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A LOWER BOUND FOR ADJACENCIES ON THE TRAVELING SALESMAN POLYTOPE*

A. SARANGARAJAN†

Abstract. We study adjacency of vertices on T_n, the asymmetric traveling salesman polytope of degree n. Applying a result of G. Boccara [*Discrete Math.*, 29 (1980), pp. 105–134] to permutation groups, we show that T_n has $\Omega((n-1)(n-2)!^2 \log n)$ edges, implying that the degree of a vertex of T_n is $\Omega((n-2)! \log n)$. We conjecture the degree to be $\Omega((n-2)!(\log n)^k)$ for any positive integer k. We compute the density function δ_n given by the fraction of the total number of vertices adjacent to a given vertex for small values of n, and conjecture that it decreases with n.

Key words. asymmetric traveling salesman polytope, adjacency

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1. Introduction. The *asymmetric traveling salesman polytope* (ATSP) is one of the widely studied polytopes in combinatorial optimization for its intrinsic relation to the traveling salesman problem. Many results are known about the facets of this polytope (see chapter 8 of [4] for a detailed survey), but not much is known about adjacency of vertices on this polytope. From an optimization point of view, studying adjacency helps in estimating the size of exact neighborhoods for local search algorithms. Such estimates have been carried out in [7] for the symmetric TSP.

The most common relaxation of the ATSP is the Birkhoff (or assignment) polytope B_n. We study the relationship between the faces of B_n and T_n, specifically the edges of T_n arising from certain two-dimensional faces of B_n. These edges are counted using a result of Boccara [2] giving us the lower bound for the number of edges of T_n. In particular, we show that T_n has $\Omega((n-1)(n-2)!^2 \log n)$ edges and thus each vertex of T_n has degree $\Omega((n-2)! \log n)$.

We define some terms that will be used for the rest of this paper. Let $\mathcal{S}_n$ be the *symmetric group* of degree n, i.e., the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. We call a permutation even (odd) if it can be expressed as a product of an even (odd) number of transpositions. Two permutations are said to have different parity if one is even and the other odd. Given $\sigma \in \mathcal{S}_n$, we define the corresponding $n \times n$ permutation matrix $X_\sigma \in \mathbf{R}^{n^2}$ by

$$(X_\sigma)_{ij} := \begin{cases} 1 & \text{if } \sigma(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by B_n the *Birkhoff polytope* of degree n, given by

$$B_n := \operatorname{conv}\{X_\sigma : \sigma \in \mathcal{S}_n\}.$$

Let

$$\mathcal{T}_n := \{\sigma \in \mathcal{S}_n : \sigma \text{ is a cycle of length } n\} \subset \mathcal{S}_n.$$

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†Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003 (sarang@math.lsa.umich.edu).

The ATSP of degree n is defined by

$$T_n := \text{conv}\{X_\sigma : \sigma \in \mathcal{T}_n\},$$

so that

$$T_n \subset B_n \subset \mathbf{R}^{n^2}.$$

Thus if F is a face of B_n, then $F \cap T_n$ is a face of T_n induced by F.

We call two vertices *adjacent* on a polytope P if they form an edge of P. The *graph* of P is a graph whose nodes are the vertices of P with two nodes adjacent if the corresponding vertices are adjacent on P.

A *partition* of n is a sequence of positive integers $\lambda := (\lambda_1, \ldots, \lambda_k)$, with $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and we indicate this by $\lambda \vdash n$. We call λ an even (respectively, odd) partition if $n - k$ is even (respectively, odd). Let $\hat{e} := (1, 1, \ldots, 1)$ be the identity partition. If $\pi \in \mathcal{S}_n$ is a product of k disjoint cycles of lengths $l_1, \ldots, l_k$ (including cycles of length 1) in nonincreasing order, then $(l_1, \ldots, l_k)$ is a partition of n. We call $(l_1, \ldots, l_k)$ the *cycle type* of π. A *composition* of n is a sequence of positive integers $\lambda := \langle \lambda_1, \ldots, \lambda_k \rangle$ with $\sum \lambda_i = n$ and we indicate this by $\lambda \models n$. Hence a partition of n into k parts can define up to $k!$ distinct compositions by permuting the parts of the partition. A k-partition (respectively, a k-composition) of n is a partition (respectively, a composition) of n into k parts.

If $f(n)$ and $g(n)$ are two positive valued functions, then we say $f(n) = \Omega(g(n))$ if there exists a positive constant c such that $g(n) \leq cf(n)$, for all allowable values of n.

2. The edges of the ATSP. We will denote the matrix X_σ by σ. We study faces of B_n induced by a pair of vertices σ, π. The following result found in [1, Proposition 2.1] and in [3] shows that these faces are in fact cubes.

PROPOSITION 2.1. *If $\sigma^{-1}\pi = \prod_{i=1}^{k} C_i \in \mathcal{S}_n$ where $C_1, \ldots, C_k$ are disjoint cycles, then the smallest face $F_{\sigma,\pi}$ of B_n containing both σ and π is a k-cube, where $k \leq \lfloor \frac{n}{2} \rfloor$. The vertices of $F_{\sigma,\pi}$ are given by $\sigma \Pi_{i \in S} C_i$, for $S \subseteq [k]$.*

The convex hull of the vertices of $F_{\sigma,\pi}$ that correspond to cycles of length n is a face of T_n. In particular, if $\sigma, \pi \in \mathcal{T}_n$, and $\sigma^{-1}\pi = C_1C_2$ is a product of two cycles of even length, then $F_{\sigma,\pi}$ is a 2-cube. Since σC_1 and σC_2 have parity different from that of σ, neither can be n-cycles. Thus σ and π are adjacent on T_n. We now find the number of such representations. To do this, we need the following result [2, Corollary 4.8].

PROPOSITION 2.2. *Let $l = (l_1, \ldots, l_k) \vdash n$. Let $g(l)$ be the number of ways of writing a permutation of cycle type l as a product of two n-cycles. Then*

$$g(l) = \frac{2(n-1)!}{n+1} \sum_{I \subseteq \{2,\ldots,k\}} (-1)^{|I|+s(I)} \binom{n}{s(I)}^{-1}, \tag{2.1}$$

if l is an even partition and zero otherwise. Here $s(I) = \sum_{i \in I} l_i$.

Thus if $l = (l_1, l_2)$, and n is even, then

$$g(l) = \frac{2(n-1)!}{n+1} \left(1 - (-1)^{l_1} \binom{n}{l_1}^{-1}\right), \tag{2.2}$$

and if $l = (l_1, l_2, l_3)$, and n is odd, then

$$g(l) = \frac{2(n-1)!}{n+1} \left(1 - (-1)^{l_1} \binom{n}{l_1}^{-1} - (-1)^{l_2} \binom{n}{l_2}^{-1} - (-1)^{l_3} \binom{n}{l_3}^{-1}\right). \tag{2.3}$$

This result is generalized in [6] to give the number of ways of writing a permutation as a product of an arbitrary number of n-cycles.

THEOREM 2.3. *Let e_n be the number of edges of T_n, $n > 3$. Then*

$$e_n = \Omega((n-1)(n-2)!^2 \log n).$$

Proof. Suppose $n = 2m$ is even. Let $\eta \in \mathcal{S}_n$ have cycle type $\lambda_r = (n-2r, 2r)$, $1 \le r \le m/2$. If $\sigma, \pi \in \mathcal{T}_n$, and $\sigma^{-1}\pi = \eta$, then by the argument before Proposition 2.2 σ and π are adjacent on T_n. By (2.2), the number of ways of writing η as a product of two n-cycles is

$$g(\lambda_r) = \frac{2(n-1)!}{n+1}\left(1 - \binom{n}{2r}^{-1}\right) \ge \frac{(n-1)!}{n+1} \quad \text{as } \binom{n}{2r} \ge 2,$$

and every such pair of n-cycles induce an edge in T_n. Now the number of permutations of cycle type λ_r is at least $n!/(4r(n-2r))$. Hence, counting each edge exactly once,

$$\begin{aligned} e_n &\ge \frac{(n-1)!}{2(n+1)} \sum_{r=1}^{\lfloor m/2 \rfloor} \frac{n!}{4r(n-2r)} = \frac{(n-1)!^2}{4(n+1)} \sum_{r=1}^{\lfloor m/2 \rfloor} \left(\frac{1}{2r} + \frac{1}{n-2r}\right) \\ &\ge \frac{(n-1)!^2}{8(n+1)} \sum_{r=1}^{m-1} 1/r \ge \frac{(n-1)!^2}{8(n+1)} \log m = \Omega((n-1)(n-2)!^2 \log n). \end{aligned}$$

If $n = 2m+1$ is odd, then let $\eta \in \mathcal{S}_n$ have cycle type $\lambda'_r = (2m-2r, 2r, 1)$, $1 \le r \le m/2$. By (2.3), the number of pairs of n-cycles whose product is η is

$$g(\lambda'_r) = \frac{2(n-1)!}{n+1}\left(1 - \binom{n}{2m-2r}^{-1} - \binom{n}{2r}^{-1} + \frac{1}{n}\right) \ge \frac{(n-1)!}{n+1},$$

and each such pair induce an edge in T_n. Hence the bound for e_n follows as before. □

The lower bound for the degree now follows from observing that T_n is a vertex symmetric polytope. If $\sigma, \pi \in \mathcal{T}_n$, then there exists $\gamma \in \mathcal{S}_n$ such that $\sigma = \gamma\pi\gamma^{-1}$. Hence vertices π and π_1 are adjacent on T_n if and only if σ is adjacent to $\gamma\pi_1\gamma^{-1}$. As a result the degree of each vertex of T_n is the same value deg(n) and

$$\text{(2.4)} \qquad \deg(n) = \frac{2e_n}{(n-1)!} \ge \frac{(n-1)!}{4(n+1)} \log m = \Omega((n-2)! \log n).$$

3. Further discussions. The degree bound shows the graph of T_n to be fairly dense. This may not seem very surprising considering that the diameter of T_n is 2 as shown in [5]. For the Birkhoff polytope B_n, the degree of a vertex is known to be $\sum_{k=0}^{n-2} \binom{n}{k}(n-k-1)!$, while for the symmetric TSP it is $\Omega(\lfloor \frac{n-1}{2} \rfloor !)$ as shown in [7].

An expression for the number of edges of T_n can be written as

$$\text{(3.1)} \qquad e_n = \sum_{\substack{\lambda \vdash n,\, \lambda \ne \hat{e} \\ \lambda \text{ even}}} \frac{n_\lambda \, g(\lambda)\, h(\lambda)}{2},$$

where n_λ is the number of permutations of cycle type λ and $h(\lambda)$ is the fraction of the pairs of n-cycles (σ, π) which are adjacent on T_n and such that $\sigma^{-1}\pi$ has cycle type λ. Hence $h(\lambda) = 1$ if λ corresponds to a cycle or a product of two cycles of even length. It is natural to ask how large e_n would be if this summation is taken over all k-partitions λ for a fixed k. We estimate this partially.

CONJECTURE 3.1. *For any positive integer k,*

$$deg(n) = \Omega((n-2)!(\log n)^k).$$

The rationale for this conjecture stems from the following argument. From (2.1) it follows that for an even partition $l = (l_1, \ldots, l_k) \vdash n$,

$$g(l) \geq \frac{2(n-1)!}{n+1}\left(1 - \frac{2^{k-1}}{n}\right) \geq \frac{(n-1)!}{n+1} \qquad \text{for } n \geq 2^k,$$

since each term in the summation in (2.1) is at least $-1/n$ except for the term corresponding to the empty set which is 1. Let n_i be the number of permutations that can be expressed as a product of i disjoint cycles (including cycles of length 1). We estimate the asymptotic growth of n_i with i *fixed.* We have

$$n_i = \sum_{\langle l_1, \ldots, l_i \rangle \models n} \frac{n!}{i! l_1 l_2 \cdots l_i} = \Omega((n-1)!(\log n)^{i-1}).$$

The above sum is taken over all i-compositions of n. The last equality follows from the lemma below.

LEMMA 3.2. *Let*

$$f_i(n) := \sum_{\langle l_1, \ldots, l_i \rangle \models n} \frac{n}{l_1 l_2 \cdots l_i},$$

the sum being taken over all i-compositions of n. Then $f_i(n) = \Omega((\log n)^{i-1})$. In particular, we show that if $n \geq 2^i$, then $f_i(n) \geq c_i(\log n)^{i-1}$, $c_i = 2^{-(i-1)(i-2)/2}$.

Proof. We prove this by induction on i. The result is straightforward for $i = 1$. Then for $i > 1$ and $n \geq 2^i$,

$$f_i(n) \geq \sum_{l_1=1}^{\lfloor n/2 \rfloor} \frac{n}{l_1(n-l_1)} f_{i-1}(n-l_1).$$

As $l_1 \leq n/2$, we have by induction $f_{i-1}(n-l_1) \geq c_{i-1}(\log(n-l_1))^{i-2} \geq c_i(\log n)^{i-2}$ since $\log(n-l_1) \geq \log(n/2) \geq 1/2 \log n$. Hence

$$f_i(n) \geq c_i(\log n)^{i-2} \sum_{l_1=1}^{\lfloor n/2 \rfloor} \left(\frac{1}{l_1} + \frac{1}{n-l_1}\right) \geq c_i(\log n)^{i-1} = \Omega((\log n)^{i-1}),$$

proving the result. □

The conjecture amounts to showing that for each k, there exists a positive constant h_k such that $h(\lambda) \geq h_k$ for any k-partition λ of n and any n such that $n-k$ is even. If this were true, then summing (3.1) over all k-partitions of n yields

$$e_n \geq \frac{h_k n_k (n-1)!}{2(n+1)} = \Omega((n-1)(n-2)!^2(\log n)^{k-1})$$

when $n - k$ is even. If $n - k$ is odd, then we sum (3.1) over all $(k+1)$-partitions of n to get a bound of $\Omega((n-1)(n-2)!^2(\log n)^k)$ for e_n. This yields the conjectured bound for $\deg(n)$.

We define the *density* δ_n of T_n to be the fraction of the total number of vertices adjacent to a given vertex, i.e., $\delta_n := \deg(n)/((n-1)! - 1)$. Our bounds on $\deg(n)$ show that $\delta_n = \Omega(\log n/n)$. It would be desirable to bound this number either away from 0 or below 1 as $n \to \infty$. Since T_3 and T_4 are simplices and T_5 is 2-neighborly, they have a density of 1. Using MAPLE, some other densities were computed by constructing the cube $F_{\sigma,\pi}$ for a fixed n-cycle σ and examining when the n-cycle π was adjacent to σ. These are tabulated below:

n	$\deg(n)$	δ_n
6	110	0.92
7	628	0.87
8	4174	0.83
9	32433	0.80

We observe that δ_n decreases with n for $n \leq 9$. We conjecture that this holds in general.

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REFERENCES

[1] L. J. BILLERA AND A. SARANGARAJAN, *The combinatorics of permutation polytopes*, in Algebraic Combinatorics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, L. Billera, C. Greene, R. Simion, and R. Stanley, eds., American Mathematical Society, Providence, RI, 1994.

[2] G. BOCCARA, *Nombre de représentations d'une permutation comme produit de deux cycles de longueurs données*, Discrete Math., 29 (1980), pp. 105–134 (in French).

[3] J. HELLER, *Neighbor relations on the convex hull of cyclic permutations*, Pacific J. Math., 6 (1956), pp. 467–477.

[4] E. L. LAWLER, J. K. LENSTRA, A. H. G. RINNOOY KAN, AND D. B. SHMOYS, EDS., *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, John Wiley, New York, 1985.

[5] M. W. PADBURG AND M. R. RAO, *The traveling salesman problem and a class of polyhedra of diameter two*, Math. Programming, 7 (1974), pp. 32–45.

[6] R. P. STANLEY, *Factorization of permutations into n-cycles*, Discrete Math., 37 (1981), pp. 255–262.

[7] P. WEINER, S. L. SAVAGE, AND A. BAGCHI, *Neighborhood search algorithms for guaranteeing optimal traveling salesman tours must be inefficient*, J. Comput. Systems Sci., 12 (1976), pp. 25–35.