In: Mathematical Programming: The State of the Art, ed. by Bachem, Grötschel & Korte, Springer-Verlag 1983?

Polyhedral Theory and Commutative Algebra

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Abstract. An expository account is presented describing the use of methods of commutative algebra to solve problems concerning the enumeration of faces of convex polytopes. Assuming only basic knowledge of vector spaces and polynomial rings, the enumeration theory of Stanley is developed to the point where one can see how the Upper Bound Theorem for spheres is proved. A briefer account is then given of the extension of these techniques which yielded the proof of the necessity of McMullen's conjectured characterization of the f-vectors of convex polytopes. The latter account includes a glimpse of the application of these methods to the study of integer solutions to systems of linear inequalities.

Introduction and Summary

There has been much progress lately on questions concerning the enumeration of faces of various dimensions in convex polytopes. Specifically, one might be interested in bounds on the number of faces of one dimension (say, the number of vertices) given information on the number of faces of some other dimension (say, the number of facets). More generally, one might ask whether there exists a polytope having a predesignated number of faces of each dimension. Questions such as these go back to Euler, and they remain of fundamental interest in the context of mathematical programming today.

An important development in this field has been the introduction by Stanley of some methods from the field of commutative algebra. He used these techniques first to obtain an extension of the Upper Bound Theorem of McMullen from polytopes to general triangulated spheres. This theorem gives tight upper bounds on the number of faces in each dimension in terms of the number of vertices (and, by polarity in the case of polytopes, also bounds in terms of the number of facets). Subsequently, he combined these methods with some recent results in algebraic geometry to complete the proof of McMullen's conjectured characterization of the face-counting vectors of simplicial convex polytopes. The latter result applies equivalently to simple (i.e., nondegenerate) polytopes and provides a complete description of the relationships between the

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* Prepared for the XI International Symposium on Mathematical Programming, Bonn, August 23-27, 1982. Supported in part by the National Science Foundation under Grant MCS81-02353.

numbers of faces of various dimensions for polytopes in either class. To date, there is no proof of this purely combinatorial result that avoids the algebraic machinery of Stanley.

This paper will describe this use of ring theoretic methods on problems of facial enumeration for convex polytopes. After introducing the problem and describing the major results for f-vectors of polytopes, some basic ideas from commutative algebra will be introduced and discussed. To each simplicial convex polytope P, we then associate a commutative ring, certain invariants of which are related to the number of faces of P in each dimension. We will see how classical algebraic results on these invariants together with some recent developments in commutative algebra related to invariant theory lead to a proof of the Upper Bound Theorem in a general setting provided by a certain class of simplicial complexes. A theorem of Reisner which characterizes complexes in this class is then discussed. Finally, after describing the construction used by the author and Lee to prove the sufficiency of McMullen's conditions, we will examine Stanley's proof of their necessity in enough detail to begin to see a striking connection between the numbers of faces of a polytope and the integer points in a related system of convex cones. The possibility of using these methods to shed light on general integer programming problems is one of the most exciting aspects of this area of research.

f-Vectors of Convex Polytopes

By a *convex polytope* P we mean the convex hull of a finite point set in a real Euclidean space. Equivalently, P can be defined as the bounded intersection of finitely many closed half-spaces. By a face F of P we mean the intersection of P with a hyperplane having the property that P is contained in one of its closed half-spaces. Thus, the empty set is always a face of P, and we call P a face of P (whether or not it arises in the above manner). All other faces will be called *proper* faces, and they are finite in number. Each face of a polytope P is again a polytope.

We define the *dimension* of a polytope P, $\dim P$, to be the dimension of $\operatorname{aff}(P)$, its affine hull, and say that P is a *d-polytope* if $\dim P = d$. In this case each face of P, except P itself, has dimension less than d. For each $i = -1, 0, 1, \ldots, d-1$, let $f_i(P)$ denote the number of i-dimensional faces of P. In particular, $f_{-1}(P) = 1$ counts the empty face, $f_0(P)$ is the number of *vertices*, $f_1(P)$ is the number of *edges* and $f_{d-1}(P)$ is the number of *facets* of P. We denote by $f(P)$ the vector $(f_{-1}(P), f_0(P), \ldots, f_{d-1}(P))$, called the *f-vector* of P. For a comprehensive treatment of the theory of convex polytopes and, in particular, of f-vectors see [11]. For survey of the latter topic which includes a discussion of the more recent results, see [19].

Let $f(P^d)$ denote the set of all f-vectors of d-polytopes. There is considerable interest in describing the set $f(P^d)$ exactly, but this remains an open problem. However, we can describe $\operatorname{aff}(f(P^d))$ and certain inequalities satisfied by each $f \in f(P^d)$. First, each $f \in f(P^d)$ satisfies the *Euler Equation*

$$f_0 - f_1 + f_2 - \ldots \pm f_{d-1} = 1 - (-1)^d,$$

and, further, this equation specifies the affine hull, namely

$$\operatorname{aff}(f(P^d)) = \{(f_{-1}, f_0, \ldots, f_{d-1}) : f_{-1} = 1, f_0 - f_1 + \ldots = 1 - (-1)^d\}.$$

The inequalities are somewhat harder to describe. To this end, consider the moment curve in $\mathbb{R}^d$ given by $x(t) = (t, t^2, t^3, \ldots, t^d)$ and choose real numbers $t_1 < t_2 < \ldots < t_n$, with $n > d$. Define $C(n, d)$ to be the convex hull of $V = \{x(t_1), x(t_2), \ldots, x(t_n)\}$. While the actual polytope obtained by this procedure depends on the choices of the t_i's, it is known that its combinatorial structure, in particular, its f-vector, is independent of the t_i's. We use the symbol $C(n, d)$ to refer to this combinatorial type. It is easily seen that $C(n, d)$ is a simplicial d-polytope, that is each facet (and thus, each proper face) is a simplex (an $(r-1)$-polytope having just r vertices).

One of the most remarkable properties of the polytope $C(n, d)$ is that it is neighborly, that is, each pair of vertices forms an edge of $C(n, d)$. In fact, for $k = 1, \ldots, [d/2]$, the convex hull of *any* k-subset of V is a face of $C(n, d)$. (Here $[x]$ denotes the largest integer less than or equal to x.) Thus among all d-polytopes with n vertices, $C(n, d)$ clearly has the maximum number of i-faces for $i = 0, 1, \ldots, [d/2] - 1$. That $C(n, d)$ has the maximum number of i-faces, among all d-polytopes with n vertices, for *all* i is the content of the *Upper Bound Theorem*, first formulated by Motzkin [27] and proved by McMullen [22]. Thus we have that for all d-polytopes P with n vertices

$$f(P) \leq f(C(n, d)).$$

Since the number of i-faces of $C(n, d)$ is known for each i as a function of n and d, this gives upper bounds for each $f_i(P)$ in terms of $n = f_0(P)$ (and d), and thus inequalities which must be satisfied for each $f \in f(P^d)$.

A particularly important component of $f(C(n, d))$ is

$$f_{d-1}(C(n, d)) = \binom{n - [(d+1)/2]}{n-d} + \binom{n - [(d+2)/2]}{n-d},$$

which is the maximum number of facets in a d-polytope with n vertices (or, by polarity, the maximum number of vertices in a d-polytope with n facets). Of course, by the above discussion we have

$$f_i(C(n, d)) = \binom{n}{i+1}$$

for $i = 0, 1, \ldots, [d/2] - 1$. See [11] or [25] for a general expression for the terms of $f(C(n, d))$ and its derivation.

A complete description of $f(P^d)$ has remained elusive for general d. There do not seem to be even reasonable conjectures as to a final set of conditions. However, if one restricts to the case of simplicial polytopes, then the situation is considerably better understood. In fact, the set $f(P_s^d)$ of all f-vectors of simplicial d-polytopes is completely known, being specified entirely by a list of linear equations, linear inequalities and nonlinear inequalities. We will describe these in turn. First note that by the usual polyhedral polarity [11], to each d-polytope P there corresponds another d-polytope P^* which has the property that $f_i(P)=f_{d-1-i}(P^*)$. In particular, when P is simplicial, then P^* is *simple* (or nondegenerate, in linear programming terms), that is, each vertex is on precisely d facets. Thus, describing $f(P_s^d)$ is equivalent to describing the f-vectors of all nondegenerate d-polytopes. Further, one proves the Upper Bound Theorem by first showing that the maximum numbers of faces must occur in simplicial polytopes, and then proving the Upper Bound Theorem for simplicial polytopes.

First, we note that each $f\in f(P_s^d)$ satisfies the *Dehn-Sommerville Equations*

$$E_k^d:\ \sum_{j=k}^{d-1}(-1)^j\binom{j+1}{k+1}f_j=(-1)^{d-1}f_k$$

for $k=-1,0,1,\dots,d-1$. The equation E_{-1}^d is just the Euler equation. It is known that $[(d+1)/2]$ of these equations are independent, and they completely determine $\mathrm{aff}(f(P_s^d))$. Thus the dimension of $\mathrm{aff}(f(P_s^d))$ is $[d/2]$. See [11] or [25] for details.

To be able to describe the remaining conditions, we must apply a change of variables to the space of f-vectors, first suggested by Sommerville [32], which recasts the Dehn-Sommerville equations in a particularly simple form. To this end, take f in $f(P_s^d)$ and define the polynomial

$$f(t)=\sum_{j=-1}^{d-1}f_jt^{j+1}$$

and let

$$h(t)=(1-t)^d f(t/(1-t)).$$

Since f is of degree d, h is again a polynomial, say

$$h(t)=\sum_{i=0}^{d}h_it^i.$$

If $f=f(P)$ then we define the h-vector of P to be the vector of coefficients of $h(t)$, that is $h(P)=(h_0,h_1,\dots,h_d)$. Note that $h_0(P)=1$ since $f_{-1}(P)=1$.

The vector $h(P)$ can be obtained from $f(P)$ by means of an invertible linear transformation, whose inverse is given by

$$f_j=\sum_{i=0}^{j+1}\binom{d-i}{d-j-i}h_i.$$

Thus f_j is a *nonnegative* linear combination of $h_0,\dots,h_{j+1}$, and so an inequality of the form $h(P)\le h(P')$ implies the corresponding inequality $f(P)\le f(P')$. In fact, the proof of the Upper Bound Theorem proceeds by showing $h(P)\le h(C(n,d))$ for the simplicial d-polytopes with n vertices. In terms of the h-vector of P, the Dehn-Sommerville equations become $h_i=h_{d-i}$, for $i=0,1,\dots,[d/2]$. (See [26] or [25] where our h_{k+1} corresponds to their $g_k^d(P)$.)

Let $h(P_s^d)$ denote the set of h-vectors of simplicial d-polytopes. By the above discussion, knowing $h(P_s^d)$ is equivalent to knowing $f(P_s^d)$. We can now describe the set of linear inequalities satisfied by all $f\in f(P_s^d)$. They were first proposed by McMullen and Walkup in the form of a *Generalized Lower Bound Conjecture*, which stated that $h_{i+1}\ge h_i$, for $i=0,1,\dots,[d/2]-1$. Thus, in light of the Dehn-Sommerville equations, these inequalities imply that the h-vector must be unimodal. In terms of the f_j's, $h_{i+1}\ge h_i$ implies a lower bound on f_i as a linear function of the f_j's for $j<i<[d/2]$; these lower bounds imply the lower bounds given in the so-called ***Lower Bound Theorem*** proved by Barnette [2] (see [26]).

To complete the description, we must establish a last bit of notation. For positive integers h and i, we note that h can always be written uniquely in the form

$$h=\binom{n_i}{i}+\binom{n_{i-1}}{i-1}+\dots+\binom{n_j}{j},$$

where $n_i>n_{i-1}>\dots>n_j\ge j\ge 1$. (Choose n_i to be the largest integer with $h\ge\binom{n_i}{i}$, etc.) Define the ith *pseudopower* of h to be

$$h^{(i)}=\binom{n_i+1}{i+1}+\binom{n_{i-1}+1}{i}+\dots+\binom{n_j+1}{j+1}.$$

Put $0^{(i)}=0$ for all i.

We state the nonlinear inequalities on the components of the h-vector (and thus the f-vector) together with the earlier conditions in the form of a characterization of $h(P_s^d)$.

Theorem (McMullen's Conditions): An integer vector $h=(h_0,h_1,\dots,h_d)$ is the h-vector of a simplicial convex d-polytope if and only if the following three conditions hold:

(i) $h_i=h_{d-i}$, $i=0,1,\dots,[d/2]$,
(ii) $h_{i+1}\ge h_i$, $i=0,1,\dots,[d/2]-1$, and
(iii) $h_0=1$ and $h_{i+1}-h_i\le(h_i-h_{i-1})^{(i)}$, $i=1,\dots,[d/2]-1$

This characterization was conjectured in 1971 by McMullen [24], [25], and proved by him for $d\le 5$ and for the case of d-polytopes having n vertices where $d<n<d+3$. The sufficiency of these conditions was proved in 1979 by Billera and Lee [3], [4], followed almost immediately by a proof of necessity by Stanley [42]. The proof of sufficiency depends heavily on insights provided by earlier work of Stanley [36] in which the Upper Bound Theorem was extended to general triangulations of spheres by means of techniques of commutative algebra. The proof of necessity extends this earlier work, introducing powerful new techniques from algebraic geometry. In the rest of this paper, we will describe the algebraic techniques developed by Stanley and outline the steps leading to the proof of this result.

Note that the f-vectors of unbounded polyhedra are not covered by any of the results discussed here. For treatments of this topic, see [17], [6], [5] and [18].

As an interesting consequence of the sufficiency proof, Lee [18a] has shown that for every simple d-polytope P with n facets there is another such polytope P' with $f(P')=f(P)$ such that the diameter of P' is at most $n-d+1$, one more than the bound given by the Hirsch conjecture. Thus if there are examples where this conjecture fails badly, it will not be due solely to the *number* of edges (or to the number of faces of any other dimension).

The Stanley-Reisner Ring of a Simplicial Complex

By a simplicial complex on a vertex set $V=\{v_1, v_2, \dots v_n\}$ we mean a subset Δ of 2^V such that $G\in\Delta$, $F\subset G$ implies $F\in\Delta$ and each $\{v_i\}\in\Delta$. In other words, Δ is an independence system for which each one-element set is independent. We define the dimension of Δ, $\dim\Delta$, to be $d-1$, where $d=\max_{F\in\Delta}|F|$, the maximum cardinality of an F in Δ. Our main example of a simplicial complex will be the boundary complex of a simplicial d-polytope P: the set V will be the vertex set of P and Δ will consist of all vertex sets of proper faces of P plus the empty set. In this case, the two uses of d are consistant.

An element F of Δ will be called a *face* (or *simplex*) of Δ, and if $|F|=i+1$, F will be called an i=face. We define $f_i(\Delta)$ to be the number of i-faces of Δ, and we call $f(\Delta)=(f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta))$ the *f-vector* of Δ (again $f_{-1}(\Delta)=1$ counts the empty face). We define the quantities $h_i(\Delta)$, $0\le i\le d$, by means of the same polynomial relation as in the case of polytopes, and let $h(\Delta)=(h_0(\Delta), \dots, h_d(\Delta))$ be the *h-vector* of Δ.

Let k be a field, and let $R=k[X_1, \dots, X_n]$ be the ring of polynomials in n indeterminates over k. If $X_1^{a_1}\dots X_n^{a_n}$ is a monomial in R (where the a_i's are nonegative integers), we define its *support* to be the set $\{v_i : a_i\neq 0\}$. We define an ideal I_Δ in R by

$$I_\Delta=\langle X_{i_1}\dots X_{i_k} : i_1<i_2<\dots<i_k,\ \{v_{i_1}, \dots, v_{i_k}\}\notin\Delta\rangle,$$

the ideal generated by all sqaure-free monomials whose support is *not* a face of Δ. Finally we define the ring $k[\Delta]=R/I_\Delta$. $k[\Delta]$ has come to be called the *Stanley-Reisner ring* of Δ. It was first considered in this form by Stanley [35], [36] and Reisner [29], although it is a special case of a more general notion considered earlier by Hochster [13].

In the case where Δ is the boundary complex of a triangle with vertices v_1, v_2, and v_3, i.e., $\Delta=\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\}$, I_Δ is the ideal generated by the monomial $X_1X_2X_3$. The ring $k[\Delta]$ consists essentially of all polynomials in X_1, X_2 and X_3 none of whose monomials is divisible by $X_1X_2X_3$.

In order to get information on the h-vector (and thus the f-vector) of the complex Δ from the ring $k[\Delta]$, we need to discuss the notion of a grading on $k[\Delta]$.

Graded Algebras and Their Hilbert Functions

Let k be a field. By a *graded k-algebra* we mean a commutative, associative ring A such that $k\subset A$ (and so A is a vector space over k), together with a collection of subspaces A_i of A indexed by the nonnegative integers such that

1) A is the vector space direct sum of the A_i, denoted $A=A_0\oplus A_1\oplus\dots\oplus A_i\oplus\dots$,
2) $A_0=k$,
3) $A_iA_j\subset A_{i+j}$, i.e., the product of an element of A_i with an element of A_j is in A_{i+j},
4) A is finitely generated as a k-algebra, i.e., there are finitely many elements $x_1, \dots, x_r$ in A such that each element of A can be written as a polynomial in $x_1, \dots, x_r$ with coefficients in k.

We note that the polynomial in (4) is not necessarily unique. If the x_i's can all be chosen from A_1, then A is called *standard.*

A simple example of a graded k-algebra is $A=k[X]$, the polynomial ring over k in one indeterminate. Here A_i is the set of all k-multiples of the monomial X^i. A bit more interesting is the ring$=k[X_1, \dots, X_n]$, where A_i is the vector space generated by all monomials of degree i. Normally, one takes the *degree* of a monomial $X_1^{a_1}, \dots, X_n^{a_n}$ to be the total degree, $a_1+a_2+\dots+a_n$. Notice, however, that by defining degree by $w_1a_1+w_2a_2+\dots+w_na_n$, where the w_i's are arbitrary positive integers, one gets different *gradings* on A, i.e., different direct summands A_i. Choosing different gradings for an algebra can prove useful for some applications. For our puposes, however, we will always consider a polynomial ring to be graded by total degree, making it a standard k-algebra.

An element in a graded k-algebra A is said to be homogeneous (of degree i) if it belongs to A_i for some i. Since the A_i's are vector subspaces of A, any k-linear combination of homogeneous elements of the same degree is again homogeneous of that degree, and by (3) above, the product of homogeneous elements of any degree is homogeneous. By (1), each element a of A can be written uniquely as a finite sum of homogeneous elements, at most one of each degree. These are called the *homogeneous components* of a. It follows from this that the x_i's in (4) can always be chosen to be homogeneous. In a polynomial ring, the notion of homogeneous element coincides with the usual notion of homogeneous polynomial

An ideal I in A is said to be homogeneous if I can be generated by homogeneous elements or, equivalently, if for each a in I, all the homogeneous components of a are also in I. Each such ideal I, considered as a k-vector space, has a direct sum decomposition

$$I=I_0\oplus I_1\oplus I_2\oplus\dots$$

where $I_j=I\cap A_j$. Note that if I is a proper ideal, that is, $I\neq A$, then $I_0=0$. (From now on the term "ideal" will necessarily mean "proper ideal".) Thus each homogeneous ideal is contained in the maximal (homogeneous) ideal

$$A_+=\bigoplus\{A_i : i>0\}.$$

sometimes called the *irrelevant* maximal ideal. The most important fact to be noted about a homogeneous ideal I in a graded ring A is that the quotient ring A/I has a natural grading given by $(A/I)_j = A_j/I_j$. Thus the quotient of a (standard) graded k-algebra by a homogeneous ideal is again a (standard) graded k-algebra.

It is immediate that the standard graded k-algebras are precisely the quotients of polynomial rings (with the standard grading) by homogeneous ideals. If we let Δ be a simplicial complex and define the ideal I_Δ and the ring $k[\Delta]$ as before, then I_Δ is a homogeneous ideal in the polynomial ring R since it is generated by monomials, which are surely homogeneous (in any of the gradings mentioned for R). Thus $k[\Delta]$ is a standard graded k-algebra for any simplicial complex Δ.

Let A be any graded k-algebra. Since each A_i is a vector space over k, it makes sense to consider its dimension, denoted $\dim_k A_i$. It follows from the fact that A is finitely generated as an algebra that $\dim_k A_i$ is finite for each i. (A is in fact a Noetherian ring). We define the *Hilbert function* of A by

$$H(A, i) = \dim_k A_i, \quad i \geq 0.$$

The *Hilbert series* of A is defined to be the formal power series

$$F(A, t) = \sum_{i \geq 0} H(A, i) t^i.$$

Since $A_0 = k$, we always have $H(A, 0) = 1$, and so the constant term of $F(A, t)$ is always 1. For $A = k[X]$, each $H(A, i) = 1$ and so in this case

$$F(A, t) = 1/(1-t) = 1 + t + t^2 + t^3 + \dots .$$

When $A = k[X_1, \dots, X_n]$, a straightforward counting argument gives

$$F(A, t) = [1/(1-t)]^n$$

since in this case $H(A, i)$ is the number of monomials of degree i in n variables. Note that for any simplicial complex D, $H(k[\Delta], 1)$ is always the number of vertices in Δ.

In the case that A is a *standard* k-algebra, it is known that for i sufficiently large, $H(A, i)$ is a polynomial function of i. This polynomial is called the *Hilbert polynomial* of A, and its degree is $d-1$ where $d = \dim A$, the *Krull dimension* of A. In general, $\dim A$ is defined to be the longest length r of strictly increasing chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_r$$

of (homogeneous) prime ideals or, equivalently, the maximum number of elements of A which are algebraically independent over the field k (i.e., satisfy no nontrivial polynomial over k). See, for example, [1] for further details. We will use this observation about the degree of the Hilbert polynomial later to show that $\dim k[\Delta] = 1 + \dim \Delta$ for a simplicial complex Δ.

As an example, let A be the standard k-algebra $k[X, Y]/(XY)$. A is the ring of a simplicial complex consisting merely of two points (and no other simplices). Then $H(A, 0) = 1$ and $H(A, i) = 2$ for $i > 0$. Thus the Hilbert polynomial is the constant polynomial 2 of degree 0 and so $\dim A = 1$. Finally, $F(A, t) = (1+t)/(1-t)$.

Let A be a graded k-algebra of Krull dimension d. A (homogeneous) *system of parameters* for A is a set $\theta_1, \theta_2, \dots, \theta_d$ of homogeneous elements in A_+ (i.e., homogeneous elements of positive degree) such that A is a finitely generated module over the subalgebra $k[\theta_1, \dots, \theta_d]$ (the algebra of all polynomials in the elements $\theta_1, \dots, \theta_d$). A system of parameters is always algebraically independent over k (see [31]) and so $k[\theta_1, \dots, \theta_d]$ is in fact a polynomial ring. An equivalent condition for $\theta_1, \dots, \theta_d$ to be a system of parameters is that $A/(\theta_1, \dots, \theta_d)$ be a finite dimensional vector space over k, where $(\theta_1, \dots, \theta_d)$ denotes the ideal generated by the θ_i's.

In the case A is a polynomial ring in n indeterminates, the Krull dimension is clearly n, and the indeterminates themselves form a system of parameters, as will any nonsingular k-linear transformation of the indeterminates. Consider $A = k[X]$ and let $\theta = X^2$. Since for every $f \in A$, $f(X) = g(X^2) + Xh(X^2)$, with g and h polynomials, A is finitely generated over $k[X^2]$ with generators 1 and X, and so θ is a system of parameters for A. Equivalently, $A/(\theta)$ is generated as a vector space over k by the images of 1 and X.

It is a consequence of the Noether Normalization Lemma that a system of parameters always exists for any graded k-algebra A (see, e.g., [1], [31] or [48]). Further, if A is standard and k is an infinite field (like the rationals), there exists a system of parameters each of whose elements is homogeneous of degree 1 (that is, each $\theta_i \in A_1$) [1]. The proof in this case is quite intuitive, proceeding somewhat like the process of obtaining a basis for a vector space from a set of generators. One starts with a set of algebra generators $x_1, \dots, x_r$ for A (which are chosen from A_1 if A is standard). If these are algebraically independent, then they form a system of parameters, and we are done. If not, then there is a polynomial relation satisfied by $x_1, \dots, x_r$. Assume for the moment that this relation is monic in x_r, that is, the coefficient of the highest power of x_r, say the m^{th}, is 1. Then x_r^m (and all higher powers of x_r) can be expressed as a sum of lower powers of x_r multiplied by polynomials in $x_1, \dots, x_{r-1}$. Thus A is a finitely generated module over $k[x_1, \dots, x_{r-1}]$. Now continue the process with $k[x_1, \dots, x_{r-1}]$; it will eventually stop with a system of parameters. While the relation obtained at each step will not always be monic, it is always possible, since the field is infinite, to replace the x_i's by linear combinations of the x_i's to make it monic.

To illustrate, consider $A = k[x, y]/(xy)$. At the first step one gets the relation $xy = 0$, which is not monic in either variable. Making the linear change

$$x' = x - y$$

one gets the relation

$$0 = xy = (y + x')y = y^2 + x'y$$

which is monic in y. We get, then, that x' is a system of parameters for A, which is generated by 1 and y as a module over $k[x']$.

Systems of parameters which are homogeneous of degree 1 need not exist for standard graded k-algebras when k is a finite field. In fact when Δ is a 1-dimensional simplicial complex (i.e., a graph) and k is a field with q elements then one can show that $k[\Delta]$ has a system of parameters which is homogeneous of degree 1 if and only if Δ is $q+1$ colorable. This is a consequence of a result of Stanley, mentioned in [50], which states that (for a d-dimensional complex Δ) homogeneous elements of degree 1, $\theta_1, \ldots, \theta_d$, form a system of parameters for $k[\Delta]$ if and only if the d by n matrix of coefficients expressing the θ_i as linear combinations of the x_j's (generators of $k[\Delta]_1$ corresponding to vertices of Δ) has the property that, for each face of Δ, the set of columns corresponding to its vertices is linearly independent.

Cohen-Macaulay Rings: *h*-Vectors as Hilbert Functions

We turn now to the task of showing how the h-vector of a simplicial convex polytope can be viewed as the Hilbert function of a certain graded k-algebra [36]. We begin by considering a general simplicial complex D and restrict later to a special class of complexes which includes boundary complexes of simplicial convex polytopes as well as general simplicial balls and spheres.

Let Δ be a $(d-1)$-dimensional simplicial complex and $k[\Delta]$ be the associated k-algebra. Recall that the f-vector $f(\Delta)=(f_{-1}, f_0, \ldots, f_{d-1})$ and h-vector $h(\Delta)=(h_0, \ldots, h_d)$ are related by the polynomial equation

$$h(t)=(1-t)^d f(t/(1-t))$$

where

$$f(t)=f(\Delta, t)=\sum_{i=-1}^{d-1} f_i t^{i+1}$$

and

$$h(t)=h(\Delta, t)=\sum_{i=0}^{d} h_i t^i.$$

Now $H(k[\Delta], m)$ is the number of monomials of degree m with support in Δ, and a straightforward counting argument proves the following result.

Lemma 1: $H(k[\Delta], 0)=1$ and for all $m>0$,

$$H(k[\Delta], m)=\sum_{i=0}^{d-1} f_i \binom{m-1}{i}.$$

Corollary 2: $\dim k[\Delta]=d=1+\dim\Delta$

Proof: For $m>0$, $H(k[\Delta], m)$ is given by a polynomial of degree $d-1$, and so $\dim k[\Delta]=d$ by earlier comments.

Lemma 3: $(1-t)^d F(k[\Delta], t)=h(\Delta, t)$.

Proof: It is sufficient to show that the Hilbert series of $k[\Delta]$ is given by $f(t/(1-t))$. But

$$f(t/(1-t))=\sum_{i=-1}^{d-1} f_i t^{i+1}(1+t+t^2+\ldots)^{i+1},$$

and it is again straightforward, using Lemma 1, to verify that the coefficient of t^m in this series is given by $H(k[\Delta], m)$.

We are now led to ask, for a standard graded k-algebra A of Krull dimension d, when $(1-t)^d F(A, t)$ is again the Hilbert series of a graded k-algebra. We have seen that this series is a polynomial when $A=k[\Delta]$ for some $(d-1)$-dimensional complex Δ. In fact, it is generally true that $(1-t)^r F(A, t)$ is a polynomial when A is a standard graded k-algebra which can be generated as an algebra by r elements of degree 1 (see, e.g. [45]). This can be proved using the following result [39, Theorem 3.1]. Recall that if I is homogeneous ideal in A, then I inherits a grading from A given by $I_j=I\cap A_j$. We can then define $H(I, j)=\dim_k I_j$ and $F(I, t)$ as before.

For $a\in A$, define $\operatorname{Ann} a=\{r\in A: ra=0\}$. $\operatorname{Ann} a$ is an ideal which is homogeneous when a is.

Lemma 4: Let A be a graded k-algebra, and let θ be a homogeneous element of degree $m>0$. Then

$$(1-t^m)F(A, t)=F(A/\langle\theta\rangle, t)-t^m F(\operatorname{Ann}\theta, t).$$

Proof: By linear algebra we get for each j and homogeneous ideal I

$$H(A, j)=H(I, j)+H(A/I, j)$$

and so

$$F(A, t)=F(I, t)+F(A/I, t).$$

Now let $I=\langle\theta\rangle$ and note that for each j we have

$$\begin{aligned} H(I, j+m) &= \dim_k(\theta A_j) \\ &= \dim_k A_j - \dim_k(A_j\cap\operatorname{Ann}\theta) \\ &= H(A, j)-H(\operatorname{Ann}\theta, j), \end{aligned}$$

and so

$$F(I, t)=t^m[F(A, t)-F(\operatorname{Ann}\theta, t)].$$

Eliminating $F(I, t)$ yields the desired result.

Lemma 4 is of special interest in the case where $\operatorname{Ann}\theta$ is the zero ideal, that is, where θ is not a *zero divisor*. For then $F(\operatorname{Ann}\theta, t)=0$ and if, further, the degree of θ is 1, we have

$$(1-t)F(A, t)=F(A/\langle\theta\rangle, t),$$

the Hilbert series of the graded k-algebra $A/\langle\theta\rangle$. The problem is now whether we can choose θ so that $A/\langle\theta\rangle$ itself has a homogeneous non-zero divisor (of degree 1), and so on, continuing the process for d steps. We next describe the class of algebras for which this process can always be carried out.

Recall that a graded k-algebra A of Krull dimension d always has a system of parameters, that is, a set $\theta_1, \ldots, \theta_d$ of homogeneous elements in A_+ such that A is a finitely generated module over the subalgebra $k[\theta_1, \ldots, \theta_d]$. A is said to be a *Cohen-Macaulay* graded k-algebra if for some (equivalently, for each) system of parameters $\theta_1, \ldots, \theta_d$, A is a *free* module over $k[\theta_1, \ldots, \theta_d]$. This means that there are homogeneous elements $\eta_1, \ldots, \eta_r$ in A so that each a in A has a *unique* representation of the form

$$a = \sum_{i=1}^{r} \eta_i p_i(\theta_1, \ldots, \theta_d)$$

where the p_i's are polynomials in $\theta_1, \ldots, \theta_d$. It is a characterizing property of Cohen-Macaulay k-algebras that every system of parameters is a *regular sequence* (or an *A-sequence*), that is, each system of parameters $\theta_1, \ldots, \theta_d$ for A has the property that for $i = 1, \ldots, d$, θ_i is not a zero divisor on the ring $A/\langle\theta_1, \ldots, \theta_{i-1}\rangle$. (In fact, the existence of a sequence having this property is normally taken as the definition of a Cohen-Macaulay ring, the freeness being a consequence in this case. See [31] or [15] for details.)

We can now state a key result of Stanley [36], [39, Corollary 3.2], whose origins trace back to work of Macaulay [20].

Theorem 5: Let A be a standard graded k-algebra of Krull dimension d, and let $\theta_1, \ldots, \theta_d$ be a system of parameters for A which is homogeneous of degree 1. Suppose A is Cohen-Macaulay. Then

$$(1-t)^d F(A, t) = F(B, t)$$

where $B = A/\langle\theta_1, \ldots, \theta_d\rangle$.

Proof: Since A is Cohen-Macaulay, $\theta_1, \ldots, \theta_d$ is a regular sequence. The assertion now follows from Lemma 4 and the subsequent discussion.

We will say that a complex Δ is a *Cohen-Macaulay complex* (over k) if its associated k-algebra $k[\Delta]$ is Cohen-Macaulay. Later we will describe conditions on Δ which will insure that $k[\Delta]$ is Cohen-Macaulay. These conditions will depend on the choice of field k, but it sufficies for now to note that if Δ is Cohen-Macaulay over any field, then it will be Cohen-Macaulay over the rationals. Thus if we omit mention of k when we specify that Δ is a Cohen-Macaulay complex, we will assume that k is the field of rational numbers, Q. Recall that for any complex Δ, $k[\Delta]$ is always a standard k-algebra, and thus, if k is an infinite field, there always exists a system of parameters which is homogeneous of degree 1.

When we refer to a vector of finite length as a Hilbert function, we mean that vector with a sequence of zeros appended to it. The following is a consequence of Theorem 5 and Lemma 3.

Corollary 6: The h-vector of a Cohen-Macaulay complex Δ (over k) is always the Hilbert function of a standard graded k-algebra B. In fact, we can always choose $k = Q$, and take B to be $k[\Delta]/\langle\theta_1, \ldots, \theta_d\rangle$, where $d = \dim\Delta + 1$, and the θ_i's are homogeneous elements of degree 1 which form a system of parameters for $k[\Delta]$. In this case, h_i counts the number of η_i of degree i, where the elements $\eta_1, \ldots, \eta_r$ are as specified above.

It follows from work of Hochster [13] that $k[\Delta]$ is Cohen-Macaulay whenever Δ is the boundary complex of a simplicial convex polytope. This uses the fact, shown by Bruggesser and Mani, that boundary complexes of polytopes are always shellable [7]. A complete characterization of when $k[\Delta]$ is Cohen-Macaulay was later supplied by Reisner [29], who showed that, in fact, $k[\Delta]$ is Cohen-Macaulay for any simplicial sphere or ball. We will discuss Reisner's result a bit later, after discussing the relevance of Corollary 6 to the Upper Bound Theorem.

O-Sequences and the Upper Bound Theorem

As with simplicial convex polytopes, one can ask for any simplicial complex Δ, having n vertices and dimension $d-1$, whether the inequality of f-vectors

$$f(\Delta) \leqslant f(C(n, d)) \tag{1}$$

holds where, as before, $C(n, d)$ is the boundary complex of the cyclic d-polytope with n vertices. As before, this inequality is implied by the inequality of h-vectors

$$h(\Delta) \leqslant h(C(n, d)). \tag{2}$$

In the presence of the Dehn-Sommerville equations

$$h_i(\Delta) = h_{d-i}(\Delta), \quad i = 0, \ldots, [d/2]$$

(2) is equivalent to

$$h_i(\Delta) \leqslant \binom{n-d+i-1}{i}, \quad i = 0, \ldots, [d/2]. \tag{3}$$

See [22] or [25] for details. We will show that (3) holds whenever Δ is a Cohen-Macaulay complex. In particular, since the Dehn-Sommerville equations hold for triangulations of $(d-1)$-spheres as well as for the boundaries of simplicial d-polytopes ([16], [11]), we get that (2) and hence (1) hold for these complexes, proving the Upper Bound Theorem for spheres as well as for polytopes.

The key to showing (3) for a Cohen-Macaulay complex Δ is Corollary 6, which says that the h-vector of such a complex is the Hilbert function of a standard graded k-algebra. Accordingly, we turn now to characterizing such Hilbert functions.

Let $X_1, \ldots, X_n$ be a list of indeterminates, and let m and m' be monomials in the X_i's. We denote $m|m'$ to mean m divides m', that is, the power of each X_i in m is at most that in m'. By an order ideal of monomials we mean a nonempty set M of monomials such that $m' \in M$, $m|m' \Rightarrow m \in M$, that is, all divisors of monomials in M are also in M. In particular, $1 \in M$ for each order ideal of monomials M. For a monomial m, we write $\deg m$ to denote the total degree of m computed in the standard fashion (with $\deg X_i = 1$). A sequence of integers $H_0, H_1, H_2, \ldots$ is called on *O-sequence* if there exists an order ideal of monomials M such that for each i, $H_i = |\{m \in M: \deg m = i\}|$.

We define an ordering on the set of all monomials in $X_1, \ldots, X_n$ by $m < m'$ if $\deg m < \deg m'$ or if $\deg m = \deg m'$ and m precedes m' in the lexicographic order induced by $X_1 < X_2 < \ldots X_n$ (so, for instance, $X_1 < X_1^2 < X_2 < X_1X_2 < X_2^2$). The following theorem is due to Stanley [39], [36], who attributes it essentially to Macaulay [21].

Theorem 7: Let $H_0, H_1, H_2, \ldots$ be a sequence of integers. The following four conditions are equivalent.

(i) There exists a standard graded k-algebra A with Hilbert function $H(A, i) = H_i$ for $i \geq 0$.

(ii) $H_0, H_1, H_2, \ldots$ is an O-sequence.

(iii) $H_0 = 1$, $H_1 \geq 0$ and for $i \geq 1$

$$0 \leq H_{i+1} \leq H_i^{\langle i \rangle}.$$

(iv) Let $n = H_1$, and for each $i \geq 0$, let M_i be the (lexicographically) first H_i monomials of degree i in n variables. Then $M = \bigcup_{i \geq 0} M_i$ is an order ideal of monomials.

The hard part of this theorem is the implication (ii) ⇒ (iv), which was proved by Macaulay. The proof of a more general version can be found in [8]. Condition (iii) was first considered by Stanley [35] [36]. A proof of the equivalence of (iii) and (iv) can be found in [4, Prop. 1]. The connection between the algebra and the combinatorics is provided by the equivalence of (i) and (ii), which we consider further.

To see (ii) ⇒ (i), suppose M is an order ideal of monomials in $X_1, \ldots, X_n$ counted by the O-sequence H_i. Let $A = k[X_1, \ldots, X_n]/I$, where I is the homogeneous ideal generated by those monomials not in M. Then $H(A, i) = H_i$ for each i. On the other hand, (i) ⇒ (ii) is a consequence of the following result [39, Theorem 2.1].

Lemma 8: Let A be a standard graded k-algebra and let $x_1, \ldots, x_n$ be a set of homogeneous elements of degree 1 which generate A as a k-algebra. Then there is an order ideal of monomials in $x_1, \ldots, x_n$ which forms a vector space basis for A over k.

The proof of Lemma 8 proceeds by a greedy algorithm. Inductively form a k-basis M for A by putting 1 in M, and continue at each step by choosing the first monomial (in the ordering defined above) which is linearly independent of those already in M. It follows easily that M is a k-basis which is an order ideal of monomials.

Theorem 7 allows us to prove Stanley's Upper Bound Theorem for Cohen-Macaulay complexes [36, Corollary 4.4].

Theorem 9: Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with n vertices. Then for $i = 0, \ldots, d$,

$$h_i(\Delta) \leq \binom{n-d+i-1}{i}.$$

Proof: We know that $h(\Delta)$ is the Hilbert function of a standard graded k-algebra. One can check directly from the definition of h that $h_1(\Delta) = n - d$. Otherwise, using the notation of Corollary 6, note that

$$\begin{aligned} h_1(\Delta) &= \dim_k B_1 \\ &= \dim_k k[\Delta]_1/I_1 \\ &= H(k[\Delta], 1) - H(I, 1) \end{aligned}$$

where $I = \langle \theta_1, \ldots, \theta_d \rangle$. We know that $H(k[\Delta], 1) = n$, and since the θ_i's are all homogeneous of degree 1 and linearly independent (the are *algebraically* independent), we have $H(I, 1) = d$, proving the assertion.

We complete the proof in two different ways. First, by (ii) of Theorem 7, there is an order ideal of monomials in $h_1(\Delta) = n - d$ variables having $h_i(\Delta)$ monomials of degree i. This can be no more than the *total* number of monomials of degree i with $n - d$ variables, which is $\binom{n-d+i-1}{i}$. Alternatively, we proceed by induction. Note that

$$h_1(\Delta) = n - d = \binom{n-d}{1},$$

and if

$$h_{i-1}(\Delta) \leq \binom{n-d+i-2}{i-1}$$

then by (iii) of Theorem 7

$$\begin{aligned} h_i(\Delta) &\leq h_{i-1}^{\langle i-1 \rangle} \\ &\leq \binom{n-d+i-2}{i-1}^{\langle i-1 \rangle} \\ &= \binom{n-d+i-1}{i}, \end{aligned}$$

and again the result follows.

Note that (due to the presence of the Dehn-Sommerville equations) the Upper Bound Theorem for spheres and polytopes requires the inequalities in Theorem 9 only for i up to $[d/2]$. The rest are redundant in this case. For general Cohen Macaulay complexes, however, the later inequalities are not necessarily consequences of the earlier ones. For Cohen-Macaulay complexes that do not satisfy the Dehn-Sommerville equations (simplicial balls, for example), Theorem 9 does provide an upper bound on the f-vector, though it is not that given by (1). See [30] for a treatment of upper bounds in even more general complexes.

Finally, we have noted that the Cohen-Macaulay property, and thus the Upper Bound Theorem, for the boundary complex of a simplicial polytope follows from the shellability of polytopes. But this is precisely the property needed in McMullen's proof for this case (see [22] or [25]), although he requires that certain special shellings exist, and this may not be true for a general shellable sphere. Further, not all spheres are shellable.

Reisner's Theorem: When $k[\Delta]$ is Cohen-Macaulay

The question of whether one could characterize topologically those complexes Δ for which $k[\Delta]$ is Cohen-Macaulay was first raised by Hochster [13], who was originally motivated by questions in invariant theory. A complete answer was given by Reisner [29, Theorem 1], whose work was done independently of that of Stanley on upper bounds [35], and who was unaware of the implications of his work in enumeration theory.

To describe Reisner's conditions, we need the following notation. If F is a face in the simplicial complex Δ, the *link* of F in Δ, denoted $lk[F,\Delta]$, is defined by

$$lk[F,\Delta]=\{G\in\Delta: G\cap F=\emptyset,\ G\cup F\in\Delta\}.$$

For each F, $lk[F,\Delta]$ is a subcomplex of Δ. (It is the *contraction* of F in the case that Δ is the complex of independent sets in a matroid.) Note that $lk[\emptyset,\Delta]=\Delta$.

Theorem 10: For a simplicial complex Δ, $k[\Delta]$ is Cohen-Macaulay if and only if for each face F of Δ, including $F=\emptyset$,

$$\tilde{H}_i(lk[F,\Delta];k)=0 \quad \text{for } i<\dim lk[F,\Delta],$$

that is, the reduced homology of $lk[F,\Delta]$ with coefficients in k vanishes except possibly in the dimension of $lk[F,\Delta]$.

It is known that if Δ is a triangulation of a manifold or a manifold with boundary, then the homology conditions on the links of nonempty faces F automatically hold, so it reduces to a requirement that the reduced homology of Δ vanishes below $\dim\Delta$. In particular, this is true for triangulations of balls and spheres. (See, for example, [12].) It is easily checked that this result implies that for 1-dimensional complexes (i.e. graphs), being Cohen-Macaulay is equivalent to being connected. A triangulation of the projective plane shows that whether $k[\Delta]$ is Cohen-Macaulay for a given Δ may depend upon the choice of k [29, Remark 3]. It is relatively easy to show using these conditions that each shellable complex is Cohen-Macaulay. (See [9] for the definition of shellable complex.)

To help understand the meaning of this result, we will briefly describe the notion of the reduced homology groups of a simplicial complex with coefficients in a field k. It will be clear that these conditions are really combinatorial and algebraic in nature.

For a d-dimensional simplicial complex Δ, let C_i, $i=-1,0,1,\ldots,d$, be the k-vector space (formally) spanned by all i-faces of Δ, the so-called space of i-*chains* of Δ. C_i is merely a vector space whose basis elements correspond to the i-faces of Δ, or, equivalently, it is the subspace of $k[\Delta]$, spanned by all square-free monomials. Let $V=\{v_1,\ldots,v_n\}$ be the set of vertices of Δ, and for an r-face $F=\{v_{i_0},v_{i_1},\ldots,v_{i_r}\}$, where $i_0<i_1<\ldots<i_r$, we define F_j to be the $(r-1)$-face $F\backslash\{v_{i_j}\}$, for $j=0,\ldots,r$. For $r=0,\ldots,d$, we define the r^{th} *boundary map*, a linear transformation

$$\partial_r: C_r\to C_{r-1},$$

by specifying it on the basis of C_r and extending by linearity. For an r-face F, define

$$\partial_r(F)=\sum_{j=0}^{r}(-1)^j F_j$$
$$=F_0-F_1+F_2-\ldots\pm F_r.$$

For convenience, we define $\partial_{d+1}=0$.

It is not hard to check that with this definition of boundary map, the composition $\partial_r\partial_{r+1}=0$, for $r=0,\ldots,d$. Thus for each such r we have $\operatorname{Im}\partial_{r+1}\subset\operatorname{Ker}\partial_r$, where Im and Ker denote the image and the kernel (nullspace) respectively. We now can define the r^{th} *reduced homology* of Δ with coefficients in k to be the quotient vector space

$$\tilde{H}_r(\Delta;k)=\operatorname{Ker}\partial_r/\operatorname{Im}\partial_{r+1}$$

for $r=0,1,\ldots,d$. (To define the regular (non-reduced) homology, one proceeds in the same way, except with $C_{-1}=0$. The only difference is in the dimension of the 0^{th} space, which is the number of connected components of Δ in the regular case, and one less in the reduced case. See, for example, [12] or [28] for details.)

A discussion of the proof of Reisner's Theorem is beyond the scope of this paper. It involves the use of a characterization of the Cohen-Macaulay property via the therory of local cohomology of rings. The homology of Δ arises out of the calculation of the cohomology of $k[\Delta]$. Other proofs of this result can be found in [14] and [49], the latter being the most elementary.

Necessity and Sufficiency of the McMullen Conditions

In light of Theorem 7, we can give the following restatement of McMullen's conditions.

Theorem 11: An integer vector $h=(h_0,h_1,\ldots,h_d)$ is the h-vector of a simplicial d-polytope if and only if it satisfies the Dehn-Sommerville equations ($h_i=h_{d-i}$) and the sequence of differences

$$H_0,H_1,\ldots,H_{[d/2]},$$

given by $H_0=h_0$ and $H_i=h_i-h_{i-1}$ for $i=1,\ldots,[d/2]$, is an O-sequence.

The proof of the sufficiency of these conditions makes use of the special order ideal of monomials provided by (iv) of Theorem 7, when the sequence of H_i's is an O-sequence, to construct a simplicial d-polytope P such that $h=h(P)$. The idea of this construction is to associate with each monomial in this order ideal a facet of the cyclic polytope $C(n,d+1)$, where $n=h_1+d$. One shows that for suitable choice of $t_1<t_2<\ldots<t_n$, there will be a point z in R^{d+1} which is beyond precisely those facets of $C(n,d+1)$ corresponding to the order ideal and beneath the rest. The desired polytope P turns out to be the intersection of $P'=\operatorname{conv}(C(n,d+1)\cup\{z\})$ with a hyperplane strictly separating z from

$C(n, d+1)$. The idea of constructing polytopes in this way over cyclic polytopes was inspired by a construction of Klee [17].

The association of facets of $C(n, d+1)$ to monomials depends on a combinatorial characterization of the facets of cyclic polytopes known as *Gale's evenness criterion*, which states roughly that between any two $x(t_i)$ not in a given facet, there must be an even number of $x(t_i)$ in the facet (with between being in the sense of the ordering of the t_i's). This induces a pairing of some of the vertices in any facet, and in the facet associated to a given monomial, the exponent of X_i in the monomial denotes the number of pairs of vertices which are to be displaced i units to the right from the position of this pair in the facet $\{x(t_1), \ldots, x(t_{d+1})\}$. Thus, considering just the incidence vector of the $x(t_i)$'s in a facet, the monomial 1 corresponds to the facet $(1, 1, \ldots, 1, 0, \ldots, 0)$, X_1 to the facet $(1, \ldots, 1, 0, 1, 1, 0, \ldots, 0)$, X_2 to $(1, \ldots, 1, 0, 0, 1, 1, 0, \ldots, 0)$ and X_1^2 to $(1, \ldots, 1, 0, 1, 1, 1, 1, 0, \ldots, 0)$. See [3] for a longer sketch of the proof, and [4] for all the details.

To show that the conditions are necessary, it is enough, by (i) of Therorem 7, to find a standard graded k-algebra with Hilbert function $H(A, i) = H_i$ for $i = 0, 1, \ldots, [d/2]$. Recall that by Corollary 6, if $h = h(P)$ for a simplicial d-polytope P, then $h_i = H(B, i)$ for the graded k-algebra $B = k[P]/\langle\theta_1, \ldots, \theta_d\rangle$, where the θ_i's form a system of parameters for $k[P]$. Now

$$B = B_0 \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_d$$

where each B_i is a k-vector space of dimension h_i. Stanley shows [42] that for a suitable choice of θ_i's there is an element $w \in B_1$ such that the linear map

$$T_i\colon B_i \to B_{i+1}$$

given by $T_i(x) = w x$ (recall that the B_i's form a grading) is injective (one-to-one) for $i < [d/2]$. Since w is homogeneous, the ring $A = B/\langle w\rangle$ is a graded k-algebra. Now take $i < [d/2]$. Since T_i is injective we have $\dim_k \operatorname{Im} T_i = \dim_k B_i$ and so

$$\begin{aligned} H(A, i+1) &= \dim_k B_{i+1}/\operatorname{Im} T_i \\ &= \dim_k B_{i+1} - \dim_k B_i \\ &= h_{i+1} - h_i \\ &= H_{i+1} \end{aligned}$$

Thus the H_i's are seen to form an O-sequence.

To find the element w, Stanley constructs from P an abstract algebraic variety which turns out to be essentially a complex projetive variety. This *toric variety* X_P [10] has the property that for a suitably chosen system of parameters $\theta_1, \ldots, \theta_d$ for $k[P]$, the spaces B_i describe the cohomology of X_P in even dimensions. The existence of w then follows from the so-called "hard Lefschetz theorem" for X_P. While the definition of X_P is beyond the scope of this paper, it is particularly intriguing to note what information about P goes into the construction of X_P. Doing so allows us to briefly discuss an aspect of this subject with potential interest to mathematical programming far beyong the topic at hand.

Let L be a rational convex polyhedral cone (i.e., given by finitely many rational inequalities) in $\mathbb{R}^n$. We associate a graded k-algebra with L in the following way. For each $z \in \mathbb{Z}^n$, where $\mathbb{Z}$ is the ring of integers, let X^z denote the monomial $X_1^{z_1} \ldots X_n^{z_n}$, $z = (z_1, \ldots, z_n)$. Now define A_L to be the subring of $k[X_1, \ldots, X_n]$ spanned by $\{X^z : z \in L \cap \mathbb{Z}^n\}$. Since L is a convex cone (so is closed under addition), it is immediate that A_L consists of all k-linear combinations of monomials X^z where z is an integer point of L. Using the fact that L is finitely generated, it is an easy argument to show that A_L is finitely generated as a k-algebra. The Krull dimension of A_L is the same as the usual dimension of the cone L. It is a theorem of Hochster [13] that A_L is Cohen-Macaulay for each L. (See also [10].) In some sense, this result says something about the regularity of the lattice points in convex cones. In general, a study of these rings may prove useful in understanding the difficult problems involving integral solutions to linear systems. For work related to this topic, see [10], [13], [33], [34], [39], [37], [40], [44], [43] and [45], the last two of which are surveys.

To "describe" the variety X_P, let us assume that the origin is an interior point of P and that all vertices of P are rational vectors. (Since P is simplicial, a small perturbation of the vertices of P can assure this without changing the combinatorial structure of P.) For each face F of P let cone(F) be the (rational) cone spanned by F (and the origin), and let L be the polar cone to cone(F), i.e., $L^* = \text{cone}(F)$. Denote by $A(F)$ the k-algebra A_L defined above. The toric variety is made up of the so-called affine schemes Spec$A(F)$, called affine toric varieties. The recipe for constructing X_P from the Spec$A(F)$ is derived directly from how the various F's fit together in P. (See [10] or [46] for more details.) We should point out here that the crucial property of X_P for this proof, its projectivity, is equivalent to the convexity of P. Thus this proof does not work for non-convex spherical complexes, and so the necessity of McMullen's conditions (other than the Dehn-Sommerville equations) remains an open question for triangulations of general spheres, piecewise linear (i.e., combinatorial [9]) spheres or even shellable spheres. This situation should be contrasted with the case of upper bounds.

The important thing to note here is that the information used to specify X_P, and then prove the necessity of last two of McMullens's conditions, come from the location of integer points in the polars of the cones specified by the faces of P. This provides a rather surprising connection between two seemingly unrelated aspects of polyhedral theory: enumeration and integrality. Whether or not this indicates the existence of a deep interplay between these subjects which will aid our understanding and ability to deal with problems of integrality is a question that can only be answered by further research.

Bibliography

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969.

[2] D. W. Barnette, A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.* 46 (1973), 349–354.

[3] L. J. Billera and C. W. Lee, Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes, *Bull. Amer. Math. Soc.* (New Series) 2 (1980), 181–185.

[4] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, *Jour. Combinatorial Theory (A)* 31 (1981), 237–255.

[5] L. J. Billera and C. W. Lee, The numbers of faces of polytope pairs and unbounded polyhedra, *Europ. J. Combin.* 2 (1981), 307–332.

[6] A. Björner, The minimum number of faces of a simple polyhedron, *Europ. J. Combin.* 1 (1980), 27–31.

[7] H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* 29 (1971), 197–205.

[8] G. Clements and B. Lindström, A generalization of a combinatorial theorem of Macauly, *J. Combinatorial Theory* 7 (1969), 230–238.

[9] G. Danaraj and V. Klee, Which spheres are shellable?, *Ann. Discrete Math.* 2 (1978), 32–52.

[10] V. I. Danilov, The geometry of toric varieties, *Russian Math. Surveys* 33:2 (1978), 97–154; translated from *Uspekhi Mat. Nauk.* 33:2 (1978), 85–134.

[11] B. Grünbaum, *Convex Polytopes*, Wiley Interscience, New York, 1967.

[12] P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge Univ. Press, 1960.

[13] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, *Annals of Math.* 96 (1972), 318–337.

[14] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, in *Ring Theory II (Proc. Second Oklahoma Conference)*, B. R. McDonald and R. Morris, eds., Marcel Dekker, New York and Basel, 1977, 171–223.

[15] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. of Chicago Press, 1974.

[16] V. Klee, A combinatorial analogue of Poincare's duality theorem, *Can. Jour. Math.* 16 (1964), 517–531.

[17] V. Klee, Polytope pairs and their relationship to linear programming, *Acta Math.* 133 (1974), 1–25.

[18] C. W. Lee, Bounding the numbers of faces of polytope pairs and simple polyhetra, *Ann. Discrete Math.* (to appear).

[18a] C. W. Lee, Two combinatorial properties of a class of simplicial polytopes, Research Report RC 8957, IBM, Yorktown Heights, New York, 1981.

[19] C. W. Lee, Characterizing the numbers of faces of a simplicial convex polytope, in *Convexity and Related Combinatorial Geometry*, D. C. Kay and M. Breen, eds., Marcel Dekker, New York and Basel, 1982.

[20] F. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 19, Cambridge Univ. Press, London, 1916.

[21] F. S. Macaulay, Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* 26 (1927), 531–555.

[22] P. McMullen, The maximum numbers of faces of a convex polytope, *Mathematika* 17 (1970), 179–184.

[23] P. McMullen, The minimum number of faces of a convex polytope, *J. London Math. Soc. (2)* 3 (1971), 350–354.

[24] P. McMullen, The numbers of faces of simplicial polytopes, *Israel J. Math.* 9 (1971), 559–570.

[25] P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, London Math Soc. Lecture Notes Series, Vol. 3, 1971.

[26] P. McMullen and D. W. Walkup, A generalized lower bound conjecture for simplicial polytopes, *Mathematika* 18 (1971), 264–273.

[27] T. S. Motzkin, Comonotone curves and polyhedra, Abstract 111, *Bull. Amer. Math. Soc.* 63 (1957), p. 35.

[28] L. S. Pontryagin, *Foundations of Combinatorial Topology*, Graylock Press, Rochester, N.Y., 1952.

[29] G. Reisner, Cohen-Macaulay quotients of polynomial rings, *Advances in Math.* 21 (1976), 30–49.

[30] P. Schenzel, On the number of faces of simplicial complexes and the purity of Frobenius, *Math. Z.* 178 (1981), 125–142.

[31] W. Smoke, Dimension and multiplicity for graded algebras, *J. Algebra* 21 (1972), 149–173.

[32] D. M. Y. Sommerville, The relations connecting the angle-sums and volume of a polytope in space of n dimensions, *Proc. Roy. Soc. London*, ser. A 115 (1927), 103–119.

[33] R. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.* 40 (1973), 607–632.

[34] R. Stanley, Combinatorial reciprocity theorems, *Advances in Math.* 14 (1974), 194–253.

[35] R. Stanley, Cohen-Macaulay rings and constructible polytopes, *Bull. Am. Math. Soc.* 81 (1975) 133–135.

[36] R. Stanley, The upper bound conjecture and Cohen-Macaulay rings, *Studies in Applied Math.* 54 (1975), 135–142.

[37] R. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings, *Duke Math. J.* 43 (1976), 511–531.

[38] R. Stanley, Cohen-Macaulay complexes, in *Higher Combinatorics*, M. Aiger, ed., Reidel, Dordrecht and Boston, 1977, 51–62.

[39] R. Stanley, Hilbert functions of graded algebras, *Advances in Math.* 28 (1978), 57–83.

[40] R. Stanley, Combinatorics and invariant theory, in *Relations between Combinatorics and Other Parts of Mathematics*, D. K. Ray-Chaudhuri, ed., Proceedings of Symposia in Pure Mathematics, Vol. XXXIV, American Mathematical Society, Providence, 1979, 345–355.

[41] R. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Math.* 6 (1980), 333–342.

[42] R. Stanley, The number of faces of a simplicial convex polytope, *Advances in Math.* 35 (1980), 236–238.

[43] R. Stanley, Interactions between commutative algebra and combinatorics, Report No. 4, Mathematiska Institutionen, Stockholms Universitet, Sweden, 1982.

[44] R. Stanley, Linear diophantine equations and local cohomology, *Inv. Math.* 68 (1982), 175–193.

[45] R. Stanley, An introduction to combinatorial algebra, *Proceedings of the Silver Jubilee Conference on Combinatorics*, University of Waterloo (to appear).

[46] B. Tessier, Varietes toriques et polytopes, *Séminaire Bourbaki, 1980/81*, no. 565, Lecture Notes in Mathematics, Vol. 901, Springer Verlag, Berlin, Heidelberg, 1981.

[47] D. W. Walkup, The lower bound conjecture for 3- and 4-manifolds, *Acta Math.* 125 (1970), 75–107.

[48] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, van Nostrand, Princeton, N.J., 1960.

[49] K. Baclawski and A. M. Garsia, Combinatorial decompositions of a class of rings, *Advances in Math.* 39 (1981), 155–184.

[50] R. Stanley, Balanced Cohen-Macaulay complexes, *Trans. Amer. Math. Soc.* 249 (1979), 139–157.