Lemma 2.2 (from Billera - Sturmfels) $\theta : \mathbb{R}^m \to \mathbb{R}^n$ and

Let $\theta : P \to Q$ be an affine map. Consider the map $Q^v \mapsto P^\psi$ where ψ spans the orthogonal complement to the hyperplane $\theta^{-1}(v^\perp)$ in $\mathbb{R}^m$.

It defines a lattice isomorphism !!!

$$F(Q) \approx \left\{ P^\psi \in F(P) \mid \psi \in \operatorname{Ker}(\theta)^\perp \right\}$$

In the map $Q^v \mapsto P^\psi$, the face P^ψ is in fact the largest face (containing $\theta^{-1}(Q^v)$) projecting to Q^v.

Suppose P^φ is the largest face of P projecting to Q^v

$$\varphi = \varphi^{\theta^{-1}(v^\perp)} + \varphi^{(\theta^{-1}(v^\perp))^\perp}$$

$$x \in P^\varphi \iff \langle x, \varphi \rangle = \langle q, \varphi \rangle \qquad \theta(q) \in Q^v$$

$$\iff \langle x - q, \varphi \rangle = 0 \implies$$

$$\langle x - q, \varphi^{\theta^{-1}(v^\perp)} + \varphi^{(\theta^{-1}(v^\perp))^\perp} \rangle = 0$$

$$\implies \langle x - q, \varphi^{\theta^{-1}(v)} \rangle + \langle x - q, \varphi^{(\theta^{-1}(v))^\perp} \rangle = 0$$

and $\langle x - q, \varphi^{(\theta^{-1}(v))^\perp} \rangle = 0$! because $\cancel{\varphi^{(\theta^{-1}(v))^\perp} = 0}$

$$\theta(x - q) \in v^\perp \iff x - q \in \theta^{-1}(v^\perp)$$

NO impossible

<u>The ONLY Danger</u> is then

that $\varphi^{(\theta^{-1}(v^\perp))^\perp}$ is $= 0$!! But that means

there <u>must</u> be another $\tilde{\varphi} \in (\theta^{-1}(v^\perp))^\perp$ that maximizes that face !

Else if $\varphi = \varphi^{(\theta^{-1}(v^\perp))}$

$\theta(P^\varphi)$ can <u>not</u> be a face of Q !!!

$\theta(P^\varphi) \cap \text{rel int}(Q) \neq \emptyset$.

<u>Simply</u> $\theta^{-1}(V^\perp) + c$ is a supporting hyperplane!!!

[illegible] Proof of lemma) We know $Q^{v_1} \subseteq Q^{v_2}$

<u>We want</u> $P^{\psi_1} \subseteq P^{\psi_2}$

Take $x \in P^{\psi_1} \Rightarrow \langle \psi_1, x \rangle = \langle \psi_1, d \rangle$

$d \in P^{\psi_1}$ and thus $\theta(d) \in Q^{v_1} \cap Q^{v_2} = Q^{v_1}$

$\Rightarrow \langle \psi_1, x - d \rangle = 0 \Rightarrow x - d \in \theta^{-1}(v_1^\perp)$

$\Rightarrow \theta(x - d) \in v_1^\perp \Rightarrow \langle v_1, \theta(x-d) \rangle = 0$

$\Rightarrow \langle v_1, \theta(x) \rangle = \langle v_1, \theta(d) \rangle$!! all

$\Rightarrow \langle v_2, \theta(x) \rangle = \langle v_2, \theta(d) \rangle$

The steps can be reversed hence

$$\Rightarrow \langle v_2, \theta(x-d) \rangle = 0$$

$$\Rightarrow \theta(x-d) \in v_2^{\perp}$$

$$\Rightarrow x-d \in \theta^{-1}(v_2^{\perp})$$

$$\Rightarrow \langle \psi_2, (x-d) \rangle = 0$$

$$\Rightarrow \underline{x \in P^{\psi_2}}$$