Modules of Piecewise Polynomials and their Freeness

Louis J. Billera*
Cornell University, Ithaca, NY 14853

and

Lauren L. Rose**
Ohio State University, Columbus, OH 43210

Abstract

For a polyhedral subdivision Δ of a region in $\mathbb{R}^d$, we study the algebra $C^r(\Delta)$ of all piecewise polynomial functions on Δ that are smooth of order r. $C^r(\Delta)$ is a module over the polynomial ring $R = \mathbb{R}[x_1,\ldots,x_d]$; we are particularly interested in determining when this module is free. Necessary conditions for freeness are derived for general d and r, and characterizations are given for $d = 1, 2$ (general r) and in the simplicial case when $r = 0$ (general d). The latter draws on a relationship between $C^0(\Delta)$ and the face ring A_Δ of Stanley and Reisner.

*Partially supported by NSF Grants DMS-8703370/DMS-8896193, DMS-8903004
**Partially supported by NSF Grants DMS-8703370, DMS-8908267

1 Introduction

Let Δ be a triangulated region in $\mathbb{R}^d$. We define $C^r(\Delta)$ to be the set of piecewise polynomial functions on Δ, i.e., functions given by a polynomial on each d-simplex, which are continuously differentiable of order r. These functions are called splines, piecewise polynomials, or finite elements. For each k in $\mathbb{N}$, we define $C^r_k(\Delta)$ to be the set of F in $C^r(\Delta)$ such that for each σ in Δ, $F|_\sigma$ has degree less than or equal to k. For each k, $C^r_k(\Delta)$ is a finite dimensional vector space over $\mathbb{R}$.

Such functions have many practical applications, including the finite element method for solving partial differential equations. More recently, these functions have been studied in computational geometry and used for surface modeling and computer graphics. For these and other applications, it is useful to know the dimension of $C^r_k(\Delta)$ as an $\mathbb{R}$-vector space, and to find bases for these spaces. For a further discussion of these problems, see [1], [4], or [9].

Our approach to this problem is to study the set $C^r(\Delta)$ where Δ is a general polyhedral d-complex in $\mathbb{R}^d$. $C^r(\Delta)$ is not a finite dimensional vector space over $\mathbb{R}$, but it has a great deal more algebraic structure than the $C^r_k(\Delta)$'s. $C^r(\Delta)$ is a ring (and thus an $\mathbb{R}$-algebra) with pointwise multiplication, and becomes a module over R = $\mathbb{R}[x_1,\ldots,x_d]$ by viewing R ⊂ $C^r(\Delta)$ as the set of global polynomial functions on Δ, i.e., functions given by the same polynomial on each simplex. Our primary interest in this paper is to study the structure of $C^r(\Delta)$ as an R-module. The advantage of studying this module is to make explicit the connections between the spaces $C^r_k(\Delta)$ for various k. Except for [4], [5], [6] and [7], these connections seem to have been noted only in [30] and [31], where the focus is essentially on the 2-dimensional local case. In [5], the algebra $C^0(\Delta)$ of continuous piecewise polynomials is studied.

In this section, we derive some basic properties of the module $C^r(\Delta)$. We see that the differentiability condition on $C^r(\Delta)$ is really an algebraic one. From this we can define a local notion of differentiablity, leading to a somewhat more general class of modules, denoted $C^\mu(\Delta)$. We also allow Δ to be embedded in K^d, where K is any ordered field.

This work is mostly a study of the freeness of $C^r(\Delta)$ as an R-module. Freeness means that $C^r(\Delta)$ has a basis over R, finite in this case, which implicitly describes all elements of the module. In Section 2 we give a local criterion for freeness when Δ is a (finite) simplicial complex. This reduces our study to complexes that are stars of vertices. In Section 3 we show that we may restrict our study to hereditary complexes, which are generalizations of connected manifolds. This enables us to make use of the characterization of $C^r(\Delta)$ from Proposition 1.5 as the kernel of a map between free modules.

In general, we look for conditions on Δ, r and d so that the freeness of $C^r(\Delta)$ will be combinatorially determined, i.e. independent of the embedding of Δ in K^d. Such conditions have proved extremely useful, for example, in the study of combinatorial properties of posets and simplicial complexes (see, e.g., [3], [19], [26]-[29]). In Section 4, we show for d=2, $C^r(\Delta)$ is free if and only if Δ is a manifold. Then in Section 5, we completely characterize those simplicial Δ for which $C^0(\Delta)$ is free; these are the complexes with Cohen-Macaulay links of vertices. What is perhaps more interesting is that this result comes from relating $C^0(\Delta)$ to the face ring of Δ, A_Δ. If $\hat{\Delta}$ is the homogenization of Δ, we show that $C^0(\hat{\Delta}) \cong A_\Delta$. In Section 6, we obtain an alternate proof of this result by viewing both rings as inverse limits over the poset Δ.

The rest of the paper deals mostly with the structure of bases when $C^r(\Delta)$ is free, and Hilbert series when $C^r(\Delta)$ is a graded module. In Section 7, we describe several tests to determine freeness, given a generating set for $C^r(\Delta)$ as an R-module. In Section 8 we generalize the concept of a homogeneous basis to the non-graded case. We call such bases *reduced*. In particular, if a free basis is a Gröbner basis under an ordering of monomials which respects total degree, it will be reduced.

A combinatorial characterization of $C^r(\Delta)$ is impossible in general for $d > 2$ and $r > 0$, and in Section 9 we give an example of a 3-complex where freeness varies with the embedding.

We introduce now some preliminary notions. Let Δ be a polyhedral complex in $\mathbb{R}^d$, that is, a finite set of convex polytopes in $\mathbb{R}^d$ such that every face of a polytope in Δ is a face of the complex, and the intersection of any two elements of the complex is a face of each. (See [13] for details about polytopes and polyhedral complexes.) If Δ is any complex, the dimension of Δ is the maximum dimension of an element of Δ. We say $\Delta \subset \mathbb{R}^d$ is pure if all maximal faces are of the same dimension. A special case of a polyhedral complex is a finite simplicial complex, where each face of Δ is a simplex.

By d-complex we will mean a pure d-dimensional polyhedral complex embedded in $\mathbb{R}^d$. In this case we may think of Δ as a partition of a region in $\mathbb{R}^d$ into finitely many d-polytopes. For a d-complex Δ and $i \leq d$, we denote the set of i-dimensional faces of Δ by Δ_i, and the set of i-dimensional interior faces of Δ by Δ_i^0. Similarly, $f_i(\Delta)$ denotes the number of i-dimensional faces of Δ, and $f_i^0(\Delta)$ the number of i-dimensional interior faces of Δ.

If Δ is simplicial, we can identify $\sigma \in \Delta$ with its set of vertices. Then $\tau \cup \sigma$ will correspond to the union of the vertex sets of τ and σ. If Δ is simplicial, recall the link of σ in Δ, $\mathrm{lk}_\Delta \sigma \equiv \{\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\}$, and the star of σ in Δ, $\mathrm{st}_\Delta \sigma \equiv \{\tau \cup \tau' : \tau \in \mathrm{lk}_\Delta \sigma, \tau' \subset \sigma\}$. For arbitrary polyhedral complexes we define the star of σ in Δ by

$$\mathrm{st}_\Delta \sigma \equiv \{\tau \in \Delta : \exists\, \tau' \in \Delta \text{ such that } \tau \subset \tau' \text{ and } \sigma \subset \tau'\}.$$

In other words, $\mathrm{st}_\Delta \sigma$ is the smallest subcomplex of Δ containing all faces which contain σ. If the complex Δ is understood, we will write $\mathrm{st}\sigma$ or $\mathrm{lk}\sigma$.

For a d-complex Δ, we consider the graph $G(\Delta)$ with vertices corresponding to the elements of Δ_d and edges defined as follows: if v, v' are vertices of $G(\Delta)$ corresponding to $\sigma, \sigma' \in \Delta_d$, then $\{v, v'\}$ is an edge of $G(\Delta)$ if and only if $\sigma \cap \sigma' \in \Delta_{d-1}$. Δ is said to be strongly connected if the graph $G(\Delta)$ is connected. A connected complex Δ is said to be hereditary if for all $\sigma \in \Delta - \{\emptyset\}$, $\mathrm{st}\sigma$ is strongly connected. (If Δ is simplicial, this is equivalent to the property that for all $\sigma \in \Delta - \{\emptyset\}$, $\mathrm{lk}\sigma$ is strongly connected.) From this condition it follows that Δ itself is strongly connected.

Let Δ be a d-complex and let $R = \mathbb{R}[x_1,\ldots,x_d]$, the polynomial ring over $\mathbb{R}$ in d variables. We now define $C^r(\Delta)$ more explicitly and give an algebraic condition for smoothness.

Definition: If $r \in \mathbb{N}$ and Δ is a d-complex, then $C^r(\Delta)$ is the set of functions $F : \Delta \longrightarrow \mathbb{R}$ such that

i. For all $\sigma \in \Delta_d$, $F|_\sigma$ is given by a polynomial in $R = \mathbb{R}[x_1,\ldots,x_d]$.

ii. F is continuously differentiable of order r.

Given a point p in Δ, and $F: \Delta \longrightarrow \mathbb{R}$, let F_p denote the set $\{F_{|\sigma} \;\; \sigma \in \Delta_d \,,\, p \in \sigma\}$. Then F will be differentiable of order r at a point p in Δ if the partial derivatives up to order r of elements of F_p agree at p. Note that since σ is d-dimensional, the polynomial $F_{|\sigma}$ is uniquely determined. We write F_σ for $F_{|\sigma}$.

For $\sigma \in \Delta$, let aff σ denote the affine span of points in σ, and for $f_1,\dots,f_n$ in R, let $(f_1,\dots,f_n)$ denote the ideal that they generate. If $T \subset R$ is any set of polynomials, recall the zero set of T, $Z(T) \equiv \{p \in \mathbb{R}^d : f(p) = 0 \text{ for all } f \in T\}$, and if $X \subset \mathbb{R}^d$ is any set, the ideal of X, $I(X) \equiv \{f \in R : f(p) = 0 \text{ for all } p \in X\}$. In addition to the fact that I and Z are inclusion reversing ($X \subset Y$ implies $I(Y) \subset I(X)$, and $T \subset S$ implies $Z(S) \subset Z(T)$), we will need the following easily verified properties.

Proposition 1.1:

(a) Let A be an affine subspace of $\mathbb{R}^d$ of codimension c and let $A = H_1 \cap \cdots \cap H_c$, where H_i are hyperplanes in $\mathbb{R}^d$, and ℓ_i are affine forms such that $H_i = Z(\ell_i)$. Then $I(A) = (\ell_1,\dots,\ell_c)$.

(b) If σ is a convex polytope in $\mathbb{R}^d$ then $I(\sigma) = I(\text{aff } \sigma)$ and $Z(I(\sigma)) = \text{aff } \sigma$. ∎

Recall that for an ideal I, I^r is the ideal generated by all r-fold products of elements of I. The next proposition is proved in [6].

Proposition 1.2 (Algebraic Criterion): Let Δ be a d-complex and let $F : \Delta \longrightarrow \mathbb{R}$ be a piecewise polynomial function. Then $F \in C^r(\Delta)$ if and only if for every pair of faces σ_1, σ_2 in Δ_d, $F_{\sigma_1} - F_{\sigma_2}$ lies in $I(\sigma_1 \cap \sigma_2)^{r+1}$. ∎

Remark and Notation: If $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{d-1}$, then $I(\sigma_1 \cap \sigma_2) = I(\tau)$ is principal and generated by an affine form, which we denote ℓ_{12} or ℓ_τ. If $\sigma \in \Delta_d$, note that Proposition 1.1(b) implies $I(\sigma) = (0)$. We simplify notation by setting $\tilde{I}(\tau) = I(\tau)^{r+1}$ and $\tilde{\ell}_\tau = \ell_\tau^{r+1}$.

The following corollary can be found in [4].

Corollary 1.3: If Δ is hereditary then $F \in C^r(\Delta)$ if and only if for every pair of faces σ_1, σ_2 in Δ_d which meet in a d-1 face τ, $F_{\sigma_1} - F_{\sigma_2} \in \tilde{I}(\tau) = (\tilde{\ell}_\tau)$. ∎

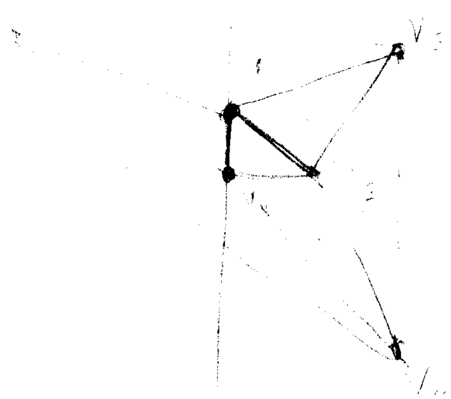
We now describe some important properties of $C^r(\Delta)$. Given an ordering $\sigma_1, \ldots, \sigma_t$ of the d-simplices of Δ, $F \in C^r(\Delta)$ can be represented as a t-tuple of polynomials in R, i.e. $F = (f_1, \ldots, f_t)$, where each f_i is just F_{σ_i}. In this way we see that $C^r(\Delta)$ is a submodule of R^t, the free R-module of rank $t = f_d(\Delta)$. The R-algebra structure of $C^r(\Delta)$ is given by pointwise multiplication.

Recall that the rank of a module M over R is the dimension of $M \otimes_R F$ as a vector space over F, the quotient field of R, i.e., $F = \mathbb{R}(x_1, \ldots, x_d)$, the field of rational functions in d variables over $\mathbb{R}$. The rank of M is also the maximal number of R-linearly independent elements in M. (See [17].) $C^r(\Delta)$ has the following structure as an R-module. See [6] for the proof.

Proposition 1.4: Let Δ be a d-complex. Then $C^r(\Delta)$ is a finitely generated torsion free R-module with rank = $f_d(\Delta)$. ■

When Δ is a hereditary d-complex, we can view $C^r(\Delta)$ as the kernel of a map between free R-modules. This representation was used in [6] to study the structure of $C^r(\Delta)$. In particular, we can use it to compute bases for $C^r(\Delta)$ when it is free [7]. The representation is constructed as follows. Let $G(\Delta)$ be the graph of Δ, as defined earlier in this section. Given an ordering of Δ_d and $\overset{\circ}{\Delta}_{d-1}$, we define the boundary matrix of Δ, $\partial(\Delta)$, to be the transpose of the node-arc incidence matrix of $G(\Delta)$, i.e.,

$$\partial(\Delta) = (a_{ij}) = \begin{cases} 1 & \text{if } v_j \text{ is the smaller vertex of } e_i \\ -1 & \text{if } v_j \text{ is the larger vertex of } e_i \\ 0 & \text{otherwise} \end{cases}$$



If Δ is hereditary, we define the matrix associated to $C^r(\Delta)$, $A(\Delta,r)$, to be

$$\left[\begin{array}{c|ccc} & \tilde{\ell}_1 & & \\ \partial & & \ddots & \\ & & & \tilde{\ell}_n \end{array}\right]$$

where $\partial = \partial(\Delta)$, $n = f^{0}_{d-1}$, and if $\sigma_i \in \Delta^{0}_{d-1}$, ℓ_i is the form defining σ_i and $\tilde{\ell}_i = \ell_i^{r+1}$. (Note that the right-hand section is a diagonal matrix.) Let $M(\Delta,r)$ denote the kernel of $A(\Delta,r)$. It is clear that $M(\Delta,r)$ does not depend on the ordering of Δ^{0}_{d-1}. The following proposition, from [6], follows easily from Corollary 1.3.

Proposition 1.5: If Δ is hereditary then for any ordering of Δ_d, $C^r(\Delta) \cong M(\Delta,r)$. ∎

Note: We can generalize the concept of $C^r(\Delta)$ in a number of ways. First, we can vary r to get a local notion of differentiability. Let $\mu : \Delta \longrightarrow \mathbb{N}$ be an order preserving function and let $\mu_\sigma \equiv \mu(\sigma)$. We can now define $C^\mu(\Delta)$ to be the set of piecewise polynomial functions F, on Δ, such that for every pair σ_1, σ_2, of d-simplices, $F_{\sigma_1}-F_{\sigma_2} \in I(\tau)^{\mu_\tau}$ where $\tau = \sigma_1 \cap \sigma_2$. Since $\mu_\tau \le \mu_\sigma$ whenever $\tau \subset \sigma$, we have that $I(\sigma)^{\mu_\sigma} \subset I(\tau)^{\mu_\tau}$. Thus $C^\mu(\Delta)$ is determined by $\{\mu_\sigma : \sigma$ is maximal with respect to lying on more than one d-face$\}$. Then elements of $C^\mu(\Delta)$ will be all piecewise polynomial functions on Δ which are differentiable of order $\mu_\sigma - 1$ on σ. If μ is the constant function equal to $r+1$, this means that $C^\mu(\Delta)$ is just $C^r(\Delta)$ as previously defined. If Δ is hereditary then $C^\mu(\Delta)$ is determined by $(\mu_\sigma : \sigma \in \Delta^{0}_{d-1})$ where Δ^{0}_{d-1} is the set of d-1 interior faces of Δ. In this paper, we will only consider $C^r(\Delta)$, but in fact all of the results are true for $C^\mu(\Delta)$ with minor modifications to the proofs.

Another generalization we can make is to assume that we are working over a general field of characteristic zero in place of $\mathbb{R}$. Polyhedra can be considered to be in an appropriate ordered subfield (say, $\mathbb{Q}$ or $\mathbb{R}$) and the relevant data extended to K (by tensor product). From now on K will denote any field of characteristic zero unless otherwise specified.

2 A Local Criterion for Freeness

In this section we give a local criterion for $C^r(\Delta)$ to be free over R, which will reduce our study of freeness to complexes that are stars of vertices. This result, however, only applies to simplicial complexes. First, we need some preliminary results.

Proposition 2.1: Let M be a finitely generated module over a Noetherian ring S. Then M is projective over S if and only if M_P is free over S_P for all maximal ideals P in S.
Proof: See [18; Chapter 2,Theorem 14]. This proves the result for finitely presented S-modules, but since S is Noetherian, all finitely generated S-modules are finitely presented. ∎

The following important result is due to Quillen and Suslin. (See [16] or [20].)
Proposition 2.2: Let $S = k[x_1,\dots,x_d]$ where k is a PID. Then projective modules over S are free. ∎

Let S be a graded ring where S_0 is a field. Let $S_+ = \oplus_{i\geq 1} S_i$ be the irrelevant maximal ideal of S. The following is Lemma 10.4 in [11].

Proposition 2.3: Let S be a Noetherian graded ring where S_0 is a field. Let M be a finitely generated graded S-module and let $P = S_+$. Then M is free over S if and only if M_P is free over S_P. ∎

The following lemma can be found in [6].
Lemma 2.4: Let $\Delta \subset K^d$ and $\tau \in \Delta$. Then $C^r(\mathrm{st}\tau)$ is a graded R-module. ∎

Recall that for a subset X of K^d, I(X) denotes the ideal of polynomials in $S = K[x_1,\dots,x_d]$ which vanish on X, and for T in S, Z(T) is the set of zeroes of T in K^d.

Lemma 2.5: Let $\sigma \subset K^d$ be a simplex and let τ_1 and τ_2 be faces of σ. Then

$$I(\tau_1) + I(\tau_2) = I(\tau_1 \cap \tau_2).$$

Proof: The lemma follows from two observations.

(1) If τ_1 and τ_2 are both faces of a simplex, then $\tau_1 \cap \tau_2$ is a face of maximal

possible dimension, i.e. $\mathrm{aff}\,(\tau_1 \cap \tau_2) = \mathrm{aff}\,\tau_1 \cap \mathrm{aff}\,\tau_2$.

(2) For an affine subspace A of K^d, I(A) is generated by affine forms (Proposition 1.1(a)). For affine subspaces A and B, it follows that I(A) + I(B) is a radical ideal, and so by [16; 1.8(c),(f)], $I(A) + I(B) = I(A \cap B)$.

Applying (1) and (2) we now have:

$$I(\tau_1 \cap \tau_2) = I(\mathrm{aff}(\tau_1 \cap \tau_2)) = I(\mathrm{aff}\,\tau_1 \cap \mathrm{aff}\,\tau_2) = I(\mathrm{aff}\,\tau_1) + I(\mathrm{aff}\,\tau_2) = I(\tau_1) + I(\tau_2). \quad \blacksquare$$

Remark: The above result is false if σ is not a simplex. For example, let σ be the trapezoid in Figure 1 below. Then $\tau_1 \cap \tau_2 = \varnothing$ but $\mathrm{aff}\,\tau_1 \cap \mathrm{aff}\,\tau_2 = \{p\}$, so $I(\tau_1) + I(\tau_2) = I(p) \neq R$ and $I(\tau_1 \cap \tau_2) = I(\varnothing) = R$.

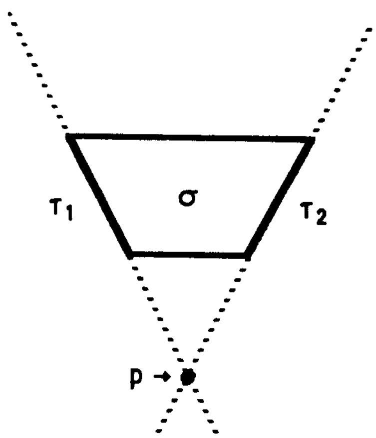


FIGURE 1

Trapezoid

Lemma 2.6: Let $\Delta \subset K^d$ and P a maximal ideal of R. Let
$S(P) = \{\tau \in \Delta : I(\tau) \subset P$, and τ is minimal with respect to this property$\}$. Then

(a) Each face $\sigma \in \Delta_d$ contains some $\tau \in S(P)$. If σ is a d-simplex, this τ is unique.

(b) Let $P = (x_1 - a_1, \ldots, x_d - a_d)$ where $v = (a_1, \ldots, a_d)$ is a vertex of Δ. Then for $\sigma \in (\mathrm{st}\,v)_d$, v is the unique face of σ in S(P).

Proof:

(a) If $\sigma \in \Delta_d$, then $I(\sigma) = (0) \subset P$. This implies there is some $\tau \subset \sigma$ (possibly $\tau = \sigma$) which is also in S(P). Suppose now that σ is a simplex. Let τ and τ' be faces of σ. Then by Lemma 2.5, $I(\tau \cap \tau') = I(\tau) + I(\tau')$. If τ and τ' are both in S(P), then this implies $I(\tau \cap \tau') \subset P$. But τ and τ' were minimal with respect to this property so $\tau = \tau'$.

(b) Suppose $\tau \in S(P)$ and $\tau \subset \sigma$ where $\sigma \in (\mathrm{st}\,v)_d$. Since $I(\tau) \subset P$ we have $Z(P) \subset Z(I(\tau))$. But $Z(P) = v$ and $Z(I(\tau)) = \mathrm{aff}\,\tau$ by Proposition 1.1(a) and (b), so this implies $v \in \mathrm{aff}\,\tau$. Let H be a supporting hyperplane for σ such that $H \cap \sigma = \tau$. Then H contains aff τ so it must contain v. Then $v \in H \cap \sigma$ so $v \in \tau$. Now by minimality, $\tau = v$. ∎

We can now prove the main result of this section.

Theorem 2.7: Let Δ be a d-complex.

(a) If $C^r(\Delta)$ is free over R then $C^r(\mathrm{st}\sigma)$ is free over R for all σ in Δ-{∅}.

(b) If Δ is simplicial then the converse is also true: If $C^r(\mathrm{st}\sigma)$ is free over R for all σ in Δ-{∅}, then $C^r(\Delta)$ is free.

Proof : Let P be a maximal ideal of R, S(P) be as in Lemma 2.6, and let $\tau \in S(P)$. Suppose that stτ has the following property:

(3) For each $\sigma \in (\mathrm{st}\tau)_d$, there is exactly one $\tau' \subset \sigma$ which is in S(P), namely $\tau' = \tau$.

Let $s = f_d(\mathrm{st}\tau)$ and $t = f_d(\Delta)$. Then $C^r(\mathrm{st}\tau) \subset R^s$ and $C^r(\Delta) \subset R^t$ as seen in Section 1. Order Δ_d so that the first s elements are in stτ. For F in $C^r(\Delta)$ we can then write $F = (f_1,\dots,f_t)$ according to this ordering. Recall that $\tilde{I}(\tau) = I(\tau)^{r+1}$ for any τ in Δ. Let $M = \{ (f_{s+1},\dots,f_t) \in R^{t-s} : f_i - f_j \in \tilde{I}(\sigma_i \cap \sigma_j)$ whenever $\sigma_i \cap \sigma_j$ contains some $\tau \in S(P)$ }.

We claim the following:

(4) As R_P-modules, $C^r(\Delta)_P \cong [C^r(\mathrm{st}\tau) \oplus M]_P$.

We prove this as follows. From above we see that both $C^r(\Delta)$ and $C^r(\mathrm{st}\tau) \oplus M$ are submodules of R^t. Since there are fewer conditions on $(f_1,\dots,f_t)$ in $C^r(\mathrm{st}\tau) \oplus M$ than in $C^r(\Delta)$, we may view $C^r(\Delta)$ as a submodule of $C^r(\mathrm{st}\tau) \oplus M$. Then by localizing at P we have the induced inclusion

$$C^r(\Delta)_P \subset [C^r(\mathrm{st}\tau) \oplus M]_P$$

since localization is exact ([2; Proposition 3.3]). We must show the reverse inclusion. Let $(f_1,\dots,f_t)/s$ be in $[C^r(\mathrm{st}\tau) \oplus M]_P$, where $s \notin P$. Let σ_i and σ_j be distinct d-simplices.

case 1: If $\sigma_i \cap \sigma_j$ contains no $\tau' \in S(P)$ then $I(\sigma_i \cap \sigma_j) \not\subset P$. (If $I(\sigma_i \cap \sigma_j) \subset P$ then the minimal τ in $\sigma_i \cap \sigma_j$ with this property would be in S(P).) Since P is prime, this implies $\tilde{I}(\sigma_i \cap \sigma_j) \not\subset P$. Thus there is some $x_{ij} \in \tilde{I}(\sigma_i \cap \sigma_j) - P$.

case 2: If $\sigma_i \cap \sigma_j$ contains some $\tau' \in S(P)$, then by (3) either both $\sigma_i, \sigma_j \in st\tau$ or both $i,j > s$. This implies $f_i - f_j \in \tilde{I}(\sigma_i \cap \sigma_j)$ by the definitions of $C^r(st\tau)$ and M. In this case, we set $x_{ij} = 1$.

Let x denote the product of all the x_{ij}'s. Then x lies in $\tilde{I}(\sigma_i \cap \sigma_j) - P$ whenever $\sigma_i \cap \sigma_j$ is as in case 1. Then $sx \notin P$ since $s \notin P$ and $x \notin P$, and

$$(f_1,\ldots,f_t)/s = x(f_1,\ldots,f_t)/xs \in C^r(\Delta)_P$$

since $x(f_1,\ldots,f_t) = (xf_1,\ldots,xf_t)$ is in $C^r(\Delta)$ by construction. This gives the reverse inclusion and proves (4).

(a) Let $v = (a_1,\ldots,a_d)$ be a vertex of Δ. Then $P = (x_1-a_1,\ldots,x_d-a_d)$ is a maximal ideal of R. Let $s = f_d(stv)$ and $t = f_d(\Delta)$. Order Δ_d so that the first s are in stv. Then by Lemma 2.6(b), stv satisfies property (3), so (4) gives:

(5) As R_P-modules, $C^r(\Delta)_P \cong [C^r(stv) \oplus M]_P$.

If $C^r(\Delta)$ is free over R then $C^r(\Delta)_P$ is free over R_P by Proposition 2.1. Then by (5), $[C^r(stv) \oplus M]_P$ is free over R_P. But $[C^r(stv) \oplus M]_P \cong C^r(stv)_P \oplus M_P$ so $C^r(stv)_P$ is projective since it is a direct summand of a free module. But finitely generated projective modules over local rings are free ([16; Corollary 3.5]) so $C^r(stv)_P$ must be free. We now apply Lemma 2.4 and Proposition 2.3 to get that $C^r(stv)$ is free over R.

Let $\sigma \in \Delta$. To see that $C^r(st\sigma)$ is free over R, let σ have vertices $v_1,\ldots,v_n$ and let $\Sigma_{i+1} = st_{\Sigma_i} v_{i+1}$, for $i \geq 1$, and $\Sigma_1 = st_\Delta v_1$. Then by the above result, $C^r(\Sigma_i)$ is free for all i. This proves (a) since $\Sigma_n = st_\Delta \sigma$.

(b) Let P be any maximal ideal in R. We claim that as R_P-modules,

$$C^r(\Delta)_P \cong [\bigoplus_{\tau \in S(P)} C^r(st\tau)]_P.$$

To prove this, let $\rho \in S(P)$. Since Δ is simplicial, Lemma 2.6(a) implies that $st\rho$ satisfies (3). Then (4) gives:

(6) As R_P-modules, $C^r(\Delta)_P \cong [C^r(st\rho) \oplus M]_P$.

By Lemma 2.6(a), we also have that $\bigoplus_{\tau \in S(P)} C^r(st\tau) \subset R^t$ since each $\sigma \in \Delta_d$ belongs to exactly one $st\tau$. But it is easy to see that

$$M = \bigoplus_{\tau \in S(P)-\{\rho\}} C^r(st\tau)$$

since they are both submodules of R^{t-s} and the conditions on $(f_1,\ldots,f_{t-s})$ are the same.

This is again because of Lemma 2.6(a). Substituting $\bigoplus_{\tau \in S(P)-\{p\}} C^r(st\tau)$ for M in (6) proves the claim.

If $C^r(st\tau)$ is free over R for all $\tau \in \Delta-\{\varnothing\}$, then $C^r(st\tau)$ is free over R for all $\tau \in S(P)$. This implies $[\bigoplus_{\tau \in S(P)} C^r(st\tau)]$ is free over R which implies $[\bigoplus_{\tau \in S(P)} C^r(st\tau)]_P$ is free over R_P by Proposition 2.1. By the claim, $C^r(\Delta)_P$ is free over R_P. But this is true for all maximal P, so Propositions 2.1 and 2.2 imply $C^r(\Delta)$ is free over R. ∎

Corollary 2.8: Let $\Delta \subset K^d$ be simplicial. Then $C^r(\Delta)$ free if and only if $C^r(stv)$ is free for all vertices v in Δ_0.

Proof: Suppose $C^r(stv)$ is free for all vertices v in Δ_0. If $\sigma \in \Delta-\{\varnothing\}$, then $st_\Delta\sigma$ is the same complex as $st_{stv}\sigma$ for any vertex v of σ. Apply (a) from the theorem to stv to get that $C^r(st_{stv}\sigma)$ and hence $C^r(st_\Delta\sigma)$ is free. Now apply (b) to get that $C^r(\Delta)$ is free.

The converse follows immediately from (a). ∎

3 A Necessary Condition for Freeness

We may restrict our study to hereditary complexes with the following result.

Theorem 3.1: Let $\Delta \subset K^d$ be a pure connected d-complex. If $C^r(\Delta)$ is free then Δ is a hereditary complex.

Proof: Suppose Δ is not hereditary. Then $st\tau$ is not strongly connected for some $\tau \in \Delta-\{\emptyset\}$. Choose τ maximal with respect to this property. Note that if $C^r(\Delta)$ is free then $C^r(st\tau)$ is free by Theorem 2.7(a). Thus we may assume without loss of generality that $\Delta = st\tau$. Since Δ is not strongly connected, $G(\Delta)$ is not connected, where $G(\Delta)$ is the graph of Δ as defined in Section 1. Let $\Sigma \subset \Delta$ be the subcomplex of Δ corresponding to one of the connected components of $G(\Delta)$. Let $t = f_d(\Delta)$ and $s = f_d(\Sigma)$. Let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be an ordering of the d-faces of Δ such that the first s are in Σ. Then for F in $C^r(\Delta)$, we may write $F = (f_1, f_2, \ldots, f_t)$ where $f_i = F_{\sigma_i}$. For any i, $\sigma_i \supset \tau$, so $I(\sigma_i) \subset I(\tau)$. Let $I = I(\tau)$. Then for any i and j, $I(\sigma_i \cap \sigma_j) \subset I$. Recall that for σ in Δ, $\tilde{I}(\sigma) = I(\sigma)^{r+1}$. Then $\tilde{I}(\sigma_i \cap \sigma_j) \subset \tilde{I}$. Define a map $\varphi\colon C^r(\Delta) \longrightarrow \tilde{I}$ by $\varphi(F) = f_1 - f_t$. Since $F \in C^r(\Delta)$, $f_1 - f_t \in \tilde{I}(\sigma_1 \cap \sigma_t)$ which is contained in $\tilde{I}$ by above. This shows that φ is well-defined. φ is clearly R-linear. Now let $\psi\colon \tilde{I} \longrightarrow C^r(\Delta)$ be defined by $\psi(f) = (f, f, \ldots, f, 0, 0, \ldots, 0)$, where the first s components are f. To show that ψ is well-defined, we must show that $\psi(f) \in C^r(\Delta)$. By Proposition 1.2, it is sufficient to check that $f = f - 0 \in \tilde{I}(\sigma_i \cap \sigma_j)$, whenever $1 \le i \le s$ and $s+1 \le j \le t$.

We claim: $\sigma_i \cap \sigma_j = \tau$ whenever $1 \le i \le s$ and $s+1 \le j \le t$. To prove the claim, first recall that $\sigma_i \cap \sigma_j \supset \tau$. Thus $st(\sigma_i \cap \sigma_j) \subset st\tau$. This implies $G(st(\sigma_i \cap \sigma_j)) \subset G(st\tau)$ since any vertex or edge of $st(\sigma_i \cap \sigma_j)$ will be a vertex or edge of $st\tau$. Since $i \le s$ and $j \ge s+1$, $\sigma_i \cap \sigma_j$ meets two connected components of $G(\Delta')$. This shows that $G(st(\sigma_i \cap \sigma_j))$ is disconnected. But τ was maximal with respect to this property, so $\sigma_i \cap \sigma_j = \tau$, proving the claim.

The claim implies $\tilde{I}(\sigma_i \cap \sigma_j) = \tilde{I}(\tau)$. But $f \in \tilde{I} = \tilde{I}(\tau)$ so $f \in \tilde{I}(\sigma_i \cap \sigma_j)$ as desired, showing that ψ is well-defined.

It is easy to verify that $\varphi\psi$ is the identity map on $\tilde{I}$, so the exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow C^r(\Delta) \xrightarrow{\varphi} \tilde{I} \longrightarrow 0$$

splits, which gives

$$(*) \qquad C^r(\Delta) \cong \tilde{I} \oplus \ker \varphi.$$

Notice that τ must have codimension $c \geq 2$. Otherwise the star of τ will consist of either one d-face or two adjacent d-faces and both of these are hereditary. Since affτ is defined by at least c affine functions, and $I(\tau) = I(\text{aff}\tau)$ by Proposition 1.1(b), $I = I(\tau)$ will be minimally generated by at least c elements. Then $\tilde{I}$ will also be minimally generated by at least c elements. In particular $\tilde{I}$ is not principal so it cannot be a free R-module. (This is because ideals in R have rank 1.) If $C^r(\Delta)$ is free, then by (*), $\tilde{I}$ is projective since it is the direct summand of a free module. Then $\tilde{I}$ must be free by Proposition 2.2. This is a contradiction, so Δ must be hereditary. ∎

Remark: If Δ is not connected, let $\Delta^1, \Delta^2, \ldots, \Delta^n$ be the connected components of Δ. Then $C^r(\Delta) = \oplus C^r(\Delta^i)$. To see this just notice that if $F \in C^r(\Delta)$, then there are no conditions on F_σ and F_τ if σ and τ lie in different components of Δ. By Proposition 2.2, this implies $C^r(\Delta)$ is free if and only if $C^r(\Delta^i)$ is free for all $1 \leq i \leq n$. Then by Theorem 3.1, we have

Corollary 3.2: If $C^r(\Delta)$ is free then Δ^i is hereditary for all $1 \leq i \leq n$. ∎

In particular, there is no loss of generality in assuming that Δ is connected.

4 Freeness when d = 1 or 2

When Δ is a 1-complex or a 2-complex freeness is completely characterized by the combinatorial properties of Δ. This means that freeness is independent of the particular embedding of Δ in K or K^2.

Theorem 4.1: If d=1, $C^r(\Delta)$ is free for any Δ and r.

Proof: R = K[x] is a PID and by Proposition 1.4, $C^r(\Delta)$ is a finitely generated torsion free R-module. By the structure theorem for modules over a PID, $C^r(\Delta)$ is free.

Remark: There are more constructive ways to prove this. (In the example below we describe an explicit basis for $C^r(\Delta)$ over R.) However, we want to note that when d=1, R is a PID and so submodules of free modules are always free, but when d>1, this is in general false.

Example: Let $\Delta \subset K$ be a connected 1-complex. Then Δ is just a line segment partitioned into $n = f_1(\Delta)$ pieces, with interior vertices (points in K) $a_1 < a_2 < \cdots < a_{n-1}$. Let $\ell_i = x - a_i$. Recall that $\tilde{\ell}_i = \ell_i^{r+1}$. Using Proposition 1.2, it is routine to verify that a free basis for $C^r(\Delta)$ as a submodule of R^n is given by

$$\{(1,1,\ldots,1),\ (0,\tilde{\ell}_1,\ldots,\tilde{\ell}_1),\ (0,0,\tilde{\ell}_2,\ldots,\tilde{\ell}_2),\ \ldots,\ (0,\ldots,0,\tilde{\ell}_{n-1})\}.$$

Lemma 4.2: If $\Delta \subset K^2$ is a connected 2-complex, then Δ is hereditary if and only if it is a manifold (with boundary).

Proof: Consider $\Delta \subset K^2$. In a plane, the star of a polygon σ is σ itself and the star of an edge consists of one polygon or two adjacent polygons (meeting in an edge). The star of a vertex v consists of a union of polygons which contain v. Since this is embedded in K^2, it will be strongly connected if and only if it is a disk. ∎

Proposition 4.3: If Δ is hereditary, then $C^r(\Delta)$ has projective dimension at most d-2.

Proof: Since Δ is hereditary, $C^r(\Delta) \cong M(\Delta,r)$ by Proposition 1.5, where $M(\Delta,r)$ is the kernel of the map $A(\Delta,r)$ as defined in Section 1. Let $M = M(\Delta,r)$, $A = A(\Delta,r)$ and

$N = \operatorname{coker} A$. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow R^{n+k} \xrightarrow{A} R^k \longrightarrow N \longrightarrow 0.$$

Then M is a 2nd syzygy module of N. By the Hilbert Syzygy Theorem [15; p.176], N has projective dimension $\leq d$, and Schanuel's Lemma [15; p.167] implies that any two 2nd syzygies are projectively equivalent so M must have projective dimension $\leq d-2$. ∎

The next theorem characterizes freeness of $C^r(\Delta)$ when Δ is a 2-complex.

Theorem 4.4: Let Δ be a 2-complex. Then $C^r(\Delta)$ is free if and only if Δ is a manifold (with boundary).

Proof: If $C^r(\Delta)$ is free then each connected component Δ^i of Δ is hereditary by Corollary 3.2. Now by Lemma 4.2, Δ^i, and hence Δ, is a manifold. Conversely, suppose Δ is a manifold. Then it is hereditary by Lemma 3.2. Since $d = 2$, Proposition 4.3 implies $C^r(\Delta)$ is projective and freeness follows from Proposition 2.2. ∎

5 Freeness when r = 0 (the Continuous Case)

In this section we introduce the face ring A_Δ, of a simplicial complex Δ. We will think of Δ as an abstract simplicial complex since A_Δ is completely determined by the combinatorial properties of Δ. We set d = dim Δ.

If $\Delta \subset K^d$ is pure of dimension d (so that $C^0(\Delta)$ is defined), we show that $C^0(\hat{\Delta})$ and A_Δ are isomorphic as graded K-algebras, where $\hat{\Delta}$ is the join of Δ and a new vertex v. Using this isomophism we can completely characterize freeness for $C^0(\Delta)$.

Definition 5.1: Let Δ be an (abstract) simplicial complex Δ with vertices $v_1,\ldots,v_n$, and let $A = k[x_1,\ldots,x_n]$ for some field k. We define I_Δ be the ideal in A generated by square-free monomials corresponding to vertex sets which are not faces of Δ, i.e.,

$$I_\Delta \equiv (x_{i_1}\cdots x_{i_m} : 1 \le i_1 < \cdots < i_m \le n \text{ and } \{v_{i_1},\ldots,v_{i_m}\} \notin \Delta).$$

Then the face ring of Δ over k, A_Δ, is defined to be the quotient A/I_Δ. (See [19] or [28] for a detailed study of the face ring, also called the Stanley-Reisner ring.)

Let S be a finitely generated graded k-algebra with Krull dimension d. A set $\{\theta_1,\ldots,\theta_d\}$ of homogeneous elements in S_+ is said to be a (homogeneous) system of parameters for S if S is a finitely generated module over $k[\theta_1,\ldots,\theta_d]$. A graded k-algebra is Cohen-Macaulay if it is a free module over $k[\theta_1,\ldots,\theta_d]$, for some (equivalently, every) system of parameters $\{\theta_1,\ldots,\theta_d\}$ for S. To see that this is equivalent to other definitions of Cohen-Macaulay, see [24; Proposition 6.8].

Definition 5.2: A simplicial complex Δ is Cohen-Macaulay over k if A_Δ is a Cohen-Macaulay ring.

A topological characterization of Cohen-Macaulay complexes due to Reisner can be found in [19].

The following result is Theorem 3.6 in [5]. It is proved there for $K = \mathbb{R}$, but the proof is the same for arbitrary fields K.

Proposition 5.3: Let Δ be a simplicial d-complex. As K-algebras,

$$C^0(\Delta) \cong A_\Delta/(\overline{y_1} + \cdots + \overline{y_n} - 1),$$

where $\overline{y_i}$ is the image of y_i in A_Δ. ∎

For $1 \le i \le n$, let $X_i : \Delta \longrightarrow K$ denote the unique piecewise (affine) linear function on Δ such that $X_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. These functions are called the Courant functions of Δ. The isomorphism from Proposition 5.3 is induced by the map $\varphi : A \longrightarrow C^0(\Delta)$ which sends y_i to X_i.

Given two simplicial complexes Δ and Δ', we define the join of Δ and Δ' by $\Delta\cdot\Delta' = \{ \sigma \cup \tau : \sigma \in \Delta, \tau \in \Delta' \}$. We may view a simplex σ as the complex consisting of all faces of σ. In this way we can define $\Delta\cdot\sigma$, the join of Δ with σ. Given Δ with vertices $\{v_1,\dots,v_n\}$, let $\hat{\Delta}$ denote the join of Δ with a new vertex v_0, and let $\hat{A} = A[y_0] \equiv K[y_0,y_1,\dots,y_n]$. The following lemma follows immediately from [19; Lemma 3].

Lemma 5.4: $\hat{A}_{\hat{\Delta}}/(\overline{y_0} - 1) \cong A_\Delta$. ∎

Theorem 5.5: Let Δ be simplicial d-complex. As graded K-algebras, $C^0(\hat{\Delta}) \cong A_\Delta$.

Proof: By Proposition 5.3, $C^0(\hat{\Delta}) \cong \hat{A}_{\hat{\Delta}}/(\overline{y_0} + \cdots + \overline{y_n} - 1)$. The map from $\hat{A}$ to $\hat{A}$ which sends y_i to y_i for $1 \le i \le n$, and y_0 to $y_0 - y_1 - \cdots - y_n$ induces an isomorphism

$$\hat{A}_{\hat{\Delta}}/(\overline{y_0} + \cdots + \overline{y_n} - 1) \cong \hat{A}_{\hat{\Delta}}/(\overline{y_0} - 1).$$

By Lemma 5.4, this gives an isomorphism $\overline{\varphi} : A_\Delta \longrightarrow C^0(\hat{\Delta})$, and $\overline{\varphi}$ is induced by the map $\varphi : A \longrightarrow R^t$ which sends y_i to X_i where $1 \le i \le n$. To see that this is a graded isomorphism, notice that for $i \ge 1$, X_i is a piecewise affine form which vanishes at the origin, so $X_i \in R^t$ will be homogeneous of degree 1. This shows φ is a homogeneous map of degree 0, so $C^0(\hat{\Delta})$, the image of the map, is a graded subalgebra of R^t, and $\overline{\varphi}$ will be an isomorphism of graded algebras. ∎

For a further discussion of homogeneous maps and graded modules, see [6] or [34]. Notice that $C^0(\hat{\Delta})$ is defined whenever $\hat{\Delta}$ is a (d+1)-complex, i.e., $\hat{\Delta}$ is pure and embedded in K^{d+1}. Then $\Delta = \mathrm{lk}_{v_0}\hat{\Delta}$ need not be embedded in K^d, but it will be pure. This gives the following corollary.

<u>Corollary 5.6</u>:

(a) As graded K-algebras, $C^0(\mathrm{stv}) \cong A_{\mathrm{lkv}}$.

(b) If P is a simplicial polytope and ∂P is the boundary complex of P, then as graded K-algebras, $C^0(\Sigma_P) \cong A_{\partial P}$, where Σ_P is the complex of convex cones (fan) generated by faces of P with respect to an origin in the interior of P. ∎

We note here that the linear forms $x_1,\ldots,x_d$ form a homogeneous system of parameters for $C^0(\Sigma_P)$, since $C^0(\Sigma_P)$ is finitely generated as a module over $R = K[x_1,\ldots,x_d]$. Thus if we form the graded algebra

$$B = C^0(\Sigma_P)/(x_1,\ldots,x_d)C^0(\Sigma_P),$$

then $B = B_0 \oplus B_1 \oplus \cdots \oplus B_d$ where $\dim_K B_i = h_i$, and $h = (h_1,\ldots,h_d)$ is the h-vector of P (see [28]). Here the elements of B_i can be interpreted as continuous piecewise polymonial functions on Σ_P of degree i (modulo functions of the form $x_1F_1 + \ldots + x_nF_n$, i.e. elements in $(x_1,\ldots,x_d)C^0(\Sigma_P)$). The element $\omega \in B_1$ crucial to the proof of necessity in the g-theorem ([27],[29]) can be interpreted as the class of a suitable piecewise linear function on Σ_P; it is reasonable to conjecture that the support function of the polar polytope P^* will always work.

From Corollary 5.6 we readily obtain the following.

<u>Corollary 5.7</u>: The following are equivalent:

i. $C^0(\mathrm{stv})$ is free

ii. $C^0(\mathrm{stv})$ is Cohen-Macaulay

iii. A_{lkv} is Cohen-Macaulay. ∎

The next theorem extends Theorem 4.7 from [5], which says that if Δ is a <u>Cohen-Macaulay</u> (simplicial) complex, then $C^0(\Delta)$ is free. We obtain a complete characterization of freeness for $C^0(\Delta)$ when Δ is a simplicial complex. We show that freeness in this case is a combinatorial property of Δ, independent of the embedding in K^d. The following lemma is straightforward.

Lemma 5.8: Let Δ be any simplicial complex. Then Δ has Cohen-Macaulay links of vertices if and only if Δ has Cohen-Macaulay stars of vertices. ∎

Theorem 5.9: Let Δ be a simplicial d-complex. Then $C^0(\Delta)$ is free over R if and only if Δ has Cohen-Macaulay links of vertices.

Proof: Suppose Δ has Cohen-Macaulay links of vertices. Then by Lemma 5.8, Δ has Cohen-Macaulay stars of vertices. By [5; Theorem 4.7], $C^0(\mathrm{st}v)$ is free for any v in Δ_0. Applying Corollary 2.8, we see that $C^0(\Delta)$ is free.

Conversely, let $v \in \Delta_0$. If $C^0(\Delta)$ is free then by Theorem 2.7, $C^0(\mathrm{st}v)$ is free. By Corollary 5.7, this implies A_{lkv} is Cohen-Macaulay. ∎

Remark: A finite simplicial complex which has Cohen-Macaulay links of vertices is also called Buchsbaum [22] or almost Cohen-Macaulay [3]. For a further discussion of these complexes see also [14] or [32].

In Section 8, we will characterize those complexes for which $C^0(\Delta)$ has a reduced basis, i.e. a free basis which is computationally useful.

6 $C^r(\Delta)$ as an Inverse Limit

For a partially ordered set (poset) P, let $\{A_\alpha\}_{\alpha\in P}$ be an inverse system of rings over P, and let $\lim_{\leftarrow} A_\alpha$ denote the inverse limit of the A_α's. (See [20; pp.49-56] for details.) As usual, let Δ be a d-complex and $R = K[x_1,\ldots,x_d]$. The faces of Δ form a poset under inclusion. We show that $C^r(\Delta)$ is the inverse limit over Δ of certain quotients of R.

Proposition 6.1: $C^r(\Delta) = \lim_{\leftarrow} R/\tilde{I}(\sigma)$, where the limit is taken over $\sigma \in \Delta$.

Proof: If $\tau \subset \sigma$, the restriction maps $\varphi_{\sigma\tau} : R/\tilde{I}(\sigma) \longrightarrow R/\tilde{I}(\tau)$ are those induced by the identity map on R. By (4.3), $\tilde{I}(\sigma) \subset \tilde{I}(\tau)$, so these maps are well defined. This shows that $\{R/\tilde{I}(\sigma), \varphi\}$ is an inverse system over Δ. By the proof of the existence of inverse limits [20; Theorem 2.22], we may view $\lim_{\leftarrow} R/\tilde{I}(\sigma)$ as a subalgebra of $\Pi_{\sigma\in\Delta} R/\tilde{I}(\sigma)$. In fact, since Δ is finite, we may view $\lim_{\leftarrow} R/\tilde{I}(\sigma)$ as a subalgebra of $\Pi\, R/\tilde{I}(\sigma)$ for σ maximal. Since Δ is pure and $\tilde{I}(\sigma) = (0)$ for $\sigma \in \Delta_d$, this means $\lim_{\leftarrow} R/\tilde{I}(\sigma)$ is actually a subalgebra of R^t, where $t = f_d(\Delta)$. Let $\sigma_1,\ldots,\sigma_t$ be an ordering of Δ_d. Then $(f_1,\ldots,f_t) \in \lim_{\leftarrow} R/\tilde{I}(\sigma)$ if and only if for all i and j, whenever $\tau \subset \sigma_i \cap \sigma_j$, $f_i \equiv f_j \bmod \tilde{I}(\tau)$; this happens if and only if for all i and j, $f_i \equiv f_j \bmod \tilde{I}(\sigma_i \cap \sigma_j)$. But by Proposition 1.2, this is equivalent to $(f_1,\ldots,f_t) \in C^r(\Delta)$. ∎

Note: Δ is not a directed set, so some of the usual properties of inverse limits will not hold here. However, since Δ is finite, we obtain some other useful properties. For example, in this case, the inverse limit commutes with tensor products.

We get immediately from the proposition (and its proof) that $C^r(\Delta)$ is an R-algebra, with pointwise multiplication and scalar multiplication as a subalgebra of R^t, where $t = f_d(\Delta)$.

We can also use the inverse limit characterization of $C^r(\Delta)$ to define $C^r(\Delta)$ for complexes which are not pure or embedded. In this case, we may still view $C^r(\Delta)$ as a subalgebra of $\Pi\, R/\tilde{I}(\sigma)$ for σ maximal. However, $C^r(\Delta)$ will only be a submodule of a free R-module if $\Delta \subset K^d$ is pure of dimension d.

Rings which arise as the inverse limit of domains over finite posets have also been studied in [10; p.108], [12], [23] and [33]. In particular, [10] gives a description of the face ring of a simplicial complex (see Section 5) as an inverse limit. In this case A_σ is just $K[y_1,\dots,y_k]$ where $\sigma = \{v_1,\dots,v_k\}$. More details about this example can be found in [23] and [33].

One benefit of the characterization of $C^r(\hat{\Delta})$ and A_Δ as inverse limits is that we can use it to give a direct proof of Theorem 5.5:

Let $R = K[x_0,\dots,x_d]$ and let R_σ denote $R/I_{\hat{\sigma}}$. For $\sigma \in \Delta$, we define a map $f_\sigma\colon A_\sigma \longrightarrow R_\sigma$ by sending y_i, where $v_i \in \sigma$, to the class in R_σ containing all linear forms in $x_0,\dots,x_d$ which have a value of 1 at the vertex v_i, and 0 at all other vertices of σ, i.e.,

$$\{\ell_{\sigma,i} : \ell_{\sigma,i}(v_i) = 1, \text{ and } \ell_{\sigma,i}(v_j) = 0 \text{ for all } v_j \in \sigma,\ j \neq i\}.$$

These maps are easily seen to be well-defined and compatible with the restriction maps on $\{A_\sigma\}$ and $\{R_\sigma\}$. To see that each f_σ is an isomorphism, note first that if ρ is a d-simplex, then $R_\rho = R = K[x_0,\dots,x_d]$ is generated by the d+1 linear forms $\{\ell_{\rho,i} : i \in \rho\}$. For any σ, choose $\rho \supset \sigma$ with $\dim\rho = d$, and note that in $R \cong K[\ell_{\rho,i} : i \in \rho]$, $I_\sigma = (\ell_{\rho,j} : j \notin \sigma)$. Then $R_\sigma \cong K[\,\overline{\ell}_{\rho,i} : i \in \rho]$, and so the map $\overline{\ell}_{\rho,i} \longrightarrow y_i$, for $v_i \in \sigma$, gives an inverse to f_σ. Since inverse limit is a functor (in this case from inverse systems to graded K-algebras), the isomorphisms $\{f_\sigma\}$ induce an isomorphism between A_Δ and $C^r(\hat{\Delta})$.

7 Freeness, Bases, and Hilbert Series

In this section, we develop some tests to determine whether $C^r(\Delta)$ is free, given a particular Δ and r. By Theorem 3.1, we may assume Δ is hereditary. The idea for the first two tests comes from Saito [21] who proved similar results for determining when derivation modules of hyperplane arrangements are free. See also [25]. The second and third tests only apply when Δ is a central complex (i.e. the star of a vertex), in which case $C^r(\Delta)$ will be a graded module.

Let $R = K[x_1,...x_d]$, and let Δ be a hereditary d-complex with $t = f_d(\Delta)$. Recall that elements of $C^r(\Delta)$ may be viewed as t-tuples of R, given an ordering of Δ_d.

Lemma 7.1: Let τ and τ' be distinct interior d-1 faces of Δ. If $\operatorname{aff}\tau = \operatorname{aff}\tau'$, then τ and τ' cannot both be faces of the same d-face σ.
Proof: If τ and τ' are faces of σ, then $\tau = \operatorname{aff}\tau \cap \sigma$ and $\tau' = \operatorname{aff}\tau' \cap \sigma$. Since τ and τ' are distinct, $\operatorname{aff}\tau$ cannot be the same hyperplane as $\operatorname{aff}\tau'$. ∎

Proposition 7.2: Let $\{F_1,...,F_t\}$ be elements of $C^r(\Delta)$. Let Q be the product of $\{(\ell_\tau)^{r+1}\}$, where τ ranges over Δ^0_{d-1}. Then $\det[F_1,...,F_t]$ is in QR.

Proof: Let $\tau \in \Delta^0_{d-1}$, say $\tau = \sigma_1 \cap \sigma_2$. Then each Let $F_i = (f_{1i},...,f_{ti})^T$. Then

$$\det[F_1,...,F_t] = \begin{vmatrix} f_{11} & \cdots & f_{1t} \\ f_{21} & \cdots & f_{2t} \\ & & \\ f_{t1} & \cdots & f_{tt} \end{vmatrix} = \begin{vmatrix} f_{11}-f_{21} & \cdots & f_{1t}-f_{2t} \\ f_{21} & \cdots & f_{2t} \\ & & \\ f_{t1} & \cdots & f_{tt} \end{vmatrix}$$

For each i, $\tilde{\ell}_\tau = (\ell_\tau)^{r+1}$ divides $f_{1i}-f_{2i}$ (by Corollary 1.3), so $\tilde{\ell}_\tau$ divides $\det[F_1,...,F_t]$. This is true for all τ. If the ℓ_τ's are distinct, then they are pairwise relatively prime, so Q must divide $\det[F_1,...,F_t]$. If the ℓ_τ's are not distinct, we can do the following. Suppose $\ell_\tau = \ell_{\tau'}$. Then if $\tau = \sigma_1 \cap \sigma_2$ and $\tau' = \sigma_i \cap \sigma_j$, we must have i and j both greater than 2, by Lemma 7.1. Assume without loss of generality that (i,j) = (3,4).

Then

$$\det [F_1,\dots,F_t] = \begin{vmatrix} f_{11}-f_{21} & \dots & f_{1t}-f_{2t} \\ f_{21} & \dots & f_{2t} \\ f_{31}-f_{41} & \dots & f_{3t}-f_{4t} \\ & & \\ f_{t1} & \dots & f_{tt} \end{vmatrix}$$

For each i, $\tilde{\ell}_\tau$ divides $f_{3i}-f_{4i}$ so $(\tilde{\ell}_\tau)^2$ divides $\det [F_1,\dots,F_t]$, etc. ∎

Theorem 7.3: $\{F_1,\dots,F_t\}$ in $C^r(\Delta)$ form a basis over R if and only if $\det [F_1,\dots,F_t]$ is in QK^*.

Proof: Suppose $\det [F_1,\dots,F_t] = Q$. Then clearly $\{F_1,\dots,F_t\}$ must be linearly independent over R. By Cramer's Rule, $QR^t \subset (F_1,\dots,F_t)$, the R-module generated by $F_1,\dots,F_t$. Let $F \in C^r(\Delta) - \{0\}$. Then $QF \in (F_1,\dots,F_t)$, so $QF = \sum_{i=1}^t r_iF_i$ for some $\{r_i\}$ in R. Then

$$\begin{aligned} r_i\, Q &= r_i\,(\det [F_1,\dots,F_t]) \\ &= \det [F_1 \dots F_{i-1}\ \ r_iF_i\ \ F_{i+1} \dots F_t] \\ &= \det [F_1 \dots F_{i-1}\ \ \textstyle\sum r_jF_j\ \ F_{i+1} \dots F_t] \\ &= \det [F_1 \dots F_{i-1}\ \ QF\ \ F_{i+1} \dots F_t] \\ &= Q\,(\det [F_1 \dots F_{i-1}\ F\ F_{i+1} \dots F_t]) \end{aligned}$$

which lies in Q^2R by Proposition 7.2. So Q divides r_i. Then $F = \sum(r_i/Q)F_i \in (F_1,\dots,F_t)$.

Conversely, suppose $\{F_1,\dots,F_t\}$ form a basis for $C^r(\Delta)$. By Proposition 7.2, $\det [F_1,\dots,F_t] = rQ$, for some $r \in R-\{0\}$. Fix τ in $\Delta_{d-1}{}^0$. Let $Q_\tau = Q/\tilde{\ell}_\tau$. Then if $\tau = \sigma_1 \cap \sigma_2$, then $(Q_\tau,Q_\tau,0,\dots,0)$ is in $C^r(\Delta)$, if the $\ell\tau$'s are distinct. Then

$$Q\, Q_\tau{}^{t-1} = \begin{vmatrix} Q_\tau & & & \\ Q_\tau & Q & & \\ 0 & 0 & Q_\tau & \\ & & & \\ 0 & 0 & \dots & Q_\tau \end{vmatrix}$$

But this determinant is equal to rsQ, for some s in $R-\{0\}$, since each column is in $C^r(\Delta)$, and so can be written as a combination of the F_i's. Thus r divides $Q_\tau{}^{t-1}$. Since τ was arbitrary, this means that if the ℓ_τ's are distinct, r must be a constant, i.e., $r \in K^*$. Suppose the ℓ_τ's are not distinct, for example, $\ell_\tau = \ell_{\tau'}$. By Lemma 7.1, we may assume that $\tau = \sigma_1 \cap \sigma_2$ and $\tau' = \sigma_3 \cap \sigma_4$. Let $\tilde{Q}_\tau = Q_\tau/\tilde{\ell}_\tau$. Then

$$Q\,\tilde{Q}_\tau^{t-1} = \begin{vmatrix} \tilde{Q}_\tau & & & & \\ \tilde{Q}_\tau & Q_\tau & & & \\ 0 & 0 & \tilde{Q}_\tau & & \\ 0 & 0 & \tilde{Q}_\tau & Q_\tau & \\ & & & & \\ 0 & 0 & \dots & & \tilde{Q}_\tau \end{vmatrix}$$

Thus r divides $\tilde{Q}_\tau^{t-1}$. Since τ was arbitrary, r must be a constant, i.e., $r \in K^*$. The proof is similar if there are more than two τ in Δ^0_{d-1} whose ideal is the same linear form. ■

Corollary 7.4: Let Δ be a central complex. A set of homogeneous elements $\{F_1,\dots,F_t\}$ in $C^r(\Delta)$ form a basis over R if and only if $\sum_{i=1}^t \deg(F_i) = f^0_{d-1}(\Delta)(r+1)$.

Proof: Since the F_i's are homogeneous, the degree of det $[F_1,\dots,F_t]$ is either 0 or the sum of the degrees of the F_i's. The proof now follows using Proposition 7.2 and Theorem 7.3. ■

Let $R = K[x_1,\dots,x_d]$ where K is a field. Let M be a finitely generated graded R-module. We will assume this means that M is graded by $\mathbb{N}$. Then $R = \oplus R_i$ where $i \geq 0$ and R_i consists of the homogeneous elements of degree i and $M = \oplus M_i$ where $R_iM_j \subset M_{i+j}$. The Hilbert function of M is defined by $H(M,i) = \dim_K M_i$. Since M is finitely generated and R is finitely generated as a K-algebra, it follows that this function takes values in $\mathbb{N}$. The Hilbert series of M, which we will denote $\mathcal{F}(M,\lambda)$, is the series $\sum H(M,i)\,\lambda^i$ in $\mathbb{N}[[\lambda]]$. $\mathcal{F}(M,\lambda)$ is known to have the following standard form. (See for example [26].)

Proposition 7.5: If M is finitely generated then $\mathcal{F}(M,\lambda)$ has the form $P(M,\lambda)/(1-\lambda)^d$ where $P(M,\lambda)$ is a polynomial in λ with integer coefficients. ■

Proposition 7.6: Let $R = K[x_1,\ldots,x_d]$ where K is a field. Suppose M is a graded R-module which is free of rank t, and let $\{G_1,G_2,\ldots,G_t\}$ be a homogeneous basis for M. Let $\deg(G_s)$ denote the degree of G_s and $P(M,\lambda) = \sum a_i\lambda^i$. Then $a_i = \#\{G_s : \deg(G_s) = i\}$. In particular, if M is free, then the coefficients of $P(M,\lambda)$ are non-negative.

Proof: $\{x_1,\ldots,x_d\}$ is clearly a regular sequence on R (R-sequence). It is also an R^t-sequence, since for $1 \le i \le d$, $R^t/x_iR^t \cong (R/x_iR)^t$. If M is free then $M \cong R^t$ so $\{x_1,\ldots,x_d\}$ is an M-sequence. (See [sta1*] for a description of regular sequences.) By [26, Cor. 3.2], we see that the Hilbert Series of M has the form

$$\mathcal{F}(M,\lambda) = \mathcal{F}(M/(x_1,\ldots,x_d)M,\lambda) \ / \ (1-\lambda)^d$$

since each x_i has degree 1. Then by the definition of $P(M,\lambda)$,

$$(*) \qquad P(M,\lambda) = \mathcal{F}(M/(x_1,\ldots,x_d)M,\lambda).$$

Let $\overline{M} = M/(x_1,\ldots,x_d)M$. Then since $P(M,\lambda) = \sum a_i\lambda^i$, (*) implies $a_i = \dim_K \overline{M}_i$. Let $\{G_1,\ldots,G_t\}$ be a homogeneous basis for M over R. Then it is easy to see that $\{\overline{G}_1,\ldots,\overline{G}_t\}$ is a basis for $\overline{M}$ as a vector space over K. Then a basis for $\overline{M}_i$ over K consists of those $\overline{G}_s$ which have degree i. Thus $a_i = \#\{G_s : \deg(G_s) = i\}$. ∎

Corollary 7.7: Let Δ be a central complex. Suppose $C^r(\Delta)$ is free with homogeneous basis $\{F_1,F_2,\ldots,F_t\}$. If $P(C^r(\Delta),\lambda) = \sum a_i\lambda^i$, then $a_i = \#\{F_s : \deg(F_s) = i\}$. In particular, if $C^r(\Delta)$ is free, then the coefficients of $P(C^r(\Delta),\lambda)$ are non-negative. ∎

8 Reduced Bases for $C^r_-(\Delta)$

Let $R = K[x_1,\ldots,x_d]$ and $\Delta \subset K^d$ be a d-complex. Recall that $C^r_k(\Delta)$ is the subspace of $C^r(\Delta)$ consisting of piecewise polynomials of degree at most k. If Δ is the star of a vertex, then $C^r(\Delta)$ is graded, and if $C^r(\Delta)$ is free over R, then given any homogeneous module basis we can write down a vector space basis for $C^r_k(\Delta)$ over K, for any k in N. We define an analagous version of a homogeneous basis in the non-graded case. Given r, we characterize those complexes for which such a basis exists. We also provide a characterization when r=0 which depends only on the combinatorial properties of Δ.

More generally, let $M \subset R^t$ be a submodule. Let $M_{(k)} \equiv \{ F \in M : \deg(F) \leq k \}$ where if $F = (f_1,\ldots,f_t)$, deg(F) denotes the maximum of the total degrees of the f_i's. The following is straightforward.

Proposition 8.1: Let $M \subset R^t$ be a graded submodule which is free over R. Let $\Lambda = \{H_1,\ldots,H_t\}$ be a homogeneous basis for M. Then

$$\Lambda_k = \{ mH_i : m \text{ is a monomial in R and } \deg mH_i \leq k \}$$

is a K-basis for $M_{(k)}$. ∎

Remark: If Δ is the star of a vertex, then $C^r(\Delta)$ is graded by Lemma 2.4 and $C^r(\Delta)_{(k)}$ is just $C^r_k(\Delta)$, so a K-basis for $C^r_k(\Delta)$ will be Λ_k.

If $C^r(\Delta)$ is free but Δ is not the star of a vertex, it is useful to know when a basis for $C^r_k(\Delta)$ over K can be constructed using a given basis for $C^r(\Delta)$ over R. With this in mind, we make the following definition.

Definition 8.2: For any module $M \subset R^t$, we say that $\Gamma = \{G_1,\ldots,G_t\}$ is a reduced basis for M if Γ is a free basis, and $F = \sum r_i G_i$ implies $\deg(r_i G_i) \leq \deg F$.

Notice that if M is graded, then any homogeneous basis for M will be reduced. A free Gröbner basis for M (if one exists) will be a reduced basis if the ordering respects the total degree ordering on R. See [6] or [7] for a description of Gröbner bases.

Lemma 8.3: Let $M \subset R^t$ be a submodule of rank n. If M contains a generating set B with n elements then M is free with basis B.

Proof: Since M is torsion free, then if S is any multiplicative set in R, the map

$$\varphi: M \longrightarrow M_S \quad \text{given by} \quad \varphi(m) = m/1$$

is injective. Thus we may assume $M \subset M_S$ as R-modules. If $B = \{b_1,\ldots,b_n\}$ generates M over R then $B/1 = \{b_1/1,\ldots,b_n/1\}$ generates $M_{(0)}$ over $R_{(0)} = F$, the quotient field of R. From [2], $M_{(0)} \cong M \otimes_R F$. Then $n = \dim_F M_{(0)}$ so B/1 is a basis for $M_{(0)}$. Since $M \subset M_{(0)}$, any non-trivial relation on B will give a non-trivial relation on B/1. Thus M is free with basis B. ∎

Proposition 8.4: Let $M \subset R^t$ be a submodule. Then $\Gamma = \{G_1,\ldots,G_t\}$ is a reduced basis if and only if for each $k \in \mathbb{N}$,

$$\Gamma_k = \{mG_j : 1 \le j \le t,\ m \text{ is a monomial in } R \text{ and } \deg mG_j \le k\}$$

is a K-basis for $M_{(k)}$.

Proof: If $F \in M_{(k)}$, then F has degree $\le k$. If Γ is a reduced basis, $F = \sum r_i G_i$, where $\deg(r_i G_i) \le k$. This shows that Γ_k generates $M_{(k)}$. The linear independence of elements from Γ_k over K follows from the linear independence of the G_i's over R and of the m's over K.

Conversely, if $F \in M$ has degree k, then F is a K-linear combination of elements of Γ_k, showing that Γ generates M (in a reduced way). That Γ is a basis follows from Lemma 8.3 ∎

Remark: If $C^r(\Delta)$ has a reduced basis, then any such basis contains the element $(c,c,\ldots,c)$, for some $c \in K - \{0\}$.

Let $\hat{\Delta}$ in K^{d+1} be the join of Δ with a new vertex v outside of affΔ. Let $\hat{R} = R[z]$. Notice that $C^r(\hat{\Delta})$ will be a graded $\hat{R}$-module.

Theorem 8.5: $C^r(\Delta)$ has a reduced basis if and only if $C^r(\hat{\Delta})$ is free over $\hat{R}$.

Proof: Let $F \in C^r(\Delta)$ be of degree n. By [6, Lemma 2.3], ${}^hF$, the homogenization of F, is in $C^r(\hat{\Delta})$ and is homogeneous of degree n. Suppose $C^r(\hat{\Delta})$ is free and let $\Lambda = \{H_1,\ldots,H_t\}$ be a homogeneous basis for $C^r(\hat{\Delta})$. Then $^hF = \sum r_i H_i$ where $r_i \in \hat{R}$, $\deg(m_i H_i) = n$. Evaluating at $z = 1$ and using well known properties of homogenization, we get

(*) $\quad F = (^hF)(1) = \sum r_i(1)H_i(1)$.

Then $\Lambda(1) = \{H_1(1),\ldots,H_t(1)\}$ generates $C^r(\Delta)$ over R, and $t = \operatorname{rank} C^r(\Delta)$, so by Lemma 8.3, $\Lambda(1)$ is a basis for $C^r(\Delta)$ over R. To see that it is reduced, note that in (*), $\deg(r_i(1)H_j(1)) \le \deg r_i H_i = n$.

Conversely, let $\Gamma = \{G_1,\ldots,G_t\}$ be a reduced basis for $C^r(\Delta)$ over R. Let $H \in C^r(\hat{\Delta})$ be homogeneous of degree k. Then $H(1) \in C^r(\Delta)$ has degree $n \le k$, and $H(1) = \sum r_i G_i$, where $\deg(r_i G_i) = n_i \le n$. By [6, Lemma 2.3], $^hG_i \in C^r(\hat{\Delta})$ for all i. Then $J = \sum z^{k-n_i} (^hr_i)(^hG_i) \in C^r(\hat{\Delta})$ and is homogeneous of degree k. Moreover, $J(1) = \sum r_i G_i = H(1)$. Since H and J are both homogeneous of degree k, they must be equal by [6, Theorem 2.6]. This shows that $^h\Gamma = \{^hG_1,\ldots,{}^hG_t\}$ generates $C^r(\hat{\Delta})$. Since the rank of $C^r(\hat{\Delta})$ is t, $C^r(\hat{\Delta})$ is free with basis $^h\Gamma$. ∎

We have the analogue of Corollary 7.4 for general Δ.

Corollary 8.7: Any reduced basis for $C^r(\Delta)$, $\Gamma = \{G_1,\ldots,G_t\}$, satisfies

$$\sum_{j=1}^{t} \deg(G_j) = f_{d-1}^{0}(\Delta)(r+1).$$

Proof: Let $\Gamma = \{G_1,\ldots,G_t\}$ be a reduced basis for $C^r(\Delta)$. In the proof of Theorem 8.5 we saw that $^h\Gamma = \{^hG_1,\ldots,{}^hG_t\}$ is a homogeneous basis for $C^r(\hat{\Delta})$. Since $\deg(G_i) = \deg(^hG_i)$ for all i, the result follows from Corollary 7.4. ∎

Let Δ be simplicial d-complex. We can characterize those complexes for which $C^0(\Delta)$ has a reduced basis, showing that when $r = 0$, the property of having a reduced basis is a combinatorial invariant.

Corollary 8.8: $C^0(\Delta)$ has a reduced basis if and only if Δ is Cohen-Macaulay.

Proof: Immediate from Theorems 5.9 and 8.5. ∎

9 Further Results and Examples

Let Δ be a hereditary d-complex.. We show that $C^r(\Delta)$ is free whenever $A(r,\Delta)$ is surjective and use this fact to prove that $C^r(\Delta)$ is free if $G(\Delta)$ is a tree. A special case of this is when $d = 1$, since the graph of any connected 1-complex in K is a tree.

We also give an example to show that Theorem 5.9 is false if $d = 3$ and $r=1$.

Theorem 9.1:

(a) If $A(\Delta,r)$ is surjective then $C^r(\Delta)$ is free.

(b) If Δ is hereditary and $G(\Delta)$ is a tree, then $C^r(\Delta)$ is free for any r.

Proof: If Δ is hereditary then $C^r(\Delta) \cong M(\Delta,r)$ by Proposition 1.5, where $M(\Delta,r)$ is the kernel of the map $A(\Delta,r)\colon R^{t+n} \longrightarrow R^n$ given by the matrix

$$\left[\begin{array}{c|ccc} & \mathfrak{I}_1 & & \\ \partial & & \ddots & \\ & & & \mathfrak{I}_n \end{array}\right].$$

Here $t = f_d(\Delta) \equiv$ the number of d-faces of Δ, and $n = \mathring{f}_{d-1}(\Delta) \equiv$ the number of interior d-1 faces of Δ. Then t = the number of vertices of $G(\Delta)$ and n = the number of edges. By [8; Corollary 7.2], ∂ has rank t-1, since Δ is hereditary. But if Δ is a tree, $t = n+1$, which means ∂ has rank n. This is the maximal rank possible, and since the entries of ∂ lie in K, it follows that A is surjective. Let $M = M(\Delta,r)$, $A = A(\Delta,r)$. Then the sequence

$$0 \longrightarrow M \longrightarrow R^{t+n} \xrightarrow{A} R^n \longrightarrow 0$$

is exact and it splits since R^n is free, giving $R^{t+n} \cong R^n \oplus M$. Then M is projective and hence free by Proposition 2.2. ∎

The following example shows that that for $d \geq 3$, the embedding of Δ is important. It also shows that freeness of $C^0(\Delta)$ does not imply freeness for $C^1(\Delta)$. Further, this gives an example where Δ is a manifold and $C^r(\Delta)$ is not free, showing Theorem 4.4 is false for general d.

Example 9.2: Let Δ be the octahedron in $\mathbb{R}^3$ with vertices at the unit vectors $\pm e_i$, triangulated by putting a vertex at the origin, as in Figure 2. $C^0(\Delta)$ is free by Theorem 5.9 In fact, $C^r(\Delta)$ is free for all r. In terms of the Courant Functions (see Section 5), a basis for $C^r(\Delta)$ is given by

$$\{1, X_3^{r+1}, X_4^{r+1}, (X_3X_4)^{r+1}, X_5^{r+1}, (X_4X_5)^{r+1}, (X_3X_5)^{r+1}, (X_3X_4X_5)^{r+1}\}.$$

Let $\tilde{\Delta}$ be the same complex again but with $v_5 = (1,1,1)$. Then $C^0(\tilde{\Delta})$ is free by Theorem 5.9. We compute that $P(C^1(\tilde{\Delta}),\lambda) = 1+\lambda^2+4\lambda^3+\lambda^5+2\lambda^6-\lambda^7$. This has a negative coefficient so by Corollary 7.7, $C^1(\tilde{\Delta})$ is not free.

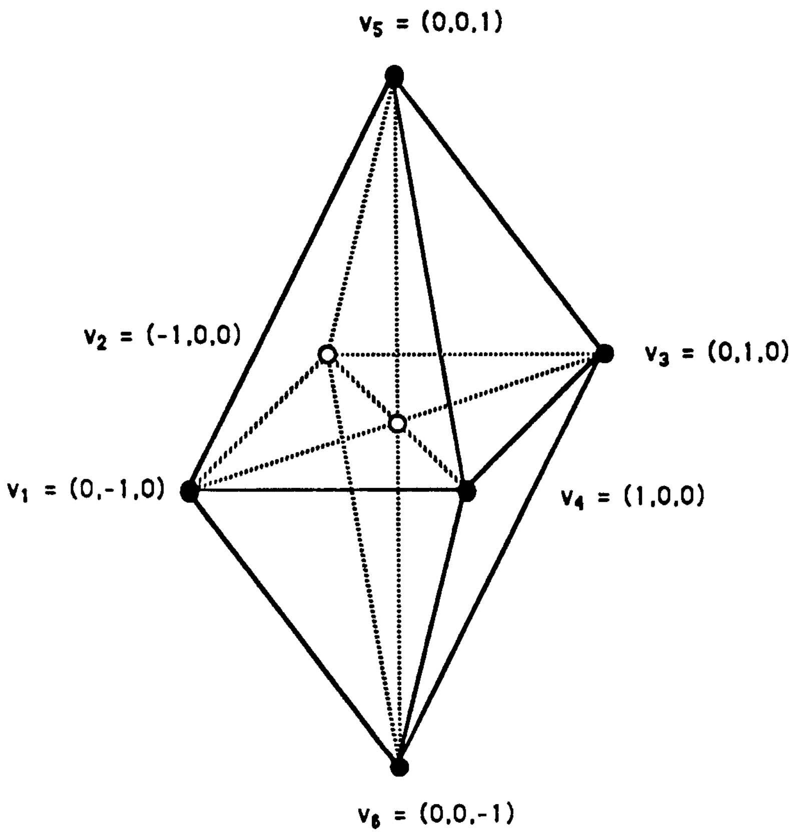


FIGURE 2
Regular octahedron

References

[1] P. Alfeld, A case study of multivariate piecewise polynomials, in Geometric Modeling: Algorithms and New Trends, G. E. Farlin, ed., SIAM, Philadelphia, 1987.

[2] M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Mass., 1969.

[3] K. Baclawski, Cohen-Macaulay ordered sets, J. Algebra 63 (1980) 226-258.

[4] L.J. Billera, Homology of smooth splines: Generic triangulations and a conjecture of Strang, Trans. Amer. Math. Soc. 310 (1988) 325-340.

[5] L.J. Billera, The algebra of continuous piecewise polynomials, Advances in Math. 76 (1989) 170-183.

[6] L.J. Billera and L.L. Rose, A dimension series for multivariate splines, Discrete and Computational Geometry (to appear).

[7] L.J. Billera and L.L. Rose, Gröbner basis methods for multivariate splines, in Mathematical Methods in Computer Aided Geometric Design, T. Lyche and L.L. Schumaker, eds., Academic Press, New York, 1989, pp. 93-104.

[8] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, MacMillan Press Ltd., London and Basingstoke, 1977.

[9] W. Dahmen and C.A. Micchelli, Recent progress in multivariate splines, in Approximation Theory IV, C. K. Chui, L. L. Schumaker and J. D. Ward, eds., Academic Press, New York, (1983) 27-122.

[10] V.I. Danilov, The geometry of toric varieties, Russian Math. Surveys 33:2 (1978) 97-154.

[11] R.M. Fossum, The Divisor Class Group of a Krull Domain, Springer-Verlag, Berlin, Heidelberg, 1973.

[12] A.V. Geramita and C.A. Weibel, On the Cohen-Macaulay and Buchsbaum property for unions of planes in affine space, Journal of Algebra 92 (1985) 413-445.

[13] B. Grünbaum, Convex Polytopes, Interscience, London, 1967.

[14] T. Hibi, Union and glueing of a family of Cohen-Macaulay partially ordered sets, Nagoya Math. J. 107 (1987) 91-119.

[15] I. Kaplansky, Fields and Rings, 2nd ed., Univ. of Chicago Press, 1972.

[16] E. Kunz, Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser, Boston, 1985.

[17] H. Matsumura, Commutative Algebra, 2nd ed., Benjamin, London, 1980.

[18] D.G. Northcott, Finite Free Resolutions, Cambridge Univ. Press, Cambridge, 1976.

[19] G.A. Reisner, Cohen-Macaulay quotients of polynomial rings, Advances in Math. 21 (1976) 30-49.

[20] J.J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[21] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Science, University of Tokyo, Sci. IA 27 (1980) 265-291.

[22] P. Schenzel, On the number of faces of simplicial complexes and the purity of Frobenius, Math. Zeit. 178 (1981) 125-142.

[23] D.E. Smith, On the Cohen-Macaulay property in commutative algebra and simplicial topology, preprint.

[24] W. Smoke, Dimension and multiplicity for graded algebras, Journal of Algebra 21 (1972) 149-173.

[25] L. Solomon and H. Terao, A formula for the characteristic polynomial of an arrangement, Advances in Math. 64 (1987) 305-325.

[26] R.P. Stanley, Hilbert functions of graded algebras, Advances in Math. 28 (1978) 57-83.

[27] R.P. Stanley, The number of faces of a simplicial convex polytope, Advances in Math. 35 (1980) 236-238.

[28] R.P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston, 1983.

[29] R.P. Stanley, The number of faces of simplicial polytopes and spheres, in Discrete Geometry and Convexity, J.E. Goodman and R. Pollack, eds., New York Acad. Sci., New York, 1985, 212-223.

[30] P.F. Stiller, Certain reflexive sheaves on $P^n_{\mathbb{C}}$ and a problem in approximation theory, Trans. Amer. Math. Soc. **279** (1983) 125-142.

[31] P.F. Stiller, Vector bundles on complex projective spaces and systems of partial differential equations I, Trans Amer. Math. Soc. **298** (1986) 537-548.

[32] J. Stückrad and W. Vogel, Buchsbaum Rings and Applications, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo, 1986.

[33] S. Yuzvinsky, Cohen-Macaulay Rings of Sections, Advances in Math. **63** (1987) 172-195.

[34] O. Zariski and P. Samuel, Commutative Algebra, Volume II, Van Nostraand, Princeton, 1960; Springer-Verlag, New York, 1975.