ADVANCES IN MATHEMATICS 76, 170–183 (1989)

The Algebra of Continuous Piecewise Polynomials

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DEDICATED TO THE MEMORY OF D. R. FULKERSON

For a triangulated d-dimensional region $\Delta \subset \mathbf{R}^d$, we consider the algebra $C^0(\Delta)$ of all continuous piecewise polynomial functions on Δ. We find generators for $C^0(\Delta)$ as an $\mathbb{R}$-algebra and use these to give an isomorphism between $C^0(\Delta)$ and a quotient of the face ring A_Δ of Δ. We then study the structure of $C^0(\Delta)$ as a module over $R = \mathbb{R}[y_1, ..., y_d]$, the polynomial ring in d indeterminates, giving generators for $C^0(\Delta)$ as an R-module. These form a free basis when Δ is a shellable complex. In general, we show that $C^0(\Delta)$ is a free R-module whenever Δ is a disk.

1. INTRODUCTION

For a finite pure d-dimensional simplicial complex Δ (rectilinearly) embedded in $\mathbb{R}^d$, we define $C^r(\Delta)$ to be the set of all C^r functions $F: \Delta \to \mathbb{R}$ such that for each maximal $\sigma \in \Delta$, $F|_\sigma$ is given by a (real) polynomial in d variables. (Here, *pure* means all maximal simplices in Δ have dimension d.) Such an F is C^r at a point $x \in \Delta$ if the value at x of any partial derivatives up to order r of $F|_\sigma$, for σ a maximal simplex containing x, is independent of the choice of σ. The elements of $C^r(\Delta)$ are called *piecewise polynomials, splines*, or *finite elements*.

The set $C^r(\Delta)$ forms a vector space over $\mathbb{R}$. Of particular interest are the subspaces $C^r_m(\Delta)$ of elements F such that each $F|_\sigma$ is of degree at most m, $m \geq 0$. In general, one would like to find the dimension and a basis for each of these, a problem originally stated in this form by Strang [13, 14]. See [3] for further discussion of the general problem as well as specific results for the case $d = 2$, especially for $r = 1$.

Additionally, $C^r(\Delta)$ forms a ring under pointwise multiplication. (If Δ is a d-pseudomanifold all of whose links are pseudomanifolds, one can use [3, Theorem 2.4] to give an easy proof of this. Otherwise, one can use a

* Supported, in part, by the National Science Foundation under Grant DMS-8403225.

0001-8708/89 $7.50

multivariate form of the product rule.) In fact, if $R = \mathbb{R}[y_1, ..., y_d]$, the polynomial ring in d indeterminates, then $C^r(\Delta)$ is an R-algebra via the diagonal embedding $R \subsetneq C^r(\Delta)$ which sends $p \in R$ to the piecewise polynomial P with $P|_\sigma = p$ for all $\sigma \in \Delta$. It is the purpose of this paper to study the R-algebra $C^0(\Delta)$ of all continuous piecewise polynomials over a d-dimensional complex $\Delta \subset \mathbb{R}^d$.

We begin in Section 2 by specifying a finite set of $\mathbb{R}$-algebra generators for $C^0(\Delta)$. In Section 3 we consider relations on these generators and use these to relate $C^0(\Delta)$ to A_Δ, the face ring of Δ. We show that as a ring, $C^0(\Delta)$ is the quotient of A_Δ by a principal ideal. As a consequence, we derive the dimensions (as vector spaces over $\mathbb{R}$) of the subspaces $C^0_m(\Delta)$. In Section 4 we consider the R-module structure of $C^0(\Delta)$, using a slight modificaton of a result of Kind and Kleinschmidt to obtain a set of R-module generators. In the case that Δ is a disk, we show that $C^0(\Delta)$ is a free R-module (i.e., has a basis over R), and for shellable Δ, we give a free basis. This basis is shown to have a triangular form that may prove useful in computation.

2. $\mathbb{R}$-ALGEBRA GENERATORS FOR $C^0(\Delta)$

We first consider the problem of identifying a set of generators for $C^0(\Delta)$ as algebra over $\mathbb{R}$. Suppose the vertices of Δ are $v_1, v_2, ..., v_n$. Let X_i be the unique piecewise linear function on Δ defined by $X_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, $i, j = 1, ..., n$. It is straightforward to see that $X_1, ..., X_n$ forms a basis for $C^0_1(\Delta)$ as a real vector space; consideration of this basis traces back at least to a 1943 paper of Courant [5]. We will refer to the functions X_i as the *Courant functions* of Δ.

The aim of this section is to show that $X_1, ..., X_n$ generate $C^0(\Delta)$ as an $\mathbb{R}$-algebra, that is, for each $F \in C^0(\Delta)$ there is a real polynomial G in n indeterminates so that as functions on Δ, $F = G(X_1, ..., X_n)$.

Suppose Δ is a d-complex in $\mathbb{R}^d$ with vertices $v_1, ..., v_n$. Let v_0 be a new vertex and consider the join of v_0 and Δ, $\hat{\Delta} = v_0 \cdot \Delta$, defined as a subset of $\mathbb{R}^{d+1}$ by considering

$$\Delta \subset \{(1; y) \mid y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1},$$

$v_0 = (0; 0) \in \mathbb{R}^d$ and maximal simplices of $\hat{\Delta}$ being the join (convex hull) of maximal simplices in Δ with the point v_0. For any $F \in C^0(\Delta)$ we associate a function $\hat{F}$ on $\hat{\Delta}$ as follows. Let $m = \deg F \equiv \max_{\sigma \in \Delta} \deg F|_\sigma$ and define

$$\hat{F}(y_0; y) = y_0^m F\left(\frac{1}{y_0} y\right), \tag{2.1}$$

when $y_0>0$, and $\hat{F}(0;y)=0$. It is easy to check that $\hat{F}\in C^0(\hat{\Delta})$, $\hat{F}(1;y)=F(y)$ and if X_i is a Courant function on Δ then $\hat{X}_i$ is the corresponding Courant function on $\hat{\Delta}$. Denote by $\hat{X}_0$ the Courant function on $\hat{\Delta}$ corresponding to v_0. We will need the following simple observation.

LEMMA 2.2. *If $C^0(\hat{\Delta})$ is generated by $\hat{X}_0, \hat{X}_1, ..., \hat{X}_n$ as an $\mathbb{R}$-algebra, then $C^0(\Delta)$ is generated by $X_1, ..., X_n$.*

Proof. This is immediate from the above discussion, since if $F\in C^0(\Delta)$ and $\hat{F}$ is given by (2.1), there is a polynomial $\hat{G}$ in $n+1$ variables so that

$$\hat{F}=\hat{G}(\hat{X}_0, \hat{X}_1, ..., \hat{X}_n).$$

But for any $y\in\Delta$,

$$\begin{aligned} F(y) &= \hat{F}(1;y) \\ &= \hat{G}(\hat{X}_0(1;y), \hat{X}_1(1;y), ..., \hat{X}_n(1;y)) \\ &= \hat{G}(0, X_1(y), ..., X_n(y)) \\ &\equiv G(X_1, ..., X_n)(y). \quad \blacksquare \end{aligned}$$

THEOREM 2.3. *For any d-dimensional complex Δ embedded in $\mathbb{R}^d$, $C^0(\Delta)$ is generated as an $\mathbb{R}$-algebra by its Courant functions $X_1, ..., X_n$.*

Proof. The proof is by induction on the number of maximal simplices in Δ. If Δ consists of a single d-simplex σ, then the functions $X_1, ..., X_{d+1}$ give the barycentric coordinates of a point $y\in\sigma$ in terms of the vertices $v_1, ..., v_{d+1}$. In particular, if v_{ij} is the jth coordinate of vertex v_i, then if $y\in\sigma$,

$$y_j=\sum_{i=1}^{d+1} v_{ij}X_i(y). \tag{2.4}$$

Thus, if $F(y)=F(y_1, ..., y_d)$ is a polynomial function on σ, then using (2.4) we can write $F(y)=G(X_1, ..., X_{d+1})(y)$ for some polynomial G.

In general, suppose $\bar{\Delta}=\Delta\cup\bar{\sigma}$, where σ is a maximal simplex in $\bar{\Delta}$ and Δ has fewer maximal simplices than $\bar{\Delta}$. Here $\bar{\sigma}$ denotes the family consisting of σ and all its subsets. (Note that we are freely moving between the notion of Δ as a family of subsets of $\{v_1, ..., v_n\}$ and Δ as triangulated region in $\mathbb{R}^d$.) By Lemma 2.2, we may assume there is a vertex v_0 in each maximal simplex of $\bar{\Delta}$. Assume $\sigma=\{v_0, ..., v_d\}$ and let

$$\bar{\sigma}\cap\Delta=\bigcup_{j=1}^{t}\bar{\tau}_j,$$

where the τ_j are the maximal simplices in $\bar{\sigma}\cap\Delta$.

By our assumption, $v_0\in\bigcap_{j=1}^{t}\tau_j$. Using an affine transformation, if necessary, we may further assume that the embedding is such that $v_0=0$ and for $i>0$, $v_i=e_i$, the ith unit vector in $\mathbb{R}^d$. Thus, for each j there is a subset $S_j\subset\{1, ..., d\}$ so that

$$\tau_j=\{y\in\operatorname{conv}\{v_0, ..., v_d\}: y_i=0 \text{ for } i\in S_j\}.$$

Now, suppose $F\in C^0(\bar{\Delta})$. By induction, we may assume $F|_\Delta=0$. Thus, as a polynomial function on σ, $F|_\sigma=0$ on τ_j for each j, i.e.,

$$F|_\sigma\in\bigcap_{j=1}^{t}\langle y_i: i\in S_j\rangle=I \tag{2.5}$$

as a polynomial in $y_1, ..., y_d$, where $I_j\equiv\langle y_i: i\in S_j\rangle$ denotes the ideal generated by that set of y_i. The I_j are "face ideals" in the terminology of Reisner [8] and correspond via the lattice anti-isomorphism of [8, Proposition 1] to the faces τ_j of σ (actually, to the faces $\tau_j\setminus\{v_0\}$ of $\sigma\setminus\{v_0\}$). By [8, Lemma 1] (and its proof), we have that $I\equiv\bigcap I_j=I_\Sigma$, where

$$\Sigma=\bigcup\bar{\tau}_j\equiv\bar{\sigma}\cap\Delta$$

and I_Σ is the ideal generated by (square-free) monomials not supported on Σ.

Thus for $y\in\sigma$

$$F(y)=\sum_{\substack{\rho\notin\Sigma\\ \rho\subset\sigma\setminus\{v_0\}}}\left(\prod_{v_i\in\rho} y_i\right)G_\rho(y_1, ..., y_d) \tag{2.6}$$

by (2.5). Define $\bar{F}\in\mathbb{R}[X_1, ..., X_n]$ by

$$\bar{F}=\sum_{\substack{\rho\notin\Sigma\\ \rho\subset\sigma\setminus\{v_0\}}}\left(\prod_{v_i\in\rho} X_i\right)G_\rho(X_1, ..., X_d). \tag{2.7}$$

By (2.6), $\bar{F}|_\sigma=F|_\sigma$. To complete the proof, we must show $\bar{F}|_\Delta=0$, and so $\bar{F}=F$ on $\bar{\Delta}$. If $\bar{F}|_\Delta\neq 0$, there must be a $\tau\in\Delta$ with $\bar{F}|_\tau\neq 0$. Thus by (2.7), $\tau\supseteq\rho$ for some $\rho\notin\Sigma$, $\rho\subset\sigma$. But then $\rho\in\Delta\cap\bar{\sigma}=\Sigma$, which is impossible. $\blacksquare$

3. RELATIONS AND THE FACE RING

We consider now relations on the generators of $C^0(\Delta)=\mathbb{R}[X_1, ..., X_n]$ and use these to relate $C^0(\Delta)$ to the face ring of Δ of Stanley [9] and Reisner [8].

For a simplicial complex Δ with vertices $v_1, v_2, ..., v_n$, the *face ring* of Δ (over $\mathbb{R}$) is the ring

$$A_\Delta = \mathbb{R}[x_1, ..., x_n]/I_\Delta, \tag{3.1}$$

where $\mathbb{R}[x_1, ..., x_n]$ is the polynomial ring in n indeterminates and, as in Section 2, I_Δ is the ideal generated by square-free monomials not supported by faces of Δ, i.e.,

$$I_\Delta = \langle x_{i_1} \cdots x_{i_k} : \{v_{i_1}, ..., v_{i_k}\} \notin \Delta \rangle. \tag{3.2}$$

A_Δ has proved to be enormously useful in dealing with questions of enumeration in certain complexes [9, 11]. Here, perhaps for the first time, it will be used to obtain structural as well as enumerative consequences, in this case, with regard to the C^0 piecewise polynomials on Δ.

To see the connection between A_Δ and $C^0(\Delta)$, note first that the Courant functions $X_1, ..., X_n$ satisfy all the defining relations of A_Δ given in (3.2). This is due to the fact that a product $X_{i_1} \cdots X_{i_k}$ is not identically zero on Δ if and only if there is a $\sigma \in \Delta$ such that none of the X_{i_j} is identically zero on σ. But such a σ must have $v_{i_1}, ..., v_{i_k}$ among its vertices, showing $X_{i_1} \cdots X_{i_k} \neq 0$ as a function on Δ if and only if $\{v_{i_1}, ..., v_{i_k}\} \in \Delta$.

By Theorem 2.3, there is a surjective $\mathbb{R}$-algebra homomorphism

$$\mathbb{R}[x_1, ..., _n] \to C^0(\Delta) \tag{3.3}$$

defined by sending x_i onto the Courant function X_i. The discussion above shows that the map (3.3) induces a well-defined surjection

$$A_\Delta \to C^0(\Delta) \tag{3.4}$$

with $\bar{x}_i$ going to X_i, $\bar{x}_i$ being the image of x_i under the canonical surjection $\mathbb{R}[x_1, ..., x_n] \to A_\Delta$. Thus as $\mathbb{R}$-algebras,

$$C^0(\Delta) \cong A_\Delta/K, \tag{3.5}$$

where the K is the kernel of the map (3.4). We wish to describe the ideal K.

Note that, in addition to the relations in (3.2), there is at least one further relation on the Courant functions $X_1, ..., X_n$. Since the function $X_1 + \cdots + X_n$ takes the value 1 identically on Δ, we have $\bar{x}_1 + \cdots + \bar{x}_n - 1$ is an element of the ideal K. We show, in fact, that it generates K as an ideal of A_Δ.

THEOREM 3.6. *As $\mathbb{R}$-algebras*

$$C^0(\Delta) \cong A_\Delta/\langle \bar{x}_1 + \cdots + \bar{x}_n - 1 \rangle.$$

Proof. We proceed by induction on the number of maximal simplices in Δ.

Suppose Δ consists of a single d-simplex σ with vertices $v_1, ..., v_{d+1}$. In this case $A_\Delta = \mathbb{R}[x_1, ..., x_{d+1}]$ and $C^0(\Delta) \cong \mathbb{R}[x_1, ..., x_d]$. Thus we must show there is an isomorphism

$$\mathbb{R}[x_1, ..., x_d] \cong \mathbb{R}[x_1, ..., x_{d+1}]/\langle x_1 + \cdots + x_{d+1} - 1 \rangle.$$

Let $S = \mathbb{R}[x_1, ..., x_d]$ and consider the surjection

$$S[x_{d+1}] \to S$$

defined by $x_{d+1} \mapsto a = 1 - x_1 - \cdots - x_d$ (this is the map (3.4) in this case). It is easy to see (by [15, Corollary 1, p. 31], for example) that the kernel of this map is the ideal generated by $x_{d+1} - a$.

In general, an element $a \in A_\Delta$ may be represented as a polynomial $p(\bar{x}_1, ..., \bar{x}_n)$, and the image of a under the map (3.4) by $p(X_1, ..., X_n)$. To prove the theorem, we must show that if $a = p(\bar{x}_1, ..., \bar{x}_n)$ and

$$\bar{a} = p(X_1, ..., X_n) = 0 \tag{3.7}$$

in $C^0(\Delta)$, then there is some $b \in A_\Delta$ so that

$$a = (\bar{x}_1 + \cdots + \bar{x}_n - 1)b \tag{3.8}$$

in A_Δ.

Suppose now that $\bar{\Delta} = \Delta \cup \bar{\sigma}$, where σ is a maximal simplex in $\bar{\Delta}$ and Δ has fewer maximal simplices than $\bar{\Delta}$. Let $\rho_1, ..., \rho_s$ be all the minimal faces of σ that are not in Δ. If $v_1, ..., v_n$ are the vertices of $\bar{\Delta}$, suppose $v_1, ..., v_k$, $k \leqslant n$, are in Δ and $v_{k+1}, ..., v_n$ are in $\bar{\sigma}\backslash\Delta$. In particular, the latter set of vertices is among the ρ_i. We can view $A_\Delta \cong A_{\bar{\Delta}}/J$, where J is the ideal generated by the square-free monomials $m_1, ..., m_s$ corresponding to $\rho_1, ..., \rho_s$. With this interpretation, $\bar{x}_{k+1}, ..., \bar{x}_n$ exist in A_Δ and are all equal to 0.

If $a = p(\bar{x}_1, ..., \bar{x}_n) \in A_{\bar{\Delta}}$ and (3.7) holds in $C^0(\bar{\Delta})$ then

$$p(X_1, ..., X_k, 0, ..., 0) = 0 \tag{3.9}$$

in $C^0(\Delta)$. By induction, (3.8) holds in A_Δ, that is, there is a $b \in A_\Delta$, so that

$$p(\bar{x}_1, ..., \bar{x}_n) = (\bar{x}_1 + \cdots \bar{x}_n - 1)b \tag{3.10}$$

in A_Δ (where we can list all n $\bar{x}_i$'s by the comment above). Lifting these elements via the canonical surjection $A_{\bar{\Delta}} \to A_\Delta$, we have that

$$p(\bar{x}_1, ..., \bar{x}_n) = (\bar{x}_1 + \cdots + \bar{x}_n - 1)b + \sum \alpha_i m_i \tag{3.11}$$

holds in $A_{\bar{\Delta}}$, for some $\alpha_i \in A_{\bar{\Delta}}$

...ce of the simplices ρ_i, they can only be faces of other faces of ...se they would have to be in Δ). Thus, we can assume that the α_i ...) are all elements of A_σ, the face ring of $\bar{\sigma}$ (otherwise $\alpha_i m_i = 0$), as ... all the m_i. So we have

$$\sum \alpha_i m_i = q(\bar{x}_l, ..., \bar{x}_n) \tag{3.12}$$

in A_σ and $q(X_l, ..., X_n) = 0$ in $C^0(\bar{\sigma})$ by (3.7) and (3.11), where $v_l, ..., v_{k+1}, ..., v_n$ are all the vertices of σ. By induction again, there is a b' in A_σ so that

$$\sum \alpha_i m_i = (\bar{x}_l + \cdots + \bar{x}_n - 1)b' \tag{3.13}$$

in A_σ.

Now A_σ is just the polynomial ring in $x_l, ..., x_n$, and each monomial appearing in the expression on the left side of (3.13) is divisible by one of the m_i, so the same must hold on the right side. This can only happen if $b' = \sum \beta_i m_i$ for some $\beta_i \in A_\sigma$. As before, we can view $A_\sigma = A_\Delta/\langle \bar{x}_1, ..., \bar{x}_{l-1}\rangle$ and lift (3.13) to A_Δ. Now (3.13) lifts to

$$\sum \alpha_i m_i = (\bar{x}_1 + \cdots + \bar{x}_n - 1)\sum \beta_i m_i + \sum_{j<l} \gamma_j \bar{x}_j \tag{3.14}$$

in A_Δ. Finally, A_Δ inherits the finest grading of the polynomial ring (by monomials), and so we can conclude that each monomial appearing in the unique representation of the term $\sum \gamma_j \bar{x}_j$ in (3.14) must be a multiple of one of the m_i as well as of one of the $\bar{x}_j$, $j < l$. Again, by the choice of ρ_i, we have $\bar{x}_j m_i = 0$ if $j < l$, and so $\sum \gamma_j \bar{x}_j = 0$ in A_Δ. Combining this, (3.14), and (3.11), we obtain the desired conclusion. ∎

In particular, since $\bar{x}_1 + \cdots + \bar{x}_n - 1$ is not a zero divisor and the Krull dimension of A_Δ is $d+1$ [9], we get that the Krull dimension of $C^0(\Delta)$ is d.

As a first application of Theorem 3.6, we compute the dimensions of the subspaces $C^0_m(\Delta)$ of continuous piecewise polynomials of degree at most m. Since by (3.2) the ideal I_Δ is homogeneous in the usual grading of $\mathbb{R}[x_1, ..., x_n]$ by total degree, A_Δ is a graded ring. We denote by A_m the subspace of all homogeneous elements of degree m (images under the map $x_i \mapsto \bar{x}_i$ of all degree m homogeneous polynomials) and let

$$A^{(m)} = A_0 \oplus A_1 \oplus \cdots \oplus A_m.$$

PROPOSITION 3.15. *As vector spaces over* $\mathbb{R}$,

$$C^0_m(\Delta) \cong A_m.$$

Proof. Restricting the surjection (3.4) to the subspace $A^{(m)}$ gives a surjection onto $C^0_m(\Delta)$ by the proof of Theorem 2.3, and so we get an exact sequence of vector spaces

$$0 \to A^{(m)} \cap K \to A^{(m)} \to C^0_m(\Delta) \to 0, \tag{3.16}$$

where $K = \langle \bar{x}_1 + \cdots + \bar{x}_n - 1\rangle$.

Now consider the map

$$A^{(m-1)} \to A^{(m)} \cap K$$

defined by $p \mapsto p(\bar{x}_1 + \cdots \bar{x}_n - 1)$. The map is clearly injective (consider the nonzero homogeneous component of p of least degree). It is surjective as well since if $q = p(\bar{x}_1 + \cdots + \bar{x}_n - 1) \in A^{(m)}$, then $p \in A^{(m-1)}$ (consider the lexicographically first monomial in p of highest degree; its product with the first $\bar{x}_i$ dividing it will not be zero and cannot be cancelled).

Thus by (3.16) we have isomorphisms

$$C^0_m(\Delta) \cong A^{(m)}/A^{(m)} \cap K \cong A^{(m)}/A^{(m-1)} \cong A_m. \quad ∎$$

Recall that the Hilbert function of the graded algebra A_Δ is defined by

$$H(m) = \dim_{\mathbb{R}} A_m$$

for $m \in \mathbb{N}$. This was explicitly computed by Stanley in [9, Proposition 3.2] (see also [12, p. 63]), and so we have the following result. We define $f_i = f_i(\Delta)$ to be the number of i-dimensional simplices in Δ.

COROLLARY 3.17. *For a pure d-dimensional simplicial complex* $\Delta \subset \mathbb{R}^d$,

$$\dim C^0_m(\Delta) = \sum_{i=0}^{d} f_i \binom{m-1}{i} \tag{3.18}$$

for $m > 0$.

Clearly, $\dim C^0_0(\Delta) = 1$. For $m = 1$, (3.18) gives $\dim C^0_1(\Delta) = f_0 = n$, a fact already illustrated by the basis $X_1, ..., X_n$ for $C^0_1(\Delta)$. For the first few values of $m > 1$, we get from (3.18) that $\dim C^0_2(\Delta) = f_0 + f_1$, $\dim C^0_3(\Delta) = f_0 + 2f_1 + f_2$, and $\dim C^0_4(\Delta) = f_0 + 3f_1 + 3f_2 + f_3$.

Up to this point, the assumption that Δ be a pure complex is probably not necessary. In particular, a direct proof of Corollary 3.17 can be given that does not require Δ to be pure.

4. $C^0(\Delta)$ AS AN R-MODULE

We use the relationship between $C^0(\Delta)$ and A_Δ to study the R-module structure of $C^0(\Delta)$, obtaining a combinatorially defined generating set for

$C^0(\Delta)$ over R and a sufficient condition for $C^0(\Delta)$ to be a free R-module. In order to do this, we must modify somewhat the treatment of the face ring for shellable complexes due to Kind and Kleinschmidt [7]. (See also Garsia [6] and Baclawski and Garsia [1] for similar treatments of the face ring for shellable complexes.)

For any ordering of the maximal simplices $\sigma_1, \sigma_2, ..., \sigma_t$ ($t = f_d(\Delta)$ = number of d-simplices in Δ) of the (pure d-dimensional) simplicial complex Δ, we denote by

$$\Delta_i = \bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_i \tag{4.1}$$

the subcomplex generated by the first i maximal simplices; $\Delta_t = \Delta$. As in the proof of Theorem 3.6, we consider the minimal faces of σ_i that are not in Δ_{i-1} (where we let $\Delta_0 = \phi$, the empty complex). Denote these faces by $\rho_1^i, ..., \rho_{s_i}^i$, where $s_i \geq 1$.

The vector $s = (s_1, ..., s_t)$ depends upon the given ordering of the maximal simplices of Δ. If $s = (1, 1, ..., 1)$ for some ordering, Δ is said to be *shellable* and the ordering is called a *shelling* of Δ. For complexes that are not shellable, there may be some interest in studying orderings that minimize s in some reasonable sense (e.g., minimum sum or lexicographically or some combination).

For our purposes, assume the ordering $\sigma_1, \sigma_2, ..., \sigma_t$ to be fixed. For each i, we let $m_1^i, ..., m_{s_i}^i$ be the square-free monomials in A_Δ corresponding to the faces $\rho_1^i, ..., \rho_{s_i}^i$. If Δ has vertices $v_1, ..., v_n$ and $\dim \Delta = d$, then suppose $C = (c_{ij})$ is a $(d+1) \times n$ matrix, with rows indexed $0, 1, ..., d$ and columns indexed by the vertices, such that the columns corresponding to any simplex of Δ are linearly independent. (This is the case, for example, if any $(d+1) \times (d+1)$ submatrix is invertible; this is the assumption of [7].) For $0 \leq i \leq d$, we define the linear form

$$\theta_i = c_{i1}\bar{x}_1 + \cdots + c_{in}\bar{x}_n \tag{4.2}$$

in A_Δ. The following is a partial generalization of the main theorem of Kind and Kleinschmidt [7].

THEOREM 4.3. *Considered as a module over the subring* $\mathbb{R}[\theta_0, ..., \theta_d]$, A_Δ *is generated by the monomials* m_j^i, $1 \leq i \leq t$, $1 \leq j \leq s_i$.

Proof. The proof is a straightforward extension of the inductive proof of generation given in [7] and will not be repeated here. We will only note that the apparent weakening of the hypothesis on the matrix C causes no problem since the proof in [7] uses only the invertibility of the subsets of columns corresponding to the maximal simplices. The fact that multiple

generators are introduced when a new maximal simplex is added is easily incorporated into the argument. ∎

We note here that a consequence of Theorem 4.3 is that the linear forms $\theta_0, \theta_1, ..., \theta_d$ form a *homogeneous system of parameters* for A_Δ since the Krull dimension of A_Δ is $d+1$ [12]. (This fact was known by Stanley [personal communication, 1976], and noted by him without proof in [10].) The full result of Kind and Kleinschmidt is that if $\sigma_1, ..., \sigma_t$ is a shelling of Δ (and so $s_i = 1$ for each i), then the t monomials $m_1^1, ..., m_1^t$ form a *free basis* for A_Δ over the (polynomial) subring $\mathbb{R}[\theta_0, ..., \theta_d]$. More generally, for any Cohen–Macaulay complex Δ (for example, Δ a d-disk), A_Δ is a free module of rank $t = f_d(\Delta)$ over $\mathbb{R}[\theta_0, ..., \theta_d]$. (See, e.g., [12] for a discussion of Cohen–Macaulay complexes and some basic references. An elementary survey of some of the relevant material can be found in [2].)

We show now that Theorem 4.3 leads to generators for $C^0(\Delta)$ over R. For disks, for example, this translates to freeness of $C^0(\Delta)$ as an R-module, and gives free generators when Δ is shellable. To this end, if m_j^i is one of the square-free monomials in $\bar{x}_1, ..., \bar{x}_n$ from the theorem, let M_j^i denote the corresponding monomial in the functions $X_1, ..., X_n$.

THEOREM 4.4. *For any pure d-complex Δ, $C^0(\Delta)$ is generated as an R-module by the piecewise polynomial functions M_j^i, $1 \leq i \leq t$, $1 \leq j \leq s_i$.*

Proof. As in (2.4), we write for $y \in \Delta$ (considered as a point set in $\mathbb{R}^d$)

$$y_j = \sum_{i=1}^{n} v_{ij} X_i(y), \tag{4.5}$$

where v_{ij} is the jth coordinate of vertex v_i of Δ. Define a $(d+1) \times n$ matrix $C = (c_{kl})$ by

$$c_{kl} = \begin{cases} 1 & \text{if } k = 0 \\ v_{lk} & \text{if } k > 0; \end{cases} \tag{4.6}$$

C clearly satisfies the property that the columns corresponding to any simplex of Δ are linearly independent. Thus the elements $\theta_0, \theta_1, ..., \theta_d$ of A_Δ defined by (4.2) define a subring $\mathbb{R}[\theta_0, \theta_1, ..., \theta_d]$ such that the monomials m_j^i give generators of A_Δ over this subring.

The result now follows from the surjection (3.4) and the fact that, by (4.5), the image of $\mathbb{R}[\theta_0, \theta_1, ..., \theta_d]$ under that map is the ring $R = \mathbb{R}[y_1, ..., y_d]$. ∎

In particular, we have shown that the θ_i's defined by (4.2), (4.5), and (4.6) form a homogeneous system of parameters for A_Δ, and so, in particular, they are algebraically independent. If Δ is Cohen–Macaulay, in particular, if Δ is a disk, this means that there will be homogeneous elements $\eta_1, ..., \eta_t$ in A_Δ (i.e., homogeneous polynomials in $\bar{x}_1, ..., \bar{x}_n$) such that the η_i form a basis for A_Δ as a free module over $\mathbb{R}[\theta_0, \theta_1, ..., \theta_d]$. (See, e.g., [9, Proposition 4.1].) We show next that, under the map (3.4), this translates to the freeness of $C^0(\Delta)$ over R.

THEOREM 4.7. *If Δ is a disk (more generally, if Δ is any Cohen–Macaulay complex), then $C^0(\Delta)$ is a free R-module of rank $t = f_d(\Delta)$.*

Proof. Let $\theta_0, \theta_1, ..., \theta_d$ be the homogeneous system of parameters for A_Δ defined above, and let $\eta_1, ..., \eta_t$ be homogeneous elements that form a free basis for A_Δ over $\mathbb{R}[\theta_0, \theta_1, ..., \theta_d]$. Let $\bar{\eta}_1, ..., \bar{\eta}_t$ be the images of these elements under the surjection (3.4). As before, $\bar{\eta}_1, ..., \bar{\eta}_t$ generate $C^0(\Delta)$ as an R-module. To show that they form a free basis, suppose there is a relation

$$\sum_{i=1}^{t} p_i(y_1, ..., y_d)\bar{\eta}_i = 0 \tag{4.8}$$

in $C^0(\Delta)$, where each $p_i \in R$. Then lifting (4.8) to A_Δ, we get that

$$\sum_{i=1}^{t} p_i(\theta_1, ..., \theta_d)\eta_i \in \langle \theta_0 - 1 \rangle$$

by Theorem 3.6, and so there exist $q_i(\theta_0, \theta_1, ..., \theta_d)$ in $\mathbb{R}[\theta_0, \theta_1, ..., \theta_d]$ so that

$$\sum_{i=1}^{t} p_i(\theta_1, ..., \theta_d)\eta_i = (\theta_0 - 1) \sum_{i=1}^{t} q_i(\theta_0, ..., \theta_d)\eta_i.$$

By the freeness of A_Δ, we get

$$p_i(\theta_1, ..., \theta_d) = \theta_0 q_i(\theta_0, ..., \theta_d) - q_i(\theta_0, ..., \theta_d) \tag{4.9}$$

for each i. Since the θ_i's are algebraically independent and the degree of θ_0 on the left of (4.9) is zero, we conclude that each $q_i = 0$, and so each $p_i = 0$. ∎

We remark here that $C^0(\Delta)$ can be shown to be a free R-module for *any* d-manifold $\Delta \subset \mathbb{R}^d$. The proof involves an application of Theorem 4.7 to various localizations of $C^0(\Delta)$ and the fact that projective modules over polynomial rings are free. It will appear as a part of a forthcoming general study of the rings $C^r(\Delta)$.

In the case that Δ is a shellable d-complex in $\mathbb{R}^d$ (and thus a shellable disk), the result of Kind and Kleinschmidt [7] is that one can take $\eta_i = m_i'$, the monomial corresponding to the unique minimal face introduced at the ith stage of the shelling. Letting M_i be the corresponding product of the appropriate Courant functions, we get the following.

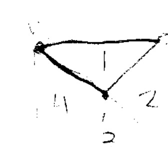
COROLLARY 4.10. *For shellable Δ, $C^0(\Delta)$ is freely generated as an R-module by the piecewise polynomial functions $M_1, ..., M_t$.*

As before, we let $f_i = f_i(\Delta)$ be the number of i-dimensional simplices in Δ ($f_{-1} \equiv 1$), and we define, for $0 \leq k \leq d+1$,

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.$$

Then we have that $t = f_d = h_0 + h_1 + \cdots + h_{d+1}$, $h_0 = 1$, $h_1 = f_0 - d - 1$, and $h_{d+1} = (-1)^{d+1}(1 - \chi(\Delta))$, where $\chi(\Delta)$ is the Euler characteristic of Δ (and so $h_{d+1} = 0$ if Δ is a d-disk). Further, if we define the degree of a piecewise polynomial to be the maximum degree of any of its components, h_k is the number of M_i that are of degree k. (See, e.g., [4, Proposition 2].) Thus there are no elements in the basis of degree larger than d. While different shellings of Δ will lead to different basis elements M_i', the h_k's are clearly invariants of Δ.

EXAMPLE 4.11. Let Δ be the triangulation of a quadrilateral (with vertices v_1, v_2, v_3, v_4) by adding a single vertex v_0 in the interior. If the maximal simplices are ordered



$$v_0v_1v_2,\ v_0v_2v_3,\ v_0v_3v_4,\ v_0v_1v_4$$

we get a shelling, which yields the free basis

$$1, X_3, X_4, X_1X_4$$

for $C^0(\Delta)$ over $R = \mathbb{R}[y_1, y_2]$. If, instead, we choose the shelling

$$v_0v_1v_2,\ v_0v_2v_3,\ v_0v_1v_4,\ v_0v_3v_4,$$

we obtain the basis

$$1, X_3, X_4, X_3X_4.$$

In either case, we have $h_0 = 1$ degree 0 element, $h_1 = 2$ degree 1 elements, and $h_2 = 1$ degree 2 elements in the basis.

Finally, we remark here that the basis given in Corollary 4.10 is triangular in the following sense. Suppose $\sigma_1, \sigma_2, ..., \sigma_t$ is the shelling of Δ that produced the basis $M_1, M_2, ..., M_t$, and let Δ_i be defined by (4.1). Then $M_1 = 1$ and for $i > 1$, the function M_i is zero on $\sigma_1, ..., \sigma_{i-1}$. The reason for this is that M_i is the product of the Courant functions corresponding to the minimal face of σ_i not in Δ_{i-1}, and so it must vanish in Δ_{i-1}.

This suggests the following scheme for obtaining a continuous piecewise polynomial approximation (of degrees at most m) to an arbitrary function f on Δ. Start with a polynomial p_1 of degree at most m that best approximates the function f (in whatever sense is of interest) on simplex σ_1. If polynomials $p_1, ..., p_{i-1}$ have been chosen, then choose p_i of degree at most $m - \deg M_i$ so that $p_i M_i$ is the best approximation of this form to the function $f - \sum_{j=1}^{i-1} p_j M_j$ on the simplex σ_i. Because of the triangular property noted above, the choice of p_i does not effect the approximation already achieved on σ_j for $j < i$.

If $F \in C^0(\Delta)$, the above scheme will produce the unique representation of F in terms of the basis $M_1, ..., M_t$; this follows from the triangular property of the basis. Thus this basis has the following reduced property: whenever $F = \sum_{i=1}^{t} p_i M_i$ then $\deg p_i M_i \leqslant \deg F$ for each i. One consequence of this is that one can obtain a direct computation of the dimensions of the spaces $C_m^0(\Delta)$ in terms of the h_i's. In particular, $\dim C_m^0(\Delta)$ is the coefficient of t^m in the power series

$$\frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{d+1}}.$$

That this is the same as (3.18) is discussed in [9, 12].

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