ for every $x \in \underline{X}\backslash y$. Since M is acyclic, $|\underline{X}| > 1$ so $\underline{X}\backslash y \neq \varnothing$. Then $\mathrm{sg}_X\, x = +$ for every $x \in \underline{X}\backslash y$, so M acyclic implies $\mathrm{sg}_X\, y = -$. Thus $\mathrm{sg}_X\, y\langle y, z\rangle = - = -\sum_{x\in \underline{X}\backslash y} \mathrm{sg}_X\, x\langle x, z\rangle$, and (I3) is satisfied.

Another example was suggested to us by Bland. Let B be any base for $M = (E, \mathcal{O})$, say $B = \{e_1, e_2, \ldots, e_\rho\}$ It is well known [4] that for each $i = 1, \ldots, \rho$, there exists a unique cocircuit Y_i of M such that $e_i \in Y_i^+$ and $\underline{Y}_i \cap B = \{e_i\}$. Let $\mathbf{A}$ be the $\rho \times |E|$ matrix formed by taking Y_i as the ith row of $\mathbf{A}$, where we consider Y_i as the $|E|$-vector with entries in $\{+, -, 0\}$. $\mathbf{A} = (a_{ij})$ is called the *standard representative matrix* of M for the base B. Note that if $e_j \in E \setminus B$, the signed set X defined by $X^+ = \{e_i \in B \mid a_{ij} = +\}$ and $X^- = \{e_j\} \cup \{e_i \in B \mid a_{ij} = -\}$ is the unique circuit of M such that $e_j \in X^-$ and $\underline{X} \subseteq B \cup \{e_j\}$. (See [4].)

Now for every $e_i, e_j \in E$, define $\langle e_i, e_j \rangle$ by

$$\langle e_i, e_j \rangle = \begin{cases} a_{ki}a_{kj} & \text{where } k \text{ is the index of the first row} \\ & \text{of } \mathbf{A} \text{ such that } a_{ki}a_{kj} \neq 0, \\ 0, & \text{if no such } k \text{ exists.} \end{cases}$$

Note that for different choices of the base B and different orderings of the elements of B we get different functions $\langle \ , \ \rangle$ on M. That $\langle \ , \ \rangle$ is an inner product will follow from more general considerations later.

It is easy to see that axiom (I3) for matroid inner products is equivalent to

(I3′) For any $X \in \mathcal{O}$ and $z \in E$, if $\mathrm{sg}_X\, x \langle x, z \rangle = \oplus$ for $x \in E$, $\mathrm{sg}_X\, x\langle x, z \rangle = 0$ for every $x \in E$.

This restatement of the axiom leads to

LEMMA 4.2. *Let $\langle \ , \ \rangle$ be an inner product on $M = (E, \mathcal{O})$. For $z \in E$, define the signed subset Y_z of E by $\mathrm{sg}_{Y_z}\, x = \langle x, z \rangle$. Then $Y_z \in \mathcal{K}(\mathcal{O}^\perp)$.*

PROOF. If z is a loop of M, then $\langle x, z \rangle = 0$ for every $x \in E$. Hence $\underline{Y}_z = \emptyset$, and $Y_z \in \mathcal{K}(\mathcal{O}^\perp)$. So assume z is not a loop. We need to show Y_z is orthogonal to every circuit of M.

Let $X \in \mathcal{O}$, and suppose Y_z is not orthogonal to X. Then without loss of generality we may assume $(X^+ \cap Y_z^+) \cup (X^- \cap Y_z^-) \neq \emptyset$, but $(X^+ \cap Y_z^-) \cup (X^- \cap Y_z^+) = \emptyset$. Then for every $x \in E$, we have $\mathrm{sg}_X\, x\, \mathrm{sg}_{Y_z}\, x = \oplus$, so for every x, $\mathrm{sg}_X\, x\langle x, z \rangle = \oplus$. Thus by axiom (I3′), $\mathrm{sg}_X\, x\langle x, z \rangle = \mathrm{sg}_X\, x\, \mathrm{sg}_{Y_z}\, x = 0$ for each x, contradicting $(X^+ \cap Y_z^+) \cup (X^- \cap Y_z^-) \neq \emptyset$. Thus X and Y_z are orthogonal.

THEOREM 4.3. *Let $M = (E, \mathcal{O})$ be an oriented matroid with no loops, $E = \{e_1, e_2, \ldots, e_n\}$. Let $\langle \ , \ \rangle$ be a function from $E \times E$ into $\{0, +, -\}$, and let $\{Y_e \mid e \in E\}$ be the set of signed subsets of E given by $\mathrm{sg}_{Y_e} f = \langle e, f \rangle$. Then $\langle \ , \ \rangle$ is an inner product on M if and only if $\mathrm{sg}_{Y_e}\, e = +$ and $Y_e \in \mathcal{K}(\mathcal{O}^\perp)$, for every $e \in E$, and the $n \times n$ matrix $\mathbf{A}$ whose ith row is the signed vector associated with Y_{e_i} is symmetric.*

PROOF. Suppose $\langle \ , \ \rangle$ is an inner product on M. By axiom (I1), $\langle e, e \rangle = +$ for every $e \in E$, so $\mathrm{sg}_{Y_e}\, e = +$. By (I2), $\langle e, f \rangle = \langle f, e \rangle$ so $\mathrm{sg}_{Y_e} f = \mathrm{sg}_{Y_f}\, e$, and hence $\mathbf{A}$ is symmetric. By Lemma 4.2, $Y_e \in \mathcal{K}(\mathcal{O}^\perp)$ for every $e \in E$.

Now suppose $\mathrm{sg}_{Y_e}\, e = +$ and $Y_e \in \mathcal{K}(\mathcal{O}^\perp)$ for every $e \in E$ and the matrix $\mathbf{A}$ is symmetric. Then $\mathrm{sg}_{Y_e}\, e = +$ implies $\langle e, e \rangle = +$, so axiom (I1) is satisfied. The symmetry of $\mathbf{A}$ implies $\langle e, f \rangle = \langle f, e \rangle$ for every $e, f \in E$, so (I2) holds. To verify axiom (I3′), suppose $\mathrm{sg}_X\, x\langle x, z \rangle = \oplus$ for every $x \in E$ and some $X \in \mathcal{O}$, $z \in E$. Then $\mathrm{sg}_X\, x\, \mathrm{sg}_{Y_z}\, x = \oplus$ for every $x \in E$, so $(X^+ \cap Y_z^-) \cup (X^- \cap Y_z^+) = \emptyset$. Since $Y_z \in \mathcal{K}(\mathcal{O}^\perp)$, $(X^+ \cap Y_z^+) \cup (X^- \cap Y_z^-) = \emptyset$, and hence $\mathrm{sg}_X\, x\, \mathrm{sg}_{Y_z}\, x = \mathrm{sg}_X x\langle x, z \rangle = 0$ for every $x \in E$.

This characterization of inner products enables us to make the following observation. Let $M = (E, \mathcal{O})$ be an oriented matroid and define $E' \equiv E \setminus \{e \in E \mid e \text{ is a loop of } M\}$. Let $Y_1, \ldots, Y_k$ be any list of elements of $\mathcal{K}(\mathcal{O}^\perp)$ such that $\bigcup_{i=1}^k \underline{Y}_i = E'$. Define $\langle \ , \ \rangle$ by $\langle x, y \rangle = \mathrm{sg}_{Y_i}\, x\, \mathrm{sg}_{Y_i}\, y$ where i is the least index such that $\{x, y\} \subseteq \underline{Y}_i$ and $\langle x, y \rangle = 0$ if no such i exists. It is clear from the definition of $\langle \ , \ \rangle$ that axioms (I1) and (I2) are satisfied.

For each $x \in E$ define $I_x \equiv \{j | x \in \underline{Y}_j\} = \{i_1, \ldots, i_k\}$ and order the elements of I_x so that $i_1 < i_2 < \cdots < i_k$. For each $i \in I_x$, let Y_i^x be $\pm Y_i$ such that $x \in (Y_i^x)^+$. Let $Y^x = Y_{i_1}^x \circ Y_{i_2}^x \circ \cdots \circ Y_{i_k}^x$. Then $Y^x \in \mathcal{K}(\mathcal{O}^\perp)$, and $\mathrm{sg}_{Y^x}\, y = \langle x, y \rangle$ for every $y \in E$. Furthermore, $\mathrm{sg}_{Y^x}\, x = +$ and $\mathrm{sg}_{Y^x}\, y = \mathrm{sg}_{Y^y}\, x$ by (I2) for every $x, y \in E$. Thus the conditions of Theorem 4.3 are satisfied, so $\langle\ ,\ \rangle$ is an inner product on M. The different possible lists of elements of $\mathcal{K}(\mathcal{O}^\perp)$ then define a whole class of inner products on M.

Both the all-positive inner product and Bland's inner product are members of this class. The all-positive inner product for an acyclic oriented matroid is defined by any list of elements $\mathcal{K}(\mathcal{O}^\perp)$ for which Y_1 is such that $Y_1^+ = E$. Bland's inner product results when Y_i is the unique cocircuit of M such that $e_i \in Y_i^+$ and $\underline{Y}_i \cap B = \{e_i\}$. It is easy to construct inner products which cannot be obtained in this way, for example, using the affine oriented matroid on the vertices of a hexagon in the plane.

As an easy corollary of Theorem 4.3, we have the following.

Corollary 4.4. *Let H be a hyperplane of $M = (E, \mathcal{O})$, and let $\langle\ ,\ \rangle$ be an inner product on M. Suppose there exists $z \in E$ such that z is not a loop of M and $\langle z, h \rangle = 0$ for every $h \in H$. Then for all $x, y \in E$, $\langle x, z \rangle \langle y, z \rangle = \mathrm{sg}_Y\, x\, \mathrm{sg}_Y\, y$, where Y is one of the signed cocircuits of M such that $\underline{Y} = E \backslash H$.*

Proof. By 4.3, $Y_z \in \mathcal{K}(\mathcal{O}^\perp)$ where $\mathrm{sg}_{Y_z}\, y = \langle z, y \rangle$ for every $y \in E$. Since z is not a loop, $\underline{Y}_z \neq \varnothing$, so $\langle z, h \rangle = 0$ for every $h \in H$ implies $Y_z \in \mathcal{O}^\perp$ with $\underline{Y}_z = E \backslash H$. Then $\langle x, z \rangle \langle y, z \rangle = \mathrm{sg}_{Y_z}\, x\, \mathrm{sg}_{Y_z}\, y = \mathrm{sg}_{-Y_z}\, x\, \mathrm{sg}_{-Y_z}\, y$ for every $x, y \in E$.

Corollary 4.4 is a matroid abstraction of the result that if $z \in \mathbb{R}^n$ is normal to a hyperplane H of $\mathbb{R}^n$ which passes through the origin, the inner product of z with any $x \in \mathbb{R}^n$ indicates the side of H on which the vector x lies. Another combinatorial property of inner products in $\mathbb{R}^n$ which can be seen to hold for matroid inner products is the analog of the fact that any non-zero element of $\mathbb{R}^n$ can be orthogonal to at most a hyperplane of $\mathbb{R}^n$.

Corollary 4.5. *Let $M = (E, \mathcal{O})$ be an oriented matroid with inner product $\langle\ ,\ \rangle$. Suppose $x \in E$ is not a loop, and $\langle x, s \rangle = 0$ for every $s \in S \subseteq E$. Then $\rho(S) \leq \rho(M) - 1$.*

Proof. By 4.3 there exists $Y_x \in \mathcal{K}(\mathcal{O}^\perp)$ such that $\mathrm{sg}_{Y_x}\, z = \langle x, z \rangle$ for every $z \in E$. Since x is not a loop, $\underline{Y}_x \neq \varnothing$. Since $\{z \in E | \langle x, z \rangle = 0\} = E \backslash \underline{Y}_x$, the corollary follows from the facts that the minimal elements of $\mathcal{K}(\mathcal{O}^\perp)$ are cocircuits of M and $\rho(E \backslash \underline{Z}) = \rho(M) - 1$ for every $Z \in \mathcal{O}^\perp$.

Proposition 4.6. *Suppose $M = (E, \mathcal{O})$ is an oriented matroid, and let $S \subseteq E$ be independent in M. Suppose there exists x not a loop of M such that $\langle x, s \rangle = 0$ for every $s \in S$. Then $S \cup \{x\}$ is independent.*

Proof. If $x \in S$, there is nothing to prove since by hypothesis S is independent. So assume $x \in E \backslash S$. If $\{x\} \cup S$ is not independent, there exists $X \in \mathcal{O}$ such that $x \in \underline{X} \subseteq S \cup \{x\}$. For every $y \in \underline{X} \backslash x$, $\mathrm{sg}_X\, y \langle x, y \rangle = 0$, so by (I3), $\mathrm{sg}_X\, x \langle x, x \rangle = 0$. But $x \in \underline{X}$ implies $\mathrm{sg}_X\, x \neq 0$, and x not a loop implies $\langle x, x \rangle \neq 0$. Thus $S \cup \{x\}$ must be independent.

Corollary 4.7. *Let $M = (E, \mathcal{O})$ and suppose $S \subseteq E$ contains no loops and is such that $\langle s, t \rangle = 0$ for every $s, t \in S$, $s \neq t$. Then S is independent.*

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$. Let $S_1 = \{s_1\}$, and apply Proposition 4.6 with s_2 as x. Then $S_2 = S_1 \cup \{s_2\} = \{s_1, s_2\}$ is independent. Continuing, $S = S_k = S_{k-1} \cup \{s_k\}$ is independent.

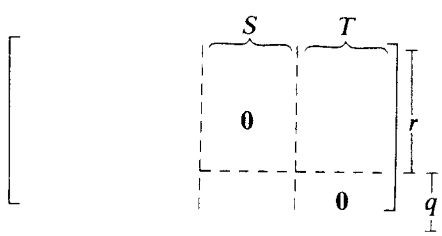
Some of the preceding results have been concerned with rank properties in relation to inner products on matroids. Unfortunately, not all of the connections found in the setting of real vector spaces hold in this matroid situation. For example, in general we do not have

(4.8) if $S \subseteq E$ and $T \subseteq E$ are such that $\langle s, t\rangle = 0$ for every $s \in S$, $t \in T$, then $\rho(S) + \rho(T) \leq \rho(M)$.

As an example of the failure of this property, consider the oriented matroid M defined by the affine dependencies of the four vertices of a convex quadrilateral. Then the only circuit in M is $(e_1, \bar{e}_2, e_3, \bar{e}_4)$. Suppose we define an inner product on M by $\langle e, e\rangle = +$ for every $e \in E$, $\langle e_i, e_j\rangle = \langle e_j, e_i\rangle$ for every $e_i, e_j \in E$, and $\langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_4\rangle = \langle e_3, e_4\rangle = 0$, $\langle e_1, e_4\rangle = \langle e_2, e_3\rangle = +$. Then if $S = \{e_2, e_3\}$ and $T = \{e_1, e_4\}$, $\langle s, t\rangle = 0$ for every $s \in S$, $t \in T$, but $\rho(S) + \rho(T) = 2 + 2 \not\leq 3 = \rho(M)$. Further, this example shows that an inner product on a representable matroid need not arise from a vector space inner product.

However, some inner products can be shown to satisfy 4.8, and it seems that this property may be necessary in order to prove other desired properties. We will call 4.8 the *rank property*. The all-positive inner product can trivially be seen to have the rank property since there do not exist s and t in E such that $\langle s, t\rangle = 0$.

Bland's inner product can also be seen to have the rank property. For suppose $M = (E, \mathcal{O})$ and $\langle\ ,\ \rangle$ is Bland's inner product for some ordered base $B = (e_1, \ldots, e_p)$. Suppose $\langle s, t\rangle = 0$ for every $s \in S$, $t \in T$. Then in any row of the standard representative matrix for **B** where there exists a non-zero entry in the column corresponding to some $s \in S$, every t must have a zero entry. Similarly, for any i such that $a_{it} \neq 0$ for some $t \in T$, $a_{is} = 0$ for every $s \in S$. Thus permuting the rows and columns of the matrix would give it the form



Since the last q rows of the columns corresponding to the elements of T have only zero entries, $t \in \mathrm{cl}\, B_r$ for every $t \in T$, where $B_r = \{b_{i_1}, \ldots, b_{i_r}\} \subseteq B$. Thus $T \subseteq \mathrm{cl}\, B_r$ and $\rho(T) \leq \rho(B_r) = r$. Similarly, $S \subseteq \mathrm{cl}(B \setminus B_r)$, so $\rho(S) \leq \rho(B \setminus B_r) = \rho(M) - r$. Thus $\rho(S) + \rho(T) \leq \rho(M)$.

In an inner product space there are also close connections between inner products and linear transformations. Not only can any linear functional on $\mathbb{R}^n$ be defined by the inner product of the elements of $\mathbb{R}^n$ with a fixed vector, but, conversely, positive linear functionals (or, equivalently, positive definite bilinear forms) can be used to define inner products, i.e., if $c_i > 0$ for every i and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are in $\mathbb{R}^n$, $\langle\ ,\ \rangle$ defined by $\langle x, y\rangle = \sum_{i=1}^{n} c_i x_i y_i$ is an inner product. In the following, we define positive linear functionals of signed vectors and use them to try to define inner products. As we will see, the approach fails to produce anything new.

DEFINITION 4.9. *A positive linear functional of order p is a function $T: \{0, +, -\}^p \to \{0, +, -\}$ satisfying*

(a) *if $S = (\oplus, \oplus, \ldots, \oplus)$, $T(S) = +$, and $T(S) = 0$ if and only if $S = 0$;*

(b) *$T(S) = -T(-S)$;*

(c) *if $\underline{R} \cap \underline{S} = \varnothing$, $T(R) = \oplus$ and $T(S) = \oplus$, then $T(S + R) = T(S) + T(R)$; similarly, if $T(S) = \ominus$ and $T(R) = \ominus$, $T(S + R) = T(S) + T(R)$.*

In (c) $\underset{\sim}{R}$ and $\underset{\sim}{S}$ denote the sets of non-zero components of R and S, respectively. As an example, define lex: $\{0, +, -\}^\rho \to \{0, +, -\}$ by

$$\operatorname{lex}(S) = \begin{cases} s_i, & \text{where } i \text{ is the least index such that } s_i \neq 0 \\ 0, & \text{if no such } i \text{ exists,} \end{cases}$$

where $S = (s_1, \ldots, s_\rho)$. It is trivial to see that lex is a positive linear functional.

Suppose $M = (E, \mathcal{O})$ is an oriented matroid, and let $\mathbf{A}$ be the standard representative matrix of M for some ordered base $\mathbf{B}$. For every $x \in E$, let $c(x)$ denote the column of $\mathbf{A}$ which is associated with x, and let $c_i(x)$ be the ith component of $c(x)$. Let T be a positive linear functional of order $\rho(\mathbf{M})$, and define $\langle , \rangle_T$ by $\langle x, y\rangle_T = T(S)$ where $S = (c_1(x)c_1(y), c_2(x)c_2(y), \ldots, c_\rho(x)c_\rho(y))$. We are interested in knowing when $\langle\ ,\ \rangle_T$ is an inner product on M. In the case where $T = \operatorname{lex}$, it is clear from the definition that $\langle\ ,\ \rangle_T$ is Bland's inner product. In the representable case, this inner product corresponds to choosing an orthogonal basis in which the first element is much longer than the second, which is much longer than the third, and so on.

Proposition 4.10. *Let T be a positive linear functional of order ρ. If $\langle\ ,\ \rangle_T$ is an inner product for every oriented matroid of rank ρ and every choice of ordered base, T must be a permutation of lex.*

Proof. Suppose T satisfies the hypotheses. For $i \neq j$, let $(i, -j)$ denote the signed ρ-vector which has $+$ as its ith entry, $-$ as its jth and 0 for all other entries. We first show that $T(i, -j) \neq 0$ for any pair i, j. Suppose otherwise, i.e., there exist i and j such that $T(i, -j) = 0$. Consider a rank ρ oriented matroid generated by the linear dependencies of a set of vectors in $\mathbb{R}^\rho$ which includes all the unit vectors e_k, as well as $a = e_i - e_j$, $b = e_j - 2e_i$, and $c = (1, 1, \ldots, 1)$. Consider the standard representative matrix for the ordered base $\{e_1, \ldots, e_\rho\}$. Clearly $X = (e_i, a, b)$ is a positive circuit. Since $T(-i, j) = -T(i, -j)$, $\langle a, c\rangle_T = \langle b, c\rangle_T = 0$. Then $\operatorname{sg}_X a\langle a, c\rangle_T = \operatorname{sg}_X b\langle b, c\rangle_T = 0$, so (13) implies $\operatorname{sg}_X e_i\langle e_i, c\rangle_T = 0$. But $\operatorname{sg}_X e_i = +$ and $\langle e_i, c\rangle_T = T(0, \ldots, 0, +, 0, \ldots, 0) = +$. Thus $T(i, -j) \neq 0$.

For every pair i, j of distinct elements of $\{1, \ldots, \rho\}$, we will say $i > j$ if $T(i, -j) = +$. Construct a directed graph G with vertices $1, 2, \ldots, \rho$ and with arc set $\{(i, j) | i > j\}$. Since $T(i, -j) \neq 0$ and $T(j, -i) = -T(i, -j)$ for every i and j, each pair of vertices is connected by precisely one arc.

Claim 1. *G has no directed cycles.*

Proof of Claim 1. Since each pair of vertices is connected by precisely one arc, there exist no directed two-cycles in G. An argument similar to that above shows there exist no directed 3-cycles.

Now assume there exists no directed k-cycle for $k < s$, $s > 3$, and suppose $i_1 > i_2 > i_3 > \cdots > i_s > i_1$ is a directed s-cycle. Since there are no directed 3-cycles and i_1 and i_3 are connected by an arc, $i_1 > i_3$. Then $i_1 > i_3 > \cdots > i_s > i_1$ is a directed $(s-1)$-cycle which is a contradiction. Thus Claim 1 is established.

G is then an acyclic orientation of the complete graph on $\{1, \ldots, \rho\}$, and so specifies a unique order $i_1 > i_2 > \cdots > i_\rho$.

Claim 2. *T is lex with the order $i_1, i_2, \ldots, i_\rho$ of the base B.*

Proof of Claim 2. Reorder the elements so that i_i is i. Suppose $S = (s_1, \ldots, s_\rho) \in$

$\{i\in\{1,\dots,\rho\}|s_i=+\}=\{i_1,\dots,i_{|I|}\}$, and let $J=\{i\in\{1,\dots,\rho\}|s_i=-$ and $i\neq i_*\}=\{j_1,\dots,j_{|J|}\}$. Let e_i, $i=1,\dots,\rho$, be the ith unit ρ-vector. For each $k=1,\dots,|I|$, let a_k be the ρ-vector $(i_*,-i_k)$. Define the ρ-vector b by $b_{i_*}=-|I|$, $b_k=1$ for $k\in I$, $b_k=-1$ for $k\in J$, and $b_k=0$ otherwise. Note that the signed vector corresponding to b is the vector S. Let c be the ρ-vector with every coordinate equal to 1. Now consider the oriented matroid defined by the real linear dependencies on the elements $\{e_1,e_2,\dots,e_\rho,a_1,\dots,a_{|I|},b,c\}$. Clearly $X=(e_{j_1},\dots,e_{j_{|J|}},a_1,\dots,a_{|I|},b)$ is a positive circuit in this matroid. Since $i_*>i_k$ for every $i_k\in I$, $\langle c,a_k\rangle=+$. We also have $\langle c,e_i\rangle=+$ for every $i=1,\dots,\rho$. Hence for every $x\in \underline{X}\backslash b$, $\mathrm{sg}_X\, x\langle c,x\rangle=+$. Thus since $\langle\ ,\ \rangle_T$ is an inner product, $\mathrm{sg}_X\, b\langle b,c\rangle=-$, and therefore $\langle b,c\rangle=T(S)=-$. Thus T is lex.

5. Inner Products and Polar Matroids

A potential application of the inner product on an oriented matroid M is to aid in the construction of the polar of M. Suppose C is an n-dimensional cone in $\mathbb{R}^n$. Then the polar cone, C^*, of C can be given by $C^*=\{x\in\mathbb{R}^n|\langle x,y\rangle\leq 0 \text{ for every } y\in C\}$. Furthermore, suppose $F_1,\dots,F_m$ are the facets of C, and for $i=1,\dots,m$ let $x_i\in\mathbb{R}^n$ be a non-zero vector such that $\langle x_i,y\rangle=0$ for every $y\in F_i$ and $\langle x_i,y\rangle\leq 0$ for every $y\in C$. Then $C^*=\mathrm{cone}\{x_i|i=1,\dots,m\}$. We wish to study the extent to which an oriented matroid polar, when it exists, can be constructed in an analogous fashion by means of a matroid inner product. Theorem 5.2 represents a first step in this direction.

Lemma 5.1. *Let L and L' be point lattices for which every element is the meet of copoints. Suppose there exists a bijective function from the set of points and copoints of L to the set of points and copoints of L' which maps the points of L onto the copoints of L' and the copoints of L onto the points of L' in such a way that if v is a point and F a copoint of L, $v\leq F$ if and only if $\phi(F)\leq\phi(v)$. Then ϕ can be extended to an anti-isomorphism between L and L'* (cf. [10; p. 41]).

Proof. Let $\{v_1,\dots,v_k\}$, $\{F_1,\dots,F_m\}$ be the sets of points and copoints of L, respectively, and $\{v'_1,\dots,v'_m\}$, $\{F'_1,\dots,F'_k\}$ the same for L', where $\phi(v_i)=F'_i$, $\phi(F_j)=v'_j$. Define a binary relation between the points of L and L' by $v_i\sim v'_j$ if and only if $v_i\leq F_j$. By the hypothesis on ϕ, $v_i\sim v'_j$ if and only if $v'_j\leq F'_i$, and so the definition of $\sim$ is symmetric in L and L'. Again, as in the proof of Lemma 3.1, we get, by [2; p. 123], an anti-isomorphism between sets of the form $\{v_i|v_i\sim v'_j\forall j\in J\}$ and those of the form $\{v'_j|v_i\sim v'_j\forall i\in I\}$. But

$$\{v_i|v_i\sim v'_j,\forall j\in J\}=\{v_i|v_i\leq F_j\forall j\in J\}$$

$$=\Big\{v_i\Big|v_i\leq\bigwedge_{j\in J}F_j\Big\}$$

and

$$\{v'_j|v_i\sim v'_j,\forall i\in I\}=\{v'_j|v'_j\leq F'_i,\forall i\in I\}$$

$$=\Big\{v'_j\Big|v'_j\leq\bigwedge_{i\in I}F'_i\Big\},$$

and so this defines an anti-isomorphism ϕ between L and L', since each element in L or L' is the meet of copoints and the join of its points.

To see that ϕ extends the original function, note that for $F\in L$,

$$\phi(F)=\bigvee\{v'_j|v'_j\sim v_i,\forall v_i\leq F\}$$

$$=\bigvee\{v'_j|v'_j\leq F'_i,\forall i \text{ s.t. } v_i\leq F\}$$

$$=\bigwedge_{i:v_i\leq F}F'_i.$$

Thus $\phi(v_i)=F'_i$ and, by the symmetry between L and L', $\phi^{-1}(v'_j)=F_j$.

If M is an oriented matroid on E with inner product $\langle\ ,\ \rangle_M$ and N is an extension of M to $E' \supset E$, then an inner product $\langle\ ,\ \rangle_N$ on N is said to be an *extension* of $\langle\ ,\ \rangle_M$ if $\langle\ ,\ \rangle_N = \langle\ ,\ \rangle_M$ on $E \times E$.

THEOREM 5.2. *Let $M = (E, \mathcal{O})$ be an acyclic oriented matroid with inner product $\langle\ ,\ \rangle_M$. Let $F^1, F^2, \ldots, F^n$ be the copoints of the lattice $L(M)$. Suppose there exists an extension N of M on $E \cup \{w_1, w_2, \ldots, w_n\}$ with $\rho(N) = \rho(M)$, and an extension $\langle\ ,\ \rangle_N$ of $\langle\ ,\ \rangle_M$, such that w_i corresponds to F^i in the sense that $\langle w_i, v\rangle_N = 0$ for every $v \in F^i$ and $\langle w_i, v\rangle_N = -$ for $v \in E\backslash F^i$. Let $\tilde{M} = N\backslash E$ and suppose, further, that for each copoint G of the lattice $L(\tilde{M})$, $\bigwedge\{F_i | w_i \in G\} \neq \varnothing$. Then $\tilde{M}$ is a polar of M.*

PROOF. We may assume M is simple and every element of E is a point of $L(M)$. We will denote both $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_N$ by $\langle\ ,\ \rangle$.

By the lemma, it is sufficient to show that there exists an injective function ϕ from the points of $L(M)$ onto the copoints of $L(\tilde{M})$ and from the copoints of $L(M)$ onto the points of $L(\tilde{M})$ such that for v a point and F a copoint of $L(M)$, $v \leq F$ if and only if $\phi(F) \leq \phi(v)$.

By the construction of $\tilde{M}$, we know that to each copoint of $L(M)$ corresponds an element of $\tilde{M}$. Define $\phi(F^i) = w_i$ for F^i a copoint of $L(M)$, and for v a point of $L(M)$ define $\phi(v) = \{w_i \in E(\tilde{M}) | v \in F^i\}$. We need to show that $\phi(v)$ is a copoint of $L(\tilde{M})$ for every $v \in E$, that w_i is a point of $L(\tilde{M})$ for every $i = 1, \ldots, n$, that ϕ is one-to-one, and onto the points and copoints of $L(\tilde{M})$, respectively, and, finally, that $v \in F^i$ if and only if $\phi(F^i) \in \phi(v)$.

By the hypothesis on $\langle\ ,\ \rangle_N$ and the definition of ϕ, $w_i \in \phi(v)$ implies $\langle w_i, v\rangle = 0$. Then by Corollary 4.5, $\rho_N(\phi(v)) \leq \rho(N) - 1$, so $\phi(v)$ spans at most a hyperplane of N. We can choose copoints $F^{i_1}, F^{i_2}, \ldots, F^{i_{\rho(N)-1}}$ of $L(M)$ such that $v = \bigcap_{j=1}^{\rho(N)-1} F^{i_j}$ and $F^{i_k} \cap (\bigcap_{j=1}^{k-1} F^{i_j}) \subsetneq \bigcap_{j=1}^{k-1} F^{i_j}$ for every $k = 2, \ldots, \rho(N) - 1$. Let $\{w_{i_1}, \ldots, w_{i_{\rho(N)-1}}\}$ be the set of corresponding elements of N. Then $\{w_{i_1}\}$ is independent since w_{i_1} is not a loop of N, and if $\{w_{i_1}, \ldots, w_{i_k}\}$ is independent, then so is $\{w_{i_1}, \ldots, w_{i_{k+1}}\}$. For suppose $\{w_{i_1}, \ldots, w_{i_{k+1}}\}$ is dependent. Then there exists $X \in \mathcal{O}(N)$ such that $w_{i_{k+1}} \in \underline{X} \subseteq \{w_{i_1}, \ldots, w_{i_{k+1}}\}$. Let $v_{k+1} \in (\bigcap_{j=1}^{k} F^{i_j})\backslash(\bigcap_{j=1}^{k+1} F^{i_j})$. Then $\langle v_{k+1}, w_{i_j}\rangle = 0$ for $j = 1, \ldots, k$, so $\mathrm{sg}_X\, w_{i_j}\langle v_{k+1}, w_{i_j}\rangle = 0$ for every $w_{i_j} \in \underline{X}\backslash w_{i_{k+1}}$. By axiom (I3), $\mathrm{sg}_X\, w_{i_{k+1}}\langle v_{k+1}, w_{i_{k+1}}\rangle = 0$. But $v_{k+1} \notin F^{i_{k+1}}$, so $\langle v_{k+1}, w_{i_{k+1}}\rangle \neq 0$, and we have a contradiction. Thus $\{w_{i_1}, \ldots, w_{i_{\rho(N)-1}}\}$ is independent, and since $\{w_{i_1}, \ldots, w_{i_{\rho(N)-1}}\} \subseteq \phi(v)$, $\rho_N(\phi(v)) \geq \rho(N) - 1$. Thus $\phi(v)$ spans a hyperplane of N.

Suppose $y \in (\mathrm{cl}_N\, \phi(v))\backslash\phi(v)$. Then there exists $X \in \mathcal{O}(N)$ such that $y \in \underline{X} \subseteq \phi(v) \cup \{y\}$. For every $x \in \underline{X}\backslash y$, $x \in \phi(v)$, so $\langle v, x\rangle = 0$. Thus for every $x \in \underline{X}\backslash y$, $\mathrm{sg}_X\, x\langle x, v\rangle = 0$, so $\mathrm{sg}_X\, y\langle y, v\rangle = 0$. Since $y \in \underline{X}$, we must have $\langle y, v\rangle = 0$. Then $y \notin \phi(v)$ implies $y \in E(N)\backslash E(\tilde{M})$. Therefore $(\mathrm{cl}_N\phi(v)) \cap \{w_1, \ldots, w_n\} = \phi(v)$.

Since no point is contained in every copoint of $L(M)$, $\phi(v) \subsetneq \{w_1, \ldots, w_n\}$ for every $v \in E$. Choose any $v \in E$, and suppose, say, $w_1 \in E(\tilde{M})\backslash\phi(v)$. Let Y_v be the cocircuit of N such that $\underline{Y}_v = E(N)\backslash\mathrm{cl}_N\phi(v)$ and $w_1 \in Y_v^+$. Since $\langle v, w_i\rangle = 0$ for every $w_i \in \phi(v)$, $\langle v, x\rangle = 0$ for every $x \in \mathrm{cl}_N\phi(v)$, and hence by Corollary 4.4, $\mathrm{sg}_{Y_v}\, w_i\, \mathrm{sg}_{Y_v} w_1 = \langle v, w_i\rangle\langle v, w_1\rangle$. If $w_i \notin \phi(v)$, $v \notin F^i$, so $\langle w_i, v\rangle = -$. Thus for every $w_i \in E(\tilde{M})\backslash(\phi(v) \bigcup \{w_1\})$, $\mathrm{sg}_{Y_v}\, w_i = \mathrm{sg}_{Y_v}\, w_i\, \mathrm{sg}_{Y_v}\, w_1 = \langle v, w_i\rangle \cdot \langle v, w_1\rangle = - \cdot - = +$. Thus $\underline{Y}_v \cap E(\tilde{M}) \subseteq Y_v^+$.

Now $\rho(\tilde{M}) = \rho_N(\{w_1, \ldots, w_n\})$ since $\{w_1, \ldots, w_n\} \cap E(M) = \varnothing$, so $\rho(N) = \rho(M)$, $\rho_N(\phi(v)) = \rho(N) - 1$, and $E(\tilde{M})\backslash\mathrm{cl}_N\, \phi(v) \neq \varnothing$ together imply $\rho(\tilde{M}) = \rho(N)$. Then $\rho_{\tilde{M}}(\phi(v)) = \rho(\tilde{M}) - 1$, so $Y_v\backslash E(M)$ is a positive cocircuit of $\tilde{M}$ for every $v \in M$. Thus $\phi(v)$ is a copoint of $L(\tilde{M})$ for every $v \in M$.

To show that w_i is a point of $L(\tilde{M})$ for every $i = 1, \ldots, n$, we show that $\{w_i\} = \bigcap_{v \in F^i} \phi(v)$. Now $v \in F^i$ implies $w_i \in \phi(v)$ by the definition of $\phi(v)$, so $\{w_i\} \subseteq \bigcap_{v \in F^i} \phi(v)$

Suppose $w_j \neq w_i$ is in $E(\tilde{M})$. Then $F^j \neq F^i$, so there exists $v' \in F^i \backslash F^j$. Since $v' \notin F^j$, $w_j \notin \phi(v')$. But $v' \in F^i$, so $w_j \notin \bigcap_{v \in F^i} \phi(v)$. Thus $\bigcap_{v \in F^i} \phi(v) = \{w_i\}$.

By the hypotheses on N, ϕ is clearly one-to-one from the copoints of $L(M)$ onto the points of $L(\tilde{M})$. Since $v \neq v'$ implies there exists a copoint F^i of $L(M)$ such that $v \in F^i$, $v' \notin F^i$, ϕ is obviously one-to-one from the points of $L(M)$ into the copoints of $L(\tilde{M})$. Also, by definition, $v \in F^i$ if and only if $w_i = \phi(F^i) \in \phi(v)$.

To show, finally, that ϕ is onto the copoints of $L(\tilde{M})$, let G be any one of these. Then $G = \bigvee \{w_i | w_i \in G\} = \bigvee \{\phi(F_i) | w_i \in G\}$. Let $H = \bigwedge \{F_i | w_i \in G\}$. By hypothesis, $H \neq \varnothing$; we show that H is a point of $L(M)$. Suppose v, v' are points of $L(M)$ such that $v, v' \leq H$. Then $v \in F_i$ for every i such that $w_i \in G$ implies that $w_i \in \phi(v)$ for every i such that $w_i \in G$, and so $G \leq \phi(v)$. Similarly, $G \leq \phi(v')$. If $v \neq v'$ then $\phi(v) \neq \phi(v')$ and so $G \leq \phi(v) \wedge \phi(v') < \phi(v)$, contradicting the assumption that G is a copoint of $L(\tilde{M})$. Thus $v = v'$ and we conclude that $H = v$, a point of $L(M)$, and $\phi(v)$ is thus a copoint of $L(\tilde{M})$. But $G \leq \phi(v)$ implies $G = \phi(v)$ as desired. Thus by Lemma 5.1, $L(M)$ is anti-isomorphic to $L(\tilde{M})$, so $\tilde{M}$ is a polar of M.

Therefore, if the required extension N with inner product $\langle\ ,\ \rangle_N$ exists, then M has a polar matroid, $\tilde{M} = N \backslash (E(M))$. As we have seen, not every M has a polar. An interesting question is whether, when M has a polar, one can always be constructed in this manner. While this works in the representable case, it is not clear what happens in the general case for we encounter problems due to the lack of general constructions of non-principal extensions.

We note that it is not clear to us whether the extra assumption on the copoints of $L(\tilde{M})$ is a consequence of the other assumptions. It is surely a consequence of the conclusion. We thank Achim Bachem for pointing out an error in an earlier version of this work in which we implicitly made this assumption.

References

1. A. Bachem and W. Kern, Adjoints of oriented matroids, Report 84.1, Mathematisches Institut, Universität zu Köln, W. Germany, 1984.
2. G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, RI, 1967.
3. R. G. Bland, Complementary orthogonal subspaces of R^n and orientability of matroids, Ph.D. thesis, Cornell University, 1974.
4. R. G. Bland, A combinatorial abstraction of linear programming, *J. Combin. Theory, Ser. B* **23** (1977), 33–57.
5. R. G. Bland and M. Las Vergnas, Orientability of Matroids, *J. Combin. Theory, Ser. B* **24** (1978), 94–123.
6. A. L. C. Cheung, Adjoints of a Geometry, *Canad. Math. Bull.* **17** (1974), 363–365.
7. R. Cordovil, M. Las Vergnas and A. Mandel, Euler's relation, Möbius functions, and matroid identities, *Geom. Dedicata* **12** (1982), 147–162.
8. J. Edmonds and A. Mandel, Topology of oriented matroids, Abstract 758-05-9, *Notices Amer. Math. Soc.* **25** (1978), A-510.
9. J. Folkman and J. Lawrence, Oriented matroids, *J. Combin. Theory, Ser. B* **25** (1978), 199–236.
10. K. Fukuda, Oriented matroid programming, Ph.D. thesis, University of Waterloo, 1982.
11. B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, New York, 1967.
12. M. Las Vergnas, Convexity in oriented matroids, *J. Combin. Theory, Ser. B* **29** (1980), 231–243.
13. M. Las Vergnas, Extensions ponctuelles d'une géométrie combinatoire orientée, in *Problèmes Combinatories et Théorie des Graphes*, Actes du Coll. Int. C.N.R.S. n. 260, Orsay 1976, Paris, 1978, pp. 263–268.
14. A. Mandel, Topology of oriented matroids, Ph.D. thesis, Univeristy of Waterloo, 1982.
15. B. S. Munson, Face lattices of oriented matroids, Ph.D. thesis, Cornell University, August, 1981.
16. J. S. Provan and L. J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra,

17. J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
18. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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