

Lemma 2.2 (from Billera-Sturmfels) $\theta: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and

Let $\theta: P \rightarrow Q$ be an affine map. Consider the map $Q^\vee \mapsto P^\Psi$ where Ψ spans the orthogonal complement to the hyperplane $\theta^{-1}(V^\perp)$ in $\mathbb{R}^m$.

It defines a lattice isomorphism !!!

$$F(Q) \approx \left\{ P^\Psi \in F(P) \mid \Psi \in \ker(\theta)^\perp \right\}$$

In the map $Q^\vee \mapsto P^\Psi$, the face P^Ψ is in fact the largest face (containing $\theta^{-1}(Q^\vee)$) projecting to $Q^\vee$.

Suppose P^Ψ is the largest face of P projecting to $Q^\vee$

$$\varphi = \varphi^{\theta^{-1}(V)} + \varphi^{(\theta^{-1}(V))^\perp}$$

$$x \in P^\Psi \Leftrightarrow \langle x, \varphi \rangle = \langle q, \varphi \rangle \quad \theta(q) \in Q^\vee$$

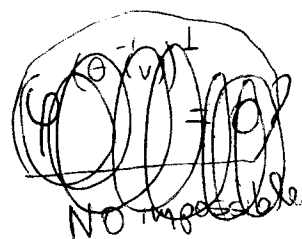
$$\Leftrightarrow \langle x - q, \varphi \rangle = 0 \Rightarrow$$

$$\langle x - q, \varphi^{\theta^{-1}(V)} + \varphi^{(\theta^{-1}(V))^\perp} \rangle = 0$$

$$\Rightarrow \langle x - q, \varphi^{\theta^{-1}(V)} \rangle + \langle x - q, \varphi^{(\theta^{-1}(V))^\perp} \rangle = 0$$

and $\langle x - q, \varphi^{(\theta^{-1}(V))^\perp} \rangle = 0$! because

$$\theta(x - q) \in V^\perp \Leftrightarrow x - q \in \theta^{-1}(V^\perp)$$



The ONLY Danger is then

that $\varphi^{(\theta^{-1}(v^\perp))^\perp}$ is $= 0$!! But that means

there must be another $\tilde{\varphi} \in (\theta^{-1}(v^\perp))^\perp$ that
maximizes that face!

Else if $\varphi = \varphi^{(\theta^{-1}(v^\perp))^\perp}$

$\theta(P^\varphi)$ can not be a
face of Q !!!

$$\theta(P^\varphi) \cap \text{rel int}(Q) \neq \emptyset.$$

Simply $\theta^{-1}(v^\perp) + c$ is a supporting hyperplane!!!

(proof of lemma) We know $Q^{v_1} \subseteq Q^{v_2}$

We want $P^{\psi_1} \subseteq P^{\psi_2}$

$$\text{Take } x \in P^{\psi_1} \Rightarrow \langle \psi_1, x \rangle = \langle \psi_1, d \rangle$$

$$d \in P^{\psi_1} \text{ and thus } \theta(d) \in Q^{v_1} \cap Q^{v_2} = Q^{v_1}$$

$$\Rightarrow \langle \psi_1, x - d \rangle = 0 \Rightarrow x - d \in \theta^{-1}(v_1^\perp)$$

$$\Rightarrow \theta(x - d) \in v_1^\perp \Rightarrow \langle v_1, \theta(x - d) \rangle = 0$$

$$\Rightarrow \langle v_1, \theta(x) \rangle = \langle v_1, \theta(d) \rangle$$

$$\Rightarrow \langle v_2, \theta(x) \rangle = \langle v_2, \theta(d) \rangle \quad \text{!! all}$$

The Steps can be reversed hence

$$\Rightarrow \langle v_2, \theta(x-d) \rangle = 0$$

$$\Rightarrow \theta(x-d) \in v_2^\perp$$

$$\Rightarrow x-d \in \theta^{-1}(v_2^\perp)$$

$$\Rightarrow \langle \psi_2, (x-d) \rangle = 0$$

$$\Rightarrow \underline{x \in P^{\psi_2}}$$