

The Stanley decomposition of the harmonic oscillator

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ABSTRACT

This paper gives a new decomposition for the ring of polynomial functions on the variety of $(n+1) \times (n+1)$ complex matrices of rank less than or equal to one. This involves decomposing the monoid

$$\mathcal{M}_n = \{(j, k) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1} \mid |j| = |k|\}$$

into a finite disjoint union of translates of \mathbb{N} cones based on certain $2n$ simplices in \mathbb{R}^{2n+2} . As a consequence we have a method for writing the normal form of a perturbed $n+1$ dimensional harmonic oscillator in a unique way.

1. INTRODUCTION

Let \mathcal{A}_n be the ring of polynomial functions on the $2n$ dimensional complex affine variety $M_{n+1,1}$ of $(n+1) \times (n+1)$ complex matrices of rank less than or equal to one. The main point of this article is to give a new decomposition of \mathcal{A}_n , which we call the Stanley decomposition.

In order to describe this decomposition we need a more explicit representation of the ring \mathcal{A}_n . In § 5 we show that \mathcal{A}_n is isomorphic to the ring

$$\mathcal{B}_n = \mathbb{C}[\mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*]^{\mathbb{C}^*}$$

of polynomials on $\mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*$ which are invariant under the \mathbb{C}^* action

$$(t, (x_0, \dots, x_n, y_0, \dots, y_n)) \mapsto (tx_0, \dots, tx_n, t^{-1}y_0, \dots, t^{-1}y_n).$$

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Clearly the monomials $\pi_{ij} = x_i y_j$, $0 \leq i, j \leq n$ are invariant under the \mathbb{C}^* action. In fact $\{\pi_{ij}\}$ generate \mathcal{B}_n . This is shown as follows. Using multi-index notation, the monomial $M = x^i y^j$ is invariant under \mathbb{C}^* iff for all $t \in \mathbb{C}^*$ $M = t^{|i| - |j|} M$, that is, iff

$$|i| = i_0 + \dots + i_n = |j| = j_0 + \dots + j_n.$$

Writing the factors of an invariant monomial as two lists

$$(1a) \quad \frac{i_0}{x_0, \dots, x_0}, \frac{i_1}{x_1, \dots, x_1}, \dots, \frac{i_n}{x_n, \dots, x_n}$$

$$(1b) \quad \frac{j_0}{y_0, \dots, y_0}, \frac{j_1}{y_1, \dots, y_1}, \dots, \frac{j_n}{y_n, \dots, y_n}.$$

which have the same number of entries since $|i| = |j|$, and pairing off the corresponding entries, shows that an invariant M is a product of suitable π_{ij} . Clearly we have the relations

$$(2) \quad \pi_{ij} \pi_{kl} = \pi_{il} \pi_{kj}$$

for all $0 \leq i, j, k, l \leq n$. In § 5 it is shown that these are the only relations among the generators of \mathcal{B}_n .

Now we are in a position to describe the Stanley decomposition of \mathcal{B}_n . Consider the \mathbb{N} monoid

$$\mathcal{M}_n = \{(i, j) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1} \mid |i| = |j|\}$$

which is the exponent monoid of \mathcal{B}_n . By a theorem of Stanley [13, p. 191], \mathcal{M}_n can be written as a finite disjoint union of translates of N commutative free \mathbb{N} monoids K_{σ_l} on $2n+1$ generators (see Theorem 3, § 3). In symbols,

$$(3) \quad \mathcal{M}_n = \bigcup_{l=1}^N (\eta_l + K_{\sigma_l}).$$

From the decomposition of \mathcal{M}_n follows the Stanley decomposition of \mathcal{B}_n ; namely,

$$(4) \quad \mathcal{B}_n = \sum_{l=1}^N \oplus x^{\alpha_l} y^{\beta_l} \mathbb{C}[\{\pi_{rs} \mid (r, s) \in \sigma_l\}]$$

where $\eta_l = (\alpha_l, \beta_l) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$. And conversely, from the Stanley decomposition of \mathcal{B}_n one obtains the decomposition of \mathcal{M}_n . Unfortunately Stanley's proof of the decomposition (3) is nonconstructive. What we do in this paper is to give an explicit constructive proof of (4).

The precise algebraic-geometric meaning of (4) is unknown to the authors. Algebraically seen the Stanley decomposition of \mathcal{B}_n is, on the one hand, weaker than the statement that \mathcal{B}_n is Cohen-Macaulay (namely, there are elements $z_1, \dots, z_r \in \mathcal{B}_n$ called polynomial generators and elements η_1, \dots, η_s called separators such that

$$\mathcal{B}_n = \sum_{l=1}^s \oplus \eta_l \mathbb{C}[z_1, \dots, z_r])$$

because the choice of polynomial generators depends on the summand. On the other hand, the Stanley decomposition is stronger than Cohen-Macaulay because the separators are monomials.

We end this introduction by writing down the Stanley decomposition of \mathcal{B}_2 . Given a monomial $M = x^i y^j$ with

$$|i| = i_0 + i_1 + i_2 = j_0 + j_1 + j_2 = |j|,$$

using the relations (2) it can be straightened into a unique monomial

$$\tilde{M} = (x_{i_1}, y_{j_1})(x_{i_2}, y_{j_2}) \cdots (x_{i_l}, y_{j_l})$$

(written also as $(i_1, j_1)(i_2, j_2) \cdots (i_l, j_l)$) where

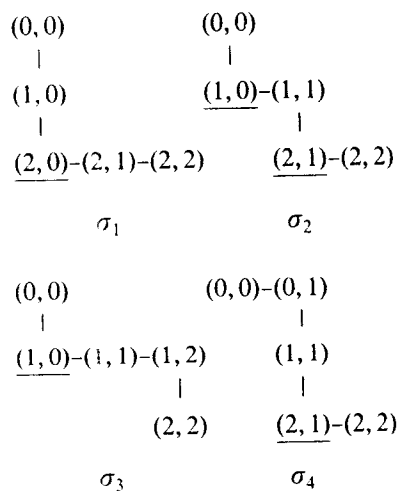
$$(5) \quad i_1 \leq i_2 \leq \cdots \leq i_l \text{ and } j_1 \leq j_2 \leq \cdots \leq j_l$$

(see § 2). Plotting the nodes (i, j) of \tilde{M} as in Figure 1, we see that (5) means that no node (i_k, j_k) in \tilde{M} lies strictly above and strictly to the right of any

(i, j)	0	1	2	j
0	•			
1				
2		•	•	
i				

Figure 1. Nodes of the monomial $(x_0 y_0)^2 (x_2 y_1) (x_2 y_2)^2$

other node of \tilde{M} . Thus all nodes of \tilde{M} lie on a **maximal monotone path**, i.e. a path beginning at $(0, 0)$ and ending at $(2, 2)$ which is made up of moves to the right or moves down. The maximal monotone paths of Figure 1 are given in Figure 2 below.



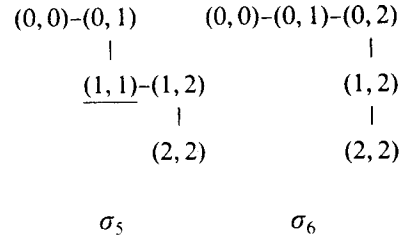


Figure 2. Maximal monotone paths. Nodes of σ_l corresponding to corners are underlined.

A **corner** of a maximal monotone path is a node which is arrived at by a move downward and left by a move to the right. Corners in Figure 2 are the nodes which are underlined. The Stanley decomposition of \mathcal{B}_2 is

$$\begin{aligned}
 \mathcal{B}_2 = & (x_2 y_0) \mathbb{C}[x_0 y_0, x_1 y_0, x_2 y_0, x_2 y_1, x_2 y_2] \\
 & \oplus (x_1 y_0)(x_2 y_1) \mathbb{C}[x_0 y_0, x_1 y_0, x_1 y_1, x_2 y_1, x_2 y_2] \\
 & \oplus (x_1 y_0) \mathbb{C}[x_0 y_0, x_1 y_0, x_1 y_1, x_1 y_2, x_2 y_2] \\
 & \oplus (x_2 y_1) \mathbb{C}[x_0 y_0, x_0 y_1, x_1 y_1, x_2 y_1, x_2 y_2] \\
 & \oplus (x_1 y_1) \mathbb{C}[x_0 y_0, x_0 y_1, x_1 y_1, x_1 y_2, x_2 y_2] \\
 & \oplus \mathbb{C}[x_0 y_0, x_0 y_1, x_0 y_2, x_1 y_2, x_2 y_2].
 \end{aligned}$$

The monomials $x_i y_j$ which are polynomial generators in a given summand correspond to the nodes (i, j) in a maximal monotone path. The separators are the products of monomials corresponding to the corners of the monotone path.

2. STANDARD MONOMIALS AND THE STRAIGHTENING PROCESS

In this section we give a proof of the Stanley decomposition for the algebra \mathcal{B}_n using the technique of straightening. A similar proof is given in [5] for the Stanley decomposition of the algebra of polynomials on the Grassmannian of 2-planes in n space.

Associate to the monomial π_{jk} the **bracket** (j, k) . We say that $(j, k) < (j', k')$ if $k < k'$ or, if $k = k'$, $j < j'$. This gives a complete ordering on the brackets. We associate to the monomial

$$M = \pi_{i_1 j_1} \pi_{i_2 j_2} \cdots \pi_{i_r j_r} \quad 0 \leq i_l, j_l \leq n \text{ for } l = 1, \dots, r$$

the **bracket monomial**

$$M = (i_1, j_1)(i_2, j_2) \cdots (i_r, j_r),$$

which we also write as a 2 column tableau

$$(6) \quad M = \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_r & j_r \end{pmatrix}.$$

Here we use the following convention on the indices:

$$(i_1, j_1) \leq (i_2, j_2) \leq \dots \leq (i_r, j_r).$$

The number r is called the **length** of M , i.e. $l(M) = r$. The pair (i_r, j_r) is called the **height** of M , i.e. $ht(M) = (i_r, j_r)$.

Thus the \mathbb{Z} algebra of \mathbb{C}^* invariant polynomials is the bracket algebra \mathbf{B} generated by the brackets (j, k) where $1 \leq j, k \leq n$. The relations (2) become the basic syzygy

$$(7) \quad (j, k)(l, m) = (j, m)(l, k), \quad 1 \leq j, k, l, m \leq n$$

among the brackets. In terms of 2×2 tableaux, (7) reads

$$(8) \quad \begin{pmatrix} j & k \\ l & m \end{pmatrix} = \begin{pmatrix} l & k \\ j & m \end{pmatrix}$$

DEFINITION. A tableau (6) is called a **standard tableau** if $i_1 \leq i_2 \leq \dots \leq i_r$.

If we represent (j, k) as nodes in $\mathbb{Z}^2 \cap ([1, n] \times [1, n])$, then the nodes $\{(j_l, k_l) | 1 \leq l \leq r\}$ represent a standard tableau if no (j_l, k_l) lies strictly above and strictly to the right of a $(j_{l'}, k_{l'})$. An arbitrary tableau can be **straightened** into a standard tableau by applying the basic syzygy (8). This is the content of the following

LEMMA 1. The \mathbb{Z} span of the standard tableau is the bracket algebra \mathbf{B} .

PROOF. We prove this by induction on the length of M . The statement is trivial for $l(M) = 1$. Suppose we can bring tableaux of length $\leq k$ into standard form. Take M with $l(M) = k + 1$. Define $M' = M/ht(M)$. Since $l(M') < l(M)$ we can bring M' into standard form \tilde{M}' . Consider the 2×2 tableau consisting of $ht(\tilde{M}')$ and $ht(M)$. Let us write this as

$$(9) \quad \begin{pmatrix} j' & k' \\ j & k \end{pmatrix}$$

There are two possibilities:

- 1 Either $j' \leq j$, in which case M is standard.
- 2 Or $j' > j$ in which case using (8) we write (9) as

$$\begin{pmatrix} j & k' \\ j' & k \end{pmatrix}$$

If we do this to M to get \tilde{M} , $ht(\tilde{M}) > ht(M)$.

Since we can increase the height of M only a finite number of times, at a certain moment after repeating the whole procedure a number of times we must end up with case 1. Then we are done. ■

LEMMA 2. The standard tableaux are linearly independent.

PROOF. Among all the nontrivial linear dependence relations among the standard tableaux $\{M_k\}_{k=1}^N$ choose one

$$(10) \quad \sum_{k=1}^N c_k M_k = 0$$

with nonzero scalars c_k such that

- (a) the number of distinct variables in the M_k is as small as possible;
- (b) subject to (a) the maximum length of the tableaux M_k is as small as possible.

Let (j, k) be the maximal bracket (with respect to the height ordering) appearing in (10). By (b) (j, k) is not a common factor of all the M_k . Put $(j, k) = 0$ in (10). Since the height of (j, k) is maximal, the standard tableaux M_k will remain standard. Thus we obtain a nontrivial relation with fewer variables. This contradicts (a). ■

A **Stanley decomposition** of the space of tableaux \mathbf{B} is a sequence of m free abelian submonoids $\mathbf{B}_1, \dots, \mathbf{B}_m$ of the multiplicative monoid \mathbf{B} , each of rank k , and m elements $\eta_1, \dots, \eta_m \in \mathbf{B}$ such that

$$\mathbf{B} = \sum_{i=1}^m \oplus \eta_i \mathbb{Z}[\mathbf{B}_i].$$

Here \oplus is the direct sum of additive abelian groups and $\mathbb{Z}[\mathbf{B}_i]$ is the free additive abelian group generated by \mathbf{B}_i .

The goal of this paragraph is to make the Stanley decomposition of \mathbf{B} explicit. Consider the figure

	1	2	3	...	n
1	(1, 1)	(1, 2)	(1, 3)	...	(1, n)
2	(2, 1)	(2, 2)	(2, 3)	...	(2, n)
...
n	(n , 1)	(n , 2)	(n , 3)	...	(n , n)

Figure 3.

where each node represents a bracket in \mathbf{B} . For each **monotone path** p_i from $(1, 1)$ to (n, n) , that is, a path which consists of moves downward or to the right, define \mathbf{B}_i to be the free abelian multiplicative monoid generated by the brackets in p_i . There are

$$N = \binom{2(n-1)}{n-1}$$

monotone paths. A **corner** of a monotone path is a node which is at the end of a move down and at the start of a move to the right. Thinking of the product of the nodes of a monotone path as a tableau, we order the monotone paths by its inverted lexicographic order: $M < M'$ if $ht(M) < ht(M')$ or, $M/ht(M) < M'/ht(M')$.

We now prove

THEOREM 1.

$$\mathbf{B}(\{(j, k) \mid 1 \leq j, k \leq n\}) = \sum_{l=1}^N \eta_l \mathbb{Z}[(r, s) \mid (r, s) \text{ is a node of } p_l]$$

where

$$\eta_l = \prod \{(u, v) \mid (u, v) \text{ is a corner of } p_l\}.$$

PROOF. By the straightening algorithm, every tableau is a unique linear combination of standard tableaux. Recall that a standard tableau does not have two nodes one of which lies strictly above and to the right of the other. Thus a monotone path p_l corresponds to a standard tableau. Since standard tableaux are linearly independent, it follows that the multiplicative monoid \mathbf{B}_l , generated by the nodes of p_l , is free. Since $\eta_l \in \mathbb{Z}[\mathbf{B}_l]$, as an additive group $\eta_l \mathbb{Z}[\mathbf{B}_l]$ is generated by standard tableaux.

Because \mathbf{B} is generated as an additive group by the standard tableaux, to prove the theorem it suffices to show that every standard tableau is in a unique $\eta_l \mathbb{Z}[\mathbf{B}_l]$. Let M be a standard tableau. Since M is standard, its factors lie on a monotone path. Let p be the largest monotone path containing the factors of M . Suppose (r, s) is a corner of p which is not a factor of M . Then $(r-1, s)$ and $(r, s+1)$ are nodes of p . Let q be the monotone path created by taking p and replacing (r, s) by $(r-1, s+1)$. Then q still contains the factors of M , and $q > p$, since $(r-1, s+1) > (r, s)$ and all other brackets are equal, thus contradicting the maximality of p . This proves that to each standard M there is a unique (because it is maximal in the complete ordering) monotone path all of whose corners are factors of M . We denote the path by p_l and the product of the corners by η_l , and we have shown that $M \in \eta_l \mathbb{Z}[\mathbf{B}_l]$. ■

3. THE STANLEY DECOMPOSITION OF \mathcal{U}_n

Here we give another proof of the Stanley decomposition of the monoid \mathcal{U}_n . This proof is more in the spirit of Stanley's original argument [13, p. 191].

We begin with some combinatorial preliminaries.

Let Σ_n be an n dimensional simplex with vertices $\{u_0, \dots, u_n\}$. Then the product $\Sigma_n \times \Sigma_n$ has vertices $\{(u_i, u_j) \mid i, j \in \{0, \dots, n\}\}$. Consider the vertices of this product as entries in the $(n+1) \times (n+1)$ dimensional array A_n . $\Sigma_n \times \Sigma_n$ is a $2n$ dimensional polytope [8]. Therefore a triangulation of $\Sigma_n \times \Sigma_n$ consists of maximal simplices of dimension $2n$, each having $2n+1$ vertices.

A triangulation Δ_n of $\Sigma_n \times \Sigma_n$ is given in [7, p. 67, def. 8.8]. We will now give another more combinatorial description of this triangulation, due, essentially, to [4]. A **monotone path** in A_n is a sequence of entries of A_n beginning with (u_0, u_0) and ending with (u_n, u_n) such that the entry following (u_i, u_j) is either (u_i, u_{j+1}) or (u_{i+1}, u_j) . In other words a monotone path is composed of moves either one step to the right or one step down. Each monotone path in A_n consists of $2n+1$ entries. The triangulation Δ_n of $\Sigma_n \times \Sigma_n$ has maximal $2n$ dimensional simplices σ consisting of the convex hull of the vertices corresponding to the entries of the monotone path p_σ in A_n . An elementary counting argument shows that there are

$$N = \binom{2n}{n}$$

monotone paths in A_n . A **corner** of a monotone path is a move down followed by a move to the right. A classical argument, which can be found in [9], shows that the number of monotone paths in A_n which have exactly i corners is

$$h_i = \binom{n}{i}^2.$$

This completes the description of the triangulation of Δ_n .

Here we give another description of Δ_n . Let ω be the ordering on the vertices (u_i, u_j) of $\Sigma_n \times \Sigma_n$, defined by

$$(u_i, u_j) \leq (u_k, u_l) \text{ if } i \leq k \text{ and } j \leq l.$$

Denote the vertex (u_i, u_j) by the node (i, j) . The collection of all d , $0 \leq d \leq 2n$, faces of $\Sigma_n \times \Sigma_n$ is the set of all $d+1$ vertices corresponding to $d+1$ distinct nodes in some monotone path. If F_d is a d -face of $\Sigma_n \times \Sigma_n$, let $\delta(F_d)$ be the smallest vertex in F_d in the ordering ω . Thus $\delta(F_d)$ is the upper left most node of the nodes corresponding to F_d . We say that the collection of faces $\Phi = (F_0, F_1, \dots, F_{2n})$ of $\Sigma_n \times \Sigma_n$ is a **full flag** if $\delta(F_i)$ is not a vertex of F_{i-1} for all $i = 1, \dots, 2n$. If Φ is a full flag then the $2n+1$ vertices $(\delta(F_0), \delta(F_1), \dots, \delta(F_{2n}))$ form a $2n$ simplex $\Delta(\Phi)$. Clearly the nodes corresponding to $\delta(F_0), \dots, \delta(F_{2n})$ form a maximal monotone path. Hence $\Gamma_\omega = \{\Delta(\Phi) \mid \Phi \text{ is a full flag of } \Sigma_n \times \Sigma_n\}$ form a triangulation of $\Sigma_n \times \Sigma_n$ which is exactly Δ_n .

In order to construct a shelling of the triangulation Δ_n we must order the maximal $2n$ dimensional simplices σ in Δ_n . This we do as follows. Encode the monotone path p_σ corresponding to σ by a binary string a_σ : a 0 in the string indicates a move to the right and a 1 a move down. Order the binary strings a_σ via the lexicographic order with the string

$$0 \dots 01 \dots 1$$

first and

$$1 \dots 10 \dots 0$$

last. Let Δ be a pure simplicial complex, that is, all the maximal simplices of Δ have $d+1$ vertices. Then a **shelling** of Δ is an ordering $\sigma_1, \dots, \sigma_k$ of all its maximal simplices such that for each j , $2 \leq j \leq k$,

$$\sigma_j \cap \left(\bigcup_{i < j} \sigma_i \right)$$

is a subcomplex Δ' of Δ determined by some number, say n_j , of $(d-1)$ -faces of σ_j . When $j=1$, we put $n_1=0$. Having a shelling of Δ is equivalent to saying that there is a *unique* minimal face τ_j of σ_j which

1. is not in

$$\bigcup_{i < j} \sigma_i$$

and

2. has n_j vertices, each of which is the vertex of σ_j omitted from one of the $(d-1)$ -faces determining Δ' .

Given a shelling of Δ , define

$$h_i(\Delta) = \# \{j \mid n_j = i\}.$$

The numbers $h_i(\Delta)$ depend only on Δ and not on the particular shelling (see e.g. [3]). In particular

$$h_i(\Delta) = \sum_{j=0}^i (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta), \quad 0 \leq i \leq d,$$

where $f_j(\Delta)$ is the number of j faces of Δ .

THEOREM 2. The lexicographic order on the $2n$ dimensional simplices of Δ_n gives a shelling of Δ_n such that

$$h_i(\Delta) = h_i = \binom{n}{i}^2.$$

In fact for each maximal simplex σ of Δ , the minimal face of σ not in the union of earlier maximal simplices has vertices corresponding to the corners of the monotone path p_σ associated to σ .

PROOF. Consider the binary string a_σ associated to σ . Each 10 substring corresponds to a corner of p_σ . Replacing a 10 substring in a_σ by a 01 substring results in a binary string $a_{\sigma'}$ where $\sigma' < \sigma$ in the lexicographic order. But $\sigma \cap \sigma'$ is a face of σ of codimension 1. Since any $a_{\sigma''}$ for $\sigma'' < \sigma$ is obtained from a_σ by a sequence of 10 substring replacements, the vertex set of $\sigma \cap \sigma''$ must lie in $\sigma \cap \sigma'$ where σ' is obtained from σ by a single 10 substring replacement. The number of such possible replacements in a_σ is just the number of corners of σ . Thus it is the number of codimension 1 faces of σ in

$$\sigma \cap \left(\bigcup_{\sigma' < \sigma} \sigma' \right),$$

which in turn is the number of vertices of the unique minimal face τ of σ not in

$$\bigcup_{\sigma' < \sigma} \sigma'.$$

In fact, the unique minimal face τ has vertices corresponding to the corners of the monotone path p_σ . ■

In the next paragraphs we construct the Stanley decomposition of \mathcal{M}_n . Consider the \mathbb{N} monoid \mathcal{M}_n defined by the set of all $(j, k) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ which satisfy

$$(11) \quad j_0 + j_1 + \cdots + j_n = k_0 + k_1 + \cdots + k_n.$$

Let K_n be the cone of nonnegative rational solutions of (11). Slicing K_n with the hyperplane

$$j_0 + j_1 + \cdots + j_n + k_0 + k_1 + \cdots + k_n = 2$$

gives the polytope $\Sigma_n \times \Sigma_n$. The triangulation Δ_n of $\Sigma_n \times \Sigma_n$ described in this section gives a decomposition of K_n into $2n+1$ dimensional simplicial cones, that is, cones having $2n+1$ linearly independent extreme rays. Let K_σ be the simplicial cone corresponding to the maximal simplex $\sigma \in \Delta_n$.

We now explicitly carry out the decomposition of \mathcal{M}_n into a disjoint union of translates of simplicial cones described by Stanley [13]. More concretely we prove

THEOREM 3.

$$(12) \quad \mathcal{M}_n = \bigsqcup_{l=1}^N (\eta_l + \bar{K}_{\sigma_l}),$$

where \bar{K}_{σ_l} is the free commutative \mathbb{N} monoid generated by the vertices of σ_l ; $\sigma_1 < \cdots < \sigma_N$,

$$N = \binom{2n}{n}$$

is the shelling of Δ_n ; η_l is the sum of the vectors associated to the vertices of σ_l which correspond to the corners of the path p_{σ_l} .

On our way toward proving (12) we will need to understand the structure of the integral points of K_n and K_σ . For this we need the notion of an extreme ray. Corresponding to the entry (u_r, u_s) of p_σ , σ a (maximal) $2n$ simplex in $\Sigma_n \times \Sigma_n$, let $v_{rs}^{(n)} \in \mathbb{R}^{2n+2}$ be the vector with 1 in the $(r+1)^{st}$ and $(n+1+s+1)^{st}$ components and 0 elsewhere. The **extreme rays** of K_σ are the vectors $v_{rs}^{(n)}$ where (u_r, u_s) is an entry in the monotone path p_σ associated to σ . We have the

LEMMA 3.

1. Any integral point of K_n lies in some K_σ .
2. Any integral point of K_σ is a unique nonnegative integral combination of the extreme rays of K_σ .

PROOF. Part (1) has already been proved (see the argument after formula (1)).

To prove part (2) first we note that any such representation is necessarily unique, since the extreme rays of K_σ are linearly independent. Next, suppose $z \in K_\sigma \cap \mathbb{Z}^{2n+2}$. We show that any representation

$$z = \sum_{r,s} \lambda_{r,s} v_{r,s}$$

To see this, note that there is a linear order on the $v_{r,s}$ induced by the order of the entries (u_r, u_s) in the path p_σ . Further, note that each (u_r, u_s) in p_σ is either the last entry in row r or the last entry in column s . If some $\lambda_{r,s} \notin \mathbb{Z}$, let (\bar{r}, \bar{s}) be the first such in this order. Then either entry $\bar{r}+1$ or entry $n+1+\bar{s}+1$ of the vector z is not integral, since for that row or column, $\lambda_{\bar{r},\bar{s}}$ is both the first noninteger weight and the last nonzero weight. Since z is all integer, this is impossible. ■

We remark that a consequence of this lemma is that ω is a compressed ordering [12]. To see that ω is compressed it suffices to show that for any rational point α in the interior of a face F_d of $\Sigma_n \times \Sigma_n$, if c is the unique rational number such that

$$\frac{1}{1-c} (\alpha - c\delta(F))$$

lies in the boundary of F_d then $l(\alpha)c$ is an integer. Here $l(\alpha)$ is the least common multiple of the denominators of α . Write

$$\alpha = \sum_{r=0}^d \lambda_r v_r,$$

where v_r are vertices of F_d and $0 < \lambda_r < 1$, $\lambda_r \in \mathbb{Q}$. Then $l(\alpha)\alpha$ is an integral linear combination of extreme rays belonging to some K_σ . Hence by the lemma, all $l(\alpha)\lambda_r$ are integers. Suppose that $\delta(F_d) = v_s$ for some s , $0 \leq s \leq d$. Then $\alpha - \lambda_s \delta(F_d)$ belongs to the boundary of F_d . Hence $c = \lambda_s$. Therefore $l(\alpha)c$ is an integer.

We now prove theorem 3. Because the geometric nature of \bar{K}_{σ_i} has been given by the lemma, we need only show that the \mathbb{N} monoid

$$\mathcal{M}_n = K_n \cap \mathbb{Z}^{2n+2}$$

is a disjoint union given by the statement of the theorem.

We proceed by induction on N , the number of maximal simplices in the shelling of Δ_n . Let

$$\Delta^i = \bigcup_{j \leq i} \sigma_j \text{ and } \bar{K}^i = \bigcup_{j \leq i} \bar{K}_{\sigma_j}.$$

We have $\Delta_n = \Delta^N$ and $\mathcal{M}_n = \bar{K}^N$. The result is true for $N=1$ since $\eta_1=0$. Suppose that

$$\bar{K}^{i-1} = \bigsqcup_{j \leq i} (\eta_j + \bar{K}_{\sigma_j}).$$

We know that

$$\bar{K}^i = \bar{K}^{i-1} \cup \bar{K}_{\sigma_i}.$$

Moreover points in

$$\bar{K}_{i-1} \cap \bar{K}_{\sigma_i}$$

are exactly those lying on the n_i codimension 1 subcones of \bar{K}_{σ_i} spanned by the n_i codimension 1 faces of σ_i which lie in Δ^{i-1} . Therefore points $\bar{K}^i - \bar{K}^{i-1}$ are just those points which do not lie on any of those codimension 1 faces of σ_i . Therefore $z \in \bar{K}^i - \bar{K}^{i-1}$ is a point whose expression in the monoid \bar{K}^i involves **positive** integer coefficients of each of the corners of p_{σ_i} . Thus $z \in \eta_i + \bar{K}_{\sigma_i}$. Hence

$$\bar{K}^i = \bar{K}^{i-1} \cup (\eta_i + \bar{K}_{\sigma_i}).$$

No point in $\eta_i + \bar{K}_{\sigma_i}$ can lie in \bar{K}^{i-1} since the corner entries of p_{σ_i} span the unique face τ_i of σ_i which does not lie in Δ^{i-1} . Thus the union is disjoint. ■

We note that everything done in the past section extends to the case $\Sigma_r \times \Sigma_s$ corresponding to the monoid $M_{r,s}$ determined by

$$j_0 + j_1 + \cdots + j_r = k_0 + k_1 + \cdots + k_s.$$

The only changes that we have to make are

$$h_i = \binom{r}{i} \binom{s}{i}$$

and

$$N = \binom{r+s}{r}.$$

We end this section by showing that the h -vector $(h_0, h_1, \dots, h_{2n})$ of $\Sigma_n \times \Sigma_n$ where

$$h_i = \binom{n}{i}^2$$

for $0 \leq i \leq n$ and $h_i = 0$ for $i > n$ does *not* depend on the triangulation Δ_n of $\Sigma_n \times \Sigma_n$.

Consider the generating function

$$J(\Sigma_n \times \Sigma_n, t^2) = 1 + \sum_{m \geq 1} \# \{ \alpha \in \Sigma_n \times \Sigma_n \subseteq \mathbb{R}^{2n+2} \mid m\alpha \in \mathbb{Z}^{2n+2} \} t^{2m}.$$

Since $\Delta_n = \Gamma_\omega$, where ω is the ordering on the vertices of $\Sigma_n \times \Sigma_n$ defined above and since ω is compressed, by a theorem of Stanley [12, p. 336] it follows that

$$J(\Sigma_n \times \Sigma_n, t^2) = \frac{\sum_{i=0}^n h_i t^{2i}}{(1-t^2)^{2n+1}}$$

the h_i is the i th component of the h vector of the triangulation Δ_n of

consider the Poincaré series for the algebra \mathcal{B}_n of \mathbb{C}^* invariant polynomials; namely

$$\mathcal{PB}_n(t) = \sum_{m \geq 0} (\dim_{\mathbb{C}} \mathcal{B}_n^{(m)}) t^{2m},$$

$\mathcal{B}_n^{(m)}$ is the vectorspace of homogeneous \mathbb{C}^* invariant polynomials of degree m . Then

$$\begin{cases} \mathcal{PB}_n(t) = \sum_{m \geq 0} \# \{(j, k) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1} \mid |j| = |k|, |j| + |k| = 2m\} t^{2m} \\ = \sum_{m \geq 0} \binom{m+n}{n}^2 t^{2m}, \end{cases}$$

the last equality follows from the fact that

$$\# \{j \in \mathbb{N}^{n+1} \mid |j| = m\} = \binom{m+n}{n}.$$

Ally \mathcal{PB}_n is a rational function in t^2 [11]. To find it we substitute the binomial coefficient identity

$$\sum_{j=0}^n \binom{N+n+j}{2n} \binom{n}{j} \binom{n}{n-j} = \binom{N+n}{n}^2$$

(13) (see [10]). Changing the variable of summation gives

$$\begin{aligned} \mathcal{PB}_n(t) &= \sum_{j=0}^n \left(\sum_{k=0}^{\infty} \binom{k+2n}{2n} t^{2k} \right) \binom{n}{j} \binom{n}{n-j} t^{2n-2j} \\ &= \frac{1}{(1-t^2)^{2n+1}} \sum_{j=0}^n \binom{n}{n-j} \binom{n}{j} t^{2j} \\ &= \frac{1}{(1-t^2)^{2n+1}} \sum_{j=0}^n \binom{n}{j}^2 t^{2j} \\ &= J(\Sigma_n \times \Sigma_n, t^2). \end{aligned}$$

Thus the h vector of $\Sigma_n \times \Sigma_n$ is independent of the triangulation Δ_n .

A \mathbb{C}^* ACTION AND THE HARMONIC OSCILLATOR

In this section we explain the relation between the normal form of a perturbed n dimensional harmonic oscillator and the algebra \mathcal{B}_n of \mathbb{C}^* invariant polynomials.

Consider the Hamiltonian function

$$\mathcal{H}_2(\xi, \eta) = \frac{1}{2} \sum_{i=0}^n (\eta_i^2 + \xi_i^2)$$

$$\zeta = \eta$$

$$\dot{\eta} = -\dot{\xi}$$

ose flow is an S^1 action on $T^*\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^*$ given by

$$1) \quad t \cdot (\xi, \eta) = (\cos t \xi + \sin t \eta, -\sin t \xi + \cos t \eta).$$

We now say what is meant by the normal form of a formal power series (perturbation of the harmonic oscillator. Let $P_m = P_m(T^*\mathbb{R}^{n+1}, \mathbb{R})$ be the space homogeneous polynomials of degree m on $T^*\mathbb{R}^{n+1}$. The formal power series Hamiltonian

$$\mathcal{H}(\xi, \eta) = \mathcal{H}_2(\xi, \eta) + \dots + \mathcal{H}_m(\xi, \eta) + \dots,$$

ere $\mathcal{H}_m \in P_m$, is in normal form with respect to \mathcal{H}_2 iff for every $m \geq 2$

$$L_{\mathcal{H}_2} \mathcal{H}_m = 0,$$

ere $L_{\mathcal{H}_2}$ is the Lie derivative with respect to the vectorfield $X_{\mathcal{H}_2}$. In other words, \mathcal{H} is in normal form iff it is invariant under the linear S^1 action given (14). Because $X_{\mathcal{H}_2}$ is a semisimple linear mapping, $L_{\mathcal{H}_2}$ is a semisimple linear mapping of P_m into itself. Therefore

$$1) \quad P_m(T^*\mathbb{R}^{n+1}, \mathbb{R}) = \ker L_{\mathcal{H}_2} | P_m \oplus \text{im } L_{\mathcal{H}_2} | P_m.$$

ing (15), standard techniques (see (1)) construct a near identity formal power series symplectic mapping which brings the formal power series \mathcal{H} into normal m.

o describe the normal form of formal power series perturbations of the monic oscillator we need to know the generators of the algebra of polynomials which are invariant under the S^1 action given by the flow of the monic oscillator. Toward this goal we introduce complex conjugate coordinates

$$z_l = \xi_l + i\eta_l \quad w_l = \xi_l - i\eta_l \quad l = 0, \dots, n$$

$T^*\mathbb{R}^{n+1}$. Then \mathcal{H}_2 becomes the quadratic Hermitian polynomial

$$\mathcal{H}_2 = \frac{1}{2} \sum_{l=0}^n z_l w_l$$

1 $X_{\mathcal{H}_2}$ becomes $X_{\mathcal{H}_2}$:

$$\dot{z}_l = -2i \frac{\partial \mathcal{H}_2}{\partial w_l} = -iz_l \text{ for } l = 0, \dots, n$$

$$\dot{w}_l = 2i \frac{\partial \mathcal{H}_2}{\partial z_l} = iw_l \text{ for } l = 0, \dots, n.$$

Let HP_m be the set of Hermitian polynomials of degree m , that is, all expressions of the form

$$\sum_{j+k=m} c_{jk} z^j w^k$$

where $c_{jk} = \bar{c}_{jk}$ and $c_{jk} \in \mathbb{C}$. Since the mapping

$$\mathcal{H}_m(\xi, \eta) \mapsto \mathcal{H}_m\left(\frac{1}{2}(z+w), \frac{1}{2i}(z-w)\right)$$

is a bijective real linear mapping between P_m and HP_m , $\ker L_{\mathcal{H}_2} | P_m$ is in bijective correspondence with $\ker L_{\mathcal{H}_2} | HP_m$. Elements of $\ker L_{\mathcal{H}_2}$ are those Hermitian polynomials which are invariant under the S^1 action (16). If we restrict the \mathbb{C}^* action on $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*$ given by

$$t \cdot (x, y) = (tx, t^{-1}y)$$

to $S^1 \subseteq \mathbb{C}^*$ and to the real subspace W_{2n+2} we obtain the S^1 action given by (16). Restricting the generators $\pi_{ij} = x_i y_j$, $0 \leq i, j \leq n$ of the algebra \mathcal{B}_n of the \mathbb{C}^* -invariant polynomials to W_{2n+2} gives the generators $\sigma_{ij} = z_i w_j$, $0 \leq i, j \leq n$ of the algebra of S^1 -invariant Hermitian polynomials. Thus $\text{Re } \sigma_{ij}$ and $\text{Im } \sigma_{ij}$ are the generators of the algebra P^{S^1} of polynomials on $T^*\mathbb{R}^{n+1}$ which are invariant under the flow of the harmonic oscillator.

5. APPENDIX. \mathcal{A}_n IS ISOMORPHIC TO \mathcal{B}_n

First we show that the ring of polynomials \mathcal{A}_n of the variety $M_{n+1,1}$ of all $(n+1) \times (n+1)$ complex matrices whose rank is less than or equal to one is isomorphic to $\mathcal{C}_n = \mathbb{C}[\mathbb{C}^{(n+1)^2}] / I_2$ where I_2 is the ideal generated by the 2×2 minors of a generic $(n+1) \times (n+1)$ matrix. Then we show that the mapping

$$\phi : \mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^* \rightarrow M_{n+1,1} \subseteq \mathbb{C}^{(n+1)^2} : (x, y) \mapsto x \otimes y$$

induces an isomorphism

$$\phi^* : \mathcal{C}_n \rightarrow \mathcal{B}_n$$

where \mathcal{B}_n is the algebra of \mathbb{C}^* invariant polynomials on $\mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*$.

Consider the algebra of polynomials $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ on the set $\mathbb{C}^{(n+1)^2}$ of $(n+1) \times (n+1)$ complex matrices. $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is generated by the linear polynomials $\{E_{ij}^*\}$ which is the dual basis corresponding to the standard basis $\{E_{ij}\}$ of $\mathbb{C}^{(n+1)^2}$. (Here E_{ij} is the matrix with 1 in the $(i, j)^{\text{th}}$ entry and 0 in the other entries). The ideal I_2 is generated by the polynomials

$$E_{ij}^* E_{kl}^* - E_{il}^* E_{kj}^* \quad 0 \leq i, j, k, l \leq n.$$

\mathcal{A}_n generation of the family of polynomial functions \mathcal{A}_n on $M_{n+1,1}$, ψ is a surjective ring homomorphism. To show that \mathcal{A}_n is isomorphic to \mathcal{C}_n , we need only show that the kernel of ψ is I_2 . Suppose that $f \in \ker \psi$, then $f|_{M_{n+1,1}} = 0$. Therefore by Hilbert's Nullstellensatz, there is a nonnegative integer r such that $f^r \in I_2$. But I_2 is a radical ideal. (This is a well known consequence of the fact that \mathcal{C}_n has a straightening law, see [2, p. 77] and [6].) For completeness we prove this in lemma 1 below). Therefore $f \in I_2$. Clearly $I_2 \subseteq \ker \psi$. Thus $\ker \psi = I_2$, which is what we wished to show.

LEMMA 1. I_2 is a radical ideal.

PROOF. Denote the linear polynomial E_{ij}^* on $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ by the bracket (i, j) . Since $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is generated by $\{E_{ij}\}$, every element of $\mathbb{C}[\mathbb{C}^{(n+1)^2}]$ is given by a bracket polynomial. Since I_2 is generated by $E_{ij}^*E_{kl}^* - E_{il}^*E_{kj}^*$, every element of \mathcal{C}_n is given by a bracket polynomial where the syzygy

$$(i, j)(k, l) = (i, l)(k, j)$$

holds. Hence we may apply the straightening process of § 2 to the bracket polynomials of \mathcal{C}_n .

Suppose that

$$(17) \quad T = \sum_{i=0}^M c_i T_i$$

is a polynomial in \mathcal{C}_n with monomials T_i such that $T^m = 0$. Then using the straightening process we may suppose that each T_i in (17) is standard and $T_1 < T_2 < \dots < T_M$ (where $T \leq S$ if $ht(T) < ht(S)$ or if $ht(T) = ht(S)$, then $T/ht(T) \leq S/ht(S)$). Now

$$T^m = \sum \frac{c_{i_1} \dots c_{i_m}}{i_1! \dots i_m!} T_{i_1}^{i_1} \dots T_{i_m}^{i_m}.$$

Since $T_1 < \dots < T_M$, T_M^m is greater than the straightened form of $T_1^{i_1} \dots T_M^{i_m}$ for all $1 \leq i_1, \dots, i_m \leq m$ with $(i_1, \dots, i_m) \neq (M, \dots, M)$. Consequently

$$(18) \quad T^m = c_M^m T_M^m + \text{lower standard monomials.}$$

Since $T^m = 0$, by linear independence of the standard monomials it follows from (18) that $c_M = 0$. Hence after finitely many repetitions of the above argument we obtain $T = 0$. Thus I_2 is a radical ideal, which is what we wished to prove. ■

Now consider the mapping

$$\phi : \mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^* \rightarrow \mathbb{C}^{(n+1)^2} ; x, y \mapsto x \otimes y.$$

LEMMA 2.

(a) ϕ is surjective;

(b) for $A \in M_{n+1,1}$, the fiber of $\phi^{-1}(A)$ is a single \mathbb{C}^* orbit, if $A \neq 0$ or is $(\mathbb{C}^{n+1} \times \{0\}) \cup (\{0\} \times (\mathbb{C}^{n+1})^*)$, if $A = 0$.

PROOF. To prove (a) suppose that A has rank equal to zero. Then $A = 0$. Hence $\phi(0, y) = 0 \otimes y = 0$ does the job. Now suppose that A is a nonzero element of $M_{n+1,1}$. Let x be a nonzero vector in $\text{im } A$. Then $v = Ax \in \text{im } A$. If $v = 0$, then there is a nonzero vector u such that

$$(19) \quad x = Au.$$

Hence $u \notin \ker A$. Since $\dim \ker A = n+1 - \dim \text{im } A = n+1-1$, there is a $y \in \mathbb{C}^{n+1}$ such that $y(u) = 1$ and $\{\lambda u | \lambda \in \mathbb{C}\} \oplus \ker A = \mathbb{C}^{n+1}$. Therefore $A = x \otimes y$ because $(x \otimes y)(u) = y(u)x = x = Au$, $x \otimes y | \ker A = 0 = A | \ker A$ and $\{\lambda u | \lambda \in \mathbb{C}\} \oplus \ker A = \mathbb{C}^{n+1}$. If $v \neq 0$, then there is a nonzero $\mu \in \mathbb{C}$ such that $v = \mu x$, since $\dim \text{im } A = 1$. Therefore $x \notin \ker A$. Hence there is a $y \in (\mathbb{C}^{n+1})^*$ such that $\ker y = \ker A$ and $y(x) = \lambda$. Because

$$(x \otimes y)(x) = y(x)x = \lambda x = Ax \text{ and } (x \otimes y) | \ker A = 0 = A | \ker A,$$

$A = x \otimes y$. Therefore ϕ is surjective.

To prove (b) suppose that $x \otimes y = 0$. Then for all $u \in \mathbb{C}^{n+1}$, $0 = (x \otimes y)(u) = y(u)x$. If $x = 0$, then $y \in (\mathbb{C}^{n+1})^*$ is arbitrary. If $x \neq 0$, then $y(u) = 0$ for all $u \in \mathbb{C}^{n+1}$ that is, $y = 0$. Therefore

$$\phi^{-1}(0) = (\mathbb{C}^{n+1} \times \{0\}) \cup (\{0\} \times (\mathbb{C}^{n+1})^*).$$

Now suppose that $A \in M_{n+1,1} - \{0\}$. To show that the fiber $\phi^{-1}(A)$ is a single \mathbb{C}^* orbit, we analyze the freedom in the choices made in the proof of (a). If $v = 0$, then all possible choices of u satisfying (19) are $u + w$ with $w \in \ker A$. But $y(u + w) = y(u) = 1$. Thus $y \in (\mathbb{C}^{n+1})^*$ does not depend on the choice of u . In other words, for a fixed $A \in M_{n+1,1}$, y depends only on the choice of basis vector $x \in \text{im } A$. There are only \mathbb{C}^* such choices. Hence $\phi^{-1}(A)$ is topologically a \mathbb{C}^* . But $\phi^{-1}(A)$ is a union of \mathbb{C}^* orbits. Hence $\phi^{-1}(A)$ is a single \mathbb{C}^* orbit. If $v \neq 0$, then again y is uniquely determined by the choice of x . Hence $\phi^{-1}(A)$ is a \mathbb{C}^* orbit. This proves (b). ■

We are now in position to show that \mathcal{A}_n and \mathcal{B}_n are isomorphic. The mapping ϕ induces a ring homomorphism

$$\phi^* : \mathcal{A}_n \rightarrow \mathbb{C}[\mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^*] : f \mapsto f \circ \phi = F.$$

Since ϕ is surjective, ϕ^* is injective. Actually, the image of ϕ^* is contained in \mathcal{B}_n . For, let y be the \mathbb{C}^* orbit which intersects either $U = \mathbb{C}^{n+1} \times \{0\}$ or

$V = \{0\} \times (\mathbb{C}^{n+1})^*$. Since U and V are invariant under the \mathbb{C}^* action, $\gamma \subseteq U$ or $\gamma \subseteq V$. Therefore $F(\gamma) \subseteq F(U \cup V) \subseteq F(\phi^{-1}(0)) = \{f(0)\}$, that is, F is constant on the \mathbb{C}^* orbit γ . Now suppose that γ is a \mathbb{C}^* orbit which does not intersect $U \cup V$. Then $\gamma \subseteq (\mathbb{C}^{n+1} - \{0\}) \times ((\mathbb{C}^{n+1})^* - \{0\})$. Since $\gamma = \phi^{-1}(A)$ for some $A \in M_{n+1,1} - \{0\}$, $F(\gamma) = F(\phi^{-1}(A)) = \{f(A)\}$. Therefore F is constant on all \mathbb{C}^* orbits. In other words F is a \mathbb{C}^* invariant polynomial, i.e. $F \in \mathcal{B}_n$. To show that ϕ^* maps \mathcal{A}_n onto \mathcal{B}_n , we recall from § 1 that the quadratic polynomials $\pi_{ij} = x_i y_j$ $0 \leq i, j \leq n$ on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ generate \mathcal{B}_n . Since

$$\begin{aligned} (\phi^*)(E_{ij}^*)(x, y) &= (e_i \otimes e_j)^*(\phi(x, y)) \\ &= (e_i^* \otimes e_j)(x \otimes y) \\ &= e_i^*(x) y(e_j) \\ &= x_i y_j = \pi_{ij}(x, y), \end{aligned}$$

ϕ^* is surjective. Thus \mathcal{A}_n is isomorphic to \mathcal{B}_n . ■

An important consequence of this isomorphism is the fact that

$$(20) \quad \begin{cases} 0 = \phi^*(I_2) \\ = \phi^*(E_{ij}^* E_{kl}^* - E_{il}^* E_{kj}^*) \\ = \pi_{ij} \pi_{kl} - \pi_{il} \pi_{kj} \end{cases}$$

are the only relations among the generators $\{\pi_{ij}\}$ of \mathcal{B}_n . For suppose that $F = 0$ is a polynomial relation among the generators of \mathcal{B}_n . Then $F = \phi^*(f)$ for some f in \mathcal{A}_n . Hence $f = 0$. This is a polynomial relation in \mathcal{A}_n . Therefore $f \in I_2$ since $\mathcal{A}_n = \mathcal{C}_n$. Consequently there are $a_{ijkl} \in \mathcal{A}_n$ such that

$$f = \sum a_{ijkl} (E_{ij}^* E_{kl}^* - E_{il}^* E_{kj}^*).$$

Hence

$$F = \phi^*(f) = \sum \phi^*(a_{ijkl}) (\pi_{ij} \pi_{kl} - \pi_{il} \pi_{kj}).$$

Thus $F = 0$ follows from (20).

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