On Periodically-cyclic Gale 4-polytopes

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Abstract

Based upon a labelling (total ordering) of vertices, a 4-polytope is Gale if the vertices satisfy a part of Gale's Evenness Condition and it is periodically-cyclic if there is an integer k such that every subset of k successive vertices generates a cyclic 4-polytope.

Among the bi-cyclic 4-polytopes introduced by Z. Smilansky, we determine which are Gale or periodically-cyclic or both.

1 Introduction

Let V be a set of points (x, y, z, w) in $\mathbb{R}^4$. Then conv Y denotes the convex hull of V and if $V = \{v_1, \ldots, v_n\}$ is finite, we set

$$[v_1, v_2, \ldots, v_n] = \text{conv } V.$$

Let $P \subset \mathbb{R}^4$ be a (convex) 4-polytope with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 6$. We set $v_i < v_j$ if, and only if, $i < j$, and call

$$v_1 < v_2 < \cdots < v_n$$

a *vertex array* of P.

We say that P is *Gale* if it has a vertex array, say, $v_1 < v_2 < \cdots < v_n$ such that for any facet F of P and any $v_i \neq v_j$ in $V \backslash F$, v_i and v_j are separated in the vertex array by an even number of vertices of F. Next, P is *periodically-cyclic* if it has a vertex array, say again, $v_1 < v_2 < \cdots < v_n$ and there is an integer k such that $6 \leq k \leq n$, $[v_{i+1}, \ldots, v_{i+k}]$ is a cyclic 4-polytope for $i = 0, \ldots, n-k$ and $[v_{i+1}, \ldots, v_{i+k}, v_{i+k+1}]$ is not a cyclic 4-polytope for any $0 \leq i \leq n-k-1$. We call k the *period* of P.

We recall that a cyclic 4-polytope C in $\mathbb{R}^4$ has a vertex array that may be chosen on an oriented 4^{th} order curve, say,

$$\Gamma(t) = (t, t^2, t^3, t^4), \qquad t \in \mathbb{R}$$

or

$$\Gamma(t) = (\cos 2\pi t, \sin 2\pi t, \cos 4\pi t, \sin 4\pi t), \qquad t \in [0, 1);$$

(cf. [3]) and as a consequence, the vertex array satisfies Gale's Evenness Condition and yields a complete description of the facial structure of C.

Clearly, a periodically-cyclic Gale 4-polytope P is a generalization of C. As such, the vertex array of P should also yield a complete description of the facial structure of P. This is a highly desired property because in general, such a description of a 4-polytope is a very difficult task and has been accomplished mostly for polytopes with "few" vertices; cf. for example [1] and [5].

In [2], we introduced a class of 4-polytopes $P_n = [v_1, \dots, v_n]$ with "many" vertices based upon the following construction: assume that $[v_1, \dots, v_k]$ is cyclic with $v_1 < \cdots < v_k$, adjoin a vertex v_{k+1} so that $[v_1, \dots, v_k, v_{k+1}]$ is Gale, but not cyclic, with $v_1 < \cdots < v_k < v_{k+1}$ and $[v_2, \dots, v_{k+1}]$ is cyclic with $v_2 < \cdots < v_{k+1}$, and repeat the process as long as the resultant polytope is combinatorially convex. This idea yields a realizable P_n that is Gale with $v_1 < \cdots < v_n$, but only for $n = 3k - 7$, and there does not appear to be a realization of P_n with the intended cyclic-subpolytope property. Thus, in order to find periodically-cyclic Gale 4-polytopes, it seems reasonable to consider directly the convex hulls of points on generalizations of 4^{th} order curves and this leads us to the work of Z. Smilansky in [7] and [8]. We present a short summary of results relevant to our presentation.

Henceforth, let $q > p \geq 2$ be relatively prime integers. Let

$$\Gamma_{pq}(t) = (\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt), \qquad t \in I = [0, 1),$$

and set $\Gamma = \Gamma_{pq}(I)$. It is well known that the curve Γ is closed, finite (any hyperplane of $\mathbb{R}^4$ intersects Γ in a finite number of points), locally of 4^{th} order and given any two points of Γ, there is an orthogonal transformation of $\mathbb{R}^4$ which maps Γ onto Γ and one point onto the other; cf. [4]. These transformations are generated by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and } T_{pq}(s) = \begin{bmatrix} \cos 2\pi ps & \sin 2\pi ps & 0 & 0 \\ -\sin 2\pi ps & \cos 2\pi ps & 0 & 0 \\ 0 & 0 & \cos 2\pi qs & \sin 2\pi qs \\ 0 & 0 & -\sin 2\pi qs & \cos 2\pi qs \end{bmatrix}.$$

We note that $\Gamma_{pq}(t)R = \Gamma_{pq}(-t)$, $\Gamma_{pq}(t)T_{pq}(s) = \Gamma_{pq}(t+s)$ and that the behaviour of Γ at any one point is the same as at any other point. In fact, it is easy to check that

(1) if $\{t_1, t_2, s_1, s_2\} \subseteq \mathbb{R}$ such that $t_1 - s_1 = t_2 - s_2 \pmod 1$ or $t_1 - s_1 \equiv s_2 - t_2 \pmod 1$ then there is an orthogonal transformation of $\mathbb{R}^4$ which maps Γ onto Γ, $\Gamma_{pq}(t_1)$ onto $\Gamma_{pq}(t_2)$ and $\Gamma_{pq}(s_1)$ onto $\Gamma_{pq}(s_2)$.

We consider now points on Γ. Let $n \geq 5$ be an integer, $b_i = \Gamma_{pq}\left(\frac{i}{n}\right)$ and

$$B(p,q,n) = [b_0, b_1, \ldots, b_{n-1}].$$

Then $B(p,q,n)$ is a *bi-cyclic* 4-polytope. Next, let $\eta : \mathbb{R}^2 \to \mathbb{R}^4$ be a function defined by

$$\eta(x,y) = (\cos\frac{2\pi}{n}x, \sin\frac{2\pi}{n}x, \frac{2\pi}{n}y, \sin\frac{2\pi}{n}y),$$

and set $\Lambda = \eta^{-1}(\{b_0, b_1, , b_{n-1}\})$. Then

$$\Lambda = \{i(p,q) \mid i = 0, \ldots, n-1\} + n\mathbb{Z}^2$$

where, as usual, Z^2 denotes the plane integer lattice. Now, there is a connection between the facial structure of $B(p,q,n)$ and the structure of the geometric lattice Λ, and this connection is the rationale for the name bi-cyclic.

Introducing some notation: two points of $\mathbb{R}^2$ are Λ-*distinct* if they are in Λ and are distinct modulo $n\mathbb{Z}^2$. A line of $\mathbb{R}^2$ is a Λ-*line* if it contains at least two Λ-distinct points. A closed parallel strip of $\mathbb{R}^2$ is a Λ-*strip* if it is bounded by two parallel Λ-lines. A parallelogram of $\mathbb{R}^2$ is a Λ-*parallelogram* if its vertices are four Λ-distinct points, its area is n, and it has an edge with positive (negative) slope. We refer to a Λ-strip or a Λ-parallelogram as a Λ-*region.* Finally, a Λ-region of $\mathbb{R}^2$ is *empty* if it contains no point of Λ in its interior.

In [8], Smilansky shows that the facial structure of $B(p,q,n)$ can be obtained from Λ. Specifically, F is a facet of $B(p,q,n)$ if, and only if,

(2) $F = \text{conv}\,\{(x,y) \mid (x,y) \in S \cap \Lambda\}$

where (a) S is an empty Λ-parallelogram,

or (b) S is an empty horizontal Λ-strip,

or (c) S is an empty vertical strip.

In case of (a), F is a simplex. In case of (b), (c), F is an antiprism over a regular q-gon (p-gon).

Next, it is important to note that because of (1), the vertex figures of $B(p,q,n)$ are equivalent up to orthogonal transformations. Thus, with $b_i = b_j$ for $i - j \in n\mathbb{Z}$,

(3) $[b_0, b_i, b_j, \ldots, b_k]$ is a facet of $B(p,q,n)$

if, and only if,

$[b_l, b_{i+l}, b_{j+l}, \ldots, b_{k+l}]$ is a facet of $B(p,q,n)$ for $l \in \mathbb{Z}$.

We are now ready to determine the values of p, q and n for which $B(p,q,n)$ is Gale, and the range of values of p, q and n for which $B(p,q,n)$ is periodically-cyclic.

2 The Gale Property

We wish to determine explicitly the set $\mathcal{F}(B(p,q,n))$ of facets of $B(p,q,n) \subset \mathbb{R}^4$. By (2), we need to determine empty Λ-regions in $\mathbb{R}^2$ for

$$\Lambda = \{i(p,q) \mid i = 0, \ldots n-1\} + nZ^2.$$

By (3), we need to determine only those empty Λ-regions that contain $(0,0)$.

With the preceding goal in mind, let L_a denote the line of $\mathbb{R}^2$ defined by $y = (qx + an)p^{-1}$. Clearly, if $(x,y) \in \Lambda$ then $(x,y) \in L_a$ for some $a \in Z$. Next, for $(x,y) \in \Lambda$ and $i \in \mathbb{Z}$, set

$$u_i = (x,y) \quad \text{if} \quad \eta(x,y) = b_i.$$

Then

$$u_i = (x,y) = (x',y') \quad \text{implies that} \quad (x-x', y-y') \in n\mathbb{Z}^2$$

and

$$u_i = u_j \quad \text{implies that} \quad i - j \in n\mathbb{Z}.$$

Finally, let $x(v)$ $(y(v))$ denote the x-coordinate (y-coordinate) of a point $v \in \mathbb{R}^2$.

Let N denote the square $[-n,n] \times [-n,n]$ in $\mathbb{R}^2$. Then $u_0 \in N$ and each $u_i \neq u_0$ appears at most once in the interior of each quadrant N_j of N, and we need to search only in N for empty Λ-regions containing $u_0 = (0,0)$; cf. Figure 1 under the assumption that $n > q$.

Lemma 1 *Let $n \geq pq$ and $B(p,q,n)$ be Gale with $b_0 < b_1 < \cdots < b_{n-1}$. Then $p > 2$ $(q > 2)$ implies that $p \mid n$ $(q \mid n)$.*

Proof. We suppose that p does not divide n, and seek a contradiction.

Let L_1 intersect $y = 0$ $(y = n)$ at $v(w)$, and L_2 intersect $y = n$ at z. Then

$$v = \left(\frac{-n}{q}, 0\right), \quad w = \left(\frac{(p-1)n}{q}, n\right) \qquad \text{and} \qquad z = \left(\frac{(p-2)n}{q}, n\right).$$

Since $\left(0, \dfrac{n}{p}\right) \in L_1 \backslash \Lambda$ and the distance from it to v or w is at least $(p^2+q^2)^{1/2}$, there is a $u_i \in L_1 \cap N_2$ such that $u_{i+1} \in L_1 \cap N_2$ and $\left(0, \dfrac{n}{p}\right) \in (u_i, u_{i+1})$. Then $u_j = u_i + u_{i+1} \in L_2 \cap \Lambda$ and $Q = [u_0, u_i, u_{i+1}, u_j]$ is clearly a parallelogram with area n, no point of Λ in its interior and a side with positive (negative) slope. Since

$$x(u_j) < x(u_{i+1}) < x(u_1) = p < (p-2)p \leq (p-2)\frac{n}{q} = x(z),$$

it follows that $u_j \in L_2 \cap (N_1 \cup N_2)$.

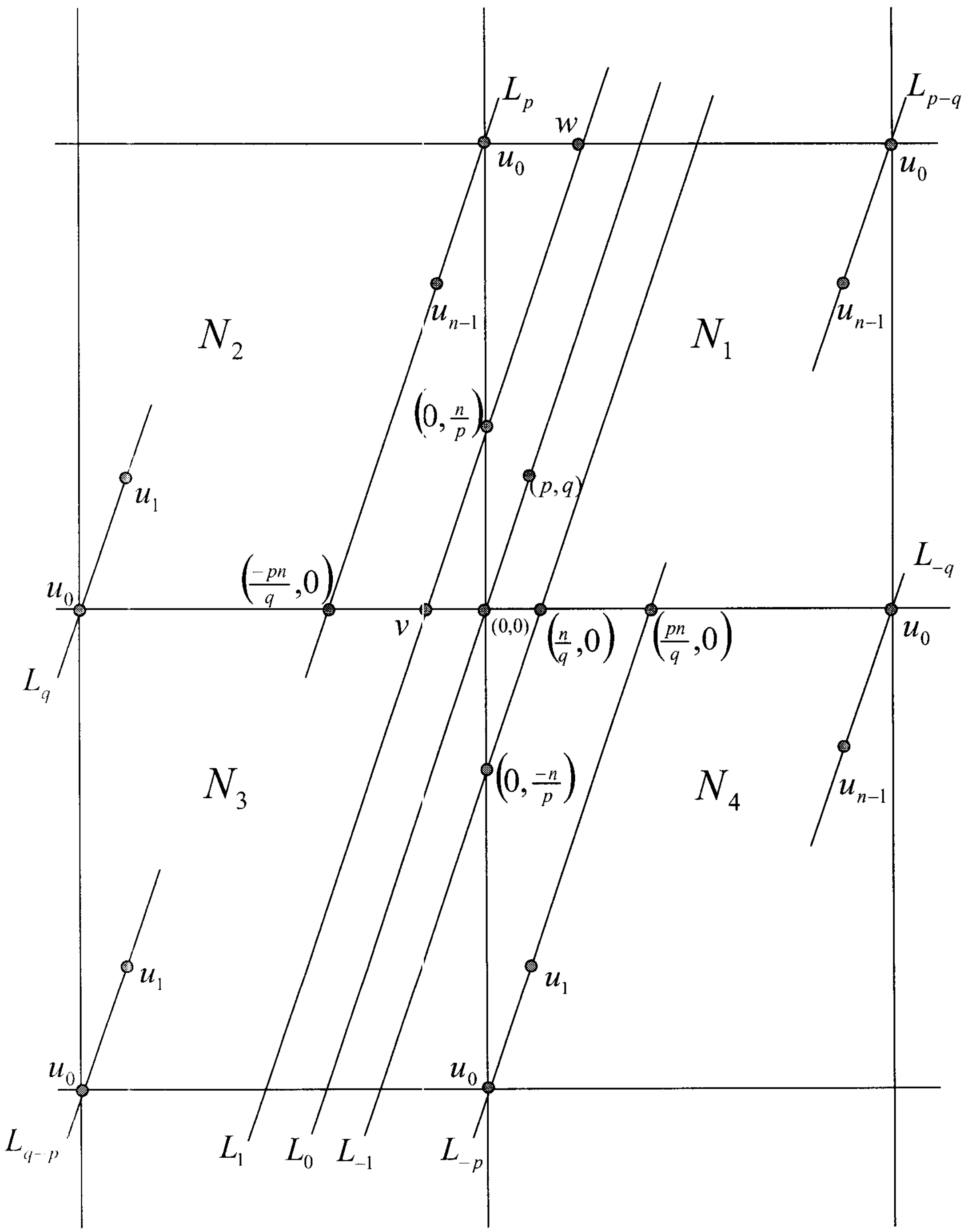
L_p
w
L_{p-q}
u_0
u_0
u_{n-1}
N_2
u_{n-1}
N_1
$\left(0,\frac{n}{p}\right)$
(p,q)
u_1
L_{-q}
u_0
$\left(\frac{-pn}{q},0\right)$
v
$(0,0)$
$\left(\frac{n}{q},0\right)$
$\left(\frac{pn}{q},0\right)$
u_0
L_q
N_3
$\left(0,\frac{-n}{p}\right)$
N_4
u_{n-1}
u_1
u_1
u_0
u_0
L_{q-p}
L_1
L_0
L_{-1}
L_{-p}

Figure 1

We observe that

$$u_k \notin L_a \cap (N_1 \cap N_2) \quad \text{for} \quad k \in \{0,1,n-1\} \text{ and } a \in \{1,\ldots,p-1\}.$$

Hence, $p > 2$ yields that Q is an empty Λ-parallelogram and $\{u_1, u_{n-1}\} \cap \{u_0, u_0, u_{i+1}, u_j\} = \emptyset$. But then $F = [b_0, b_i, b_{i+1}, b_j] \in \mathcal{F}(B(p,q,n))$, $F \cap \{b_1, b_{n-1}\} = \emptyset$ and, b_1 and b_{n-1} are separated in $b_0 < b_1 \cdots < b_{n-1}$ by three vertices of F, a contradiction.

We argue similarly that $q \mid n$. ■

Theorem 1 *Let $n > pq$. Then $B(p,q,n)$ is Gale with $b_0 < b_1 < \cdots < b_{n-1}$ if, and only if,*

a) $p = 2$ and $n = mq$ for some $m \geq 3$; or

b) $n = hpq$ for some $h \geq 2$.

Proof. Let $B = B(p,q,n)$ be Gale with $b_0 < b_1 < \cdots < b_{n-1}$. Then $q > 2$ and Lemma 1 yield that $n = mq > 2q$. If $p > 2$ then Lemma 1 yields also that $m = hp$, and so $n = hpq > pq$.

Assuming a) or b), let $F \in \mathcal{F}(B)$. We observe that it is sufficient to show that if $b_i \in F$ then either $b_{i-1} \in F$ or $b_{i+1} \in F$. By (3), we need only to show that if $b_0 \in F$ then either $b_{n-1} \in F$ or $b_1 \in F$. Let $Q \subset \mathbb{R}^2$ be an empty Λ-region containing $u_0 = (0,0)$. By (2), we need to verify that either $u_{n-1} \in Q$ or $u_1 \in Q$.

Let Q be a horizontal strip. Then Q is bounded by the lines $y = 0$ and either $y = q$ or $y = -q$. In case of the former (latter), Q contains $u_1 = (p,q)$ $(u_{n-1} = (-p,-q))$ and it is easy to show that if $u_{n-1}(u_1)$ is in Q then $n \mid q$ or $n \mid 2q$. Since $n > 2q$, it follows that this is not possible.

Let Q be a vertical strip. Then Q is bounded by $x = 0$ and either $x = p$ or $x = -p$. Since $n > 2p$, we obtain as above that Q contains $u_1(u_{n-1})$ in case of the former (latter). It should be noted that if $p = 2$ then L_a intersects $x = 0$ at $\left(0, \frac{an}{2}\right)$, which is congruent mod $n\mathbb{Z}^2$ to $(0,0)$ or $\left(0, \frac{n}{2}\right)$. Thus, Q exists only if n is even.

Let Q be a parallelogram, say, $Q = [u_0, u_i, u_j, u_i + u_j]$. We recall that area $(Q) = n$ and that Q has a side with negative (positive) slope. Thus, referring to Figure 1 and the fact that $\eta(x,y)R = \eta(-x,-y)$, we may assume that

i) $u_i \in N_1$ and $u_j \in N_4$, or

ii) $u_i \in N_1$ and $u_j \in N_2$.

We note that $n = mq$ implies that $q \mid y(u)$ for $u \in \Lambda$, and $n = hpq$ implies that $p \mid x(u)$ for $u \in \Lambda$. Finally, as p and q are relatively prime, there are integers k and l such that

(4) $kp + lq = -1$.

Case i) We observe that $u_j \in L_a$ for some $a \leq -1$, and either $p \mid x(u)$ for all $u \in \Lambda$ or $p = 2$ and n is odd. Also, if $u_i = u_1 = (p, q)$ then area $(Q) = n$ and $y(u_j) \leq -q$ imply that

$$u_j \in L_{-1} \quad \text{and} \quad u_i + u_j = u_{j+1} \in L_{-1} \cap N_4.$$

Since $u_{n-1} \in L_{-q} \cap N_4$, it follows that $u_{n-1} \notin Q$.

We suppose that $u_i \neq (p, q)$, and seek a contradiction.

If $p \mid x(u)$ for $u \in \Lambda$ then $(p, q) \notin Q$ implies that $u_i \in L_b$ for some $b \leq -1$, and as a consequence, $\left(\frac{n}{q}, 0\right) \in L_{-1} \cap \operatorname{int} Q$. But $p = 2$ implies that q is odd, $n = mq$ and (modulo $n\mathbb{Z}^2$)

$$\frac{m(q+1)}{2}(2, q) = \left(mq + m, \frac{(q+1)}{2}mq\right) \equiv (m, 0) = \left(\frac{n}{q}, 0\right),$$

and $n = hpq$ implies that

$$-hkp(p, q) = (hp(lq+1), -hkpq) \equiv (hp, 0) = \left(\frac{n}{q}, 0\right);$$

that is, $\left(\frac{n}{q}, 0\right) \in \Lambda$ and Q is not empty; a contradiction.

Let $p = 2$ and n be odd. Then (cf. Figure 2), it is easy to check that any vertical line contains at most one Λ-distinct point of Λ and that for each $u \in \Lambda$,

$$u \in V_c : y = \frac{(q+n)}{2}x + nc \quad \text{for some} \quad c \in \mathbb{Z}$$

and

$$u \in W_d : y = \frac{(q-n)}{2}x + nd \quad \text{for some} \quad d \in \mathbb{Z}.$$

Since q and n are odd, we have that for $\bar{n} = \frac{n+1}{2}$

$$u_{\bar{n}} = \bar{n}(2, q) \equiv (1, \bar{n}q) - n\left(0, \frac{q-1}{2}\right) = \left(1, \frac{q+n}{2}\right) \equiv \left(1, \frac{q-n}{2}\right).$$

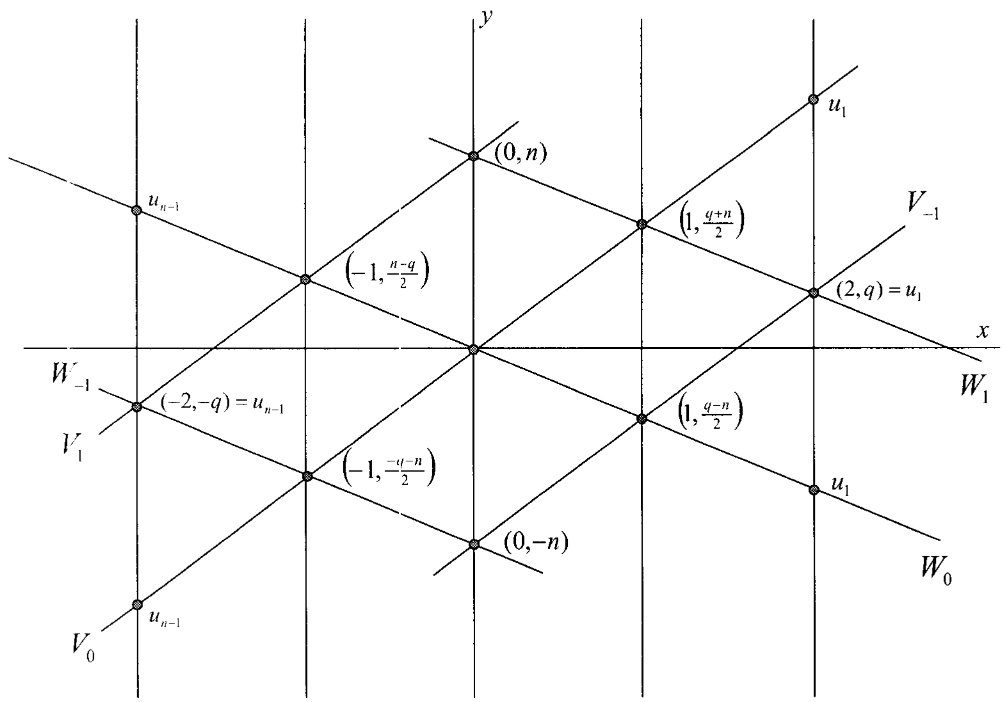

Figure 2

Thus, $u_i \neq (2,q)$ implies that either $u_i = \left(1, \dfrac{q+n}{2}\right)$ and $\left(1, \dfrac{q-n}{2}\right) \notin Q$ or $u_j = \left(1, \dfrac{q-n}{2}\right)$ and $\left(1, \dfrac{q+n}{2}\right) \notin Q$. But the former (latter) implies that $u_j \in W_d$ ($u_i \in V_c$) for some $d \geq 1$ ($c \leq -1$), and as a consequence $(2,q) \in \operatorname{int} Q$; a contradiction.

Case ii) We observe that $u_j \in L_a$ for some $a \geq 1$, $y(u_j) \geq q$ and $y(u_i) \geq q$. Also, if $u_i = u_1 = (p,q)$ then area$(Q) = n$ implies that $u_j \in L_1$ and $u_i + u_j = u_{j+1} \in L_1 \cap (N_1 \cup N_2)$. Since $u_{n-1} \in (L_p \cup L_{p-q}) \cap (N_1 \cup N_2)$, it follows that $u_{n-1} \notin Q$.

We suppose again that $u_i \neq (p,q)$. Since $(p,q) \notin Q$, it follows from $y(u_i) \geq q$ and $y(u_j) \geq q$ that $u_i \in L_b$ for some $b \geq 1$ and $\bar{w} = \left(0, \dfrac{n}{p}\right) \in L_1 \cap \operatorname{int} Q$. Since $n = hpq$ yields that

$$h(kp+1)(p,q) = h(-lq)p, hkpq + hq) \equiv (0, hq) = \left(0, \frac{n}{p}\right),$$

we may assume that $p = 2$ and n is odd. Then $\bar{w} = \left(0, \frac{n}{2}\right)$ and u_0 is the only point u of Λ such that $x(u) = 0$. Next, $\bar{w} \in [u_0, u_i, u_j]$ implies that

$$\frac{n}{2} = \text{area}\,([u_0, u_i, u_j]) \geq \text{area}\,([u_0, \bar{w}, u_i]) + \text{area}\,([u_0, \bar{w}, u_j]).$$

Since $x(u_j) \leq -1$, $x(u_i) \geq 1$ and $\text{area}\,([u_0, \bar{w}, z]) = \frac{n}{4}$ for $u_0 = (0,0)$ and any $z \in \mathbb{R}^2$ with $|x(z)| = 1$, it follows that

$$u_i = \left(1, \frac{q+bn}{2}\right) \quad \text{and} \quad u_j = \left(-1, \frac{-q+an}{2}\right).$$

But then $u_i + u_j = \left(0, \frac{(a+b)n}{2}\right)$ and $u_i + u_j \equiv u_0$; a contradiction. ■

Let $B = B(p,q,n)$ be Gale with $b_0 < b_1 < \cdots < b_{n-1}$. It is now clear that in order to completely describe $\mathcal{F}(B(p,q,n))$, we need only to determine all the empty Λ-regions $Q \subset \mathbb{R}^2$ that contain $u_0 = (0,0)$ and $u_1 = (p,q)$.

Let Q be a horizontal strip. Recalling that $n = mq$ for some $m \geq 3$, it is easy to check that $u_0, u_m, u_{2m}, \ldots, u_{(q-1)m}$ are the only points of Λ on the x-axis. Thus,

(5) $Q = [u_0, u_1, u_m, u_{m+1}, \ldots, u_{(q-1)m}, u_{(q-1)m+1}]$.

We note that if $p > 2$ then $m = hp$ for some $h \geq 2$.

Let Q be a vertical strip. If $p = 2$ then (cf. the proof of Theorem 1) n is even and u_0 and $u_{\frac{n}{2}} \equiv \left(0, \frac{n}{2}\right)$ are the points of Λ on the y-axis. Thus,

(6) $Q = [u_0, u_1, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}]$.

If $p > 2$ then $n = (hq)p$, $u_0, u_{hq}, \ldots, u_{(p-1)hq}$ are the points of Λ on the y-axis and

(7) $Q = [u_0, u_1, u_{hq}, u_{hq+1}, \ldots, u_{(p-1)hq}, u_{(p-1)hq+1}]$.

Let Q be a parallelogram. Then $Q = [u_0, u_1, u_j, u_{j+1}]$ and either $u_j \in L_1 \cap N_2$ or $u_j \in L_{-1} \cap N_4$ (cf. the proof of Theorem 1). Referring to Figure 1, we note that if $p = 2$ then

$$v = \left(-\frac{n}{q}, 0\right) = (-m, 0) \equiv u_{\left(\frac{q-1}{2}\right)m}$$

and

$$\left(0, \frac{n}{p}\right) = \left(0, \frac{n}{2}\right) = (-m, 0) + \frac{m}{2}(2, q).$$

Thus, $u_j \in L_1 \cap N_2$ yields that (with $[a]$, the integer part of $a > 0$)

(8) $Q = [u_0, u_1, u_j, u_{j+1}]$ for $j = \left(\frac{q-1}{2}\right) m + 1, \ldots, \left(\frac{q-1}{2}\right) m + \left[\frac{m+1}{2}\right] - 1.$

We note that $\frac{(q-1)m}{2} + \left[\frac{m+1}{2}\right] - 1 = \left[\frac{n+1}{2}\right] - 1.$ If $p > 2$ then

$$v = \left(\frac{-n}{q}, 0\right) = (-hp, 0) = u_{hkp}$$

and

$$\left(0, \frac{n}{p}\right) = (0, hq) = (-hp, 0) + h(p, q) \equiv u_{hkp+h}.$$

Thus, $u_j \in L_1 \cap N_2$ yields that

(9) $Q = [u_0, u_1, j_j, u_{j+1}]$ for $j = hkp + 1, \ldots, hkp + h - 1.$

Since (3) implies that $[b_0, b_1, b_j, b_{j+1}] \in \mathcal{F}(B)$ if, and only if, $[b_{n-j}, b_{n-j+1}, b_0, b_1] \in \mathcal{F}(B)$, we have shown also that $Q = [u_0, u_1, u_{n-j}, u_{n-j+1}]$ for the values of j noted in (8) and (9). It is easy to check that we obtain these Q when $u_j \in L_{-1} \cap N_4$ and thus, we may consider (8) and (9) as the complete list of parallelograms.

We are now ready to describe $\mathcal{F}(B)$. For convenience of notation, we denote the vertices of a facet only by their subscripts.

$$B = B(2, q, n); \quad q \text{ odd}, \ n = mq, \ m \geq 3 \text{ and odd}.$$

$$\begin{aligned}
\mathcal{F}(B) &= \{[i, i+1, i+m, i+m+1, \ldots, i+(q-1)m, i+(q-1)m+1] \mid i = 0, \ldots, m-1\} \\
&\cup \bigcup_{i=0}^{n-1} \left\{[i, i+1, i+j, i+j+1] \mid j = \frac{n-m}{2} + 1, \ldots, \frac{n-1}{2}\right\} \\
B &= B(2, q, n); q \text{ odd}, n = mq, \ m \geq 3 \text{ and even.}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(B) &= \{[i, i+1, i+m, i+m+1, \ldots, i+(q-1)m, i+(q-1)m+1] \mid i = 0, \ldots, m-1\} \\
&\cup \left\{\left[i, i+1, i+\frac{n}{2}, i+\frac{n}{2}+1\right] \mid i = 0, \ldots, \frac{n-2}{2}\right\} \\
&\cup \bigcup_{i=0}^{n-1} \left\{[i, i+1, i+j, i+j+1] \mid i = \frac{n-m}{2} + 1, \ldots, \frac{n-2}{2}\right\}. \\
B &= B(p, q, n); \ n = hpq; \ h > 2, \ kp \equiv -1 (\text{mod } q), \ 1 \leq k \leq n-1.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}(B) \;=\; & \{[i,i+1,i+hp,i+hp+1,\ldots,i+(q-1)hp,i+(q-1)hp+1] \mid i=0,\ldots,hp-1\} \\
& \cup\{[i,i+1,i+hq,i+hq+1,\ldots,i+(p-1)hq,i+(p-1)hq+1] \mid i=0,\ldots,hq-1\} \\
& \cup\bigcup_{i=0}^{n-1}\{[i,i+1,i+j,i+j+1] \mid j=hkp+1,\ldots,hkp+h-1\}.
\end{aligned}
$$

As an example, we present $B = B(3,4,24)$. We note that $h = 2$, $k = 5$ and $hkp = 30 \equiv 6 (\mathrm{mod}\ 24)$.

$$
\begin{aligned}
\mathcal{F}(B) \;=\; & \{[0,1,6,7,12,13,18,19],[1,2,7,8,13,14,19,20],[2,3,8,9,14,15,20,21], \\
& [3,4,9,10,15,16,21,22],[4,5,10,11,16,17,22,23],[0,5,6,11,12,17,18,23]\} \\
& \cup\{[0,1,8,9,16,17],[1,2,9,10,17,18,[2,3,10,11,18,19],[3,4,11,12,19,20], \\
& [4,5,12,13,20,21],[5,6,13,14,21,22],[6,7,14,15,22,23],[0,7,8,15,16,23]\} \\
& \cup\{[0,1,7,8],[1,2,8,9],[2,3,9,10],[3,4,10,11][4,5,11,12],[5,6,12,13], \\
& [6,7,13,14],[7,8,14,15],[8,9,15,16],[9,10,16,17],[10,11,17,18],[11,12,18,19], \\
& [12,13,19,20],[13,14,20,21],[14,15,21,22],[15,16,22,23],[0,16,17,23], \\
& [0,1,17,18],[1,2,18,19],[2,3,19,20],[3,4,20,21],[4,5,21,22],[5,6,22,23],[0,6,7,23]\}.
\end{aligned}
$$

3 The Periodically-cyclic Property

We recall that

$$\Gamma(t) = \Gamma_{pq}(t) = (\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt), \quad t \in I = [0,1),$$

$\Gamma = \Gamma(I)$ is closed, finite, locally of order 4 and that the behaviour of Γ at any one point is the same as at any other point. Next, the vertices of $B(p,q,n)$ are n evenly spaced points on Γ, and k successive vertices determine a cyclic 4-polytope only if they are on a 4^{th} order subarc of Γ. Thus, we need to determine the size of the maximal subarc of order 4.

Let $t \in I$. Since Γ is locally of order 4, the vectors $\Gamma'(t), \Gamma''(t)$ and $\Gamma'''(t)$ are linearly independent and it is well known that the osculating i-space $\Gamma_i(t)$ of Γ at t exists for $i = 0,1,2,3$. We note that $\Gamma_0(t) = \Gamma(t)$ and that for $i = 1,2,3$,

> $\Gamma_i(t)$ is the affine space of dimention i that contains $\Gamma(t)$ and is spanned by the first i derivatives of $\Gamma(t)$.

Let $J \subset \mathbb{R}$ be an open segment. Then (cf. [6]) $\Gamma(J)$ is of order 4 if, and only if, it satisfies Sauter's Condition:

$$\Gamma_{3-i}(s) \cap \Gamma_i(t) = \emptyset$$

for every $s \neq t$ in J and $i = 0, 1, 2, 3$.

In order to simplify our arguments, we identify segments of R, modulo 1, with $\left(\frac{-1}{2}, \frac{1}{2}\right]$. Let $\frac{-1}{2} < s \neq t \leq \frac{1}{2}$. We note by (1) that

$$\Gamma_3(s) \cap \Gamma(t) = \emptyset \Leftrightarrow \Gamma_3(0) \cap \Gamma(t-s) = \emptyset \Leftrightarrow \Gamma_3(0) \cap \Gamma(s-t) = \emptyset$$

and

$$\Gamma_2(s) \cap \Gamma_1(t) = \emptyset \Leftrightarrow \Gamma_2\left(\frac{s-t}{2}\right) \cap \Gamma_1\left(\frac{t-s}{2}\right) = \emptyset \Leftrightarrow \Gamma_2\left(\frac{t-s}{2}\right) \cap \Gamma_1\left(\frac{s-t}{2}\right) = \emptyset.$$

Thus, we need to determine the least positive $t^*(\tilde{t})$ such that

$$\Gamma(t^*) \in \Gamma_3(0) \quad \text{and} \quad \Gamma_2(-\tilde{t}) \cap \Gamma_2(\tilde{t}) \neq \emptyset.$$

Then, with t_{pq} as the minimum of t^* and $2\tilde{t}$, $\Gamma(0, t_{pq}) := \Gamma((0, t_{pq}))$ is a maximal subarc of order 4.

Before determining the range of values for t^* and $\tilde{t}$, we wish to examine the graph of $y = \tan x$ in $\mathbb{R}^2$.

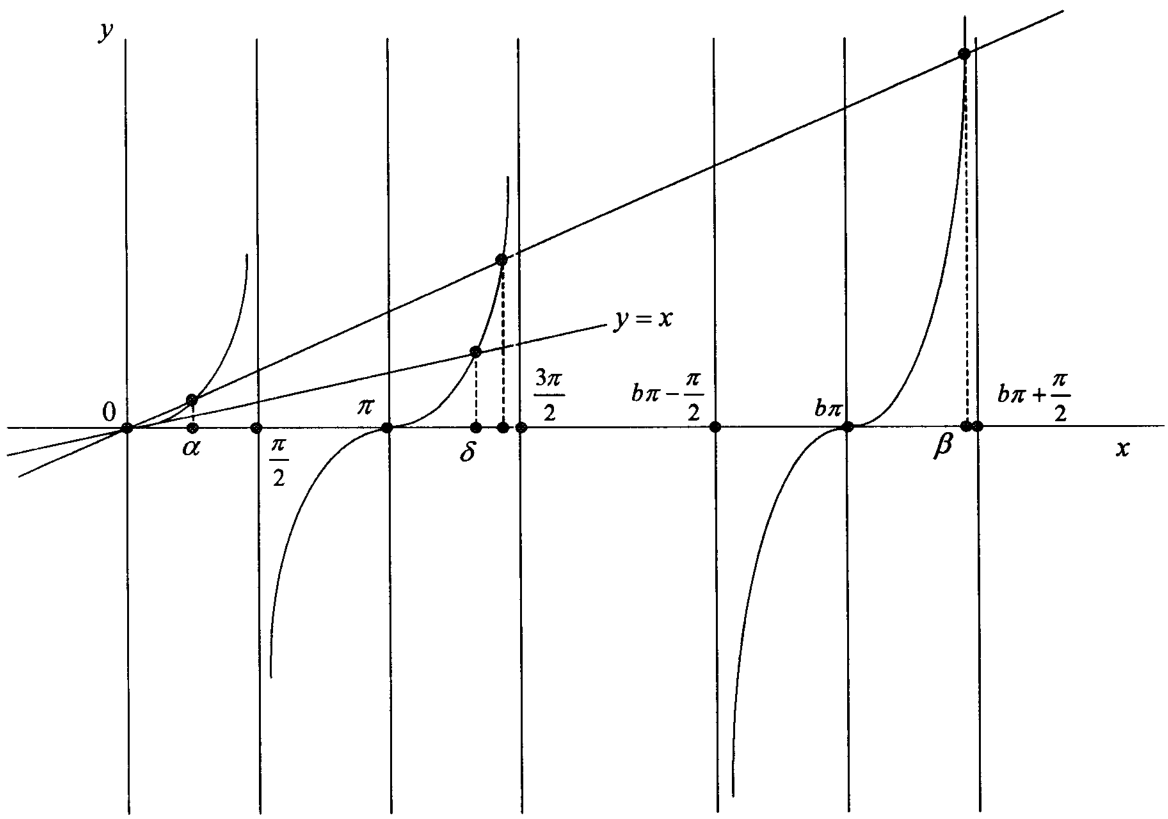

Figure 3

Lemma 2 *Let $a \geq 0$ be an integer and $\alpha > 0$ be an angle such that $a\pi < \alpha < a\pi + \frac{\pi}{2}$ or $a\pi - \frac{\pi}{2} < \alpha < a\pi$. Let $\beta > \alpha$ be an angle such that $\frac{\tan\beta}{\tan\alpha} = \frac{\beta}{\alpha}$. Then $\beta > \delta = 1.42\pi$, β is a function of α and as α ranges from $a\pi$ to $a\pi + \frac{\pi}{2}$ ($a\pi - \frac{\pi}{2}$ to $a\pi$), $\frac{\beta}{\alpha}$ is strictly monotonic.*

Proof. Referring to Figure 3, we note that $y = x$ is the tangent line to $y = \tan x$ at $x = 0$ and that $\delta = 1.42\pi$ is the smallest positive angle such that $\delta = \tan\delta$. Since $\beta > \alpha > 0$, and $\frac{\tan\beta}{\tan\alpha} = \frac{\beta}{\alpha}$ implies that $(0, 0)$, $(\alpha, \tan\alpha)$ and $(\beta, \tan\beta)$ are collinear, it follows that $\beta > \delta$.

Next, we may assume that $\tan\alpha > 0$. Then $\tan\beta > 0$ and there is a $b > a$ such that $b\pi < \beta < b\pi + \frac{\pi}{2}$. Let γ_a (γ_b) denote the graph of $y = \tan x$ for $a\pi \leq x < a\pi + \frac{\pi}{2}$ $\left(b\pi \leq x < b\pi + \frac{\pi}{2}\right)$.

Let $c > 1$. Clearly, it is sufficient to show that there is at most one α such that $a\pi \leq \alpha < a\pi + \frac{\pi}{2}$ and $\frac{\beta}{\alpha} = c$. Since $c = \frac{\beta}{\alpha} = \frac{\tan\beta}{\tan\alpha}$ implies that

$$c(\alpha, \tan\alpha) = (c\alpha, c\tan\alpha) = (\beta, \tan\beta)$$

is a common point of $c\gamma_a$ and γ_b, we need to show that there is at most one such common point.

We remark that γ_a and γ_b are translates and that γ_a and $c\gamma_a$ are homothets. Let $(x, y) \in (c\gamma_a) \cap \gamma_b$. We note that the tangent of $c\gamma_a$ at (x, y) is parallel to the tangent of γ_a at $\left(\frac{x}{c}, \frac{y}{c}\right)$, which in turn is a parallel to the tangent of γ_b at $\left(\frac{x}{c} + (b - a)\pi, \frac{y}{c}\right)$. Next, $c\gamma_a$ and γ_b are convex curves and the graphs of strictly increasing functions. Thus, it follows that the slope of the tangent to $c\gamma_a$ at (x, y) is strictly less than the slope of the tangent to γ_b at (x, y), and as a consequence, the two curves have at most one common point. ■

Lemma 3 *Let $t^* > 0$ be the smallest solution to $\Gamma(t) \in \Gamma_3(0)$. Then $\frac{1}{2}p < t^* < \frac{1}{p}$ and in addition,*

3.1 $\frac{3}{4}p < t^*$ for $2 < \frac{q}{p}$ and

3.2 $\frac{2}{q} < t^*$ for $2 < \frac{q}{p} < 3$.

Proof. From $\Gamma(t) = (\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt)$, we readily obtain that $\Gamma_3(0)$ is the 3-space defined by

$$q^2x - p^2z = q^2 - p^2.$$

Let H denote the plane $y = w = 0$ (recall $(x, y, z, w) \in \mathbb{R}^4$). Then $\Gamma_3(0)$ is perpendicular to H and $L = H \cap \Gamma_3(0)$ is a line. Let $\widehat{\Gamma}$ denote the orthogonal projection of Γ on H. Then

$$\widehat{\Gamma}(t) = (\cos 2\pi pt, \cos 2\pi qt)$$

and

$$\Gamma(t) \in \Gamma_3(0), \text{ if, and only if, } \widehat{\Gamma}(t) \in L.$$

We wish to analyze $\widehat{\Gamma}$. Since $\widehat{\Gamma}(-t) = \widehat{\Gamma}(t)$, we may assume that $0 \le t \le \frac{1}{2}$. We note that $\widehat{\Gamma}\left[0, \frac{1}{2}\right]$ is contained in the square $S = [-1, 1] \times [-1, 1]$, and that $\widehat{\Gamma}(0)$, $\widehat{\Gamma}\left(\frac{1}{2q}\right)$, $\widehat{\Gamma}\left(\frac{1}{2p}\right)$, $\widehat{\Gamma}\left(\frac{1}{q}\right)$ and $\widehat{\Gamma}\left(\frac{1}{p}\right)$ are points on the side of S; cf. Figure 4. Since slope$(L) = \frac{q^2}{p^2} > 0$, we have that $L \cap \operatorname{int} S \ne \emptyset$.

Let $T(t)$ denote the tangent line of $\widehat{\Gamma}$ at t. We show that $L = T(0)$ and that $\widehat{\Gamma}\left(0, \frac{1}{2q}\right)$ is locally convex. Then, referring to Figure 4 and the fact that $x = \cos 2\pi pt$ is strictly decreasing as t ranges from 0 to $\frac{1}{2p}$, it follows that $\widehat{\Gamma}\left[0, \frac{1}{2q}\right]$ is convex, $L \cap \widehat{\Gamma}\left(0, \frac{1}{2q}\right] = \emptyset$ and L strictly separates $\widehat{\Gamma}\left(\frac{1}{2p}\right)$ and $\widehat{\Gamma}\left(\frac{1}{p}\right)$. Thus, $L \cap \widehat{\Gamma}\left(\frac{1}{2p}, \frac{1}{p}\right) \ne \emptyset$ and $\frac{1}{2p} < t^* < \frac{1}{p}$.

Since $\widehat{\Gamma}'(t) = (-2\pi p \sin 2\pi pt, -2\pi q \sin 2\pi qt)$ implies that

$$\text{slope}\,(T(t)) = \frac{q \sin 2\pi qt}{p \sin 2\pi pt} \qquad \text{for} \qquad 0 < t < \frac{1}{2p}$$

and

$$\text{slope}\,(T(0)) = \lim_{t \to 0} \text{slope}\,(T(t)) = \frac{q^2}{p^2} = \text{slope}\,(L),$$

it follows that $L = T(0)$. Next, we recall that $\widehat{\Gamma}\left(0, \frac{1}{2q}\right)$ is locally convex if the curvature $\left|\widehat{\Gamma}'(t) \times \widehat{\Gamma}''(t)\right| / \left|\widehat{\Gamma}'(t)\right|^3$ is not zero for $0 < t < \frac{1}{2q}$. It is easy to check that

$$\begin{aligned} \left|\widehat{\Gamma}'(t) \times \widehat{\Gamma}''(t)\right| = 0 \quad &\Leftrightarrow \quad p \sin 2\pi pt \cos 2\pi qt = q \sin 2\pi qt \cos 2\pi pt \\ &\Leftrightarrow \quad \frac{\tan 2\pi qt}{\tan 2\pi pt} = \frac{q}{p}, \qquad 0 < t < \frac{1}{2q}. \end{aligned}$$

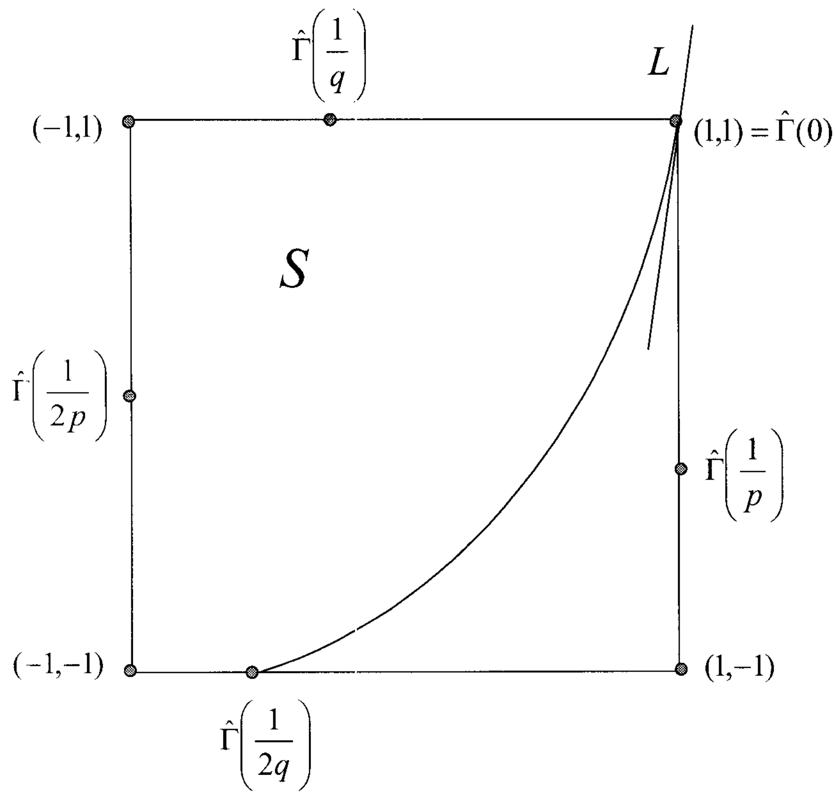

Figure 4

Let $\alpha = 2\pi pt$ and $\beta = 2\pi qt$. Since $0 < \alpha < \beta$, $\dfrac{\tan\beta}{\tan\alpha} = \dfrac{\beta}{\alpha}$ and Lemma 2 imply that $\beta > \pi$, it follows from $\beta < \pi$ for $0 < t < \dfrac{1}{2q}$ that $\left|\widehat{\Gamma}'(t) \times \widehat{\Gamma}''(t)\right| \neq 0$ and $\widehat{\Gamma}\left(0, \dfrac{1}{2q}\right)$ is locally convex.

3.1 Let $\dfrac{q}{p} > 2$. Since L intersects $x = 0$ at $\left(0, 1 - \dfrac{q^2}{p^2}\right)$ and $1 - \dfrac{q^2}{p^2} < -1$, it follows that $x > 0$ for $(x,y) \in L \cap S$. Thus, $\widehat{\Gamma}(t^*) \in L \cap S$ implies that $\cos 2\pi pt^* > 0$, and $\dfrac{1}{2p} < t^* < \dfrac{1}{p}$ implies that $t^* > \dfrac{3}{4p}$.

3.2 Let $2 < \dfrac{q}{p} < 3$. Then $\dfrac{2}{q} < \dfrac{1}{p}$ and $\widehat{\Gamma}\left(\dfrac{2}{q}\right)$ is a point on the upper side of S. Also, $\dfrac{1}{2q} < \dfrac{1}{4p}$ and the x-coordinate of $\widehat{\Gamma}\left(\dfrac{1}{2q}\right)$ is positive. Finally, $\widehat{\Gamma}\left(\dfrac{3}{4p}\right)$ is on the line $x = 0$. Altogether, these imply that $\widehat{\Gamma}\left[0, \dfrac{1}{2q}\right]$ separates in S the points, $\widehat{\Gamma}\left(\dfrac{2}{q}\right)$ and $\widehat{\Gamma}\left(\dfrac{3}{4p}\right)$, from $L \cap S$.

Since $\frac{q}{p} \geq \frac{8}{3}$ implies that $\frac{2}{q} \leq \frac{3}{4p} < t^*$, we may assume that $2 < \frac{q}{p} < \frac{8}{3}$. Then by the preceding, if $\widehat{\Gamma}\left[0, \frac{1}{2q}\right] \cap \widehat{\Gamma}\left[\frac{3}{4p}, \frac{2}{q}\right] = \emptyset$ then $t^* > \frac{2}{q}$.

Suppose that there exist $0 \leq t_1 \leq \frac{2}{q}$ and $\frac{3}{4p} \leq t_2 \leq \frac{2}{q}$ such that $\widehat{\Gamma}(t_1) = \widehat{\Gamma}(t_2)$. Then $\cos \pi p t_1 = \cos 2\pi p t_2$ with

$$0 \leq 2\pi p t_1 \leq \pi\frac{p}{q} < \frac{\pi}{2} \qquad \text{and} \qquad \frac{3\pi}{2} \leq 2\pi p t_2 \leq 4\pi\frac{p}{q} < 2\pi$$

yield that $2\pi p t_1 + 2\pi p t_2 = 2\pi$ and $t_1 + t_2 = \frac{1}{p}$. Next, $\cos 2\pi q t_1 = \cos 2\pi q t_2$ with

$$0 \leq 2\pi q t_1 \leq \pi \qquad \text{and} \qquad 3\pi < 2\pi q t_2 \leq 4\pi$$

yield that $2\pi q t_1 + 2\pi q t_2 = 4\pi$ and $t_1 + t_2 = \frac{1}{2q} = \frac{1}{p}$; a contradiction. ■

Lemma 4 *Let $\tilde{t} > 0$ be the smallest solution to $\Gamma_2(-t) \cap \Gamma_1(t) \neq \emptyset$.*

4.1 If $\frac{q}{p} > 3$ then $\frac{0.71}{q} < \tilde{t} < \frac{3}{4q}$.

4.2 If $2 < \frac{q}{p} < 3$ then $\frac{3}{4q} < \tilde{t} < \frac{1}{p}$.

4.3 If $1 < \frac{q}{p} < 2$ and $\tilde{t}$ exists then $\frac{1}{p} < \tilde{t}$.

Proof. We observe that $\Gamma_2(-t) \cap \Gamma_1(t) \neq \emptyset$ implies that $\Gamma_2(-t)$ and $\Gamma_1(t)$ span a 3-space, the vectors $\Gamma(t) - \Gamma(-t)$, $\Gamma'(t)$, $\Gamma'(-t)$ and $\Gamma''(-t)$ are linearly dependent, and as a consequence,

$$p \sin 2\pi p t \cos 2\pi q t = q \sin 2\pi q t \cos 2\pi p t.$$

Since p and q are relatively prime, it follows readily that

$$\sin 2\pi p t \neq 0 \neq \sin 2\pi p t$$

and that if $\cos 2\pi p t = 0 = \cos 2\pi q t$ then p and q are both odd and $\tilde{t} = \frac{1}{4} > \frac{1}{q}$.

Let $\cos 2\pi p t \neq 0 \neq \cos 2\pi p t$. Then with $\alpha = 2\pi p t < \beta = 2\pi q t$ and $\frac{\tan\beta}{\tan\alpha} = \frac{\beta}{\alpha}$, we need to solve for the smallest α and β such that $\frac{\beta}{\alpha} = \frac{q}{p}$.

Let $\frac{q}{p} > 3$. With $0 < \alpha < \frac{\pi}{2}$ and $\pi < \beta < \frac{3\pi}{2}$, we refer to Figure 3 and Lemma 2. We note that $\beta > 1.42\pi$ and that as α ranges from $\frac{\pi}{2}$ to 0, $\frac{\beta}{\alpha}$ ranges monotonically from 3 to ∞. Hence there are α_0 and β_0 in the given domains such that $\frac{\tan\beta_0}{\tan\alpha_0} = \frac{q}{p}$. As $1.42\pi < \beta_0 = 2\pi q\tilde{t} < \frac{3\pi}{2}$, 4.1 follows.

Let $2 < \frac{q}{p} < 3$. With $\frac{\pi}{2} < \alpha < \pi$ and $\frac{3\pi}{2} < \beta < 2\pi$, we have that as α ranges from π to $\frac{\pi}{2}$, $\frac{\beta}{\alpha}$ ranges monotonically from 2 to 3. Thus $\frac{3\pi}{2} < 2\pi q\tilde{t} < 2\pi$ and 4.2 follows.

Let $1 < \frac{q}{p} < 2$. Now if $\tilde{t}$ exists, then by the preceding cases, $2\pi q\tilde{t} > 2\pi$ and $\tilde{t} > \frac{1}{q}$. ■

Since $t^* < \frac{1}{p} \leq \frac{1}{2}$, we set

$$t_{pq} = \begin{cases} \min\{t^*, \tilde{2t}\} & \text{if } \tilde{t} \text{ exists} \\ t^* & \tilde{t} \text{ doesn't exist.} \end{cases}$$

Then $\Gamma(0, t_{pq})$ is of order 4 and by (1), $\Gamma(s, s + t_{pq})$ is of order 4 for each $s \in I$.

Theorem 2 *$B(p, q, n)$ is periocially-cyclic with the period $k = [t_{pq}n] \geq 6$ and*

$$t_{pq} = \begin{cases} 2t & \text{if } \frac{q}{p} > 2 \\ t* & \frac{q}{p} < 2. \end{cases}$$

Proof. Let $\frac{q}{p} > 3$. Then by 4.1 and 3.1,

$$2\tilde{t} < \frac{3}{2q} < \frac{3}{2} \cdot \frac{1}{3p} = \frac{1}{2p} < \frac{3}{4p} < t^*.$$

Let $2 < \frac{q}{p} < 3$. Then by 4.2 and 3.2,

$$2\tilde{t} < \frac{2}{q} < t^*.$$

Finally, if $\frac{q}{p} < 2$ and $\tilde{t}$ exists then by Lemma 3 and 4.1,

$$t^* < \frac{1}{p} < \frac{2}{q} < 2\tilde{t}. \quad ■$$

As a final remark, we note that either $\widehat{\Gamma}(t_{pq}) \in L$ or $\widehat{\Gamma}(t_{pq})$ is the first inflection point of $\widehat{\Gamma}\left[0, \frac{1}{2}\right]$.

In regard to specific values, we know that $t_{23} \approx 0.419569$, $t_{25} \approx 0.31$, $t_{27} \approx 0.2110$ and $t_{29} \approx 0.1618$. Thus, for example, $B(2,3,30)$ is Gale and periodically-cyclic with the period $k = [t_{23}30] = 12$.

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