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DECOMPOSITIONS OF SIMPLICIAL COMPLEXES RELATED TO DIAMETERS OF CONVEX POLYHEDRA*†

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We introduce the property of k-decomposability for simplicial complexes which, if satisfied by the dual simplicial complex of a convex polyhedron, implies that the diameter of the polyhedron is bounded by a polynomial of degree $k+1$ in the number of facets. These properties form a hierarchy, each one implying the next. The strongest, vertex decomposability, implies the Hirsch conjecture; the weakest is equivalent to shellability. We show that several cases in which the Hirsch conjecture has been verified can be handled by these methods, which also give the shellability of a number of simplicial complexes of combinatorial interest. We conclude with a strengthened form of the property of shellability which would imply the Hirsch conjecture for polytopes.

1. Introduction and definitions. In this paper, we use the notion of the dual complex to a polyhedron, a straightforward extension of the usual polar dual of a (bounded) polytope, to study questions related to the diameter of that polyhedron. We develop properties of these complexes which will imply good (i.e., polynomial) bounds on the diameter. In the case of simple polytopes, the study of diameter via a related simplicial complex (abstract polytope) was begun by Adler and Dantzig [1].

The *distance* between two vertices in a polyhedron is defined to be the least number of edges in an edge path joining them. In linear programming terms, it is the least number of feasible pivots needed to solve a linear programming problem for which the feasible region is the given polyhedron, assuming that the initial feasible solution is one of these vertices and the optimal solution is the other. The *diameter* of a polyhedron is the maximum distance between two vertices of that polyhedron; it represents the worst case performance of the best possible edge-following procedure for solving linear programming problems having that feasible region. The *Hirsch conjecture* ([9, p. 168], [12], [16]) states that if P is a d-dimensional polyhedron with n facets, then the diameter of P is at most $n-d$. While it is known that the Hirsch conjecture fails for unbounded polyhedra of dimension $d \geqslant 4$, it has been proved for polytopes satisfying $n-d \leqslant 5$ (see [16]) and for all polyhedra of dimension 3 or less [15]. Even in the unbounded case, there is no known example of a polyhedron having a diameter, exceeding, say, $2(n-d)$. On the other hand, proven upper bounds for the diameter of a d-polytope with n facets are like $2^{d-3}n$ [17], [24]. Thus any sort of bound which is a polynomial in n and d would be an important result.

We remark here that if one changes the definition of diameter to require that all paths be monotonic with respect to a specified linear (objective) function, then the resulting monotonic Hirsch conjecture fails for polytopes of dimension at least 4 [22].

Our approach to diameter bounds for simple d-polyhedra (each vertex on exactly d facets; for diameter purposes, it is enough to consider just these [16, Theorem 2.8]) is through decomposition properties of the associated dual simplicial complexes. In §2, we define and study the notion of a k-decomposable simplicial complex, for a nonnegative integer k. These give a hierarchy of complexes, with k-decomposable complexes always being $(k+1)$-decomposable. We show that k-decomposable complexes have simplicial diameters (defined later in this section) bounded above by a polynomial of degree $k+1$ in the number of vertices (Corollary 2.12), which implies that if a simple polyhedron P has a k-decomposable dual complex, then the diameter of P is bounded by a polynomial of degree $k+1$ in the number of facets. A special case of this is that if P has a 0-decomposable (called vertex decomposable) dual complex, then P satisfies the Hirsch conjecture. To relate k-decomposability to an already studied decomposition property, we show (Theorem 2.8) that a d-dimensional complex is d-decomposable if and only if it is shellable, a property known to hold for (simple) polyhedra and their dual simplicial complexes. (In [5], it is shown that all polytopes, and thus their duals, are shellable; the proof there can be easily extended to unbounded polyhedra and to their dual simplicial complexes, which are not polyhedra.)

In §3, we concentrate on vertex decomposable complexes, which must all be shellable and satisfy a simplicial form of the Hirsch conjecture. In particular, we discuss vertex decomposability of (i) dual complexes to simple polydedra of dimension at most 3 (obtaining, as a consequence, Klee's result [15] on the Hirsch conjecture to such polyhedra); (ii) independent set and broken circuit complexes of matroids and complexes of chains in distributive lattices (settling questions about their shellability posed by Stanley [21]); (iii) dual complexes of simple totally Leontief substitution polyhedra [10] (generalizing a result of Grinold [11] that such polyhedra satisfy the Hirsch conjecture); (iv) the boundary complex of a cyclic polytope (obtaining a result of Klee [14]; for proof, see [3]); and (v) the complexes of chains of faces in strongly shellable cell complexes, which include the face lattice complexes of convex polyhedra and, in particular, any barycentric subdivision of an arbitrary convex polytope.

Finally, in §4 we consider an extension of k-decomposability, weak k-decomposability, which still implies polynomial bounds on diameter. In particular weak vertex decomposability of the dual complex of an arbitrary polyhedron, which would imply a diameter bound of twice the number of facets, is still an open question. We also discuss a direct geometric formulation for the property of (weak) k-decomposability of the dual complex of a polyhedron (face decomposability), and a decomposition property of the dual complex which corresponds to shellability of a polyhedron (dual shellability). Most important is a strengthening of the notion of shellability of a polyhedron (deep shellability) which turns out to be equivalent to vertex decomposability of the dual complex, and hence implies the Hirsch conjecture. We note that deep shellability of simple polytopes, and hence vertex decomposability of their dual complexes (which are the boundary complexes of simplicial polytopes), remain open questions.

We now define some terms. A *simplicial complex* is any nonempty collection Σ of subsets of a finite set E with the property that if σ is an element of Σ, and τ is any subset of σ, then τ is also an element of Σ. Elements of Σ are called *faces* (or simplices) and are identified, when necessary, by listing their elements, $\sigma = v_1 \ldots v_k$. The *dimension* of a face σ is $\dim \sigma = |\sigma| - 1$; the dimension of Σ is the maximum of the dimensions of the faces of Σ, and if all maximal faces of Σ have the same dimension then Σ is called *pure dimensional*. A k-dimensional face is called, simply, a *k-face*, and 0-faces, $(d-1)$-faces, and d-faces of a d-dimensional complex Σ are referred to, respectively, as *vertices*, *ridges*, and *facets* of Σ. The number of vertices of Σ is denoted $V(\Sigma)$. For any set σ of cardinality $d+1$, the *d-simplex* $\bar{\sigma}$ is defined to be the collection of all subsets of σ (so that $\{\varnothing\}$ is a (-1)-simplex).

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Let Σ be pure d-dimensional. Σ is said to be *shellable* if the d-faces of Σ can be ordered $\sigma_1, \ldots, \sigma_s$ so that, for $i = 2, \ldots, s$, $\bar{\sigma}_i \cap (\bigcup_{j=1}^{i-1} \bar{\sigma}_j)$ is a pure $(d-1)$-dimensional complex, in other words a union of $(d-1)$-simplices. The ordering $\sigma_1, \ldots, \sigma_s$ is called a *shelling* of Σ. The *distance* between two d-faces Δ_1 and Δ_2 of Σ is the length k of a shortest simplicial path $\Delta_1 = \sigma_0, \sigma_1, \ldots, \sigma_k = \Delta_2$ between Δ_1 and Δ_2, where σ_i are each d-faces and $\sigma_i \cap \sigma_{i-1}$ is a $(d-1)$-face for $i = 1, \ldots, k$. The *diameter* of Σ, $\operatorname{diam} \Sigma$, is the maximum of the distance between any two d-faces of Σ, and Σ is said to satisfy the *Hirsch conjecture* if $\operatorname{diam} \Sigma \leqslant V(\Sigma) - \dim \Sigma - 1$.

The relation between simplicial complexes and convex polyhedra is established as follows. Let P be a *pointed* d-dimensional polyhedron (i.e., P has at least one vertex; we restrict consideration to these). P is said to be *simple* if each vertex of P is contained in exactly d facets ($(d-1)$-faces). In this case, the *dual simplicial complex* to P is defined on the set $E = \{v_i \mid f_i \text{ is a facet of } P\}$ to be

$$\Sigma^*(P) = \{v_{i_1} \cdots v_{i_k} \mid f_{i_1} \cap \cdots \cap f_{i_k} \neq \emptyset\}.$$

$\Sigma^*(P)$ is thus a pure $(d-1)$-dimensional simplicial complex. In the case that P is a simple polytope, $\Sigma^*(P)$ is the boundary complex of the simplicial polytope which is its polar dual [12, §3.4], and so is a shellable combinatorial $(d-1)$-sphere (see [7]). If P is unbounded, then $\Sigma^*(P)$ is a combinatorial $(d-1)$-ball, which can be seen as follows. Intersect P with a closed half-space containing all the vertices of P which is determined by a hyperplane H parallel to a supporting hyperplane to one of the vertices of P. This yields a polytope $\bar{P}$ having a facet $\bar{f}$ determined by H. It can be seen that $\Sigma^*(P)$ can be obtained from the combinatorial sphere $\Sigma^*(\bar{P})$ by deleting (see the definition below) the vertex $\bar{v}$ corresponding to $\bar{f}$, and so must be a combinatorial ball. This same idea can be used to show P is shellable (a shelling of P is obtained from a shelling of $\bar{P}$ in which $\bar{f}$ appears last) and $\Sigma^*(P)$ is shellable (a shelling of $\Sigma^*(P)$ is obtained from a shelling of $\Sigma^*(\bar{P})$ in which all the facets containing $\bar{v}$ appear last), see [5], [7], and [19, §2.9].

A crucial observation to make at this point is that the diameter of the polyhedron P is equal to $\operatorname{diam} \Sigma^*(P)$ and P satisfies the conditions of the Hirsch conjecture if and only if $\Sigma^*(P)$ satisfies the simplicial form of the Hirsch conjecture defined above.

The following operations on simplicial complexes will be of importance in this paper. Many correspond, via the above duality, to standard operations on polyhedra. For a simplicial complex Σ and face σ of Σ, the *deletion* of σ from Σ is

$$\Sigma \backslash \sigma = \{\tau \in \Sigma \mid \sigma \not\subseteq \tau\}$$

and the *link* of σ in Σ is

$$lk_\Sigma \sigma = \{\tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\}.$$

For complexes Σ_1 and Σ_2 defined on disjoint sets, the *join* of Σ_1 and Σ_2 is

$$\Sigma_1 . \Sigma_2 = \{\sigma \cup \tau \mid \sigma \in \Sigma_1, \tau \in \Sigma_2\}.$$

The *boundary* complex of a complex Σ is

$$\partial \Sigma = \operatorname{cl}\{\sigma \mid \sigma \text{ is a ridge of } \Sigma \text{ contained in at most one facet of } \Sigma\},$$

where, for $\Delta \subset 2^E$,

$$\operatorname{cl} \Delta = \cup \{\bar{\sigma} \mid \sigma \in \Delta\}.$$

For vertex u of complex Σ, the simplicial *wedge* of Σ on u is the complex

$$w(\Sigma, v) = (\partial \overline{ab}).(\Sigma \backslash u) \cup \overline{ab}.lk_\Sigma u$$

and for face X of Σ and arbitrary symbol a, the *stellar subdivision* of X in Σ is

$$\operatorname{st}(a, X)[\Sigma] = (\Sigma \backslash X) \cup \bar{a}.\partial X.lk_\Sigma X.$$

(In the above two definitions, $a, b \notin \Sigma$ are "new" vertices.)

In terms of the duality between simple polyhedra and complexes, links correspond to faces, joins correspond to products of polyhedra, wedges correspond to the polyhedral operation of taking a wedge with a facet as foot, and deletions correspond to removal of a face (see [16, §1] for the definitions). The operation of stellar subdivision corresponds to "cutting off" a face by making a small parallel shift of a supporting hyperplane and intersecting the polyhedron with the closed half-space not containing the face. We note here that all these operations on polyhedra—except removal of a face—preserve the property of being a polyhedron. While any face can be removed from a polytope, yielding (combinatorially) the face structure of an unbounded polyhedron [16, Corollary 1.2], the result of removal of a face from a polyhedron cannot generally be realized by another polyhedron. For this reason, we have chosen to work, instead, with the dual complexes.

To end the section we state a lemma—tacitly used throughout the remainder of the paper—concerning manipulation of links, deletions, joins, and unions. The proof (see [18]) is straightforward and omitted here. As a technicality we extend the definition of deletion and link to arbitrary subsets, so that $\Sigma \backslash \sigma = \Sigma$, $lk_\Sigma \sigma = \emptyset$ for $\sigma \notin \Sigma$ and $\Sigma \backslash \emptyset = \emptyset$.

LEMMA 1.1. *Let $\Sigma, \Sigma_1, \Sigma_2$ be simplicial complexes. If $\tau \in \Sigma_1 \cup \Sigma_2$ then*
(i) $(\Sigma_1 \cup \Sigma_2) \backslash \tau = (\Sigma_1 \backslash \tau) \cup (\Sigma_2 \backslash \tau)$, *and*
(ii) $lk_{\Sigma_1 \cup \Sigma_2} \tau = lk_{\Sigma_1} \tau \cup lk_{\Sigma_2} \tau$.
If $\tau = \tau_1 . \tau_2 \in \Sigma_1 . \Sigma_2$, with $\tau_i \in \Sigma_i$, then
(iii) $(\Sigma_1 . \Sigma_2) \backslash \tau = (\Sigma_1 \backslash \tau_1) . \Sigma_2 \cup \Sigma_1 . (\Sigma_2 \backslash \tau_2)$, *and*
(iv) $lk_{\Sigma_1 . \Sigma_2} \tau = (lk_{\Sigma_1} \tau_1) . (lk_{\Sigma_2} \tau_2)$.
If $\sigma, \tau \in \Sigma$ then
(v) $(\Sigma \backslash \sigma) \backslash \tau = (\Sigma \backslash \tau) \backslash \sigma$,
(vi) $(lk_\Sigma \sigma) \backslash \tau = lk_{\Sigma \backslash \tau} \sigma$, *if* $\sigma \cap \tau = \emptyset$,
(vii) $lk_{lk_\Sigma \sigma}(\tau \backslash \sigma) = lk_\Sigma(\tau \cup \sigma)$, *and*
(viii) $lk_{\Sigma \backslash (\sigma \backslash \tau)} \tau = lk_{\Sigma \backslash \sigma} \tau$.

2. k-decomposable complexes. We now introduce the main concept of this paper—that of k-decomposability—and discuss several of its properties.

DEFINITION 2.1. A d-dimensional complex Σ is *k-decomposable* if it is pure dimensional and either Σ is a d-simplex, or there exists a face τ of Σ, $\dim \tau \leqslant k$ (called a *shedding face*), so that
(1) $\Sigma \backslash \tau$ is d-dimensional and k-decomposable, and
(2) $lk_\Sigma \tau$ is $(d - |\tau|)$-dimensional and k-decomposable.

Note. This definition, as it stands, has some redundancies. It turns out that we can drop either (i) the pure dimensionality requirement on Σ, or (ii) the dimensionality requirements on $\Sigma \backslash \sigma$ and $lk_\Sigma \sigma$, and still obtain an equivalent definition. (In the second case we ask that either $\Sigma = \{\emptyset\}$ or there exists a shedding face. See [18].) These alternative defintions become useful in §3 and in Theorem 4.1.11.

k-decomposability, then, forms a hierarchy, with k-decomposability implying $(k+1)$-decomposability for $0 \leqslant k < \dim \Sigma$, and k-decomposability equivalent to $(k+1)$-decomposability for $k \geqslant \dim \Sigma$. The most restrictive case, $k = 0$, is of special importance, and is called *vertex decomposability*.

Simplices, then, are trivially vertex decomposable, and, as the next proposition shows, so are their boundaries.

PROPOSITION 2.2. *The boundary of the d-simplex is vertex decomposable.*

PROOF. Let $\bar{\sigma} = \overline{v_0 \cdots v_d}$ be a d-simplex, so that $\partial\bar{\sigma} = \overline{v_0 \cdots v_d} \backslash v_0 \cdots v_d$. Certainly $\partial\bar{\sigma}$ is pure d-dimensional. If $d = 0$, then $\partial\bar{\sigma} = \bar{v}_0 \backslash v_0 = \{\emptyset\}$ which is a (-1)-simplex. Otherwise proceed by induction on d, and choose any vertex v_i in $\partial\bar{\sigma}$. We have

$$\partial\bar{\sigma}\backslash v_i = \overline{v_0 \cdots v_{i-1}v_{i+1} \cdots v_d}$$

which is a $(d-1)$-simplex, and

$$lk_{\partial\bar{\sigma}} v_i = \overline{v_0 \cdots v_{i-1}v_{i+1} \cdots v_d} \backslash v_0 \cdots v_{i-1}v_{i+1} \cdots v_d$$

which is the boundary of a $(d-1)$-simplex, and so is vertex decomposable by induction. Hence v_i satisfies the properties of a shedding vertex and $\partial\bar{\sigma}$ is therefore vertex decomposable.

k-decomposability also tends to behave nicely with respect to the operations defined in §1.

PROPOSITION 2.3. *The link of every face of a k-decomposable complex is itself k-decomposable.*

PROOF. Let Σ be a k-decomposable complex, σ a face of Σ. We have $lk_\Sigma \sigma$ pure dimensional, since Σ is. If Σ is a simplex, then $lk_\Sigma\sigma$ is also a simplex, hence k-decomposable. Therefore, assume that Σ is not a simplex so that Σ has shedding face τ with $\Sigma\backslash\tau$ k-decomposable of dimension $d = \dim \Sigma$, and $lk_\Sigma\tau$ is k-decomposable of dimension $d - |\tau|$. We proceed by induction on $|\Sigma|$ = the number of faces of Σ.

Case 1 ($\tau \cup \sigma \notin \Sigma$). We have $lk_\Sigma\sigma = lk_{\Sigma\backslash\tau}\sigma$ which is k-decomposable by induction on $|\Sigma\backslash\tau| < |\Sigma|$.

Case 2 ($\tau \subseteq \sigma$). We have

$$lk_\Sigma\sigma = lk_{lk_\Sigma\tau}(\sigma\backslash\tau)$$

which is k-decomposable by induction on $|lk_\Sigma\tau| < |\Sigma|$.

Case 3 ($\sigma \cup \tau \in \Sigma$, $\tau \not\subseteq \sigma$). We prove that $\tau\backslash\sigma$ is a shedding face for $lk_\Sigma\sigma$. By Lemma 1.1, (vi), (vii) and (viii),

$$(lk_\Sigma\sigma)\backslash(\tau\backslash\sigma) = lk_{\Sigma\backslash\tau}\sigma,$$

which is $(d - |\sigma|)$-dimensional and k-decomposable as in Case 1, and

$$lk_{lk_\Sigma\sigma}(\tau\backslash\sigma) = lk_\Sigma(\tau \cup \sigma) = lk_{lk_\Sigma\tau}(\sigma\backslash\tau),$$

which is $(d - |\sigma| - |\tau\backslash\sigma|)$-dimensional and k-decomposable as in Case 2. Since this exhausts all possible choices of τ, the theorem is proved.

PROPOSITION 2.4. *Let Σ_1, Σ_2 be simplicial complexes with disjoint sets of vertices. Then $\Sigma_1.\Sigma_2$ is k-decomposable if Σ_1 and Σ_2 are. Further, $\Sigma_1.\Sigma_2$ is vertex decomposable if and only if Σ_1 and Σ_2 are.*

PROOF. We first note that $\Sigma_1.\Sigma_2$ is pure dimensional if and only if Σ_1 and Σ_2 are, and that $\Sigma_1.\Sigma_2$ is a simplex if and only if Σ_1 and Σ_2 are. Now we observe that for any face τ of $\Sigma_1.\Sigma_2$, letting τ_i be the set of vertices of τ in Σ_i, $i = 1, 2$,

$$(\Sigma_1.\Sigma_2)\backslash\tau = (\Sigma_1\backslash\tau_1).\Sigma_2 \cup \Sigma_1.(\Sigma_2\backslash\tau_2)$$

and

$$lk_{\Sigma_1.\Sigma_2}\tau = (lk_{\Sigma_1}\tau_1).(lk_{\Sigma_2}\tau_2)$$

It follows from Lemma 1.1, (iii) and (iv), that if τ_1 is a shedding face for Σ_1, then τ_1 is also a shedding face for $\Sigma_1.\Sigma_2$, since then $(\Sigma_1.\Sigma_2)\backslash\tau_1 = (\Sigma_1\backslash\tau_1).\Sigma_2$ and $lk_{\Sigma_1.\Sigma_2}\tau_1 =$ $lk_{\Sigma_1}\tau_1$ are k-decomposable complexes; similarly for τ_2. k-decomposability of $\Sigma_1.\Sigma_2$ now follows by induction. Conversely if $\Sigma_1.\Sigma_2$ is vertex decomposable with shedding vertex v, then v is a vertex of exactly one Σ_i, say Σ_1. Hence

$$(\Sigma_1.\Sigma_2)\backslash v = (\Sigma_1\backslash v).\Sigma_2$$

and

$$lk_{\Sigma_1.\Sigma_2} v = lk_{\Sigma_1} v,$$

and both are vertex decomposable complexes with $\dim((\Sigma_1.\Sigma_2)\backslash.v) = \dim \Sigma_1.\Sigma_2 = d$ and $\dim lk_{\Sigma_1.\Sigma_2} v = d - 1$. Again by induction, Σ_2 and $\Sigma_1\backslash v$ are vertex decomposable with $\dim(\Sigma_1\backslash v) = \dim \Sigma_1$, and so Σ_1 is vertex decomposable. This completes the proof of Proposition 2.4.

A similar equivalence holds for wedging of vertex decomposable complexes.

PROPOSITION 2.5. *Let Σ be a simplicial complex, u a vertex in Σ. Then $w(\Sigma, u)$ is vertex decomposable if and only if Σ is.*

PROOF. Recall the definition

$$w(\Sigma, u) = \{a, b, \emptyset\}.(\Sigma\backslash u) \cup \overline{ab}\,.lk_\Sigma u.$$

Clearly $w(\Sigma, u)$ is pure dimensional if and only if Σ is, and $w(\Sigma, u)$ is a simplex if and only if Σ is. Further, by elementary applications of Lemma 1.1 we have the following.

(i) For any vertex v in $\Sigma\backslash u$

$$\begin{aligned} w(\Sigma, u)\backslash v &= \{a, b, \emptyset\}.[(\Sigma\backslash u)\backslash v] \cup \overline{ab}\,.[(lk_\Sigma u)\backslash v] \\ &= \{a, b, \emptyset\}.[(\Sigma\backslash v)\backslash u] \cup \overline{ab}\,.lk_{\Sigma\backslash v} u = w(\Sigma\backslash v, u) \end{aligned}$$

and

$$\begin{aligned} lk_{w(\Sigma, u)} v &= \{a, b, \emptyset\}.lk_{\Sigma\backslash u} v \cup \overline{ab}\,.lk_{lk_\Sigma u} v \\ &= \{a, b, \emptyset\}.[(lk_\Sigma v)\backslash u] \cup \overline{ab}\,.lk_{lk_\Sigma v} u = w(lk_\Sigma v, u). \end{aligned}$$

(ii)

$$\begin{aligned} w(\Sigma, u)\backslash a &= (\{a, b, \emptyset\}\backslash a).(\Sigma\backslash u) \cup (\overline{ab}\backslash a).lk_\Sigma u \\ &= \bar{b}.(\Sigma\backslash u) \cup \bar{b}.lk_\Sigma u = \bar{b}.(\Sigma\backslash u) \end{aligned}$$

and

$$\begin{aligned} lk_{w(\Sigma, u)} a &= (lk_{\{a,b,\emptyset\}} a).(\Sigma\backslash u) \cup (lk_{\overline{ab}} a).lk_\Sigma u \\ &= (\Sigma\backslash u) \cup \bar{b}.lk_\Sigma u = \Sigma \quad \text{(replacing } u \text{ with } b\text{).} \end{aligned}$$

From the above facts we get

(1) for $v \neq u$, v is a shedding vertex for Σ if and only if v is a shedding vertex for $w(\Sigma, u)$;

(2) if u is a shedding vertex for Σ, then a is a shedding vertex for $w(\Sigma, u)$; and from Proposition 2.3 we get

(3) if $w(\Sigma, u)$ is vertex decomposable, then $\Sigma = lk_{w(\Sigma, u)} a$ is vertex decomposable.

These cases establish the equivalence.

Propositions 2.4 and 2.5 give Klee–Walkup type results (see [16, Proposition 2.9]) for vertex decomposability with respect to dual complexes to convex polytopes. Let $\Gamma(d, n)$ denote the class of complexes of dimension $d - 1$ with n vertices which are dual to simple polytopes (equivalently, dual complexes to simple polytopes of dimension d with n facets). Then for any element of $\Gamma(d,n)$, its wedge is in $\Gamma(d+1,n+1)$, and its product with the (vertex decomposable) complex $\{v_1, v_2, \emptyset\}$ is in $\Gamma(d+1,n+2)$. Then by applying Propositions 2.5 and 2.4, respectively, we have

THEOREM 2.6. *For $n \geq d \geq 1$*
(i) *if $\Gamma(d+1,n+1)$ is a vertex decomposable class, then so is $\Gamma(d,n)$;*
(ii) *if $\Gamma(d+1,n+2)$ is a vertex decomposable class then so is $\Gamma(d,n)$.*

In particular, to prove vertex decomposability for all complexes dual to simple polytopes, it is sufficient to consider the classes $\Gamma(d,n)$, $n \leq 2d$, for if $n > 2d$, then $n-2d$ applications of Theorem 2.6(i) shows that it is sufficient to consider the class $\Gamma(n-d, 2(n-d))$.

THEOREM 2.7. *k-decomposability is preserved under stellar subdivisions.*

PROOF. Let Σ be k-decomposable, $X \neq \emptyset$ a simplex in Σ, and a the additional vertex. We have $|\Sigma| \geq 2$, and for $\Sigma = \bar{\sigma}$ a simplex,

$$\mathrm{st}(a,X)[\Sigma] = \bar{a}.\partial X.(\overline{\sigma \backslash X})$$

which is k-decomposable since each component is. Now proceed by induction on $|\Sigma| > 2$, Σ not a simplex, and let τ be a shedding simplex for Σ, $\dim \tau \leq k$, so that $\Sigma \backslash \tau$ is k-decomposable and $lk_\Sigma \tau$ is k-decomposable.

The following facts can be derived from Lemma 1.1. (The proofs are tedious and will be omitted. See [18] for details.)

A. If $\tau \cup X \notin \Sigma$, then
(1) $lk_{\mathrm{st}(a,X)[\Sigma]}\tau = lk_\Sigma \tau$, and
(2) $\mathrm{st}(a,X)[\Sigma]\backslash\tau = \mathrm{st}(a,X)[\Sigma\backslash\tau]$.

B. If $\tau \in lk_\Sigma X$, then
(1) $lk_{\mathrm{st}(a,X)[\Sigma]}\tau = \mathrm{st}(a,X)[lk_\Sigma\tau]$, and
(2) $\mathrm{st}(a,X)[\Sigma]\backslash\tau = \mathrm{st}(a,X)[\Sigma\backslash\tau]$.

C. If $\tau \cup X \in \Sigma$, $X \not\subseteq \tau$, $X \cap \tau \neq \emptyset$, then
(1) $lk_{\mathrm{st}(a,X)[\Sigma]}\tau = \mathrm{st}(a, X\backslash\tau)[lk_\Sigma\tau]$,
(2) $lk_{\mathrm{st}(a,X)[\Sigma]\backslash\tau}(a.(\tau\backslash X)) = (\overline{X\backslash\tau}).lk_\Sigma(X \cup \tau)$, and
(3) $\mathrm{st}(a,X)[\Sigma]\backslash\tau\backslash(a.(\tau\backslash X)) = \mathrm{st}(a,X)[\Sigma\backslash\tau]$ (where we define $\mathrm{st}(a,X)[\Sigma\backslash\tau] \equiv \Sigma\backslash\tau$ for $X \notin \Sigma\backslash\tau$).

D. If $X \subseteq \tau$, then
(1) $lk_{\mathrm{st}(a,X)[\Sigma]}(a.(\tau\backslash X)) = \partial X.lk_\Sigma\tau$, and
(2) $\mathrm{st}(a,X)[\Sigma]\backslash(a.(\tau\backslash X)) = \mathrm{st}(a,X)[\Sigma\backslash\tau]$.

The right-hand sides are all k-decomposable by induction on the number of simplices and applications of Propositions 2.3 and 2.4 (noting in C and D that $\dim(a.(\tau\backslash X)) \leq k$), and since A, B, C, D, cover all cases for τ, the theorem follows.

Some of the most striking results on k-decomposability deal with shellability and diameters.

THEOREM 2.8. *A d-dimensional complex Σ is d-decomposable if and only if Σ is shellable.*

PROOF. (If): Let Σ be a d-dimensional complex, $\sigma_1, \ldots, \sigma_m$ an ordering of the d-faces of Σ which shells Σ, so that $\Sigma = \mathrm{cl}(\bigcup_{i=1}^m \sigma_i)$. If $m = 1$ then Σ is a d-simplex, and so Σ is d-decomposable. For $m > 1$ we proceed by induction on $|\Sigma|$. We have by definition that $\bar{\sigma}_m \cap (\bigcup_{i=1}^{m-1} \bar{\sigma}_i)$ is a pure $(d-1)$-dimensional complex generated by the $(d-1)$-faces $\tau_1, \ldots, \tau_k$. Let $\tau = \bigcup_{j=1}^k (\sigma_m \backslash \tau_j)$. Then $\tau \subseteq \sigma_m$ (hence $\tau \in \Sigma$), but $\tau \not\subseteq \sigma_i$, $i < m$, since it cannot be that $\tau = \tau \cap \sigma_i \subseteq \sigma_m \cap \sigma_i \subseteq \tau_j$, for some j, while $\emptyset \neq \sigma_m \backslash \tau_j \subseteq \tau$. Therefore σ_m is the only d-face containing τ. So

$$\begin{aligned}\Sigma\backslash\tau &= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup (\bar{\sigma}_m \backslash \tau) \\ &= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup (\overline{\sigma_m \backslash \tau}) = \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup \mathrm{cl}\left(\sigma_m \backslash \bigcup_{j=1}^{k} (\sigma_m \backslash \tau_j)\right) \\ &= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup \mathrm{cl}\left(\bigcup_{j=1}^{k} \tau_j\right) = \bigcup_{i=1}^{m-1} \bar{\sigma}_i \qquad (1)\end{aligned}$$

which is a shellable complex, with shelling order $\sigma_1, \ldots, \sigma_{m-1}$, hence d-decomposable by induction on $|\Sigma\backslash\tau| < |\Sigma|$, and

$$lk_\Sigma \tau = lk_{\bar{\sigma}_m} \tau = (\overline{\sigma_m \backslash \tau}) \qquad (2)$$

which is a simplex, hence d-decomposable. Therefore τ is a shedding face for Σ, and Σ is d-decomposable.

(Only if): Let Σ be d-dimensional and d-decomposable. If Σ is a d-simplex, then Σ is shellable. Otherwise there must exist a face $\tau \in \Sigma$ such that (1) $\Sigma\backslash\tau$ is d-dimensional and d-decomposable, and (2) $lk_\Sigma\tau$ is $(d-|\tau|)$-dimensional and d-decomposable, implying $lk_\Sigma\tau$ is $(d-|\tau|)$-decomposable. We proceed by induction on $|\Sigma|$. (1) implies $\Sigma\backslash\tau$ is shellable, by induction on $|\Sigma\backslash\tau| < |\Sigma|$, with shelling order $\sigma_1, \ldots, \sigma_p$ of the d-faces of Σ not containing τ. (2) implies $lk_\Sigma\tau$ is shellable, by induction on $|lk_\Sigma\tau| < |\Sigma|$, with shelling order $\tau_1, \ldots, \tau_l$. Let $\sigma_{p+i} = \tau \cup \tau_i$, $i = 1, \ldots, l$. We claim that $\sigma_1, \ldots, \sigma_{p+l}$ is a shelling for Σ. For $2 \leq k \leq p$, $\bar{\sigma}_k \cap (\bigcup_{i=1}^{k-1} \bar{\sigma}_i)$ is a pure $(d-1)$-dimensional complex by (1). For $k > p$, we have

$$\begin{aligned}\bar{\sigma}_k \cap \left(\bigcup_{i=1}^{k-1} \bar{\sigma}_i\right) &= \left[\bar{\sigma}_k \cap \left(\bigcup_{i=1}^{p} \bar{\sigma}_i\right)\right] \cup \left[\bar{\sigma}_k \cap \left(\bigcup_{i=p+1}^{k-1} \bar{\sigma}_i\right)\right] \\ &= \left[\bar{\sigma}_k \cap (\Sigma\backslash\tau)\right] \cup \left[(\bar{\tau}.\bar{\tau}_{k-p}) \cap \left(\bigcup_{i=1}^{k-p} (\bar{\tau}.\bar{\tau}_i)\right)\right] \\ &= \left[\bar{\sigma}_k \backslash \tau\right] \cup \left[\bar{\tau}.\left(\bar{\tau}_{k-p} \cap \left(\bigcup_{i=1}^{k-p} \bar{\tau}_i\right)\right)\right].\end{aligned}$$

Now $\bar{\sigma}_k \backslash \tau = \bigcup_{v \in \tau} \overline{\sigma \backslash v}$, which is pure of dimension $d-1$, and $\bar{\tau}_{k-p} \cap (\bigcup_{i=1}^{k-p} \bar{\tau}_i)$ is pure of dimension $d - |\tau| - 1$, since $\tau_1, \ldots, \tau_l$ is a shelling order. Hence $\bar{\sigma}_k \cap (\bigcup_{i=1}^{k-1} \sigma_i)$ is a pure $(d-1)$-dimensional complex for $k = 2, \ldots, p+l$, and therefore $\Sigma = \bigcup_{i=1}^{p+l} \bar{\sigma}_i$ is shellable.

We observe the important special case that the simplicial duals of simple convex polyhedra and polytopes fall at least into the bottom of the hierarchy of k-decomposable complexes, that is, they are shellable (see §1). It remains to be seen how far up in the hierarchy they lie.

As an immediate corollary of Theorem 2.8 and Propositions 2.3, 2.4 and 2.7 we have the following.

COROLLARY 2.9. *The following are shellable complexes:*
(1) *any k-decomposable complex, $k \geq 0$,*
(2) *the link of any simplex in a shellable complex,*
(3) *any stellar subdivision of a shellable complex, and*
(4) *the join of any two shellable complexes.*

THEOREM 2.10. *If Σ is a d-dimensional k-decomposable complex, $0 \leqslant k \leqslant d$, then*

$$\operatorname{diam}\Sigma \leqslant f_k(\Sigma) - \binom{d+1}{k+1}$$

where $f_k(\Sigma)$ is the number of k-faces of Σ.

PROOF. If Σ is a d-simplex, then $f_k(\Sigma) = \binom{d+1}{k+1}$, and so $\operatorname{diam}\Sigma = 0 \leqslant f_k(\Sigma) - \binom{d+1}{k+1}$. Otherwise we proceed by induction on $|\Sigma|$. Let Δ_0, Δ_1 be two d-faces of Σ and τ a shedding simplex for Σ.

Case 1 ($\tau \not\subseteq \Delta_0 \cap \Delta_1$). Then at least one of Δ_0, Δ_1 is in $\Sigma\backslash\tau$. say Δ_0. Further, if $\tau \subseteq \Delta_1$, then the pure dimensionality of $\Sigma\backslash\tau$ insures that there is a d-face $\Delta_1' \in \Sigma\backslash\tau$ adjacent to Δ_1. Since $\Sigma\backslash\tau$ is d-dimensional and k-decomposable, then by induction we can join Δ_0 and Δ_1' by a simplicial path of length at most $f_k(\Sigma\backslash\tau) - \binom{d+1}{k+1} \leqslant f_k(\Sigma) - 1 - \binom{d+1}{k+1}$ and so Δ_0 can be joined to Δ_1 by a path of length at most $f_k(\Sigma) - \binom{d+1}{k+1}$.

Case 2 ($\tau \subseteq \Delta_0 \cap \Delta_1$). We have $lk_\Sigma\tau$ ($d - |\tau|$)-dimensional and k-decomposable, and so by Proposition 2.4 $\bar{\tau}.lk_\Sigma\tau$ is d-dimensional and k-decomposable. Further, since $\Sigma\backslash\tau$ is d-dimensional, then $|\bar{\tau}.lk_\Sigma\tau| < |\Sigma|$. Hence again by induction Δ_0 and Δ_1, both d-faces of $\bar{\tau}.lk_\Sigma\tau$, can be joined by a simplicial path in $\bar{\tau}.lk_\Sigma\tau \subseteq \Sigma$ of length at most $f_k(\bar{\tau}.lk_\Sigma\tau) - \binom{d+1}{k+1} \leqslant f_k(\Sigma) - \binom{d+1}{k+1}$. This establishes the theorem.

Two immediate corollaries are:

COROLLARY 2.11. *Vertex decomposable complexes satisfy the Hirsch conjecture.*

COROLLARY 2.12. *k-decomposable complexes have diameters bounded above by a polynomial of degree $k+1$ in the number of vertices.*

3. Classes of vertex decomposable complexes.

3.1 *Complexes of dimension* $\leqslant 3$. The first result of this section is clear.

PROPOSITION 3.1.1. *All 0-dimensional complexes are vertex decomposable.*

For 1-dimensional complexes, we have the following.

THEOREM 3.1.2. *Let Σ be a 1-dimensional complex, i.e., a loopless graph with at least one edge and no multiple edges. Then the following are equivalent.*

(1) *Σ is connected.*
(2) *Σ is vertex decomposable.*
(3) *Σ is 1-decomposable.*

PROOF. Clearly (2) implies (3). We prove (3) implies (1) and (1) implies (2).

($3 \Rightarrow 1$): Proceed by induction on $|\Sigma| \geqslant 4$. If $|\Sigma| = 4$ then Σ is a single edge, which is connected. Otherwise let $|\Sigma| > 4$, so that Σ has a shedding simplex τ. Suppose that Σ contains two nonempty components Σ_1 and Σ_2.

Case 1 ($\tau \notin \Sigma_1 \cup \Sigma_2$). Here $\Sigma\backslash\tau$ still has components Σ_1 and Σ_2, and $|\Sigma\backslash\tau| < |\Sigma|$. But $\Sigma\backslash\tau$ is 1-decomposable, and so by induction is connected. This contradicts the existence of Σ_1 and Σ_2.

Case 2 ($\tau \in \Sigma_1$). Σ pure 1-dimensional implies that τ must be contained in an edge e of Σ_1, and so e must contain a vertex v which is not τ. But then $v \in \Sigma_1\backslash\tau$, and hence $\Sigma\backslash\tau$ still has nonempty components $\Sigma_1\backslash\tau, \Sigma_2$. But again $\Sigma\backslash\tau$ is 1-decomposable, contradicting the existence of Σ_1 and Σ_2.

The case $v \in \Sigma_2$ is handled similarly.

($1 \Rightarrow 2$): We have automatically that Σ connected implies Σ pure dimensional, since there can be no isolated points in Σ. We proceed again by induction on $|\Sigma| \geqslant 4$. If $|\Sigma| = 4$ then Σ is a single edge, and so is vertex decomposable. Otherwise, suppose $|\Sigma| > 4$. Let T be a spanning tree for Σ, and choose v_0 any terminal vertex of T. Then $\Sigma\backslash v_0$ contains $T\backslash v_0$ which is a spanning tree for $\Sigma\backslash v_0$, hence $\Sigma\backslash v_0$ is 1-dimensional and connected with $|\Sigma\backslash v_0| < |\Sigma|$. Therefore $\Sigma\backslash v_0$ is vertex decomposable, and $lk_{v_0}\Sigma$ is vertex decomposable by Theorem 3.1.1. Hence v_0 is a shedding simplex of Σ, and therefore Σ is vertex decomposable.

THEOREM 3.1.3. *2-spheres and 2-balls (simplicial complexes whose realizations are homeomorphic to S^2 or B^2) are vertex decomposable. Hence the dual complex of a simple 3-polyhedron is vertex decomposable.*

PROOF. We refer the reader to [18] for clarification of terms and also for the proofs of several facts which, by their topological nature, have been omitted. They are:

(1) Simplicial 2-spheres and 2-balls are pure dimensional.

(2) If Σ if a 2-sphere or 2-ball, and v is any vertex of Σ, then $lk_\Sigma v$ is either a 1-sphere or a 1-ball, which corresponds to a graph which is a single open or closed non-intersecting path.

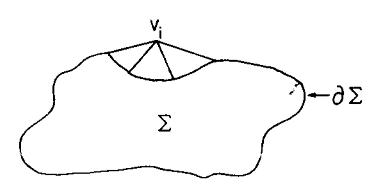
(3) If Σ is a 2-sphere, and v is any vertex in Σ, then $\Sigma\backslash v$ is a 2-ball.

(4) If Σ is a 2-ball or 2-sphere and v is any vertex in Σ, then $\Sigma\backslash v$ is a 2-ball iff $lk_\Sigma v \cap \partial\Sigma = \partial lk_\Sigma v$.

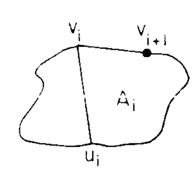
We prove the theorem by showing that every 2-sphere and every 2-ball with more than one 2-simplex contains a vertex v for which $\Sigma\backslash v$ is a 2-ball. Further, from (2) we have $lk_\Sigma v$ is connected and hence vertex decomposable by Theorem 3.1.2. Hence we have, by an induction argument, that v is a shedding vertex for Σ, and so Σ is vertex decomposable.

If Σ is a sphere, then from (3) any vertex in Σ can be removed to form a 2-ball. Assume then, that Σ is a 2-ball, so that Σ can be placed on to the plane as a graph with an unbounded region and triangular interior regions. Choose v_0 on the boundary of Σ and proceed as follows (assuming vertex v_i has been defined):

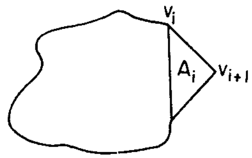
(1) If no edge of Σ containing v_i cuts entirely throught the center of Σ, then stop.



(2) Otherwise, let $\{v_i, u_i\}$ be the cutting edge, and continue clockwise around the boundary of Σ to adjacent vertex v_{i+1}. Go to 1.



First we show this procedure stops. Let A_i be that part of Σ to the "right" of $\{v_i, u_i\}$. Then A_i must contain at least one triangle (since $\{v_i, u_i\}$ cuts through the interior of Σ), and A_{i+1} is strictly contained in A_i (since u_{i+1} is to the "right" of u_i, and v_{i+1} is strictly to the right of v_i). So A_i must eventually be a single triangle, and v_{i+1} stops the procedure.



What we get from the procedure is some v for which the only vertices u adjacent to v which are on the boundary of Σ are those for which $\{v, u\}$ is also on the boundary. Hence

$$lk_\Sigma v \cap \partial\Sigma = \partial(lk_\Sigma v)$$

and so $\Sigma \backslash v$ is a 2-ball. This completes the proof of Theorem 3.1.3.

Finally we comment that by exhaustive search through the complexes in [13], it can be shown that the boundary complexes of all simplicial 4-polytopes with at most 8 vertices are vertex decomposable. We will not enumerate the decompositions here. Thus,

Corollary 3.1.4. *Dual complexes to simple polyhedra of dimension* 1, 2, *and* 3, *and to simple polytopes of dimension* 4 *with at most* 8 *facets are vertex decomposable.*

3.2 *Matroid complexes and broken circuit complexes.* A *matroid* is any nonempty collection M of subsets, called *independent sets*, of a finite set E satisfying (1) every subset of an independent set is also independent, and (2) for any subset A of E, each maximal independent subset of A has the same cardinality $r(A)$, called the *rank* of A. We have immediately that M forms a pure $(r(E) - 1)$-dimensional simplicial complex. It is also true (see [25, Chapter 4], for instance) that for any vertex v of M, the two complexes $M \backslash v$ and $lk_M v$ are both matroids on E (called the *deletion* and *contraction* matroids of M with respect to v).

We call the *circuits* of M that collection of minimal subsets of E which are not members of M. Given a particular ordering $e_1, \ldots, e_m$ of the elements of E, the *broken circuits* of M (with respect to this ordering) are those sets obtained from the circuits of M by deleting from each circuit that element of highest index. Finally, we define the *broken circuit complex* of M to be

$$B(M) = \{\sigma \subseteq E \mid \sigma \text{ contains no broken circuit}\}.$$

Brylawski [6] has studied these complexes to some length. He found that $B(M)$ is a pure $(r(E) - 1)$-dimensional complex, and that for the vertex v *of highest index* in E, $B(M) \backslash v$ and $lk_{B(M)} v$ are also broken circuit complexes.

We can derive immediately from the above discussion, Corollary 2.9, and Theorem 2.10 the following results.

Theorem 3.2.1. *Matroid complexes and broken circuit complexes are vertex decomposable.*

Corollary 3.2.2. *Matroid complexes and broken circuit complexes are shellable and satisfy the Hirsch conjecture.*

That matroid complexes satisfy the Hirsch conjecture is a well-known property of matroids. The shellability results in Corollary 3.2.2 provide affirmative answers to questions raised by Stanley [21].

For matroids we can provide a partial converse to Theorem 3.2.1. Referring to the equivalent form (ii) to Definition 2.1, we say that a *shedding order* for a vertex decomposable complex Σ is an ordering $v_1, \ldots, v_n$ of the vertices of Σ so that, defining $\Sigma_0 = \Sigma$ and $\Sigma_i = \Sigma_{i-1} \backslash v_i$, we have v_{i+1} a shedding vertex for Σ_i.

Proposition 3.2.3. *A pure dimensional complex Σ is a matroid complex if and only if any ordering of the vertices of Σ is a shedding order for Σ.*

Proof. Recall that if Σ is a matroid, then for any vertex v of Σ, $\Sigma \backslash v$ and $lk_\Sigma v$ are matroids. Hence at any stage any vertex can be chosen as the shedding vertex, and so every ordering of the vertices is a shedding order. Conversely, let every ordering of the vertices of Σ be a shedding order. Then for any set A of vertices of Σ and any ordering $v_1, \ldots, v_k$ of the remaining vertices of Σ, $\Sigma \backslash v_1 \backslash \cdots \backslash v_k$ is vertex decomposable, and hence pure dimensional. Therefore since any two faces of Σ which are maximal with respect to being contained in A are in fact maximal faces of $\Sigma \backslash v_1 \backslash \cdots \backslash v_k$, they must have the same cardinality. Thus Σ is a matroid.

3.3 *The face lattice of a cell complex.* We define a cell complex to be a finite collection $\mathcal{C}$ of convex polyhedra, called *cells* of $\mathcal{C}$, with the properties that every face of a cell is a cell and the nonempty intersection of two cells is a face of each. Each element K of $\mathcal{C}$ then has its own associated cell complex $\mathcal{C}(K)$ whose cells consist of all *proper* (not K or $\emptyset$) faces of K. Maximal faces are called *facets*, and we will call $\mathcal{C}$ a *d-dimensional* complex if *all* of its facets have the same dimension d. Two particular cell complexes which will be of interest to us are (1) the complex $\mathcal{C}(P)$ of a single polyhedron P, called a *polyhedral boundary complex*, and (2) complexes whose cells are geometric simplices, called a *simplicial cell complex*. Finally, the *face lattice complex* of $\mathcal{C}$, $L(\mathcal{C})$, is the simplicial complex whose vertices correspond to the cells of $\mathcal{C}$, and whose faces correspond to chains (totally ordered sets under inclusion) of cells of $\mathcal{C}$. If $\mathcal{C}$ is d-dimensional then $L(\mathcal{C})$ is a pure d-dimensional simplicial complex.

There is an interesting property of cell complexes which insures vertex decomposability of the associated face lattice complex. We will call a d-dimensional cell complex $\mathcal{C}$ *shellable* if $d = 0$ or, inductively, the facets of $\mathcal{C}$ can be ordered $F_1, \ldots, F_n$ so that, for $i = 2, \ldots, n$, the complex

$$S_i = \mathcal{C}(F_i) \cap \left(\bigcup_{j=1}^{i-1} \mathcal{C}(F_j) \right)$$

is $(d-1)$-dimensional and shellable. (This is a slightly more general definition than that in [5].) $F_1, \ldots, F_n$ is then called a *shelling* for $\mathcal{C}$. $\mathcal{C}$ will be called $*$*-shellable* if, in addition, the shelling above has the property that there is, inductively, a $*$-shelling of $\mathcal{C}(F_i)$ which, for $i = 2, \ldots, n$, begins with the facets of S_i. Our main result is

Theorem 3.3.1. *Let $\mathcal{C}$ be a d-dimensional cell complex. If $\mathcal{C}$ is $*$-shellable, then $L(\mathcal{C})$ is vertex decomposable.*

Proof. We prove the following stronger result.

Claim. Let $\mathcal{C}$ be strongly shellable, with $F_1, \ldots, F_n$ the $*$-shelling for $\mathcal{C}$. Set $\mathcal{C}_i = \mathcal{C}(F_i)$ (if $\dim F_i = 0$, then set $\mathcal{C}_i = \{\emptyset\}$). Then $L(\mathcal{C})$ is vertex decomposable, and the shedding order can be chosen to correspond to faces in the sets

$$\mathcal{C}_n - \left(\bigcup_{j=1}^{n-1} \mathcal{C}_j\right), \{F_n\}, \ldots, \mathcal{C}_i - \left(\bigcup_{j=1}^{i-1} \mathcal{C}_j\right), \{F_i\}, \ldots, \mathcal{C}_1, \{F_1\}$$

as ordered. (See the discussion prior to Proposition 3.2.3 for the definition of shedding order.)

We prove the claim by induction on $|\mathcal{C}|$ = the number of faces of $\mathcal{C}$. The case $\dim \mathcal{C} = 0$ is trivial and begins the induction. Let $F_1, \ldots, F_n$ be the $*$-shelling, and consider next the case where $n = 1$. Then $L(\mathcal{C}) = v_0.L(\mathcal{C}_1)$, where v_0 is the vertex corresponding to the cell F_1, and since $|\mathcal{C}_1| < |\mathcal{C}|$, then $L(\mathcal{C}_1)$ is vertex decomposable. But by the proof of Proposition 2.4, $v_0.L(\mathcal{C}_1)$ is then vertex decomposable with shedding order ending with v_0, and this establishes the case $n = 1$. Finally, consider the case $n > 1$. We have $*$-shelling $F_1, \ldots, F_{n-1}$ of $\bigcup_{i=1}^{n-1}[\mathcal{C}_i \cup \{F_i\}]$, and so, by induction on $|\bigcup_{i=1}^{n-1}[\mathcal{C}_i \cup \{F_i\}]| < |\mathcal{C}|$, $\Sigma = L(\bigcup_{i=1}^{n-1}[\mathcal{C}_i \cup \{F_i\}])$ is vertex decomposable with shedding order as required by the claim. Further, by definition $\mathcal{C}_n$ has a $*$-shelling which begins with the facets of $S_n = \mathcal{C}_n \cap (\bigcup_{i=1}^{n-1} \mathcal{C}_i)$. Hence again by induction on $|\mathcal{C}_n| < |\mathcal{C}|$, $L(\mathcal{C}_n)$ is vertex decomposable with shedding order which begins with the vertices $v_1, \ldots, v_k$ corresponding to the faces in $\mathcal{C}_n - (\bigcup_{i=1}^{n-1} \mathcal{C}_i)$. Let v_0 be the vertex corresponding to F_n. Now, $L(\mathcal{C}_n)\backslash v_1 \backslash \cdots \backslash v_j$ is pure $(d-1)$-dimensional for $j < k$, since it contains as a $(d-1)$-dimensional subcomplex $L(S_n)$, and Σ is pure d-dimensional, since it contains as a d-dimensional subcomplex $L(\mathcal{C}_1 \cup \{F_1\})$. Hence the complex

$$L(\mathcal{C})\backslash v_1 \backslash \cdots \backslash v_j = \Sigma \cup v_0.(L(\mathcal{C}_n)\backslash v_1 \cdots \backslash v_j)$$

is pure d-dimensional for $j = 1, \ldots, k$. Further, since no v_j is in Σ, then

$$lk_{L(\mathcal{C})\backslash v_1 \cdots \backslash v_{j-1}} v_j = lk_{v_0.(L(\mathcal{C}_n)\backslash v_1 \backslash \cdots \backslash v_{j-1})} v_j$$
$$= v_0.lk_{L(\mathcal{C}_n)\backslash v_1 \backslash \cdots \backslash v_{j-1}} v_j$$

which is vertex decomposable for $j = 1, \ldots, k$ by the choice of $v_1, \ldots, v_k$ and Proposition 2.4. Similarly we have

$$lk_{L(\mathcal{C})\backslash v_1 \backslash \cdots \backslash v_k} v_0 = L(\mathcal{C}_n)\backslash v_1 \backslash \cdots \backslash v_k = L(S_n),$$

which is vertex decomposable by induction, S_n being $*$-shellable. Therefore, by starting with the vertices $v_1, \ldots, v_k, v_0$ and continuing with the indicated shedding order for $\Sigma = L(\mathcal{C})\backslash v_1 \backslash \cdots \backslash v_k \backslash v_0$, we obtain a shedding order for $L(\mathcal{C})$ which satisfies the claim, and hence the theorem.

COROLLARY 3.3.2. *If $\mathcal{C}$ is a shellable simplicial (cell) complex, then $L(\mathcal{C})$ is vertex decomposable.*

PROOF. For a simplicial (cell) complex, shellings are always $*$-shellings, since the facets of simplex can be shelled in any order. The corollary then follows.

COROLLARY 3.3.3. *If $\mathcal{C}$ is a polyhedral boundary complex, then $L(\mathcal{C})$ is vertex decomposable.*

PROOF. One can make use of the technique of Brugesser and Mani [5], used to prove the shellability of boundary complexes of polytopes, to prove that any polyhedral boundary complex is in fact $*$-shellable (whether or not the underlying polyhedron is bounded).

COROLLARY 3.3.4. *The face lattice complexes of shellable simplicial (cell) complexes and of polyhedral boundary complexes are shellable and satisfy the Hirsch conjecture.*

In particular, the first barycentric subdivision of any convex polytope satisfies the Hirsch conjecture as a simplicial complex.

Björner [4] has independently considered the notion of $*$-shellability (the name is his), using it to conclude the shellability of the face lattice complex of a polytope. Recent unpublished results by Edmonds and Mandel (see *Notices Amer. Math. Soc.* 25 (1978) A-510) have extended the shellability of polytopes to the more general class of oriented matroids, and their shelling can be shown to be a $*$-shelling. Thus the results in Corollaries 3.3.3 and 3.3.4 can be extended to the face lattice complex of an oriented matroid.

3.4 *Further examples and counterexamples.* We first discuss three further examples of vertex decomposable complexes. In each case, the proof of vertex decomposability can be found elsewhere.

3.4.1 *Dual complexes of totally Leontief substitution systems.* If P is a (bounded) simple polytope of the form $\{x \geq 0 \mid Ax = b\}$ where $b \geq 0$ and A has at most one positive element in each column (see [10] and [23]), then $\Sigma^*(P)$ can be shown to be a matroid complex, hence vertex decomposable (see [18]). Since facets in $\Sigma^*(P)$ are complements of feasible bases of $Ax = b$, $x \geq 0$, it follows also that the feasible bases form the bases of a matroid.

3.4.2 *Distributive lattice complexes.* These are lattices L for which the meet and join operation satisfy the distributive law $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. The complex Σ_L, whose simplices consist of chains (totally ordered subsets) of L, can be shown to be derived from a simplex by a series of stellar subdivisions [18], and thus, by Theorem 2.7, must be vertex decomposable (and hence shellable, settling another question raised by Stanley [21]. This latter result has been since generalized by Björner [4] to finite admissible lattices, although it is not known whether these more general complexes must be vertex decomposable.).

3.4.3 *Boundary complexes of cyclic polytopes.* These simplicial polytopes (see [12, §4.7]) can be seen to be vertex decomposable [3], and thus satisfy the dual form of the Hirsch conjecture, yielding a result of Klee [14, Theorem 4.3].

We now discuss three examples of non k-decomposable complexes—a combinatorial 3-ball which is not k-decomposable for any k, a triangulation of the 27-sphere which is not vertex decomposable, and a dual complex to an unbounded 4-polyhedron which is not vertex decomposable.

3.4.4 *The Rudin unshellable ball.* This is a triangulation of the geometric 3-simplex constructed in [20] with 14 vertices and 41 3-faces which is not shellable, and therefore not k-decomposable for any k.

3.4.5 *The Walkup counterexample.* Walkup [24] has constructed a 27-sphere with 56 vertices, and more than 8,000 27-simplices, which fails to satisfy the Hirsch conjecture. It thus fails to be vertex decomposable, although it can be shown to be shellable.

3.4.6 *The Klee-Walkup counterexample.* This is a simple unbounded 4-polyhedron which fails to satisfy the Hirsch conjecture, and therefore its dual simplicial complex is not vertex decomposable. This complex is, however, 1-decomposable (see [18] for the shedding order).

One reason for interest in matroid, broken circuit and distributive lattice complexes is that these complexes can easily be shown to be constructible complexes. A d-dimensional complex Σ is *constructible* if Σ is a d-simplex, or, inductively, Σ can be written $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are d-dimensional constructible complexes and $\Sigma_1 \cap \Sigma_2$ is a $(d-1)$-dimensional constructible complex. Stanley [21, p. 57] has asked whether all constructible complexes are shellable. (Clearly the converse is true.) We note here that examples by Rudin [20] and Grunbaum (unpublished) of unshellable complexes can be shown to be constructible (we will not show the construction here).

Thus, although the constructible classes presented by Stanley are shellable, shellability is not a general property of constructible complexes.

The examples in §3, then, indicate to some extent the application and limits of k-decomposability for certain classes of simplicial complexes. There are several open questions concerning k-decomposability which are stated to end the section:

(1) Are the dual complexes of convex polytopes vertex decomposable?

(2) Are the dual complexes of convex polyhedra 1-decomposable?

(3) Are combinatorial spheres k-decomposable for some k? If so, what is the smallest such k? (The last question is that of shellability of combinatorial spheres, which has been an outstanding open question for some time.)

4. Some variations of shellability and k-decomposability. We examine briefly variations of these properties obtained by asking more of a shelling, less of a k-decomposition, or simply reinterpreting these notions in the dual setting. The most important of the new properties obtained is deep shellability, which for a simple polyhedron directly implies the diameter bounds of the Hirsch conjecture.

4.1 *Dual shellability and deep shellability.* We look first at the dual complex of a polyhedron in relation to the concept of shellability of the polyhedron itself and derive two interesting dual interpretations. One concerns the dual notion of shellability in the simplicial complex, the other the dual notion of vertex decomposability as a strong form of shelling in the original complex.

We say a collection $\mathfrak{A}$ of facets of a polyhedron P is *shellable* if the complex $\mathfrak{A} \cup [\cup\{\mathcal{C}(f) \mid f \in \mathfrak{A}\}]$ is shellable (see §3.3).

If P is a simple polyhedron, this is equivalent to saying that the facets of P can be arranged $f_1, \ldots, f_n$ so that, for $i = 2, \ldots, n$, the collection

$$\mathfrak{A}_i' = \{f_i \cap f_j \mid j < i, f_i \cap f_j \neq \emptyset\}$$

is a nonempty shellable collection of facets of f_i (since here $f_i \cap f_j \neq \emptyset$ if and only if $f_i \cap f_j$ is a facet of f_i).

To make the connection between this definition and the dual complex, we need to define the notion of the dual complex to a collection of facets of a simple polyhedron.

DEFINITION 4.1.1. Let $\mathfrak{A}$ be an arbitrary collection of facets of a d-polyhedron P. Then the *dual complex* to $\mathfrak{A}$, $\Sigma^*(\mathfrak{A})$ is the collection of subsets of the set $E = \{v_i \mid f_i \in \mathfrak{A}\}$ defined by

$$\Sigma^*(\mathfrak{A}) = \{\sigma_F = v_{i_1} \cdots v_{i_k} \mid F = f_{i_1} \cap \cdots \cap f_{i_k} \text{ is a (nonempty) face of } P\}.$$

If P is simple, then $\Sigma^*(\mathfrak{A})$ is simplicial, and the k-faces of $\Sigma^*(\mathfrak{A})$ then correspond to the $(d-k-1)$-faces of P which are contained in the interior of $\cup\mathfrak{A}$. Further, if $\mathfrak{A}$ is the collection of all facets of a simple polyhedron P, then $\Sigma^*(\mathfrak{A}) = \Sigma^*(P)$.

We begin by noting the connection between the constructions used in shellings and those used in simplicial complexes. The proof is straightforward and is omitted.

LEMMA 4.1.2. *With $\mathfrak{A}$ and $\Sigma^*(\mathfrak{A})$ as above (for P simple), let $f_i \in \mathfrak{A}$ and v_i the corresponding vertex in $\Sigma^*(\mathfrak{A})$. Then*

(1) $\Sigma^*(\mathfrak{A})\backslash v_i = \Sigma^*(\mathfrak{A}\backslash\{f_i\})$

(2) $lk_{\Sigma^*(\mathfrak{A})} v_i = \Sigma^*(\mathfrak{A}_i)$

where $\mathfrak{A}_i = \{f_i \cap f_j \mid f_j \in \mathfrak{A}, j \neq i, f_i \cap f_j \neq \emptyset\}$, taken as a collection of facets of the (simple) polyhedron f_i.

We can now obtain an equivalent definition of the shellability of $\mathfrak{A}$ in terms of $\Sigma^*(\mathfrak{A})$.

DEFINITION 4.1.3. A simplicial complex Σ is *dual shellable* if either Σ comprises a single vertex, or, inductively, there is a vertex v in Σ so that $\Sigma\backslash v$ and $lk_\Sigma v$ are both dual shellable.

Note the similarity between this definition and that of vertex decomposable complexes. This is a weaker definition, in that the dimensionality requirements are omitted.

Oddly enough, dual shellability is only a property of *proper* collections of facets of a polytope.

LEMMA 4.1.4. *If P is a simple polytope, then $\Sigma^*(P)$ is not dual shellable.*

PROOF. If $\dim P = 1$, then $\Sigma^*(P)$ comprises two vertices, and since each vertex has link $\{\emptyset\}$, neither can be a dual shelling vertex. Otherwise proceed by induction on $\dim P$, and note that for any vertex v_i in $\Sigma^*(P)$ corresponding to facet f_i of P, $lk_{\Sigma^*(P)} v_i = \Sigma^*(f_i)$, which by induction on $\dim f_i < \dim P$ is not dual shellable. Hence $\Sigma^*(P)$ cannot be dual shellable, and this proves the lemma.

For a proper collection $\mathfrak{A}$ of faces, however, dual shellability of $\Sigma^*(\mathfrak{A})$ is a characterization of shellability of $\mathfrak{A}$. Before we prove this we state a result shown by Danaraj and Klee [7, Proposition 1.2 and its proof].

LEMMA 4.1.5. *If $f_1, \ldots, f_n$ is a shelling of a proper collection of facets of a simple polytope P, then for $i = 2, \ldots, n$, U_i' comprises a proper collection of facets of f_i.*

Our main result is

PROPOSITION 4.1.6. *Let $\mathfrak{A}$ be a proper subset of facets of a simple polytope P, and $\Sigma^*(\mathfrak{A})$ its dual simplicial complex. Then $\mathfrak{A}$ is a shellable collection if and only if $\Sigma^*(\mathfrak{A})$ is dual shellable.*

PROOF. $|\mathfrak{A}| = 1$ if and only if $\Sigma^*(\mathfrak{A})$ comprises a single vertex, so we assume $|\mathfrak{A}| > 1$ and proceed by induction on $|\Sigma^*(\mathfrak{A})|$. We have from the definition that $\mathfrak{A}$ is shellable if and only if there is a facet f_n of $\mathfrak{A}$ (the last facet in the shelling) so that $\mathfrak{A}\backslash\{f_n\}$ is a shellable collection and $\mathfrak{A}_n' = \mathfrak{A}_n$ is a nonempty shellable collection of facets of f_n. Since $\mathfrak{A}$ is a proper set of facets of P, then certainly $\mathfrak{A}\backslash\{f_n\}$ is, and so by induction $\mathfrak{A}\backslash\{f_n\}$ is shellable if and only if $\Sigma^*(\mathfrak{A}\backslash\{f_n\}) = \Sigma^*(\mathfrak{A})\backslash v_n$ is dual shellable. For $\mathfrak{A}_n$ we have the following three cases:

Case 1 ($\mathfrak{A}_n = \emptyset$). Then f_n cannot be the final shelling facet, and since $\Sigma^*(\mathfrak{A}_n) = \emptyset$, it is likewise not dual shellable.

Case 2 ($\mathfrak{A}_n$ comprises all facets of f_n). Since $\mathfrak{A}$ is a proper collection of facets of P, then by Lemma 4.1.5, f_n cannot be the final shelling facet. Likewise, by Lemma 4.1.2, $\Sigma^*(\mathfrak{A}_n) = \Sigma^*(f_n)$ is not dual shellable.

Case 3 ($\mathfrak{A}_n$ is a nonempty proper collection of facets of f_n). By induction on $|\Sigma^*(\mathfrak{A}_n)| < |\Sigma(\mathfrak{A})|$, $\mathfrak{A}_n$ is shellable if and only if $\Sigma^*(\mathfrak{A}_n)$ is dual shellable.

Thus f_n is the final facet of a shelling of $\mathfrak{A}$ if and only if v_n is a dual shelling vertex, and this completes the proof of the proposition.

COROLLARY 4.1.7. *If P is an unbounded simple polyhedron, then $\Sigma^*(P)$ is dual shellable.*

PROOF. Consider the polytope $\bar{P}$ obtained from P by intersection with a closed half-space, as described in §1. Choose a shelling of the collection $\mathfrak{A}$ of facets of $\bar{P}$ so that $\bar{f}$, the "new" facet, is the final facet in the shelling. Then $\Sigma^*(\mathfrak{A}\backslash\{\bar{f}\}) = \Sigma^*(P)$, and therefore by Proposition 4.1.6, $\Sigma^*(P)$ is dual shellable.

COROLLARY 4.1.8. *If P is a simple polytope, then for all $v \in \Sigma^*(P)$, $\Sigma^*(P)\backslash v$ is dual shellable.*

PROOF. Let $v = v_f$. By [5], we can choose a shelling $f_1, \ldots, f_n$ of P with $f_n = f$. Now $f_1, \ldots, f_n = f$ is a shelling of P if and only if $f_1, \ldots, f_{n-1}$ is a shelling for the set $\mathcal{U} = \{f_1, \ldots, f_{n-1}\}$ (since the set U_n is the union of all of the facets of f, which is always a shellable collection). But by Proposition 4.1.6, this is true if and only if $\Sigma^*(\{f_1, \ldots, f_{n-1}\}) = \Sigma^*(P)\backslash v_f$ is dual shellable.

The final corollary may be of interest in testing whether a simplicial ball or sphere is the dual complex to a polyhedron.

COROLLARY 4.1.9. *If Σ is a simplicial ball or sphere, then two necessary conditions for Σ to be the dual complex to a polyhedron are*:

(1) *Σ is shellable*;

(2a) *if Σ is a ball, then Σ is dual shellable*;

(2b) *if Σ is a sphere, then for each vertex v of Σ, $\Sigma\backslash v$ is dual shellable.*

We now give the dual concept to vertex decomposability in terms of shellings.

DEFINITION 4.1.10. A collection $\mathcal{U}$ of facets of a simple d-polyhedron P is called a *deeply shellable* collection if either $d = 0$, or, inductively, $|\mathcal{U}| \geq d$ and the facets of $\mathcal{U}$ can be arranged $f_1, \ldots, f_n$ so that $f_1, \ldots, f_d$ contain a common vertex and for $i = d+1, \ldots, n$, $S_i = f_i \cap (\bigcup_{j=1}^{i-1} f_j)$ is a union of facets of f_i which themselves form a deeply shellable collection.

THEOREM 4.1.11. *A collection $\mathcal{U}$ of facets of a simple d-polyhedron P is deeply shellable if and only if $\Sigma^*(\mathcal{U})$ is vertex decomposable.*

PROOF. We use the equivalent definition of vertex decomposability (remark (i) at the end of Definition 2.1), that is, for Σ a $(d-1)$-dimensional complex, Σ is vertex decomposable if either Σ is a $(d-1)$-simplex or there exists a vertex v in Σ so that (1) $\Sigma\backslash v$ is a $(d-1)$-dimensional vertex decomposable complex, and (2) $lk_\Sigma v$ is a $(d-2)$-dimensional vertex decomposable complex. If $d = 0$ then $\Sigma^*(\mathcal{U}) = \{\emptyset\}$, which is a (-1)-simplex. Otherwise, we proceed by induction on $|\Sigma^*(\mathcal{U})|$. First note that the statement $f_1, \ldots, f_d$ contain a common vertex is equivalent to saying that $\Sigma^*(\mathcal{U})$ is $(d-1)$-dimensional, since it establishes at least one $(d-1)$-face of $\Sigma^*(\mathcal{U})$. Now, if $|\mathcal{U}| = d$ then $\mathcal{U}$ is deeply shellable if and only if all the facets of $\mathcal{U}$ contain the same vertex, or equivalently, $\Sigma^*(\mathcal{U})$ is a $(d-1)$-simplex. Otherwise, let f_i be a facet of $\mathcal{U}$ and v_i the corresponding vertex of $\Sigma^*(\mathcal{U})$. Then f_i is the last facet of a deep shelling of $\mathcal{U}$ if and only if S_i is the union of facets of f_i which is a deeply shellable collection and $\mathcal{U}\backslash\{f_i\}$ is a deeply shellable collection. By the same argument as Proposition 4.1.6, this is equivalent to the statement that $\mathcal{U}_i$ and $\mathcal{U}\backslash\{f_i\}$ are deeply shellable complexes, and by induction on $|\Sigma^*(\mathcal{U}_i)|$ and $|\Sigma^*(\mathcal{U}\backslash\{f_i\})|$, is equivalent to saying that $\Sigma^*(\mathcal{U}_i) = lk_{\Sigma^*(\mathcal{U})} v_i$ and $\Sigma^*(\mathcal{U}\backslash\{f_i\})$ are $(d-2)$- and $(d-1)$-dimensional vertex decomposable complexes, respectively. Therefore f_i is the last facet of a deep shelling of $\mathcal{U}$ if and only if v_i is a shedding vertex of $\Sigma^*(\mathcal{U})$, and so $\mathcal{U}$ is deeply shellable if and only if $\Sigma^*(\mathcal{U})$ is vertex decomposable.

COROLLARY 4.1.12. *If P is a polyhedron whose facets form a deeply shellable collection, then P satisfies the Hirsch conjecture.*

We remark again that there are no known examples of simple polytopes which are not deeply shellable, although the Klee–Walkup unbounded polyhedron in Example 3.4.6 is not deeply shellable. Further, Danaraj and Klee [7, Theorem 3.1] have established that there always exists a shelling of a polytope for which the first d facets contain a common vertex, although the sets S_i may not themselves be deeply shellable. It seems reasonable, then, if one wishes to prove the Hirsch conjecture for some class of polytopes, to attempt to find deep shellings for elements of the class.

4.2 *Weak k-decomposability.* We can define a broader class of complexes than k-decomposable complexes by deleting the condition of k-decomposability for links, while still retaining several nice properties, most notably good bounds on diameters of elements in the class.

DEFINITION 4.2.1. A d-dimensional complex Σ is *weakly k-decomposable* if it is pure dimensional and either Σ is a d-simplex, or there exists a face τ of Σ, $\dim \tau \leq k$, so that $\Sigma\backslash\tau$ is a d-dimensional and weakly k-decomposable complex.

Weak k-decomposability is a strictly weaker property than k-decomposability, in that there exist weakly vertex decomposable complexes which are not even shellable. An example is $\overline{v_1v_2v_3} \cup \overline{v_2v_3v_4} \cup \overline{v_3v_4v_5} \cup \overline{v_4v_5v_1}$ which has shedding order v_1, v_2. Of course all the examples of vertex decomposable complexes are weakly vertex decomposable, and further, Example 3.4.6, the dual complex to a polyhedron which is not vertex decomposable, is weakly vertex decomposable. Hence there is no known example of a polyhedron whose dual complex is not weakly vertex decomposable, although the property does not carry to combinatorial balls (see [18, Example 4.5.3] which is derived from a construction of Barnett [2]).

We list now a few of the properties of weakly k-decomposable complexes. By essentially the same proofs as in Propositions 2.4, 2.5, and 2.7 we have

PROPOSITION 4.2.2. *Weak k-decomposability is preserved under joins and stellar subdivisions, and weak vertex decomposability is preserved under wedging.*

With regard to diameter, by a proof similar to that of Theorem 2.10, we have

THEOREM 4.2.3. *If Σ is a d-dimensional weakly k-decomposable complex, $0 \leq k \leq d$, then*

$$\operatorname{diam} \Sigma \leq 2 f_k(\Sigma)$$

where $f_k(\Sigma)$ is the number of k-faces of Σ.

So again, the diameter of a weakly k-decomposable complex is bounded above by a polynomial of degree $k+1$ in the dimension and the number of vertices.

4.3 *Face decomposability.* If one attempts to translate the property of k-decomposability of the dual complex of a polyhedron into a property of the polyhedron itself, one encounters the problem, mentioned in §1, that the deletion operation in the complex corresponds to removal of a face in the polyhedron, which does not necessarily lead to consideration of another polyhedron. However if one broadens the class of convex sets considered to be those defined by systems of both strong and weak linear inequalities, then the translation is possible, resulting in a property called *face decomposability*. This involves the removal of a face (and all its subfaces) in such a way that the removed face (corresponding to the link) and the remaining faces (corresponding to the deletion) both contain a vertex and are themselves face decomposable. ("Containing a vertex" corresponds to "being of the right dimension.") Here the removal of a face is obtained by changing one of the weak inequalities to a strong one. By requiring that the removed face is always of dimension at least k, one has the notion of *k-face decomposability*, which implies the diameter of a d-polyhedron to be bounded by a polynomial in n (the number of facets) of degree $d-k$. There is also a weakening of this notion, corresponding to that in §4.2, which yields the same bounds. See [18, Appendix 3] for details.

References

[1] Adler, I. and Dantzig, G. (1974). Maximum Diameter of Abstract Polytopes. *Pivoting and Extensions*, M. L. Balinski, ed., Mathematical Programming Study 1, North Holland, Amsterdam, 20–40.

[2] Barnette, D. (1970). Diagrams and Schlegel Diagrams. *Combinatorial Structures and Their Applications*. Gordon and Breach, NY, 1–4.

[3] Billera, L. J. and Provan, J. S. (1979). A Decomposition Principle for Simplicial Complexes and Its Relation to Diameters and Shellings. *Ann. N.Y. Acad. Sci.* **319** 82–85.
[4] Björner, A. (1978). Shellable and Cohen–Macaulay Partially Ordered Sets. Tech. Report No. 11, Department of Mathematics, University of Stockholm, Stockholm, Sweden.
[5] Brugesser, H. and Mani, P. (1971). Shellable Decompositions of Cells and Spheres. *Math. Scand.* **29** 197–205.
[6] Brylawski, T. (1977). The Broken Circuit Complex. *Trans. Amer. Math. Soc.* **234** 417–433.
[7] Danaraj, G. and Klee, V. (1974). Shellings of Spheres and Polytopes. *Duke Math. J.* **41** 443–451.
[8] ——— and ———. (1978). Which Spheres are Shellable? *Ann. Disc. Math.* **2** 33–52.
[9] Dantzig, G. B. (1963). *Linear Programming and Extensions*. Princeton University Press, Princeton, N. J.
[10] ——— and Veinott, A. F., Jr. (1978). Discovering Hidden Totally Leontief Substitution Systems. *Math. Oper. Res.* **3** 102–103.
[11] Grinold, R. C. (1971). The Hirsch Conjecture for Leontief Substitution Systems. *SIAM J. Appl. Math.* **21** 483–485.
[12] Grünbaum, G. (1967). *Convex Polytopes*. John Wiley, London.
[13] ——— and Sreedharan, V. P. (1967). An Enumeration of Simplicial 4 Polytopes with 8 Vertices. *J. Combinatorial Theory* **2** 437–465.
[14] Klee, V. (1964). Diameters of Polyhedral Graphs. *Canad. J. Math.* **16** 602–614.
[15] ———. Paths on Polyhedra I. *SIAM J. Appl. Math.* **13** 946–956.
[16] ——— and Walkup, D. W. (1967). The *d*-step Conjecture for Polyhedra of Dimension $d < 6$. *Acta Math.* **117** 53–78.
[17] Larman, D. G. (1970). Paths on Polytopes. *Proc. London Math. Soc.* **20** 161–178.
[18] Provan, J.S. (August 1977). Decompositions, Shellings, and Diameters of Simplicial Complexes and Convex Polyhedra. Ph.D. thesis, Cornell University.
[19] ——— and Billera, L. J. (January 1979). Simplicial Complexes Associated with Convex Polyhedra I: Constructions and Combinatorial Examples. Technical Report No. 402, School of OR&IE, Cornell University.
[20] Rudin, M. (1958). An Unshellable Triangulation of a Tetrahedron. *Bull. Amer. Math. Soc.* **64** 90–98.
[21] Stanley, R. P. (1977). Cohen-Macaulay Complexes. *Higher Combinatorics*, M. Aigner, ed. Reidel Publishing Co., 51–62.
[22] Todd, M. J. (1980). The Monotonic Bounded Hirsch Conjecture is False for Dimension at Least 4. *Math. Oper. Res.* 599–601.
[23] Veinott, A. F., Jr. (1968). Extreme Points of Leontief Substitution Systems. *Linear Algebra and Appl.* **1** 181–194.
[24] Walkup, D. W. (1978). The Hirsch Conjecture Fails for Triangulated 27-Spheres. *Math Oper. Res.* **3** 224–230.
[25] Welch, D. (1976). *Matroid Theory*. Academic Press, London.

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A 3-SPHERE COUNTEREXAMPLE TO THE W_v-PATH CONJECTURE*

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A triangulation D of the 3-sphere with 16 vertices and 90 3-simplices is exhibited. The cell complex D^* dual to D has the property that each edge-path between two specified vertices visits some one of the 16 3-cells of D^* at least twice. Thus D^* is a counterexample to the W_v-path conjecture for S^3 and consequently implies a counterexample to the Hirsch conjecture for S^{11}. Previously known examples are of much larger size or dimension.

An unresolved case of the W_v-path conjecture proposed by Klee and Wolfe [4] asserts the following: Any pair of vertices of a simple polytope can be joined by an edge-path which does not revisit any facet. (A *d-polytope* P is a closed bounded convex polyhedron of dimension d, a *facet* of P is a face of dimension $d-1$, and P is *simple* if every vertex is on exactly d facets.) The W_v-path conjecture is known to be equivalent (for simple polytopes) to the following conjecture of W. M. Hirsch: The diameter of the edge-graph of a simple d-polytope with n facets is at most $n-d$. In particular, a simple non-Hirsch d-polytope is itself necessarily a simple non-W_v-path polytope, and a simple non-W_v-path d-polytope with n facets can be used to construct a simple non-Hirsch $(n-d)$-polytope with $2n-2d$ facets by the method given in [5]. By a well-known duality, both conjectures have obvious equivalent formulations in terms of simplicial polytopes (polytopes with all proper faces simplicies), dual paths (sequences of $(d-1)$-faces with adjacent members having a $(d-2)$-face in common), revisiting of vertices by dual paths, and dual diameter.

Although the W_v-path and Hirsch conjectures remain unresolved for simple and simplicial polytopes, several generalizations are known to be false. For example, the generalization to unbounded polyhedra is disproved in [5]. The simplicial forms of the W_v-path and Hirsch conjectures generalize immediately to triangulations of spheres, but Walkup [8] recently constructed a dual-non-Hirsch triangulation of S^{27} with 54 vertices. In an earlier, unpublished paper [7], Mani described a counterexample to the W_v-path conjecture for simple cell decompositions of a d-sphere. His example has low dimension, $d = 3$, but a large number of cells.

The purpose of this note is to give a simplicial, explicit, and much reduced version of Mani's counterexample. Specifically, let C be the simplicial complex on 20 vertices $a, b, \ldots, s, t$ consisting of 106 3-simplices listed in Table 1, together with their faces. Further, let D be the closed complex obtained from C by identifying the pairs $(e,k), (f,l), (g,i), (h,j)$ and deleting the degenerate 3-simplices marked with + in Table 1.

THEOREM 1. *C is a shellable, dual-non-W_v-path triangulation of S^3.*

THEOREM 2. *D is a dual-non-W_v-path triangulation of S^3 with* 16 *vertices and* 90 *3-simplices.*

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