A CLASS OF FOUR DIMENSIONAL GALE POLYTOPES

T. BISZTRICZKY

ABSTRACT. For each $k \geq 6$, we introduce a convex 4-polytope with $3k-7$ vertices such that there is a complete description of each of its facets based upon a labelling (total ordering) of the vertices that satisfies a part of Gale's Evenness Condition.

DEFINITIONS

Let Y be a set of points in R^d. Then conv Y and aff Y denote, respectively, the convex hull and affine hull of Y. If $Y = \{y_1, ..., y_m\}$ is finite, we set

$$[y_1, ..., y_m] = \text{conv } Y \text{ and } \langle y_1, ... y_m \rangle = \text{ aff } Y.$$

Thus, as usual, $[y_1, y_2]$ is the closed segment with end points y_1 and y_2.

Let P be a (convex) d-polytope. For $-1 \leq i \leq d$, let $\mathcal{F}_i(P)$ denote the set of i-faces of P and $f_i(P) = |\mathcal{F}_i(P)|$. For convenience, we usually let $\mathcal{F}(P) = \mathcal{F}_{d-1}(P)$, the set of facets of P. We recall that the *face lattice* of P is the collection of all faces of P ordered by inclusion, and that two polytopes are (*combinatorially*) *equivalent* if their face lattices are isomorphic.

Next, a *facet system* of P is a pair $(\mathcal{C}, N)$ where N is a finite set, $\mathcal{C} \subset 2^N$ and there is a bijection $h : N \to \mathcal{F}_o(P)$ such that

$$\mathcal{F}(P) = \{\text{conv } (\{h(n) | n \in C\}) | C \in \mathcal{C}\}.$$

We recall the following form [8] or [9], p. 71:

1. *If* $(\mathcal{C}, N)$ *and*$(\mathcal{C}', N)$ *are facet systems of convex d-polytopes then* $\mathcal{C} \subseteq \mathcal{C}'$ *implies that* $\mathcal{C} = \mathcal{C}'$, *and consequently the two polytopes are equivalent.*

Finally, let H be a hyperplane of R^d such that $H \cap (\text{int } P) = \emptyset$, and let y be a point of $R^d \backslash H$. If y and int P are (are not) in the same open half-space determined by H, we say that y is *beneath* (*beyond*) H *with respect to* P. If $H =$ aff F for some $F \in \mathcal{F}(P)$, we say that y is *beneath* F (*beyond* F) provided y is beneath (beyond) H with respect to P.

As noted above, we are interested in totally ordering the vertices of a polytope.

Let $V = \mathcal{F}_0(P) = \{x_1, x_2, ..., x_n\}$, $n \geq d{+}1$. We set $x_i < x_j$ if, and only if, $i < j$, and call $x_1 < x_2 < ... < x_n$ a *vertex array* of P. We say that P is a *Gale polytope* if there exists a vertex array of P such that for any $F \in \mathcal{F}(\mathcal{P})$ and any $x_i \neq x_j$ in $V \backslash F$, x_i and x_j are separated in that vertex array by an even number of vertices of F. If $x_1 < x_2 < ... < x_n$ is the vertex array in the preceding definition, we say also that P is Gale with $x_1 < x_2 < ... < x_n$.

We note that a convex polygon that is Gale with $x_1 < x_2 < ... < x_n$ has the edges $[x_n, x_1]$ and $[x_i, x_{i+1}]$ for $i = 1, ..., n-1$. Since such a labelling of vertices is always possible, convex polygons are Gale.

As other examples of Gale polytopes, we cite the cyclic d-polytopes (cf. [5],[6]) the ordinary 3-polytopes (cf. [2]) and the d-multiplices for all odd d (cf. [3]). For each of these polytopes, there is a complete description of all the facets irregardless of how many vertices they may have. In general and already for $d = 4$, such a description is a very difficult task and has been accomplished mostly for polytopes with "few" vertices, cf. for example [1], [4] and [7].

The preceding three articles are concerned with neighbourly or simplicial d-polytopes for $d = 4$ and 6 and the enumeration of their combinatorial types. We recall that a neighbourly d-polytope is a generalization of a cyclic d-polytope $\tilde{P}$ because any $[d/2]$ vertices of $\tilde{P}$ determine a face of $\tilde{P}$. We present a generalization of $\tilde{P}$ that focuses on Gale's Evenness Condition. The importance of this approach is that there is necessarily more information about the facial structure of the resultant polytope, and that this information is available even if the polytopes has "many" vertices.

We begin this investigation by introducing for each $k \geq 6$ and $n = 3k - 7$, a 4-polytope P_n that is Gale with $x_1 < x_2 < ... < x_n$. Now, the origin of these P_n is combinatorial. In particular, they are determined in the following manner:

Start with a 4-polytope that is not a simplex and that is cyclic with the vertex array $x_1 < x_2 < ... < x_k$ (this accounts both for k and for $k \geq 6$). For $i = 1, 2, ...$, adjoin a vertex x_{i+k} so that with $x_1 < x_2 < ... < x_{i+k}$, $[x_1, ..., x_{i+k}]$ is Gale, $[x_{i+1}, ..., x_{i+k}]$ is cyclic and $[x_i, x_{i+1}, ..., x_{i+k}]$ is not cyclic (call such a 4-polytope *k-sequentially cyclic*). Always choosing x_{i+k} so that as many vertices as possible may be adjoined to $[x_1, ..., x_{i+k}]$, we obtained that $i + k \leq 3k - 7$.

Clearly, such a combinatorial approach presents us with two problems:

Is P_n realizable in R^4 for each n?

If P_n is realizable in R^4, is there a realization which is k-sequentially cyclic?

In this article, we deal with the first problem. To be precise, we describe P_n in terms of all of its facets. Next, we show how to construct geometrically a 4-polytope Q_n in R^4 that is equivalent to P_n. Finally, we prove that such a construction is possible.

In regard to the second problem, we know already that there is a k-sequentially cyclic realization of P_n for $k = 6$, 7 and 8. As a general proof seems to be long and complicated, we hope to present it in another paper.

THE POLYTOPES P_n; $n = 3k - 7$, $k \geq 6$

Before describing P_n, we note that it is Gale with $x_1 < x_2 < ... < x_n$ and that it has $k^2 - 3k - 1$ facets. With the hope that the facial pattern becomes more evident as k increases, we present first P_{11}, P_{14} and P_{17}. Also, we simplify our notation by setting

$$\mathcal{F}(i) = \{F \in \mathcal{F}(P_n) | \ x_i \text{ is the least vertex of } F\}$$

and

$$\mathcal{C}(i) = \{[i, ..., j] \mid F = [x_i, ..., x_j] \in \mathcal{F}(i)\}.$$

We remark that $\mathcal{F}(P_n) = \bigcup_{i=1}^{k-2} \mathcal{F}(i)$, and set $\mathcal{C}(P_n) = \bigcup_{i=1}^{k-2} \mathcal{C}(i)$. Clearly, $\mathcal{C}(P_n)$ represents a facet system of P_n.

P_{11}: $\mathcal{C}(1) = \{[1,2,4,5,9,10], [1,2,5,6], [1,2,6,7], [1,2,7,8], [1,2,8,9], [1,3,4,8,9], [1,3,4,7,8], [1,4,5,6,7,11]\}$,

$\mathcal{C}(2) = \{[2,3,5,6,10,11], [2,3,6,7], [2,3,7,8], [2,3,8,9], [2,3,9,10]\}$,

$\mathcal{C}(3) = \{[3,4,6,7,11], [3,4,9,10], [3,4,10,11]\}$,

$\mathcal{C}(4) = \{[4,5,10,11]\}$.

P_{14}: $\mathcal{C}(1) = \{[1,2,4,5,11,12], [1,2,5,6], [1,2,6,7], [1,2,7,8], [1,2,8,9], [1,2,9,10], [1,2,10,11], [1,3,4,10,11], [1,3,4,9,10], [1,4,5,8,9], [1,5,6,7,8,14]\}$,

$\mathcal{C}(2) = \{[2,3,5,6,12,13], [2,3,6,7], [2,3,7,8], [2,3,8,9], [2,3,9,10], [2,3,10,11], [2,3,11,12]\}$,

$\mathcal{C}(3) = \{[3,4,6,7,13,14], [3,4,7,8], [3,4,8,9], [3,4,11,12], [3,4,12,13]\}$,

$\mathcal{C}(4) = \{[4,5,7,8,14], [4,5,12,13], [4,5,13,14]\}$,

$\mathcal{C}(5) = \{[5,6,13,14]\}$.

P_{17}: $\mathcal{C}(1) =$ $\{[1,2,4,5,13,14], [1,2,5,6], [1,2,6,7], [1,2,7,8], [1,2,8,9],$
$[1,2,9,10], [1,2,10,11], [1,2,11,12], [1,2,12,13], [1,3,4,12,13],$
$[1,3,4,11,12], [1,4,5,10,11], [1,5,6,9,10], [1,6,7,8,9,17]\},$

$\mathcal{C}(2) =$ $\{[2,3,5,6,14,15], [2,3,6,7], [2,3,7,8], [2,3,8,9], [2,3,9,10],$
$[2,3,10,11], [2,3,11,12], [2,3,12,13], [2,3,13,14]\},$

$\mathcal{C}(3) =$ $\{[3,4,6,7,15,16], [3,4,7,8], [3,4,8,9], [3,4,9,10], [3,4,10,11],$
$[3,4,13,14], [3,4,14,15]\},$

$\mathcal{C}(4) =$ $\{[4,5,7,8,16,17], [4,5,8,9], [4,5,9,10], [4,5,14,15], [4,5,15,16]\},$

$\mathcal{C}(5) =$ $\{[5,6,8,9,17], [5,6,15,16], [5,6,16,17]\},$

$\mathcal{C}(6) =$ $\{[6,7,16,17]\}.$

$P_n : \mathcal{C}(1)$ $= \{[1,2,4,5,2k-3,2k-2], [1,2,5,6], [1,2,6,7],...,[1,2,2k-4,2k-3],$
$[1,3,4,2k-4,2k-3], [1,3,4,2k-5,2k-4],...$
$[1,3+i,4+i,2k-5-i,2k-4-i],...,[1,k-3,k-2,k+1,k+2],$
$[1,k-2,k-1,k,k+1,3k-7]\},$

$\mathcal{C}(2)$ $= \{[2,3,5,6,2k-2,2k-1], [2,3,6,7], [2,3,7,8],...,[2,3,2k-3,2k-2]\},$

$\mathcal{C}(i)$ $= \{[i,i+1,i+3,i+4,i+2k-3], [i,i+1,i+4,i+5], [i,i+1,i+5,i+6],$
$...,[i,i+1,2k-3-i,2k-2-i], [i,i+1,i+2k-6,i+2k-5],$
$[i,i+1,i+2k-5,i+2k-4\}$ for $i = 3,...,k-4$,

$\mathcal{C}(k-3)$ $= \{[k-3,k-2,k,k+1,3k-7], [k-3,k-2,3k-9,3k-8],$
$[k-3,k-2,3k-8,3k-7]\},$

$\mathcal{C}(k-2)$ $= \{[k-2,k-1,3k-8,3k-7]\}.$

THE CONSTRUCTION

Clearly, in order to determine how to construct a 4-polytope Q_n in R^4 that is equivalent to P_n, we need to understand the relationship between P_n and its subpolytopes

$$P_m = [x_1, x_2, ..., x_m],\ m \leq n.$$

In particular, it is important to know which facets of P_m determine supporting hyperplanes of P_n. With this in mind and assuming a priori that P_n is realizable, we set

$$\mathcal{F}'(m) = \{F' \in \mathcal{F}(P_m) \mid x_m \in F' \text{ and aff } F' \text{ supports } P_n\}$$

and

$$C'(m) = \{[i, j, ..., m] \mid F' = [x_i, x_j, ..., x_m] \in \mathcal{F}'(m)\},$$

and claim the following.

2.1 $C'(m) = \{[1,2,m-1,m], [2,3,m-1,m], ..., [m-4,m-3,m-1,m]\}$ *for* $m = 5, ..., k-1$.

2.2 $C'(k) = \{[1,2,k-1,k], [2,3,k-1,k], ..., [k-4,k-3,k-1,k], [1,k-2,k-1,k]\}$.

2.3 $C'(m) = \{[1,2,m-1,m], [2,3,m-1,m], ..., [k-j-2, k-j-1, m-1, m],$ $[1, k-j-2, k-j-1, m], [1, k-j-1, k-j, m-1, m]\}$ *for* $m = k+j$ *and* $j = 1, ..., k-5$.

2.4 $C'(2k-4) = \{[1,2,2k-5,2k-4], [1,3,4,2k-5,2k-4], [2,3,2k-5,2k-4]\}$.

2.5 $C'(2k-3) = \{[1,2,4,5,2k-3], [1,2,2k-4,2k-3], [1,3,4,2k-4,2k-3],$ $[2,3,2k-4,2k-3]\}$.

2.6 $C'(m) = \{[j, j+1, j+3, j+4, m-1, m], [j+1, j+2, j+4, j+5, m],$ $[j+1, j+2, m-1, m], [j+2, j+3, m-1, m]\}$ *for* $m = 2k-3+j$ *and* $j = 1, ..., k-5$.

2.7 $C'(3k-7) = \{[1, k-2, k-1, k+1, 3k-7], [k-4, k-3, k-1, k, 3k-8, 3k-7],$ $[k-3, k-2, k, k+1, 3k-7], [k-3, k-2, 3k-8, 3k-7],$ $[k-2, k-1, 3k-8, 3k-7]\}$.

Proof. (2.1,2.2) From $C(1)$ and $C(2)$, it is clear that we need only to verify that $[i, i+1, m-1, m] \in C'(m)$ for $3 \leq i \leq m-4$ and $5 \leq m \leq k$, and that follows form $C(i)$ since $2k-2-i \geq 2k-2+4-k = k+2$.

As neither $C(k-3)$ nor $C(k-2)$ contributes any element to $C'(m)$, the assertions follow.

(2.3) Let $m = k+j$ and $1 \leq j \leq k-5$. Then $k+1 \leq m \leq 2k-5$ and $k-j-2 \leq k-3$.

From $C(1)$ and $C(2)$, $\{[1,2,m-1,m], [2,3,m-1,m]\} \subset C'(m)$. If $3 \leq i \leq \min\{k-4, k-j-2\}$ then $j \leq k-i-2$, $m \leq 2k-i-2$ and $[i, i+1, m-1, m] \in C'(m)$ from $C(i)$. Finally, if $k-j-2 = k-3$ then

$$[k-j-2, k-j-1, m-1, m] = [k-3, k-2, k, k+1] \in C'(m)$$

from $C(k-3)$.

Since

$$\begin{aligned}&\{[1, k-j-1, k-j, k+j-1, k+j] \mid j = 1, ..., k-5\}\\ &= \{[1, k-2, k-1, k, k+1], [1, k-3, k-2, k+1, k+2], ..., [1, 4, 5, 2k-6, k-5]\},\end{aligned}$$

it follows from $C(1)$ that $[1, k-j-1, k-j, m-1, m] \in C'(m)$ for $k+1 \leq m \leq 2k-5$ Then $[1, k-j-2, k-j-1, m, m+1] \in C'(m+1)$ for $k \leq m \leq 2k-6$ and $[1,3,4,2k-5,2k-4] \in C(1)$ yield that $[1, k-j-2, k-j-1, m] \in C'(m)$ for $k+1 \leq m \leq 2k-5$.

The assertion now readily follows.

(2.4,2.5) Immediate from $\mathcal{C}(1)$ and $\mathcal{C}(2)$, and the fact that for $i \geq 3$, $2k-2-i \leq 2k-5$ and $i+2k-5 \geq 2k-2$.

(2.6) Let $m = 2k-3+j$. It is clear from $\mathcal{C}(1)$, $1 \leq i \leq k-4$, that $[j, j+1, j+3, j+4, m-1, m] \in \mathcal{C}'(m)$ for $2k-2 \leq m \leq 3k-7$. From whence we argue as above to obtain that $[j+1, j+2, j+4, j+5, m] \in \mathcal{C}'(m)$ for $2k-2 \leq m \leq 3k-8$.

We note that with $j = i-1$,

$$\{[j+1, j+2, m-1, m] \mid j = 1, ..., k-5\} = \{[i, i+1, i+2k-5, i+2k-4] \mid i = 2, ..., k-4\}$$

and that with $j = i-2$,

$$\{[j+2, j+3, m-1, m] \mid j = 1, ..., k-5\} = \{[i, i+1, i+2k-6, i+2k-5] \mid i = 3, ..., k-3\}.$$

It is now easy to verify the assertion from $\mathcal{C}(P_n)$.

(2.7) Immediate. □

Now that we have an idea of the relationship between P_m and P_n, we wish to approximate a facet system of P_m by eliminating the extraneous representations of its facets from $\bigcup_{i=5}^{m} \mathcal{C}'(i)$. Accordingly, let $\mathcal{F}^*(P_m)$ denote the set of all $F^* \in \bigcup_{i=5}^{m} \mathcal{F}'(i)$ with the property that if $F' \in \bigcup_{i=5}^{m} \mathcal{F}'(i)$ and aff F' = aff F^* then F^* contains F', and set

$$\mathcal{C}^*(P_m) = \{[i, ..., j] \mid F^* = [x_i, ..., x_j] \in \mathcal{F}^*(m)\}.$$

We remark that $\mathcal{C}^*(P_n) = \mathcal{C}(P_n)$ and that from 2.1 to 2.7 and $\mathcal{C}(Pn)$, it is easy to obtain

3.1 $\mathcal{C}^*(P_k) = \bigcup_{i=5}^{k} \mathcal{C}'(i)$,

3.2 $\mathcal{C}^*(P_m) = (\mathcal{C}^*(P_{m-1}) \backslash \{[1, k-j-1, k-j, m-1]\}) \cup \mathcal{C}'(m)$ *for* $m = k+j$ *and* $j = 1, ..., k-5$,

3.3 $\mathcal{C}^*(P_{2k-4}) = (\mathcal{C}^*(P_{2k-5}) \backslash \{[1, 3, 4, 2k-5]\}) \cup \mathcal{C}'(2k-4)$,

3.4 $\mathcal{C}^*(P_{2k-3}) = (\mathcal{C}^*(P_{2k-4}) \backslash \{[1, 2, 4, 5]\}) \cup \mathcal{C}'(2k-3)$,

3.5 $\mathcal{C}^*(P_m) = (\mathcal{C}^*(P_{m-1}) \backslash \{[j, j+1, i+3, j+4, m-1], [j+1, j+2, j+4, j+5]\}) \cup \mathcal{C}'(m)$ *for* $m = 2k-3+j$ *and* $j = 1, ..., k-5$, *and*

3.6 $\mathcal{C}^*(P_n) = (\mathcal{C}^*(P_{n-1}) \backslash \{[1, k-2, k-1, k, k+1], [k-4, k-3, k-1, k, 3k-8], [k-3, k-2, k, k+1]\}) \cup \mathcal{C}'(3k-7)$.

Proceeding with the construction, let Q_k be a cyclic 4-polytope in R^4 with the vertex array $y_1 < y_2 < \dots < y_k$, $k \geq 6$, and choose points $y_{k+1}, y_{k+2}, \dots, y_{3k-7}$ in R^4 in the order presented so that y_m has the following position with respect to

$$Q_{m-1} = [y_1, y_2, \dots, y_m]:$$

4.1 *For $m = k+j$ and $j = 1, \dots, k-5$, y_m is on $\langle y_1, y_{k-j-1}, y_{k-j}, y_{m-1}\rangle$ and beyond exactly $[y_1, y_i, y_{i+1}, y_{m-1}]$ for $i = 2, \dots, k-j-2$.*

4.2 *y_{2k-4} is on $\langle y_1, y_3, y_4\rangle = \langle y_1, y_2, y_3, y_4\rangle \cap \langle y_1, y_3, y_4, y_{2k-5}\rangle$ and beyond exactly $[y_1, y_2, y_3, y_{2k-5}]$.*

4.3 *y_{2k-3} is on $\langle y_2, y_4, y_5\rangle = \langle y_1, y_2, y_4, y_5\rangle \cap \langle y_2, y_3, y_4, y_5\rangle$ and beyond exactly $[y_1, y_2, y_3, y_4, y_{2k-4}]$.*

4.4 *For $m = 2k-3+j$ and $j = 1, \dots, k-4$, y_m is on $\langle y_{j+2}, y_{j+4}, y_{j+5}\rangle = \langle y_{j+1}, y_{j+2}, y_{j+4}, y_{j+5}\rangle \cap \langle y_{j+2}, y_{j+3}, y_{j+4}, y_{j+5}\rangle$ and $\langle y_j, y_{j+1}, y_{j+3}, y_{j+4}, y_{m-1}\rangle$, and beyond exactly $[y_{j+1}, y_{j+2}, y_{j+3}, y_{j+4}, y_{m-1}]$.*

Assuming again that such a construction is possible, we set

$$\mathcal{C}(Q_m) = \{[i, \dots, j] \mid [y_i, \dots, y_j] \in \mathcal{F}(Q_m)\}$$

and compare $\mathcal{C}(Q_m)$ and $\mathcal{C}^*(P_m)$ for $m = k, \dots, n = 3k-7$.

5. $\mathcal{C}(Q_m) = \mathcal{C}^*(P_m) \cup \{[s-3, s-2, s-1, s] \mid s = 4, \dots, k\} \cup \{[1, i, i+1, m] \mid i = 2, \dots, k-j-3\}$ *for $m = k+j$ and $j = 0, \dots, k-5$.*

Proof. From 2.1, 2.2 and 3.1,

$$\mathcal{C}^*(P_k) = \{[i, i+1, s-1, s] \mid i = 1, \dots, s-4 \text{ and } s = 5, \dots, k\} \cup \{[1, k-2, k-1, k]\}.$$

Since Q_k is cyclic with $y_1 < y_2 < \dots < y_k$,

$$\mathcal{C}(Q_k) = \{[i, i+1, s-1, s] \mid i = 1, \dots, s-3 \text{ and } s = 4, \dots, k\} \cup \{[1, i, i+1, k] \mid i = 2, \dots, k-2\}$$

and the assertion follows for $j = 0$.

We assume that $1 \leq j \leq k-5$ and that

$$\mathcal{C}(Q_{m-1}) = \mathcal{C}^*(P_{m-1}) \cup \{[s-3, s-2, s-1, s] \mid s = 4, \dots, k\} \cup \{[1, i, i+1, m-1] \mid i = 2, \dots, u\}$$

where $u = k - (j-1) - 3 = k - j - 2$.

We note that

$$[1, u+1, u+2, m-1] = [1, k-j-1, k-j, m-1] \in \mathcal{C}^*(P_{m-1})$$

and that by 4.1, y_m is on $\langle y_1, y_{u+1}, y_{u+2}, y_{m-1}\rangle$ and beyond only $[y_1, y_i, y_{i+1}, y_{m-1}]$ for $i = 2, ..., u$.

We recall that if $F \in \mathcal{F}(Q_m)\backslash\mathcal{F}(Q_{m-1})$ then

$$F = \text{ conv } (\{y_m\} \cup G)$$

where either $G \in \mathcal{F}(Q_{m-1})$ and $y_m \in \text{aff } G$ or $G \in \mathcal{F}_2(Q_{m-1})$ and y_m is beyond (beneath) at least one facet of Q_{m-1} that contains G. Accordingly, we need only to examine the 2-faces of $[y_1, y_i, y_{i+1}, y_{m-1}]$ for $i = 2, ..., u$. Such an examination readily yields that $[y_1, y_i, y_{i+1}]$, $2 \leq i \leq u$, and $[y_i, y_{i+1}, y_{m-1}]$, $1 \leq i \leq u$, are the only ones from these 2-faces that generate facets through y_m. Thus by the induction, 2.3 and 3.2,

$$\begin{aligned}\mathcal{C}(Q_m) &= (\mathcal{C}(Q_{m-1})\backslash\{[1,i,i+1,m-1] \mid i=2,...,u+1\}) \cup \{[1,i,i+1,m] \mid i=2,...,u\} \\ &\cup \{[i,i+1,m-1,m] \mid i=1,...,u\} \cup \{[1,u+1,u+2,m-1,m]\} \\ &= (\mathcal{C}^*(P_{m-1})\backslash\{[1,u+1,u+2,m-1]\}) \cup \{[s-3,s-2,s-1,s] \mid s=4,...,k\} \\ &\cup \{\mathcal{C}'(m) \bigcup \ \{[1,i,i+1,m] \mid i=2,...,u-1\} \\ &= \mathcal{C}^*(P_m) \cup \{[s-3,s-2,s-1,s] \mid s=4,...,k\} \cup \{[1,i,i+1,m] \\ &\mid i=2,...,k-j-3\}\end{aligned}$$

□

6. $\mathcal{C}(Q_{2k-4}) = \mathcal{C}^*(P_{2k-4}) \cup \{[s-3,s-2,s-1,s] \mid s=5,...,k\} \cup \{[1,2,3,4,2k-4]\}$.

Proof. From 2.3, 4.2 and 5, we have that
$\mathcal{C}(Q_{2k-5}) = \mathcal{C}^*(P_{2k-5}) \cup \{[s-3,s-2,s-1,s] \mid s=4,...,k\} \cup \{[1,2,3,2k-5]\}$,
$[1,3,4,2k-5] \in \mathcal{C}^*(P_{2k-5})$ and y_{2k-4} is on both $\langle y_1, y_2, y_3, y_4\rangle$ and $\langle y_1, y_3, y_4, y_{2k-5}\rangle$, and beyond only $[y_1, y_2, y_3, y_{2k-5}]$. Thus, it is immediate by 2.4 and 3.3 that

$$\begin{aligned}\mathcal{C}(Q_{2k-4}) &= (\mathcal{C}(Q_{2k-5})\backslash\{[1,2,3,4], [1,3,4,2k-5], [1,2,3,2k-5]\} \cup \{[1,2,3,4,2k-4], \\ &[1,3,4,2k-5,2k-4], [1,2,2k-5,2k-4], [2,3,2k-5,2k-4]\} \\ &= (\mathcal{C}^*(P_{2k-5})\backslash\{[1,3,4,2k-5]\}) \cup \{[s-3,s-2,s-1,s] \mid s=5,...,k\} \\ &\cup \mathcal{C}'(2k-4) \cup \{[1,2,3,4,2k-4]\} \\ &= \mathcal{C}^*(P_{2k-4}) \cup \{[s-3,s-2,s-1,s] \mid s=5,...,k\} \cup \{[1,2,3,4,2k-4]\}.\end{aligned}$$

We remark that $[y_1, y_3, y_4, y_{2k-4}] \in \mathcal{F}_2(Q_{2k-4})$ and that $[y_4, y_{2k-4}] \in \mathcal{F}_1(Q_{2k-4})$. □

7. $\mathcal{C}(Q_m) = \mathcal{C}^*(P_m) \cup \{[s-3,s-2,s-1,s] \mid s=j+6,...,k\} \cup \{[j+2,j+3,j+4,j+5,m]\}$ *for* $m = 2k-3+j$ *and* $j = 0,1,...,k-5$.

Proof. Noting that $[y_1, y_3, y_4, y_{2k-4}]$, $[y_1, y_2, y_4]$, $[y_1, y_2, y_{2k-4}]$, $[y_2, y_3, y_4]$ and $[y_2, y_3, y_{2k-4}]$ are the 2-faces of $[y_1, y_2, y_3, y_4, y_{2k-4}]$, we argue as above to obtain that

$$\mathcal{C}(Q_{2k-3}) = \mathcal{C}^*(P_{2k-3}) \cup \{[s-3,s-2,s-1,s] \mid s=6,...,k\} \cup \{[2,3,4,5,2k-3]\},$$

$[y_2, y_4, y_5, y_{2k-3}] \in \mathcal{F}_2(Q_{2k-3})$ and $[y_5, y_{2k-3}] \notin \mathcal{F}_1(Q_{2k-3})$.

Proceeding by induction; the assertion, $[y_{j+2}, y_{j+4}, y_{j+5}, y_m] \in \mathcal{F}_2(Q_m)$ and $[y_{j+5}, y_m] \notin \mathcal{F}_1(Q_m)$ follow as above from 2.6, 3.6 and 4.4.

We remark that $\{[s-3, s-2, s-1, s] \mid k+1 \leq s \leq k\}$ is empty and thus,

$$\mathcal{C}(Q_{3k-8}) = \mathcal{C}^*(P_{3k-8}) \cup \{[k-3, k-2, k-1, k, 3k-8]\}.$$

□

8. $\mathcal{C}(Q_{3k-7}) = \mathcal{C}^*(P_{3k-7}) = \mathcal{C}(P_{3k-7})$.

Proof. Noting that $y_1 \in \langle y_{k-2}, y_{k-1}, y_k, y_{k+21}\rangle$ from 4.1, the assertion is immediate by 3.6, 4.4 and 6. □

We remark that P_n and Q_n are equivalent and that for $k = k, ..., n$, Q_m is Gale with $y_1 < y_2 < ... < y_m$.

REALIZATION

Recalling that $Q_k \subset R^4$ is a cyclic 4-polytope with the vertex array $y_1 < y_2 < ... < y_k$, $k \geq 6$, we verify the existence of points $y_{k+1}, ..., y_{3k-7}$ in R^4 satisfying 4.1 to 4.4.

Set $F = [y_1, y_{k-2}, y_{k-1}, y_k]$, $F^*[y_1, y_2, y_{k-1}, y_k]$ and $F_i = [y_1, y_i, y_{i+1}, y_k]$ for $i = 2, ..., k-3$. These are the facets of Q_k containing $[y_1, y_k]$. With

$$H = \text{ aff } F,\ H^* = \text{ aff } F^* \text{ and } H_i = \text{ aff } F_i,$$

we have that

$$M^* = H^* \cap H \text{ and } M_i = H_i \cap H$$

are planes containing the line $L = \langle y_1, y_k\rangle$. From 4.1, we need intitially to find a point $y \in H$ such that, with respect to F, it is beneath M^* and beyond each M_i.

Let $2 \leq i \leq k-3$. Since $[y_1, y_i, y_k] \in \mathcal{F}_2(Q_k)$, there is a supporting hyperplane $\tilde{H}_i \subset R^4$ such that

$$\tilde{H}_i \cap Q_k = [y_1, y_i, y_k].$$

We note that $[y_1, y_2, y_k] = F^* \cap F_2$ implies that $\tilde{H}_2$ ranges between H^* and H_2, and that for $i > 2$, $[y_1, y_i, y_k] = F_{i-1} \cap F_i$ implies that $\tilde{H}_i$ ranges between H_{i-1} and H_i. Thus the plane $\tilde{H}_i \cap H$, containing L, ranges between M^* and M_2 for $i = 2$, and between M_{i-1} and M_i for $i > 2$. Now if

$$\tilde{H}_u \cap H = \tilde{H}_v \cap H$$

for some $2 \leq u < v \leq k-3$ then through the plane $M = \tilde{H}_u \cap \tilde{H}_v \cap H$ with the property that $M \cap Q_k = [y_1, y_k]$, there passes three distinct supporting hyperplanes of Q_k such that each of them intersects $Q_k \setminus M$; a contradiction. Accordingly, the arrangement of $M^*, M_2, \ldots, .M_{k-3}$ in H is as indicated in Figure 1.

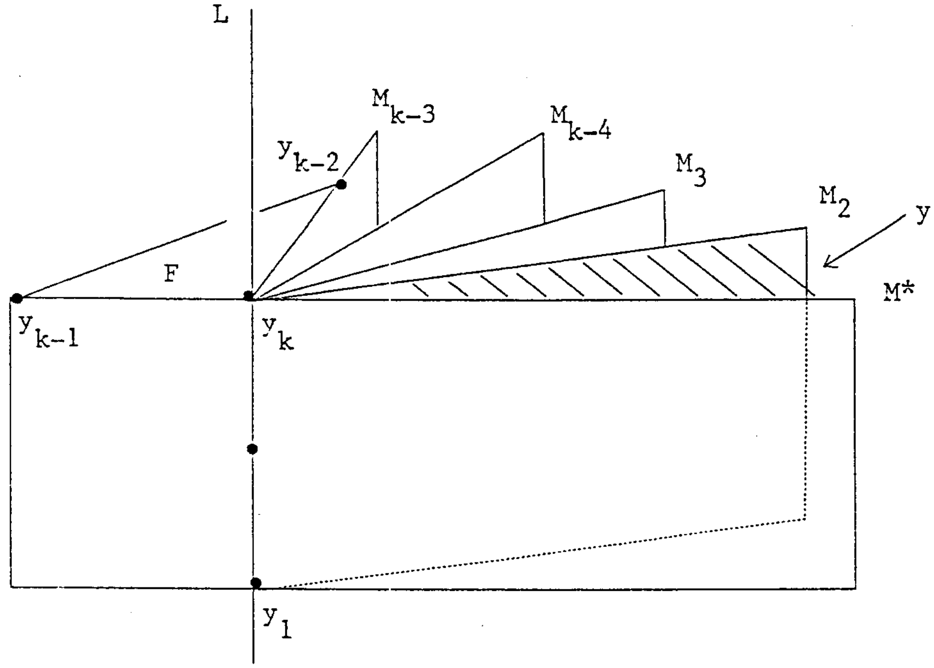

Figure 1

It is now clear that there are points $y \in H$ that are, with respect to F, beneath M^* and beyond each M_i. With respect to Q_k, such y are on aff F, beneath F^* and beyond each F_i. Now if y is arbitrarily close to say the mid-point of $[y_1, y_k]$ then it is obviously beneath all the other facets of Q_k, and we label it y_{k+1}.

Let $2 \leq j \leq k-5$, $u = k - j$ and assume that $y_{k+1}, \ldots, y_{m-1}$ exist satisfying 4.1 for $m = k + j$. From 2.3 and 5., the facets of $Q_{m-1} = [y_1, \ldots, y_{m-1}]$ containing $[y_1, y_{m-1}]$ are $F = [y_1, y_{u-1}, y_u, y_{m-1}]$, $F' = [y_1, y_u, y_{u+1}, y_{m-2}, y_{m-1}]$, $F^* = [y_1, y_2, y_{m-2}, y_{m-1}]$, and $F_i = [y_1, y_i, y_{i+1}, y_{m-1}]$ for $i = 2, \ldots, k-j-2 = u-2$. Let $H =$ aff F, $H' \doteq$ aff F', $H^* =$ aff F^* and $H_i =$ aff F_i, as well as $M' = H' \cap H$, $M^* = H^* \cap H$ and $M_i = H_i \cap H$. Now we argue as above and obtain that the arrangement of the planes in H is as indicated in Figure 2.

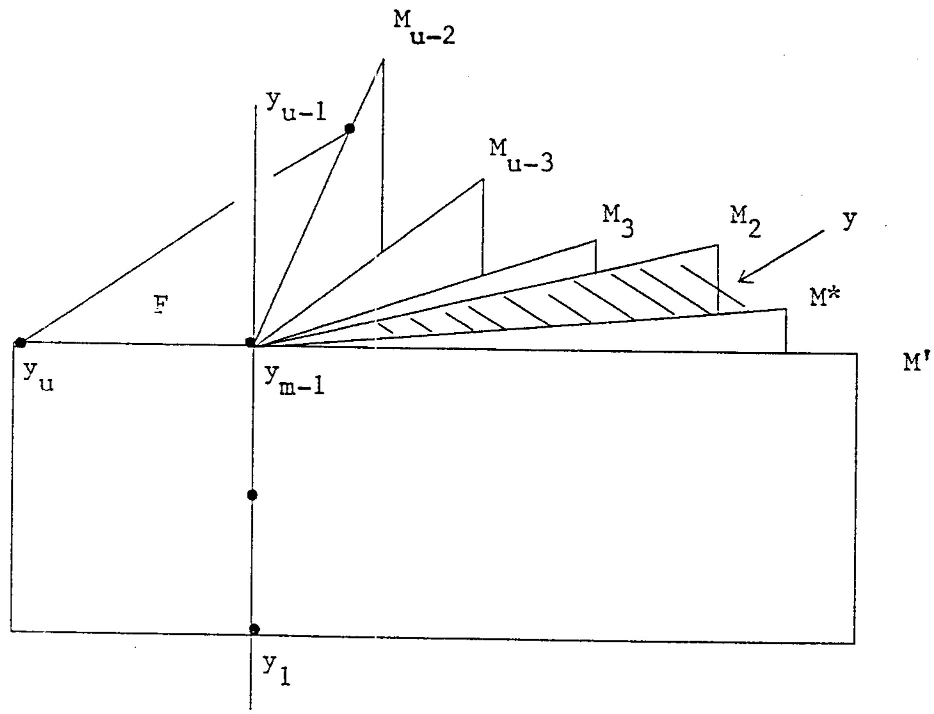

Figure 2

Again, we choose as y_m a point $y \in H$ that is beyond $M_1, ..., M_{u-2}$ and beneath M^* (and M') with respect to F, and arbitrarily close to say the mid-point of $[y_1, y_{m-1}]$.

Next, we need a point on the plane $M = \langle y_1, y_3, y_4 \rangle$ that is beyond only $[y_1, y_2, y_3, y_{2k-5}]$ with respect to Q_{2k-5}. Since

$$\langle y_1, y_3 \rangle = M \cap \langle y_1, y_2, y_3, y_{2k-5} \rangle,$$

it is immediate that a point $y \in M \backslash [y_1, y_3, y_4]$ that is arbitrarily close to say the mid-point of $[y_1, y_3]$ is suitable choice for y_{2k-4}.

We choose y_{2k-3} in a similar manner.

Finally, let $1 \leq j \leq k-4$ and assume that $y_{2k-3}, ..., y_{m-1}$ exist satisfying 4.3 to 4.4 for $m = 2k-3+j$. Set

$$\begin{aligned} M &= \langle y_{j+2}, y_{j+4}, y_{j+5} \rangle, \\ F_0 &= [y_j, y_{j+1}, y_{j+3}, y_{j+4}, y_{m-1}], \\ F_1 &= [y_{j+1}, y_{j+2}, y_{j+3}, y_{j+4}, y_{m-1}], \\ L_i &= M \cap \text{ aff } F_i, \end{aligned}$$

and let $H \subset R^4$ be a supporting hyperplane of Q_{m-1} such that

$$H \cap Q_{m-1} = F_0 \cap F_1 = [y_{j+1}, y_{j+3}, y_{j+4}, y_{m-1}].$$

From 4.4, we need a point on L_0 that is beyond L_1 with respect to $[y_{j+2}, y_{j+4}, y_{j+5}]$, and beneath all the other facets of Q_{m-1} with respect to Q_{m-1}; cf. Figure 3 with explanation to follow.

We note that if $M \cap$ aff $(F_0 \cap F_1)$ is a line then

$$\text{aff } (M \cup (F_0 \cap F_1)) = \text{ aff } (\{y_{j+2}\} \cup (F_0 \cap F_1)) = \text{ aff } F_1$$

and $y_{j+5} \in F_1$; a contradiction. Hence, $M \cap$ aff $(F_0 \cap F_1) = \{y_{j+4}\}$ and $L = H \cap M$ is a supporting line of $[y_{j+2}, y_{j+4}, y_{j+5}]$ that passes through y_{j+4} and is distinct from L_0 and L_1. Since H ranges between aff F_0 and aff F_1, and $L_0 \neq L \neq L_1$, it follows that L ranges between L_0 and L_1.

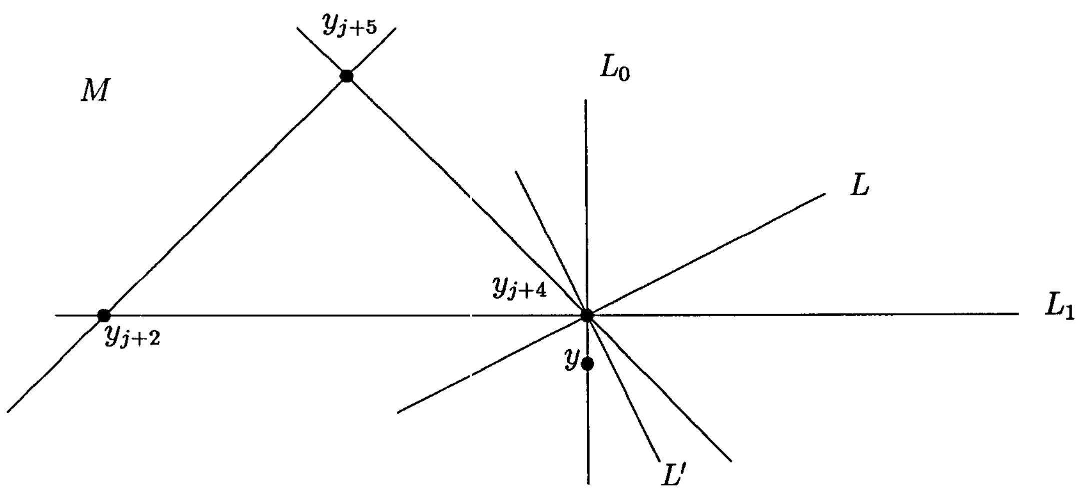

Figure 3

Let $F' \in \mathcal{F}(Q_{m-1})$ such that $y_{j+4} \in F'$ and $F_0 \neq F' \neq F_1$. Now if $L' = (\text{aff } F') \cap M = L$ then $y_{j+5} \notin F'$, and it follows from the Gale property of Q_{m-1} that $y_{j+3} \in F'$. Hence, through the plane $M' = \text{aff } (L \cup \{y_{j+3}\})$, there passes three supporting hyperplanes aff F_0, aff F_1 and aff F' of Q_{m-1} such that each of them intersects $Q_{m-1} \backslash M'$; a contradiction.

Since $L' \neq L$ and L ranges between L_0 and L_1, we have the explanation for Figure 3. It is now clear that a point $y \in L_0$ that is beyond L_1 with respect to $[y_{j+2}, y_{j+4}, y_{j+5}]$ and is arbitrarily close to y_{j+4} is a suitable choice for y_m.

REMARKS

In summary, we have the existence of a 4-polytope $Q_n(n = 3k - 7)$ and $k \geq 6$ in R^4 that is not only Gale with $y_1 < y_2 < ... < y_n$ but also has the property that for $k \leq m < n$, the subpolytpes $Q_m = [y_1, y_2, ..., y_m]$ are Gale with the induced vertex array. We remark that the f-vector of Q_n is

$$(3k - 7, k^2 + 3k - 19, 2k^2 - 3k - 13, k^2 - 3k - 1).$$

In addition to the earlier question of whether Q_n is k-sequentially cyclic, it is also natural to ask if there exists a point $y_{n+1} \in R^4$ such that $Q_{n+1} = [y_1, ..., y_n, y_{n+1}]$ is Gale with $y_1 < y_2 < ... < y_n < y_{n+1}$? All we know at present is that our method of construction does not yield y_{n+1} and the reason seems to be the following: the construction pretends that the points y_i lie on simple, locally of order four, oriented curve in R^4 that itself lies on the boundary of a strictly convex body. As such a curve may be closed then, of course, after choosing sufficiently many points as move along the curve, we return to the starting point y_1. The construction yields the $3k - 7$ points are sufficiently many.

Next, we make no claim that we have presented the only or even the best way of constructing Gale 4-polytopes. As food for thought, we present the following combinatorial 4-polytope $P = [x_1, x_2, ..., x_{28}]$ that is Gale with the natural ordering. With our usual notation for facets,

$$\begin{aligned} \mathcal{C}(P) = \; & \{[1,2,5,6,9,10,13,14,17,18,21,22,25,26], [1,2,8,9,15,16,22,23], \\ & [1,4,5,8,9,12,13,16,17,20,21,24,25,28], [1,7,8,14,15,21,22,28], \\ & [2,3,6,7,10,11,14,15,18,19,22,23,26,27], [2,3,9,10,16,17,23,24], \\ & [3,4,7,8,11,12,15,16,19,20,23,24,27,28], [3,4,10,11,17,18,24,25], \\ & [4,5,11,12,18,19,25,26], [5,6,12,13,19,20,26,27], \\ & [6,7,13,14,20,21,27,28]\}. \end{aligned}$$

It is noteworthy that P is simple, with no triangular 2-faces and the f-vector $(28, 56, 39, 11)$.

It is evident that Gale polytopes have an inherent interest and it is our belief that, as a class of explicit examples, they will further the understanding of polytopes in general.

The author gratefully acknowledges that this research was partially supported by the Natural Sciences and Engineering Council of Canada.

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Department of Mathematics
Universityof Calgary,
Calgary, Alberta, T2N 1N4
Canada