A CLASS OF PERIODICALLY-CYCLIC 6-POLYTOPES

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– the general 2m-case is done, though not yet typed. TB

Proposed running head: *periodically-cyclic 6-polytopes*

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Abstract

For each $v > k \geq 8$, we introduce a convex 6-polytope with v vertices such that there is a complete description of each of its facets based upon a labelling (total ordering) of the vertices so that every subset of k successive vertices generates a cyclic 6-polytope.

1 INTRODUCTION.

In a series of articles, we have been examining the problem of how to generalize a cyclic d-polytope $C \subset R$. We recall that C has a totally ordered set of vertices (a vertex array) that

a) may be chosen on an oriented d^{th} order curve.

As a consequence, the vertex array

b) satisfies Gale's Evenness Condition, and

c) yields a complete description of the facial structure of C.

Thus, any generalization should be a d-polytope in R^d with vertices that may be chosen from an oriented generalized d^{th} order curve. We would label the vertices in the order of appearance on the oriented curve to obtain a vertex array. Then, based upon this ordering, we would like to determine a "global" condition and a "local" condition that should be satisfied by the vertices. We call the "global" condition a Gale property and it is inspired by the fact that the hyperplane determined by the vertices of a facet of C cuts the d^{th} order curve at each of those vertices. The problem is with the "local" condition.

A d^{th} order curve in R^d meets any hyperplane in at most d points (hence, any line in at most two points) and lies on the boundary of its convex hull. Thus as a paradigm, we would like to consider a simple ordinary (locally of d^{th} order) curve that lies on the boundary of a strictly convex body in R^d as a generalized d^{th} order curve. The difficulty is that there is no workable definition of such a convex ordinary space curve in R^d for $d > 3$. Also, it seems that such curves have properties that are independent of d as long as the parity of d is the same. Thus, there should be one generalization for odd d, and one for even d.

In [1] and [2], we presented a generalization for odd dimensions. Given that the behaviour of convex ordinary space curves in R^3 is known, we were able to determine a "local" condition for our "generalized cyclic $(2n+1)$-polytopes".

In [3], we started an examination of a generalization for even dimensions by introducing a class of Gale 4-polytopes. Given that the behaviour of convex ordinary space curves in R^{2n} is not well understood, we used the idea that there is a vertex away such that for a fixed number k, all k successive vertices (corresponding to points on d^{th} order subarcs as we trace the curve) generate a cyclic subpolytope of the "generalized cyclic $2n$-polytope". This idea translated into a combinatorial "local" condition which in fact yielded a class of 4-polytope P_v with v vertices in R^4. The idea was not completely successful because it yielded P_v only for $v = 3k - 7$ and there does not appear to be a realization of P_v with the intended cyclic-subpolytope property.

In the present article, we verify that the preceding idea does yield a 6-polytope in R^6 with v vertices which, for each $k \geq 8$ and each $v > k$, has a realization with the intended cyclic-subpolytope property.

Finally, we remark that Z. Smilansky was the first to consider generalizations of cyclic d-polytopes. In [8] and [9], he considered the $d = 4$ case using an algebraic approach.

2 DEFINITIONS.

Let Y be a set of points in R^d. Then conv Y and aff Y denote, respectively, the convex hull and the affine hull of Y. If $Y = \{y_1, \ldots, y_m\}$ is finite, we set

$$[y_1, \ldots, y_m] = \text{conv } Y \qquad \text{and} \qquad \langle y_1, \ldots, y_m \rangle = \text{aff } Y.$$

Let $Q \subset R^d$ be a (convex) d-polytope. For $-1 \leq i \leq d$, let $\mathcal{F}_i(Q)$ denote the set of i-faces of Q and $f_i(Q) = |\mathcal{F}_i(Q)|$. For convenience, we set

$$\mathcal{F}(Q) = \mathcal{F}_{d-1}(Q), \text{ the set of facets of } Q.$$

We recall that the *face lattice* of Q is the collection of all faces of Q ordered by inclusion, and that two polytopes are (*combinatorially*) *equivalent* if their face lattices are isomorphic. A *facet system* of Q is a pair $(\mathcal{S}, N)$ where N is a finite set, $\mathcal{S} \subset 2^N$ and there is a bijection $h : N \rightarrow \mathcal{F}_0(Q)$ such that

$$\mathcal{F}(Q) = \{\text{conv } \{h(n) \mid n \in S\} \mid S \in \mathcal{S}\}.$$

We cite from [6] and [10], p. 71:

Lemma 1 *If $(\mathcal{S}^*, N)$ and $(\mathcal{S}, N)$ are facet systems of convex d-polytopes such that $\mathcal{S}^* \subseteq \mathcal{S}$ then $\mathcal{S}^* = \mathcal{S}$ and the two polytopes are equivalent.*

Let F be a facet of Q and y be a point of R^d such that $y \notin \text{aff } F$. Then (cf. [5], p. 78) y is either beneath F or beyond F with respect to Q. From Grünbaum's book, we cite also

Lemma 2 *Let Q and Q^* be two d-polytopes in R^d such that $Q^* = conv\ (Q \cup \{y\})$ for some point $y \in R^d \backslash Q$. Let G be a face of Q. Then*

a) G is a face of Q^* if, and only if, there is a facet F of Q such that $G \subseteq F$ and y is beneath F, and

b) $G^* = \text{conv}\ (G \cup \{y\})$ is a face of Q^* if, and only if, either $y \in \text{aff}\ G$ or among the facets of Q containing G, there is at least one such that y is beneath it and at least one such that y is beyond it.

We deal now with the total ordering of $V = \mathcal{F}_0(Q) = \{y_1, y_2, \ldots, y_v\}$, $v \geq d+1$. We set $y_i < y_j$ if, and only if, $i < j$, and call $y_1 < y_2 < \cdots < y_v$ a *vertex array* of Q. We call y_i and y_{i+1} *successive* vertices, and say that y_j *separates* y_i and y_k if $y_i < y_j < y_k$.

Let $Y \subset V$ and assume that $y_1 < y_2 < \cdots < y_v$. We say that Y is an *even set* if it is the union of mutually disjoint subsets $\{y_i, y_{i+1}\}$; otherwise, Y is an *odd set.* Next, Y is a *Gale set* if any two points of $V \backslash Y$ are separated by an even number of points in Y. We extend these concepts in the obvious way to the facets of Q, and note that even facets are Gale and that an odd Gale facet contains y_1 or y_v. Let $\mathcal{F}^e(Q)$ and $\mathcal{F}^0(Q)$ denote, respectively, the set of even and the set of odd facets of Q. Then $\mathcal{F}(Q) = \mathcal{F}^e(Q) \cup \mathcal{F}^0(Q)$.

We say that Q is a *Gale polytope* if each facet of Q is Gale with respect to a fixed vertex array of Q. If $y_1 < y_2 < \cdots < y_v$ is the vertex array, we say also that Q is Gale with $y_1 < y_2 < \cdots < y_v$. As examples of Gale polytopes, we cite the cyclic d-polytopes (cf. [4]), the ordinary $(2n+1)$-polytopes (cf. [1] and [2]) and the 4-polytopes introduced in [3]. We note that for each of these polytopes, there is a complete description of all the facets.

We remark that $C \subset R^d$ is a cyclic d-polytope if with respect to a fixed vertex array of C : $Y \subset \mathcal{F}_0(C)$ determines a facet of C if, and only if, Y is a d element Gale set (Gale's Evenness Condition).

From [7], we note the following important and useful property of cyclic $2n$-polytopes.

Lemma 3 *Let $C \subset R^d$ be a cyclic d-polytope with the vertex array $y_1 < y_2 < \cdots < y_v$, $d = 2n \geq 4$. Then each d-subpolytope of C is cyclic with respect to the vertex array induced by $y_1 < y_2 < \cdots < y_v$.*

Finally, we say that Q is a *periodically-cyclic* d-polytope if there is a vertex array, say $y_1 < y_2 < \cdots < y_v$ and an integer k such that $v \geq k \geq d+2$, $[y_{i+1}, \ldots, y_{i+k}]$ is a cyclic d-polytope with $y_{i+1} < y_{i+2} < \cdots < y_{i+k}$ for $i = 0, \ldots, v-k$ and $[y_{i+1}, \ldots, y_{i+k}, y_{i+k+1}]$ is not cyclic for any $0 \leq i \leq v-k-1$. We call k the *period* of Q.

3 THE POLYTOPES.

In this section, we restrict our attention to R^6 and begin with another useful property of cyclic 6-polytopes.

Theorem 4 *Let $C \subset R^6$ be a cyclic 6-polytope with the vertex array $y_1 < y_2 < \cdots < y_{v-1}$; $v \geq 8$. Let $C \subset R^6$ be a 6-polytope such that $Q = \text{conv}\,(C \cup \{y_v\})$, $y_v \notin C$ and*

$$\mathcal{F}^e(C) \cup \{[y_r, y_{r+1}, y_s, y_{s+1}, y_{v-1}, y_v] \mid r = 2, \ldots, v-5;\ s = r+2, \ldots, v-3\} \subset \mathcal{F}(Q).$$

Then Q is cyclic with $y_1 < y_2 < \cdots < y_{v-1} < y_v$.

Proof. We note that there is a point $y^* \in R^6$ such that $C^* = \text{conv}\,(C \cup \{y^*\})$ is a cyclic 6-polytope with $y_1 < \cdots < y_{v-1} < y^*$. By Gale's Evenness Condition, $\mathcal{F}(C^*) = \mathcal{F}^e(C^*) \cup \mathcal{F}^0(C^*)$ where

$$\mathcal{F}^e(C^*) = \mathcal{F}^e(C) \cup \{[y_r, y_{r+1}, y_s, y_{s+1}, y_{v-1}, y^*] \mid r = 2, \ldots, v-5;\ s = r+2, v-3\}$$

and

$$\mathcal{F}^0(C^*) = \{[y_1, y_{i-1}, y_i, y_j, y_{j+1}, y^*] \mid i = 3, \ldots, v-3;\ j = i+2, \ldots, v-2\}$$

Thus, if $\{[y_1, y_{i-1}, y_i, y_j, y_{j+1}, y_v] \mid 3 \le i < j \le v-2\} \subset \mathcal{F}(Q)$ then a facet system of C^* is contained in a facet system of Q, C^* and Q are equivalent by Lemma 1, and the assertion of the theorem follows.

Let $3 \le i < j \le v-2$. Since $\{y_{i-1}, y_i, y_j, y_{j+1}\}$ is an even set, it is an easy consequence of Gale's Evenness Condition that

$$[y_1, y_{i-1}, y_i, y_j, y_{j+1}] = F^e_{ij} \cap F^0_{ij}$$

where $F^e_{ij}(F^0_{ij})$ is an even (odd) facet of C, moreover, F^0_{ij} is

a) $[y_1, y_{i-1}, y_i, y_j, y_{j+1}, y_{v-1}]$ if $j \le v-3$,

b) $[y_1, y_{i-1}, y_i, y_{v-3}, y_j, y_{j+1}]$ if $i \le v-4$ and $j = v-2$, and

c) $[y_1, y_{v-5}, y_{i-1}, y_i, y_j, y_{j+1}]$ if $i = v-3$ and $j = v-2$.

Let G_{ij} denote the 4-face of C such that

$$F^0_{ij} = \text{conv}\,(\{y_1\} \cup G_{ij}).$$

We note that

$$\tilde{F}_{ij} = \text{conv}\,(G_{ij} \cup \{y_v\}) \subset \mathcal{F}(Q)$$

by the hypothesis, and that

$$G_{ij} = \tilde{F}_{ij} \cap \hat{F}_{ij}$$

where $\hat{F}_{ij} \in \mathcal{F}^e(C)$. In the case of a), $\hat{F}_{ij}$ is

$$\begin{array}{lll} & [y_{i-1}, y_i, y_j, y_{j+1}, y_{v-2}, y_{v-1}] & \text{if } j \le v-4, \\ & [y_{i-1}, y_i, y_{v-4}, y_j, y_{j+1}, y_{v-1}] & \text{if } i \le v-5 \text{ and } j = v-3, \\ \text{and} & [y_{v-6}, y_{i-1}, y_i, y_j, y_{j+1}, y_{v-1}] & \text{if } i = v-5 \text{ and } j = v-3. \end{array}$$

In the case of b), $\hat{F}_{ij}$ is

$$\begin{array}{lll} & [y_{i-1}, y_i, y_{v-4}, y_{v-3}, y_j, y_{j+1}] & \text{if } i \le v-5 \\ \text{and} & [y_{v-6}, y_{i-1}, y_i, y_{v-3}, y_j, y_{j+1}] & \text{if } i = v-4. \end{array}$$

In the case of c),

$$\hat{F}_{ij} = [y_{v-6}, y_{v-5}, y_{i-1}, y_i, y_j, y_{j+1}].$$

In summary, G_{ij} is contained in two facets of Q with the property that neither of them contains y_1.

Since $F^e_{ij} \in \mathcal{F}(Q)$, it follows that there is a $F_{ij} \in \mathcal{F}(Q)$ such that

$$[y_1, y_{i-1}, y_i, y_j, y_{j+1}] = F^e_{ij} \cap F_{ij}.$$

Clearly, either $F^0_{ij} \subset F_{ij}$ or

$$F_{ij} = [y_1, y_{i-1}, y_i, y_j, y_{j+1}, y_v]$$

and we are done. Since $G_{ij} \subset F^0_{ij}$ and $\tilde{F}_{ij} \neq F_{ij} \neq \hat{F}_{ij}$, we have that $F^0_{ij} \not\subset F_{ij}$. ■

We are ready now to introduce periodically-cyclic 6-polytopes.

Let $v \geq k \geq 8$ and $V = \{x_1, x_2, \ldots, x_v\}$ be a set of v points in R^6. By way of notation, let

$x_i = x_1$ for $i < 1$,

$P_i^j = [x_i, x_{i+1}, \ldots, x_j]$ for $i < j$,

$X_i = \{[x_1, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+k-3}, x_{i+k-2}, x_{i+k-1}, x_{i+k}]\}$

$\cup\{[x_1, x_{i+1}, x_{i+2}, x_s, x_{s+1}, x_{s+1}, x_{i+k-1}, x_{i+k}] \mid s = i+4, \ldots, i+k-4\}$,

$Y_i = \{[x_r, x_{r+1}, x_s, x_{s+1}, x_t, x_{t+1}] \mid r = i+1, \ldots, i+k-6;\ s = r+2, \ldots, i+k-4;\ t = s+2, s, i+k-2\}$,

$Z_i = \{[x_1, x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+k-2}, x_{i+k-1}, x_{i+k}],\ [x_1, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+k-1}, x_{i+k}]\}$

$\cup\{[x_1, x_{i+2}, x_{i+3}, x_s, x_{s+1}, x_{i+k}] \mid s = i+4, \ldots, i+k-3\}$ and

$W_i = \{[x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_{i+k}] \mid r = i+3, \ldots, i+k-4;\ s = r+2, \ldots, i+k-2\}$.

Theorem 5 *For $v = k+m$, $k \geq 8$ and $m \geq 0$, there is a 6-polytope $P_1^v = [x_1, x_2, \ldots, x_v]$ in R^6 such that*

$$\mathcal{F}(P_1^v) = \left(\bigcup_{i=0}^{m} X_i\right) \cup \left(\bigcup_{i=0}^{m+1} Y_i\right) \cup Z_m \cup W_m.$$

Proof. Let P_1^k be a cyclic 6-polytope in R^6 with the vertex array $x_1 < x_2 < \cdots < x_k$. Then $\mathcal{F}(P_1^k) = \mathcal{F}^e(P_1^k) \cup \mathcal{F}^0(P_1^k)$ with

$$\begin{aligned}\mathcal{F}^e(P_1^k) &= \mathcal{F}^e(P_1^{k-1}) \cup \mathcal{F}^e(P_2^k) \cup \{[x_1, x_2, x_s, x_{s+1}, x_{k-1}, x_k] \mid s = 3, \ldots, k-3\} \\ &= Y_0 \cup Y_1 \cup X_0 \cup \{[x_1, x_2, x_3, x_4, x_{k-1}, x_k]\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^0(P_1^k) &= \{[x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_k] \mid r = 2, \ldots, k-4;\ s = r+2, \ldots, k-2\} \\ &= W_0 \cup \{[x_1, x_2, x_3, x_s, x_{s+1}, x_k] \mid s = 4, \ldots, k-2\}.\end{aligned}$$

Since

$$\begin{aligned}Z_0 &= \{[x_1, x_2, x_3, x_{k-2}, x_{k-1}, x_k],\ [x_1, x_2, x_3, x_4, x_{k-1}, x_k]\} \\ &\quad \cup \{[x_1, x_2, x_3, x_s, x_{s+1}, x_k] \mid s = 4, \ldots, k-3\},\end{aligned}$$

it follows that

$$\mathcal{F}(P_1^k) = X_0 \cup Y_0 \cup Y_1 \cup Z_0 \cup W_0.$$

Specifically, we remark that $Y_0 = \mathcal{F}^e(P_1^{k-1})$, $Y_1 = \mathcal{F}^e(P_2^k)$ and

a$_0$) $G_0 = [x_1, x_2, x_3, x_k] \in \mathcal{F}_3(P_1^k)$,

b$_0$) $X_0 \cup Z_0 \cup W_0 = \{F \in \mathcal{F}(P_1^k) \mid [x_1, x_k] \subset F\}$,

c$_0$) $Z_0 = \{F \in \mathcal{F}(P_1^k) \mid G_0 \subset F\}$,

d$_0$) $M(F) = (\text{aff } F) \cap (\text{aff } G)$ is a plane through $\langle x_1, x_k \rangle$ for each $F \in X_0 \cup W_0$, and

e$_0$) $M(\tilde{F}) = \langle x_1, x_2, x_k \rangle$ for each $\tilde{F} \in X_0$.

Let $u = v - 1 = k + m - 1 \geq k$ and assume that $P_1^u = [x_1, x_2, \ldots, x_u]$ is a 6-polytope in R^6 such that

$$\mathcal{F}(P_1^u) = \left(\bigcup_{i=0}^{m-1} X_i\right) \cup \left(\bigcup_{i=0}^{m} Y_i\right) \cup Z_{m-1} \cup W_{m-1}.$$

It is easy to check that

a_{m-1}) $G_{m-1} = [x_1, x_{m+1}, x_{m+2}, x_u] \in \mathcal{F}_3(P_1^u)$,

b_{m-1}) $X_{m-1} \cup Z_{m-1} \cup W_{m-1} = \{F \in \mathcal{F}(P_1^u) \mid [x_1, x_u] \subset F\}$,

c_{m-1}) $Z_{m-1} = \{F \in \mathcal{F}(P_1^u) \mid G_{m-1} \subset F\}$,

d_{m-1}) $M(F) = (\text{aff } F) \cap (\text{aff } G)$ is a plane through $\langle x_1, x_u \rangle$ for each $F \in X_{m-1} \cup W_{m-1}$, and

e_{m-1}) $M(\tilde{F}) = \langle x_1, x_{m+1}, x_u \rangle$ for each $\tilde{F} \in X_{m-1}$.

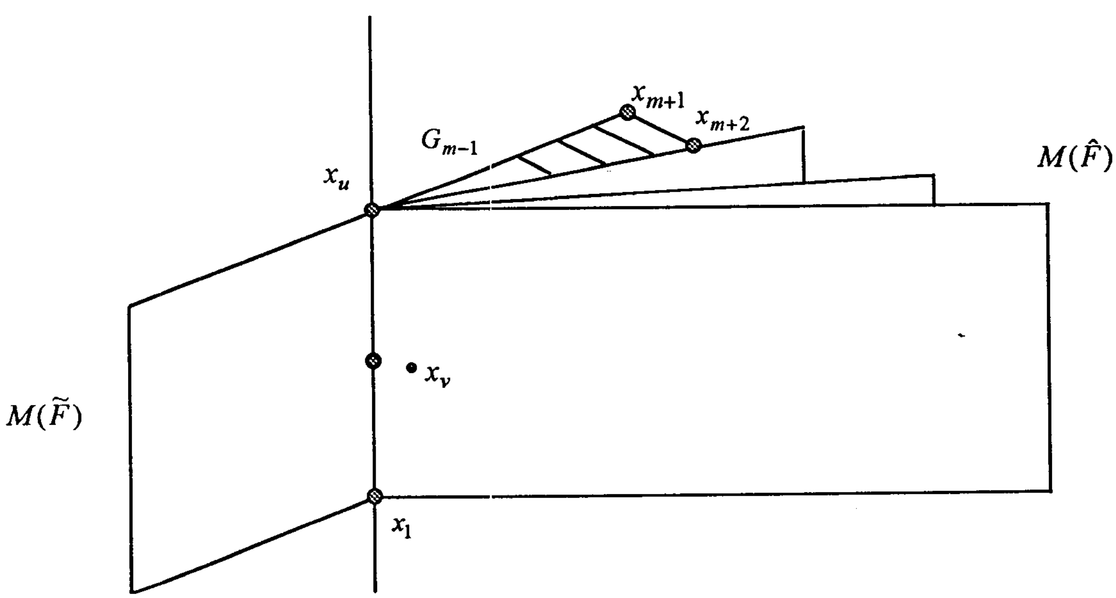

Figure 1

Referring to Figure 1, it is clear that there is a point $x \in \text{aff } G_{m-1}$ such that with respect to G_{m-1} , x is beneath $M(\tilde{F})$ and beyond $M(\hat{F})$ for each $\hat{F} \in W_{m-1}$. Let x be arbitrarily close to say, the mid-point of $[x_1, x_u]$, and label it x_v. Then, as G_{m-1} is a 3-face of P_1^u, we readily obtain that $x_v \in \text{aff } \bar{F}$ for each $\bar{F} \in Z_{m-1}$ and that with respect to P_1^u, x_v is beyond $\hat{F}$ for each $\hat{F} \in W_{m-1}$ and beneath the remaining facets of P_1^u.

Let $P_1^v = \text{conv } (P_1^u \cup \{x_v\}) = [x_1, \ldots, x_{v-1}, x_v]$ and recall that each $F \in \mathcal{F}(P_1^v) \backslash \mathcal{F}(P_1^u)$ contains x_v.

First, it is clear that $\mathcal{F}(P_1^v)$ contains $\bigcup_{i=0}^{m-1} X_i$, $\bigcup_{i=0}^{m} Y_i$ and

$$\begin{aligned} A &= \{\text{conv } (\bar{F} \cup \{x_v\}) \mid \bar{F} \in Z_{m-1}\} \\ &= \{[x_1, x_{m-1}, x_m, x_{m+1}, x_{m+2}, x_{v-3}, x_{v-2}, x_{v-1}, x_v],\ [x_1, x_m, x_{m+1}, x_{m+2}, x_{m+3}, x_{v-2}, x_{v-1}, x_v]\} \\ &\quad \cup \{[x_1, x_{m+1}, x_{m+2}, x_s, x_{s+1}, x_{v-1}, x_v] \mid s = m+3, \ldots, m+k-4 = v-4\} \\ &= X_m \cup \{[x_1, x_m, x_{m+1}, x_{m+2}, x_{m+3}, x_{v-2}, x_{v-1}, x_v],\ [x_1, x_{m+1}, x_{m+2}, x_{m+3}, x_{m+4}, x_{v-1}, x_v]\}. \end{aligned}$$

Next, it is an easy exercise to determine the 4-faces of $\hat{F} \in W_{m-1} = \{[x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_{v-1}] \mid r = m+2, \ldots, v-5;\ s = r+2, \ldots, v-3\}$ that are contained in a facet in $\mathcal{F}(P_1^u) \backslash (Z_{m-1} \cup W_{m-1})$. Then by Lemma 2, the remaining facets of P_1^v are the elements of

$$\begin{aligned} B &= \{[x_r, x_{r+1}, x_s, x_{s+1}, x_{v-1}, x_v] \mid r = m+2, \ldots, v-5;\ s = r+2, \ldots, v-3\} \\ &= Y_{m+1} \backslash Y_m, \end{aligned}$$

and

$$\begin{aligned} C &= \{[x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_v] \mid r = m+2, \ldots, v-5;\ s = r+2, \ldots, v-3\} \\ &\quad \cup \{[x_1, x_r, x_{r+1}, x_{v-2}, x_{v-1}, x_v] \mid r = m+3, \ldots, v-4\} \\ &= W_m \cup \{[x_1, x_{m+2}, x_{m+3}, x_s, x_{s+1}, x_v] \mid s = m+4, \ldots, v-3\}. \end{aligned}$$

Since $Y_{m+1} = (Y_{m+1} \backslash Y_m) \cup Y_m$ and $Z_m = (A \backslash X_m) \cup (C \backslash W_m)$, we obtain that

$$\begin{aligned} \mathcal{F}(P_1^v) &= \left(\bigcup_{i=0}^{m-1} X_i\right) \cup \left(\bigcup_{i=0}^{m} Y_i\right) \cup A \cup B \cup C \\ &= \left(\bigcup_{i=0}^{m} X_i\right) \cup \left(\bigcup_{i=0}^{m+1} Y_i\right) \cup Z_m \cup W_m. \qquad \blacksquare \end{aligned}$$

We show now that any 6-polytope that is equivalent to P_1^v in Theorem 5 may be constructed in the same manner.

Theorem 6 *Let $P_1^v = [x_1, x_2, \ldots, x_v]$ be a 6-polytope in R^6 such that*

$$\mathcal{F}(P_1^v) = \left(\bigcup_{i=0}^{m} X_i\right) \cup \left(\bigcup_{i=0}^{m+1} Y_i\right) \cup Z_m \cup W_m, \quad v = k + m > k \geq 8.$$

Then for each subpolytope $P_1^u = [x_1, x_2, \ldots, x_u]$, $v > u = k + l > k$,

$$\mathcal{F}(P_1^u) = \left(\bigcup_{i=0}^{l} X_i\right) \cup \left(\bigcup_{i=0}^{l+1} Y_i\right) \cup Z_l \cup W_l.$$

Proof. Clearly, it is sufficient to verify the assertion for $u = v - 1 = k + m - 1$. Since there is a 6-polytope in R^6 with u vertices and the desired set of facets, it is sufficient by Lemma 1 to show that

$$\left(\bigcup_{i=0}^{m-1} X_i\right) \cup \left(\bigcup_{i-0}^{m} Y_i\right) \cup Z_{m-1} \cup W_{m-1} \subseteq \mathcal{F}(P_1^u).$$

We recall that $x_v \in F \in \mathcal{F}(P_1^v)$ only if $F \in X_m \cup (Y_{m+1} \backslash Y_m) \cup Z_m \cup W_m$ and so,

$$\left(\bigcup_{i=0}^{m-1} X_i\right) \cup \left(\bigcup_{i=0}^{m} Y_i\right) \subset \mathcal{F}(P_1^u).$$

Next, we observe that

$$\{[x_1, x_{m-1}, x_m, x_{m+1}, x_{m+2}, x_{u-2}, x_{u-1}, x_u, x_v]\}$$
$$\cup\{[x_1, x_{m+1}, x_{m+2}, x_s, x_{s+1}, x_u, x_v] \mid s = m+4, \ldots, v-4\}$$
$$\cup\{[x_1, x_m, x_{m+1}, x_{m+2}, x_{m+3}, x_{u-1}, x_u, x_v],\ [x_1, x_{m+1}, x_{m+2}, x_{m+3}, x_{m+4}, x_u, x_v]\}$$

is the set of non-simplicial facets in $X_m \cup Z_m$. For such a facet F, we have that

$$F = \text{conv}\,(G \cup \{x_v\})$$

where $x_v \notin G$, $\{x_1, x_{v-1}\} \subset G$ and either $G \in \mathcal{F}(P_1^u)$ and $x_v \in \text{aff}\ G$ or $G \in \mathcal{F}_4(P_1^u)$ and there exist $\{\tilde{F}, \hat{F}\} \subset \mathcal{F}(P_1^u)$ such that

$$G = \tilde{F} \cap \hat{F},$$

x_v is beneath $\tilde{F}$ and x_v is beyond $\hat{F}$. We note that if $\tilde{F}$ exists then $\tilde{F} \in \mathcal{F}(P_1^v)$, and in particular $\tilde{F} \in X_{m-1}$. It is now easy to check that $\tilde{F}$ does not exist for any G and so, $\mathcal{F}(P_1^u)$ contains

$$\begin{aligned}&\{[x_1, x_{m-1}, x_m, x_{m+1}, x_{m+2}, x_{u-2}, x_{u-1}, x_u]\}\\&\quad\cup\{[x_1, x_{m+1}, x_{m+2}, x_s, x_{s+1}, x_u]\} \mid s = m+4, \ldots, u-3\}\\&\quad\cup\{[x_1, x_m, x_{m+1}, x_{m+2}, x_{m+3}, x_{u-1}, x_u],\ [x_1, x_{m+1}, x_{m+2}, x_{m+3}, x_{m+4}, x_u]\},\end{aligned}$$

which is Z_{m-1}.

Next, we observe that

$$Y_{m+1} \backslash Y_m = \{[x_r, x_{r+1}, x_s, x_{s+1}, x_u, x_v] \mid r = m+2, \ldots, v-5 = u-4;\ s = r+2, \ldots, u-2\}$$

and that for each r and s,

$$G_{rs} = [x_r, x_{r+1}, x_s, x_{s+1}, x_u] \in \mathcal{F}_4(P_1^v) \cap \mathcal{F}_4(P_1^u).$$

Since $W_{m-1} = \{\text{conv}\ (\{x_1\} \cup G_{rs}) \mid r = m+2, \ldots, u-4;\ s = r+2, \ldots, u-2\}$, we need only to show that each conv $(\{x_1\} \cup G_{rs}) \in \mathcal{F}(P_1^u)$. For simplicity, let $G = G_{rs}$.

Since $G \in \mathcal{F}_4(P_1^v) \cap \mathcal{F}_4(P_1^u)$, we obtain as above that there is a $\hat{F} \in \mathcal{F}(P_1^u)$ such that

$$G \subset \hat{F} \quad \text{and} \quad x_v \quad \text{is beyond} \quad \hat{F}.$$

We wish to show that $\hat{F} = \text{conv}\,(\{x_1\} \cup G)$, that is,

$$x_j \notin \hat{F} \quad \text{whenever} \quad x_j \notin \{x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_u\}.$$

Let $2 \leq j \leq m$. We note that there is at most two facets of P_1^v that contain $\{x_j, x_v\}$, and thus $[x_j, x_v]$ is not an edge of P_1^v. Since there is a $F_j \in \mathcal{F}(P_1^v)$ such that $F_j \cap \{x_j, x_v\} = \{x_j\}$, it follows that $F_j \in \mathcal{F}(P_1^u)$ and x_v is beneath F_j. Hence, Lemma 2 implies that x_v is beneath each facet of P_1^u that contains x_j. Thus, $x_j \notin \hat{F}$.

Let $j > m$ and assume that $x_j \notin \{x_1, x_r, x_{r+1}, x_s, x_{s+1}, x_u\}$. Then

$$G^* = \text{conv}\,\{x_j, x_r, x_{r+1}, x_s, x_{s+1}\}$$

is a 4-polytope. We claim that there is $\{F_1, F_2\} \subset Y_{m-1}$ such that $G^* \subseteq F_1 \cap F_2$ and $x_u \notin F_1 \cup F_2$. Then $F_1 \neq \hat{F} \neq F_2$ yields that $x_j \notin \hat{F}$.

Case 1: $r = u - 4$ and $s = u - 2$.

Then $G = [x_{u-4}, x_{u-3}, x_{u-2}, x_{u-1}, x_u]$ and $u \geq 7$ imply that $G = \hat{F} \cap [x_{u-5}, x_{u-4}, x_{u-3}, x_{u-2}, x_{u-1}, x_u]$ and $x_{u-5} \notin \hat{F}$. For $m + 1 \leq j \leq u - 6$ we have

$$F_1 = [x_{j-1}, x_j, x_{u-4}, x_{u-3}, x_{u-2}, x_{u-1}] \quad \text{and} \quad F_2 = [x_j, x_{j+1}, x_{u-4}, x_{u-3}, x_{u-2}, x_{u-1}].$$

Case 2: $m + 2 \leq r \leq u - 5$ and $s = u - 2$.

Then $G = [x_r, x_{r+1}, x_{u-2}, x_{u-1}, x_u] = \hat{F} \cap [x_r, x_{r+1}, x_{u-3}, x_{u-2}, x_{u-1}, x_u]$ and $x_{u-3} \notin \hat{F}$. Now, F_1 and F_2 are as follows:

$$m+1 \le j \le r-2: \quad [x_{j-1}, x_j, x_r, x_{r+1}, x_{u-2}, x_{u-1}] \text{ and } [x_j, x_{j+1}, x_r, x_{r+1}, x_{u-2}, x_{u-1}],$$

$$j = r-1: \quad [x_{j-1}, x_j, x_r, x_{r+1}, x_{u-2}, x_{u-1}] \text{ and } [x_j, x_r, x_{r+1}, x_{r+2}, x_{u-2}, x_{u-1}],$$

$$j = r+2: \quad [x_{r-1}, x_r, x_{r+1}, x_j, x_{u-2}, x_{u-1}] \text{ and } [x_r, x_{r+1}, x_j, x_{j+1}, x_{u-2}, x_{u-1}],$$

$$j \ge r+3: \quad [x_r, x_{r+1}, x_{j-1}, x_j, x_{u-2}, x_{u-1}] \text{ and } [x_r, x_{r+1}, x_j, x_{j+1}, x_{u-2}, x_{u-1}].$$

Case 3: $m+2 \le r \le u-5$ and $r+2 \le s \le u-3$.

Then $G = [x_r, x_{r+1}, x_s, x_{s+1}, x_{u-1}, x_u] \cap \hat{F}$, $x_{u-1} \notin \hat{F}$ and F_1 and F_2 are as follows:

$$s+2 \le j \le u-2: \quad [x_r, x_{r+1}, x_s, x_{s+1}, x_j, x_{j+1}] \text{ and one of } [x_r, x_{r+1}, x_s, x_{s+1}, x_{j-1}, x_j] \quad (j \ne s+2)$$

$$\text{or } [x_r, x_{r+1}, x_{s-1}, x_s, x_{s+1}, x_j] \quad (j = s+2 \ne r+4)$$

$$\text{or } [x_{r-1}, x_r, x_{r+1}, x_s, x_{s+1}, x_j] \quad (j = s+2 = r+4),$$

$$r+2 \le j \le s-1: \quad \text{one of } [x_r, x_{r+1}, x_j, x_s, x_{s+1}, x_{s+2}] \quad (j = s-1)$$

$$\text{or } [x_r, x_{r+1}, x_j, x_{j+1}, x_s, x_{s+1}] \quad (j \ne s-1), \text{ and}$$

$$\text{one of } [x_r, x_{r+1}, x_{j-1}, x_j, x_s, x_{s+1}] \quad (j \ne r+2)$$

$$\text{or } [x_{r-1}, x_r, x_{r+1}, x_j, x_s, x_{s+1}] \quad (j = r+2),$$

$$j = r-1: \quad [x_{j-1}, x_j, x_r, x_{r+1}, x_s, x_{s+1}] \text{ and one of } [x_j, x_r, x_{r+1}, x_{r+2}, x_s, x_{s+1}] \quad (s \ne r+2)$$

$$\text{or } [x_j, x_r, x_{r+1}, x_s, x_{s+1}, x_{s+2}] \quad (s = r+2),$$

$$m+1 \le j \le r-2: \quad [x_{j-1}, x_j, x_r, x_{r+1}, x_s, x_{s+1}] \text{ and } [x_j, x_{j+1}, x_r, x_{r+1}, x_s, x_{s+1}]. \quad \blacksquare$$

Theorem 7 *Let $P_1^v = [x_1, x_2, \ldots, x_v]$ be a 6-polytope in R^6 such that*

$$\mathcal{F}(P_1^v) = \left(\bigcup_{i=0}^{m} X_i\right) \cup \left(\bigcup_{i=0}^{m+1} Y_i\right) \cup Z_m \cup W_m, \qquad v = k + m \geq k \geq 8.$$

Then P_1^v is periodically-cyclic with the vertex array $x_1 < x_2 < \cdots < x_v$ and the period k.

Proof. Let $v \geq u = k + l \geq k$. Then by Theorem 6,

$$\mathcal{F}(P_1^u) = \left(\bigcup_{i=0}^{l} X_i\right) \cup \left(\bigcup_{i=0}^{l+1} Y_i\right) \cup Z_l \cup W_l.$$

Since $\mathcal{F}(P_1^k) = X_0 \cup Y_0 \cup Y_1 \cup Z_0 \cup W_0$, we obtain from the proof of Theorem 5 that P_1^k is a cyclic 6-polytope with $x_1 < x_2 < \cdots < x_k$ and $Y_i = \mathcal{F}^e(P_{i+1}^j)$ for $j = i + k - 1$ and $i = 0, 1$. Since

$$[x_1, x_2, x_3, x_4, x_5, x_k, x_{k+1}] \in Z_1 \subset \mathcal{F}(P_1^{k+1}),$$

P_1^{k+1} is not simplicial and hence, it is not cyclic.

Let $v > u$ and assume that P_1^u is periodically-cyclic with $x_1 < x_2 < \cdots < x_u$ and the period k. Then P_{i+1}^j is cyclic with $x_{i+1} < x_{i+2} < \cdots < x_j$ for $j = i + k - 1$ and $i = 0, 1, \ldots, l + 1$ by Lemma 3, and we note that $Y_i = \mathcal{F}^e(P_{i+1}^j)$.

We recall that $X_{l+1} \cup Z_{l+1} \subset \mathcal{F}(P_1^{u+1})$,

$$\begin{aligned} X_{l+1} &= \{[x_1, x_l, x_{l+1}, x_{l+2}, x_{l+3}, x_{u-2}, x_{u-1}, x_u, x_{u+1}]\} \\ &\cup \{[x_1, x_{l+2}, x_{l+3}, x_s, x_{s+1}, x_u, x_{u+1}] \mid s = l + 5, \ldots, u - 3\} \end{aligned}$$

and $[x_1, x_{l+2,} x_{l+3}, x_{l+4}, x_{l+5}, x_u, x_{u+1}] \in Z_{l+1}$. Since

$$\{[x_{l+2}, x_{l+3}, x_s, x_{s+1}, x_{u-1}, x_u] \mid s = l+4, \ldots, u-3\} \subset Y_{l+1} \subset \mathcal{F}(P_1^{u+1}),$$

we have that each element of

$$\{[x_{l+2}, x_{l+3}, x_s, x_{s+1}, x_u] \mid s = l+4, \ldots, u-2\}$$

is a 4-face of P_1^{u+1}, and hence of P_{l+2}^{u+1}. Thus it follows from the indicated facets in $X_{l+1} \cup Z_{l+1}$ that

$$\bar{Y} = \{[x_{l+2}, x_{l+3}, x_s, s_{s+1}, x_u, x_{u+1}] \mid s = l+4, \ldots, u-2\} \subset \mathcal{F}(P_{l+2}^{u+1}).$$

In summary, $\mathcal{F}(P_{l+2}^{u+1})$ contains

$$\begin{aligned}
&Y_{l+1} \cup Y_{l+2} \cup \bar{Y} \\
&\qquad = Y_{l+1} \cup (Y_{l+2} \backslash Y_{l+1}) \cup \bar{Y} \\
&\qquad = Y_{l+1} \cup \{[x_r, x_{r+1}, x_s, x_{s+1}, x_u, x_{u+1}] \mid r = l+3, u-4;\ s = r+2, \ldots, u-2\} \cup \bar{Y} \\
&\qquad = \mathcal{F}^e(P_{l+2}^u) \cup \{[x_r, x_{r+1}, x_s, x_{s+1}, x_u, x_{u+1}] \mid r = l+2, \ldots, u-4;\ s = r+2, \ldots, u-2\}
\end{aligned}$$

and P_{l+2}^u is cyclic with $x_{l+2} < x_{l+3} < \cdots < x_u$. Hence by Theorem 4, P_{l+2}^{u+1} is cyclic with $x_{l+2} < x_{l+3} < \cdots < x_u < x_{u+1}$.

Finally, as $[x_{l+1}, x_{l+2}, x_{l+3}, x_{u-2}, x_{u-1}, x_u, x_{u+1}]$ is a facet of P_{l+1}^{u+1}, the latter is not a cyclic subpolytope of P_1^{u+1}. ■

In order to highlight the importance of the period k, we let $P_1^v(k)$ denote the periodically-cyclic 6-polytope in Theorem 7.

4 REMARKS.

We observe that for $v \geq k \geq 8$,

$$\mathcal{F}(P_1^v(k)) = \left(\bigcup_{i=0}^{v-k} X_i\right) \cup Y_0 \cup \left(\bigcup_{i=1}^{v-k+1} (Y_i \backslash Y_{i-1})\right) \cup Z_{v-k} \cup W_{v-k}$$

and the sets on the right are mutually disjoint. From this decomposition and the fact that $\begin{pmatrix} n-m \\ m \end{pmatrix}$ is the number of paired subsets of $2m$ elements of a totally ordered set of n elements, we obtain readily that

$$f_5(P_1^v(k)) = \frac{(v-k)(k-6)(k-3)}{2} + \frac{k}{k-3}\begin{pmatrix} k-3 \\ 3 \end{pmatrix}.$$

Next, as examples, we present $P_1^9(8)$ and $P_1^{10}(8)$. For convenience, we use only subscripts of the x_i in the lists of facets.

$$P_1^9(8)$$

$$X_0 \cup X_1 = \{[1,2,5,6,7,8],\ [1,2,4,5,7,8],\ [1,2,3,6,7,8,9],\ [1,2,3,5,6,8,9]\},$$

$$Y_0 \cup Y_1 \cup Y_2 = \{[1,2,3,4,5,6],\ [1,2,3,4,6,7],\ [1,2,4,5,6,7],\ [2,3,4,5,6,7],$$
$$[2,3,4,5,7,8],\ [2,3,5,6,7,8],\ [3,4,5,6,7,8],\ [3,4,5,6,8,9],$$
$$[3,4,6,7,8,9],\ [4,5,6,7,8,9]\},$$

$$Z_1 = \{[1,2,3,4,7,8,9],\ [1,2,3,4,5,8,9],\ [1,3,4,5,6,9],\ [1,3,4,6,7,9]\},$$

$$W_1 = \{[1,4,5,6,7,9],\ [1,4,5,7,8,9],\ [1,5,6,7,8,9]\}.$$

$$P_1^{10}(8)$$

$$X_0 \cup X_1 \cup X_2 = X_0 \cup X_1 \cup \{[1,2,3,4,7,8,9,10],\ [1,3,4,6,7,9,10]\},$$

$$Y_0 \cup Y_1 \cup Y_2 \cup Y_3 = Y_0 \cup Y_1 \cup Y_2 \cup \{[4,5,6,7,9,10],\ [4,5,7,8,9,10],\ [5,6,7,8,9,10]\},$$

$$Z_2 = \{[1,2,3,4,5,8,9,10],\ [1,3,4,5,6,9,10],\ [1,4,5,6,7,10],\ [1,4,5,7,8,10]\},$$

$$W_2 = \{[1,5,6,7,8,10],\ [1,5,6,8,9,10],\ [1,6,7,8,9,10]\}.$$

Finally, we observe that $P_1^v(k) = [x_1, x_2, \ldots, x_v]$ is Gale with $x_1 < x_2 < \cdots < x_v$, and that the subpolytopes $[x_i, x_{i+1}, \ldots, x_j]$, $2 \le i < j \le v$ and $j - i \ge k$, are periodically-cyclic with $x_i < x_{i+1} < \cdots < x_j$ and the period k. This raises the question: Is $[x_i, x_{i+1}, \ldots, x_j]$ necessarily equivalent to $P_1^u(k)$ for $u = j - i + 1$? The answer is no. From below, we see that $P_2^{10}(8) = [x_2, \ldots, x_{10}]$ is not equivalent to $P_1^9(8)$. It is an easy exercise to determine that

$$\begin{aligned}\mathcal{F}(P_2^{10}(8)) = \{&[2,3,4,5,6,7],\ [2,3,4,5,7,8],\ [2,3,4,5,8,9,10],\ [2,3,4,7,8,9,10],\\ &[2,3,5,6,7,8],\ [2,3,5,6,8,9],\ [2,3,5,6,9,10],\ [2,3,6,7,9,10],\\ &[2,3,6,7,9,10],\ [2,3,4,5,6,10],\ [2,3,4,6,7,10],\ [2,4,5,6,7,10],\\ &[2,4,5,7,8,10],\ [2,5,6,7,8,10],\ [2,5,6,8,9,10],\ [2,6,7,8,9,10],\\ &[3,4,5,6,7,8],\ [3,4,5,6,8,9],\ [3,4,5,6,9,10],\ [3,4,6,7,8,9],\\ &[3,4,6,7,9,10],\ [4,5,6,7,8,9],\ [4,5,6,7,9,10],\ [4,5,7,8,9,10],\\ &[5,6,7,8,9,10]\}.\end{aligned}$$

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