ISRAEL JOURNAL OF MATHEMATICS **102** (1997), 101–123

ORDINARY $(2m+1)$-POLYTOPES

BY

T. BISZTRICZKY

Department of Mathematics and Statistics, The University of Calgary
Calgary, Canada T2N 1N4
e-mail: tbisztri@math.ucalgary.ca

ABSTRACT

For each k, m and n such that $n \geq k \geq 2m+1 \geq 5$, we present a convex $(2m+1)$-polytope with $n+1$ vertices and $2\binom{k-m}{m} + (n-k)\binom{k-m-2}{m-1}$ facets with the property that there is a complete description of each of the facets based upon a total ordering of the vertices.

Introduction

We introduce a class of convex $(2m+1)$-polytopes P, via a total ordering of the vertices of P, which contains the cyclic $(2m+1)$-polytopes and which has the property that there is a complete description of the facets of each P. These polytopes, which we call ordinary, have been defined for $m = 1$ in [1] and we present them here for $m > 1$. In fact, we define an ordinary d-polytope for any $d \geq 3$ but show that the polytope is not cyclic only if $d = 2m+1$ (Theorem A).

As guide-posts, we indicate the central concepts and results of our theory.

Let P be a convex d-polytope in E^d, $d = 2m+1 \geq 5$, with a totally ordered set of vertices, say, $x_0 < x_1 < \cdots < x_n$. Then P is ordinary if each of its facets satisfies a global condition (the necessary part of Gale's Evenness Condition) and a local one (a specific relation among the vertices of a facet). Then there exist integers k and l (see Lemma 4 for the existence of k) such that $d \leq k$, $l \leq n$, $\text{conv}\{x_0, x_i\}$ is an edge of P if and only if $1 \leq i \leq k$, and $\text{conv}\{x_{n-i}, x_n\}$ is an edge of P if and only if $1 \leq i \leq l$. In fact, k is equal to l (Corollary 13) and we

Received July 11, 1994

call it the characteristic of P. Given k and l, we list the facets of P containing x_0 or x_n in Lemmas 8 and 9, and the other facets of P in Lemma 11. In Theorem B and its Corollary, we describe completely these facets and show that if k is the characteristic of P then

$$f_{2m}(P) = 2\binom{k-m}{m} + (n-k)\binom{k-m-2}{m-1},$$

and that if $k = n$ then P is cyclic.

Finally, we note that ordinary 3-polytopes were inspired by the idea of choosing, as vertices, points on a convex ordinary space curve in E^3. Unfortunately, there is as yet no definition of a convex ordinary space curve in E^d for $d > 3$. However, certain types of curves in E^d (for example, curves of order d) have properties that are independent of d, as long as the parity of d is the same. Thus our expectation, in generalizing the definition of an ordinary 3-polytope, is that there is a new class of d-polytopes only if $d = 2m + 1$. As this is the case, our approach seems to be a reasonable one.

1. Definitions

Let Y be a set of points in E^d, $d \geq 3$. Then conv Y is the convex hull of Y and if $Y = \{y_1, \ldots, y_s\}$ is finite, we set

$$[y_1, \ldots, y_s] = \text{conv}\{y_1, \ldots, y_s\}.$$

Thus, $[y_1, y_2]$ is the closed segment with end points y_1 and y_2.

Let $V = \{x_0, x_1, \ldots, x_n\}$ be a totally ordered set of $n+1$ points in E^d with $x_i < x_j$ if and only if $i < j$. We say that x_i and x_{i+1} are **successive** points, and if $x_i < x_j < x_k$ then x_j **separates** x_i and x_k or x_j is **between** x_i and x_k.

Let $Y \subset V$. Then Y is **connected** (in V) if $x_i < x_j < x_k$ and $\{x_i, x_k\} \subset Y$ imply that $x_j \in Y$. If Y is not connected then clearly it can be written uniquely as the union of maximal connected subsets, which we call **components** of Y. A component X of Y is **even** or **odd** according to the parity of $|X| = \text{card } X$. Next, Y is a **Gale set** (in V) if any two points of $V \backslash Y$ are separated by an even number of points of Y. Finally, Y is a **paired set** if it is the union of mutually disjoint subsets $\{x_i, x_{i+1}\}$.

We note that V, $\emptyset$ and all paired subsets of V are Gale sets. Conversely, let $Y \subset V$ be a Gale set. If $Y \cap \{x_0, x_n\} = \emptyset$ then Y is a paired set. Thus if Y is not connected then Y has at most two odd components, each of which contains x_0 or x_n.

We acknowledge that a connected set is an adaptation of Shephard's contiguous set in [5], and that Gale sets stem from the article [2] by Gale.

Let r and s be integers such that $0 < 2r \leq s$, and let $Y \subset V$ be a connected set with $|Y| = s$. Let $p(r,s)$ be the number of paired subsets X of Y such that $|X| = 2r$; that is, X is the union of r mutually disjoint pairs.

Since $p(1,s) = s-1 = \binom{s-1}{1}$, we assume that $r \geq 2$ and that $p(r-1,s) = \binom{s-r+1}{r-1}$. Noting that $p(r,s) = p(r,s-1) + p(r-1,s-2)$,

$$\begin{aligned} p(r,s) &= \sum_{i=2}^{s-2(r-1)} p(r-1,s-i) \\ &= \sum_{i=2}^{s-2r+2} \binom{s-i-r+1}{r-1} = \sum_{j=s-r-1}^{r-1} \binom{j}{r-1} \\ &= \sum_{j=r-1}^{s-r-1} \binom{j}{r-1} = \binom{s-r}{r}; \end{aligned}$$

cf. formula 1.52 in [3]. We shall use $p(r,s)$ to calculate the number of facets of an ordinary polytope.

Let $P \subset E^d$ be a (convex) d-polytope. For $-1 \leq i \leq d$, let $\mathcal{F}_i(P)$ denote the set of i-faces of P and $f_i(P) = |\mathcal{F}_i(P)|$. When there is no danger of confusion, we set $\mathcal{F}_i = \mathcal{F}_i(P)$ and $\mathcal{F} = \mathcal{F}_{d-1}$. Let $V = \mathcal{F}_0(P) = \{x_0, x_1, \ldots, x_n\}$, $n \geq d$. We set $x_i < x_j$ if and only if $i < j$, and call $x_0 < x_1 < \cdots < x_n$ a **vertex array** of P. If we reverse the ordering, we call $x_n < x_{n-1} < \cdots < x_0$ a **reverse vertex array** of P. Let $G \in \mathcal{F}_i(P)$, $1 \leq i \leq d$, such that $G \cap V = \{y_0, y_1, \ldots, y_s\}$ (each y_j is some x_i) and $y_0 < y_1 < \cdots < y_s$ is the ordering induced by $x_0 < x_1 < \cdots < x_n$. We call $y_0 < y_1 < \cdots < y_s$ an (induced) vertex array of G, and set $y_j = y_0$ for $j < 0$ and $y_j = y_s$ for $j > s$.

We recall from [2] and [4] that a d-polytope P with the vertex array $x_0 < x_1 < \cdots < x_n$ is **cyclic** if P is simplicial and satisfies Gale's Evenness Condition: A d element subset Y of V determines a facet of P if and only if Y is a Gale set.

Furthermore, if P is cyclic then $p(r,s) = \binom{s-r}{r}$ readily yields that

$$f_{d-1}(P) = \begin{cases} \dfrac{n+1}{n+1-m}\dbinom{n+1-m}{m} & \text{for } d = 2m, \\ 2\dbinom{n-m}{m} & \text{for } d = 2m+1. \end{cases}$$

Let P be a d-polytope with the vertex array $x_0 < x_1 < \cdots < x_n$, $n \geq d \geq 3$. Then P is **ordinary** if for each facet F of P,

(01) $F \cap V$ is a Gale set, and

(02) if $y_0 < y_1 < \cdots < y_s$ is the (induced) vertex array of F then the $(d-2)$-faces of F are $[y_0, y_1, \dots, y_{d-2}]$, $[y_{s-d+2}, \dots, y_{s-1}, y_s]$ and $[y_{i-d+2}, \dots, y_{i-1}, y_{i+1}, \dots, y_{i+d-2}]$ for $i = 1, \dots, s-1$.

We emphasize the convention that in the description of faces as in (02), the terms y_j are to be ignored if $j < 0$ or $j > s$.

Since cyclic d-polytopes are simplicial, they are clearly ordinary. Next, and this is the reason why $f_0(P) = n+1$ and $f_0(F) = s+1$, if P is ordinary with the vertex array $x_0 < x_1 < \cdots < x_n$ then it is ordinary with the reverse vertex array $x_n < x_{n-1} < \cdots < x_0$.

Finally, if P is an ordinary 3-polytope and $F \in \mathcal{F}_2(P)$ has the vertex array $y_0 < y_1 < \cdots < y_s$ then F is a polygon with the edges $[y_0, y_1]$, $[y_{s-1}, y_s]$ and $[y_j, y_{j+2}]$ for $j = 0, \dots, s-2$. For a description of ordinary 3-polytopes, we refer to [1]. As we shall see, there are differences between the theories of ordinary 3-polytopes and ordinary d-polytopes, $d \geq 4$.

2. Preliminaries

Henceforth, we assume that P is an ordinary d-polytope with the vertex array $x_0 < x_1 < \cdots < x_n$, $d \geq 4$. We list some of the consequences of our definition, and note that Lemmas 4, 8 and 9, and Theorem A are particularly significant.

1. LEMMA: *Let $F \in \mathcal{F}$ with the vertex array $y_0 < y_1 < \cdots < y_s$, and let $G \in \mathcal{F}_{d-2}$ with the vertex array $z_1 < z_2 < \cdots < z_t$.*

1.1 *$f_{d-2}(F) = s+1$ and $f_0(G) \leq 2d-4$.*

1.2 *The vertices $y_i, y_{i+1}, \dots, y_{i+d-1}$ are affinely independent, $i = 0, \dots, s-d+1$.*

1.3 *If $s \geq d$ then $[y_0, y_1, \dots, y_{d-2}]$, $[y_0, y_2, \dots, y_{d-1}]$, $[y_{s-d+1}, \dots y_{s-2}, y_s]$ and $[y_{s-d+2}, \dots, y_{s-1}, y_s]$ are the only $(d-2)$-faces of F that are simplices.*

1.4 *If* $G \subset F$ *then* $|F \cap \{x_i \mid z_1 \leq x_i \leq z_t\}| \leq t+1$, *with equality for* $t \geq d$; *furthermore, if* $t \leq 2d-5$ *then* $y_0 = z_1$ *or* $y_s = z_t$.

1.5 $[y_0, y_j] \in \mathcal{F}_1$ *if and only if* $1 \leq j \leq d-1$ *if and only if* $[y_{s-j}, y_s] \in \mathcal{F}_1$.

1.6 *If* $s \geq d$ *then for* $j = 0, \dots, s-d$, $[y_j, y_{j+1}, y_{j+d-1}, y_{j+d}] \in \mathcal{F}_2$ *and* $[y_j, y_{j+d}] \notin \mathcal{F}_1$.

Proof: The first four observations readily follow from (02).

5. If $1 \leq j \leq d-1$ then 1.3 yields that $[y_0, y_j]$ is an edge of P. Let $d \leq j \leq s$ and $\tilde{G} \in \mathcal{F}_{d-2}(F)$ such that $\{y_0, y_j\} \subset \tilde{G}$. Clearly,

$$\tilde{G} = [y_{i-d+2}, \dots, y_{i-1}, y_{i+1}, \dots, y_{i+d-2}]$$

for some i such that $i-d+2 \leq 0$ and $d \leq j \leq i+d-2$. Hence, $2 \leq i \leq d-2$ and it follows that $y_1 \in \tilde{G}$. But then $[y_0, y_j]$ is not the intersection of $(d-2)$-faces of F, and it is not an edge of P.

By the reverse vertex array, we obtain the second part of 1.5.

6. Let $0 \leq j \leq s-d$. Since $d \geq 4$, we have that

$$\bigcap_{i=j+2}^{j+d-2} [y_{i-d+2}, \dots, y_{i-1}, y_{i+1}, \dots, y_{i+d-2}] = [y_j, y_{j+1}, y_{j+d-1}, y_{j+d}]$$

is a face of P. It is now easy to check that if $\{y_j, y_{j+d}\} \subset \tilde{G} \in \mathcal{F}_{d-2}(F)$ then $\{y_{j+1}, y_{j+d-1}\} \subset \tilde{G}$. Thus, $[y_j, y_{j+d}] \notin \mathcal{F}_1$ and from this it follows that $[y_j, y_{j+1}, y_{j+d-1}, y_{j+d}] \in \mathcal{F}_2$. ∎

2. LEMMA: *Let* $F \in \mathcal{F}$ *with the vertex array* $y_0 < \cdots < y_r < y_{r+1} < \cdots < y_{t-1} < y_t < \cdots < y_s$, $\{y_r, y_{r+1}\} = \{x_j, x_{j+1}\}$ *and* $\{y_{t-1}, y_t\} = \{x_{l-1}, x_l\}$.

2.1 *If* $r \geq 1$ *and* $s \geq r+d-1$ *then* $y_{r-1} = x_{j-1}$.

2.2 *If* $t \leq s-1$ *and* $d-1 \leq t$ *then* $y_{t+1} = x_{l+1}$.

Proof: 1. Let $r \geq 1$ and $s \geq r+d-1$. Then $2 \leq r+1 \leq s-d+2 \leq s-2$ and

$$G = [y_{r-d+3}, \dots, y_r, y_{r+2}, \dots, y_{r+d-1}] \in \mathcal{F}_{d-2}.$$

Let $F' \in \mathcal{F}$ with the vertex array $z_0 < z_1 < \cdots < z_u$ such that $F' \cap F = G$. Then $F' \cap \{x_j, x_{j+1}\} = \{x_j\}$, $x_j > x_0$ and (01) imply that x_{j-1} and x_j are successive vertices of F'. Clearly

$$G = [z_{i-d+2}, \dots, z_{i-1}, z_{i+1}, \dots, z_{i+d-2}]$$

for some $1 \le i \le u-1$. Since $|\{y_{r+2}, \dots, y_{r+d-1}\}| = d-2$, it follows that $\{y_{r-1}, y_r\} \subset \{z_{i-d+2}, \dots, z_{i-1}\}$. Hence, y_{r-1} and $y_r = x_j$ are successive vertices of F' and $y_{r-1} = x_{j-1}$.

2. Let $d-1 \le t \le s-1$. Then

$$G = [y_{t-d+1}, \dots, y_{t-2}, y_t, \dots, y_{t+d-1}] \in \mathcal{F}_{d-2}$$

and, with F' defined as above, x_l and x_{l+1} are successive vertices of F'. Now, $|\{y_{t-d+1}, \dots, y_{t-2}\}| = d-2$ yields $\{y_t, y_{t+1}\} \subset \{z_{i+1}, \dots, z_{i+d-2}\}$ and $y_{t+1} = x_{l+1}$. ∎

Let $V^0 = \{x_i \in V \,|\, [x_0, x_i] \in \mathcal{F}_1\}$ and $\mathcal{F}^0 = \{F \in \mathcal{F} \,|\, x_0 \in F\}$.

3. LEMMA: *Let $x_0 \ne x_i \in F \in \mathcal{F}^0$. Then $|F \cap V^0| = d-1$, and $x_i \in V^0$ if and only if $|F \cap \{x_0, \dots, x_i\}| \le d$.*

Proof: Apply 1.5. ∎

4. LEMMA: *There is an integer k such that $d \le k \le n$ and $V^0 = \{x_1, \dots, x_k\}$.*

Proof: Let $k \le n$ be the largest integer such that $x_k \in V^0$. Clearly, $k \ge d$. We show that $i \ge 2$ and $x_i \in V^0$ imply that $x_{i-1} \in V^0$.

Let $\mathcal{F}' = \{F \in \mathcal{F} \mid \{x_0, x_i\} \subset F\}$. Then the edge $[x_0, x_i]$ is the intersection of all the $F \in \mathcal{F}'$, and by 3., $|F \cap \{x_0, \dots, x_i\}| \le d$ for each $F \in \mathcal{F}'$. Thus, if $x_{i-1} \in F \in \mathcal{F}'$ then $|F \cap \{x_0, \dots, x_{i-1}\}| \le d$ and $x_{i-1} \in V^0$.

If $2 \le i \le n-1$ then for any $F \in \mathcal{F}'$, $F \cap \{x_{i-1}, x_{i+1}\} \ne \emptyset$ by (01). Since there must be an $F \in \mathcal{F}'$ such that $x_{i+1} \notin F$, we have that $x_{i-1} \in F$.

If $i = n$ then each $F \in \mathcal{F}'$ is a $(d-1)$-simplex by 3. Let r be the largest integer such that $r < n$ and there is an $F_r \in \mathcal{F}'$ with $x_r \in F_r$. Let $y_0 < y_1 < \cdots < y_{d-1}$ be the vertex array of F_r. Then $y_0 = x_0$, $y_{d-2} = x_r$, $y_{d-1} = x_n$ and

$$G = [y_0, \dots, y_{d-4}, x_r, x_n] \in \mathcal{F}_{d-2}.$$

Let $F' \in \mathcal{F}'$ such that $F' \cap F_r = G$. If $x_{r+1} \ne x_n$ then $x_{r-1} \in F' \cap F_r$ by (01). Since $x_{r-1} \in F_r$ implies $x_{r-1} = y_{d-3}$, and $x_{r-1} \in G$ implies $x_{r-1} = y_{d-4}$, it follows that $x_{r+1} = x_n$ and $x_r = x_{n-1}$. ∎

5. Lemma: *Let $V^0 = \{x_1, \ldots, x_k\}$. Let $F \in \mathcal{F}^0$ with the vertex array $x_0 < y_1 < \cdots < y_s$ and $x_d \le y_{d-1}$.*

5.1 *If $d = 2m$ then either $x_k = y_{d-1}$ and $\{y_1, \ldots, y_{d-2}\}$ is a paired subset of $\{x_1, \ldots, x_{k-1}\}$ or $x_1 = y_1$ and $\{y_2, \ldots, y_{d-1}\}$ is a paired subset of $\{x_2, \ldots, x_k\}$.*

5.2 *If $d = 2m+1$ then either $\{y_1, \ldots, y_{d-1}\}$ is a paired subset of $\{x_1, \ldots, x_k\}$ or $x_1 = y_1$, $x_k = y_{d-1}$ and $\{y_2, \ldots, y_{d-2}\}$ is a paired subset of $\{x_2, \ldots, x_{k-1}\}$.*

Proof: We note that by 1.5 and 4., $y_{d-1} \le x_k$. Next, $y_{d-1} \ge x_d$ implies that $\{x_0, y_1, \ldots, y_{d-1}\}$ is not connected. Thus, the two assertions in both 5.1 and 5.2 are mutually exclusive.

1. Let $d = 2m$. If $\{x_0, y_1, \ldots, y_{d-1}\}$ is paired then $x_1 = y_1$ and $\{y_2, \ldots, y_{d-1}\}$ is paired. If $\{x_0, y_1, \ldots, y_{d-1}\}$ is not paired then because it is not connected, it has exactly two odd components. One component contains x_0 and the other contains y_{d-1}. By (01), the latter is not possible if $y_{d-1} < x_k$. Hence, $y_{d-1} = x_k$ and $\{y_1, \ldots, y_{d-2}\}$ is paired.

2. Let $d = 2m+1$. Since $\{x_0, y_1, \ldots, y_{d-1}\}$ is not connected and contains an odd number of elements, it has exactly one odd component which contains either x_0 or y_{d-1}. In case of the former, $\{y_1, \ldots, y_{d-1}\}$ is paired. In case of the latter, we have $x_1 = y_1$, $\{y_2, \ldots, y_{d-2}\}$ is paired and, as above, $y_{d-1} = x_k$. ∎

We note that while the assertions in 5 are somewhat repetitive, they make it easier to list the facets in $\mathcal{F}^0$. Our goal now is to list the d element subsets of $V^0 \cup \{x_0\}$ that by 1.2 and 3., determine the facets in $\mathcal{F}^0$.

6. Lemma: *Let $V^0 = \{x_1, \ldots, x_k\}$. For each integer r such that $d-1 \le r \le k$, there is an $F \in \mathcal{F}^0$ such that $x_r \in F$ and $|F \cap \{x_0, \ldots, x_r\}| = d$; that is, $x_r \in F \cap V^0 \subseteq \{x_1, \ldots, x_r\}$.*

Proof: Since the assertion is true for $r = k$, we show that if it is true for r, $d \le r \le k$, then it is true for $r-1$. Let $d \le r \le k$ and let $F \in \mathcal{F}^0$ with the vertex array $x_0 < y_1 < \cdots < y_s$, $x_r = y_{d-1}$.

If $r = n$ then $F = [x_0, y_1, \ldots, y_{d-2}, x_n]$ is a $(d-1)$-simplex by 3. From the proof of 4., we may assume that $x_{n-1} = y_{d-2}$. We note that $G = [x_0, y_1, \ldots, y_{d-3}, x_{n-1}] \in \mathcal{F}_{d-2}$ and so, there is an $F' \in \mathcal{F}$ such that $F' \cap F = G$. Then $x_n \notin F'$, $F' \in \mathcal{F}_0$ and $x_{n-1} \in F' \cap V^0 \subseteq \{x_1, \ldots, x_{n-1}\}$.

Let $r \leq n-1$. Since $r \geq d$ and $|F \cap \{x_1, \ldots, x_r\}| = d-1$, it follows that there is an integer j such that $2 \leq j \leq r$ and

$$F \cap \{x_{j-1}, \ldots, x_r\} = \{x_j, \ldots, x_r\}.$$

If $x_{r+1} \notin F$ then $x_{j-1} \notin F$ and (01) yield that $\{x_j, \ldots, x_r\}$ is an even component of $F \cap V$, $j \leq r-1$ and $x_{r-1} = y_{d-2}$. By (02),

$$G = [x_0, y_1, \ldots, y_{d-3}, x_{r-1}] \in \mathcal{F}_{d-2}.$$

Let $F' \in \mathcal{F}$ such that $F' \cap F = G$. Then $F' \in \mathcal{F}^0$, $x_r \notin F'$, $\{x_j, \ldots, x_{r-1}\} \subset F'$ and by 1.4, $\left|F' \cap \{x_0, \ldots, x_{r-1}\}\right| \leq d$. Since $|\{x_j, \ldots, x_{r-1}\}|$ is odd, it follows that $x_{j-1} \in F'$ and $\left|F' \cap \{x_0, \ldots, x_{r-1}\}\right| = d$.

If $x_{r+1} \in F$ then $x_{r+1} \notin V^0$ and 4. imply that $r = k$. Since

$$\tilde{G} = [x_0, y_2, \ldots, y_{d-1}] = [x_0, y_2, \ldots, y_{d-2}, x_k] \in \mathcal{F}_{d-2},$$

there is an $\tilde{F} \in \mathcal{F}$ such that $\tilde{F} \cap F = \tilde{G}$. We note that $\tilde{F} \in \mathcal{F}^0$, $x_{k+1} \notin \tilde{F}$ and $x_k \in \tilde{F} \cap V^0 \subseteq \{x_1, \ldots, x_k\}$ by 4. We argue now as in the preceding paragraph to verify the assertion for $k-1$. ∎

7. LEMMA: *Let $V^0 = \{x_1, \ldots, x_k\}$, $d \leq k \leq n$. Let $d-1 \leq r \leq k$ and $F \in \mathcal{F}^0$ such that $x_r \in F$ and $|F \cap \{x_0, \ldots, x_r\}| = d$. Let $\{x_j, x_{j+1}\} \subset F \cap V$ for some $1 \leq j \leq r-2$.*

7.1 If $j > 1$ and $x_{j-1} \notin F$ then there is an $\tilde{F} \in \mathcal{F}^0$ such that

$$\tilde{F} \cap V^0 = ((F \cap V^0) \backslash \{x_{j+1}\}) \cup \{x_{j-1}\}.$$

7.2 If $j < r-2$ and $x_{j+2} \notin F$ then there is an $\tilde{F} \in \mathcal{F}^0$ such that

$$\tilde{F} \cap V^0 = ((F \cap V^0) \backslash \{x_j\}) \cup \{x_{j+2}\}.$$

Proof: Let $y_0 < y_1 < \cdots < y_s$ be the vertex array of F. Then $x_0 = y_0$, $x_r = y_{d-1}$ and $F \cap V^0 = \{y_1, \ldots, y_{d-1}\}$. For $2 \leq i \leq d-2$,

$$G_i = [y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{i+d-2}] \in \mathcal{F}_{d-2}$$

and there is an $F_i \in \mathcal{F}$ such that $F_i \cap F = G_i$. We note that $F_i \in \mathcal{F}^0$ and $(F \cap V^0) \backslash \{y_i\} \subseteq F_i \cap V^0$.

If $j > 1$ and $x_{j-1} \notin F$ then with $\{x_j, x_{j+1}\} = \{y_{i-1}, y_i\}$, (01) yields that $x_{j-1} \in F_i$. If $j < r-2$ and $x_{j+2} \notin F$ then with $\{x_j, x_{j+1}\} = \{y_i, y_{i+1}\}$, (01) yields that $x_{j+2} \in F_i$. Now by 3. and 4., $\tilde{F} = F_i$ in each case. ∎

In view of the preceding lemmas, we can now list all the facets in $\mathcal{F}^0$. Henceforth, we let S_j denote a paired set of vertices of cardinality $j > 0$, and set $S_0 = \emptyset$.

8. LEMMA: *Let* $V^0 = \{x_1, \dots, x_k\}$.

8.1 *If* $d = 2m$ *then*

$$\begin{aligned}\mathcal{F}^0 =& \{F_{0,1}(S_{d-2}) \,|S_{d-2} \subset \{x_2, \dots, x_k\}\} \\ & \cup \{F_0^k(S_{d-2}) \big|\, S_{d-2} \subset \{x_1, \dots, x_{k-1}\}\}\end{aligned}$$

where

$$F_{0,1}(S_{d-2}) \cap \{x_0, \dots, x_k\} = \{x_0, x_1\} \cup S_{d-2}$$

and

$$F_0^k(S_{d-2}) \cap \{x_0, \dots, x_k\} = \{x_0\} \cup S_{d-2} \cup \{x_k\}.$$

8.2 *If* $d = 2m+1$ *then*

$$\begin{aligned}\mathcal{F}^0 =& \{F_0(S_{d-1}) \,|S_{d-1} \subset \{x_1, \dots, x_k\}\} \\ & \cup \{F_{0,1}^k(S_{d-3}) \big|\, S_{d-3} \subset \{x_2, \dots, x_{k-1}\}\}\end{aligned}$$

where

$$F_0(S_{d-1}) \cap \{x_0, \dots, x_k\} = \{x_0\} \cup S_{d-1}$$

and

$$F_{0,1}^k(S_{d-3}) \cap \{x_0, \dots, x_k\} = \{x_0, x_1\} \cup S_{d-3} \cup \{x_k\}.$$

We note that 8. states simply that if Q is the vertex figure of P at x_0 determined by a hyperplane H and if $\{z_i\} = H \cap [x_0, x_i]$ for $i = 1, \dots, k$ then Q is a cyclic $(d-1)$-polytope with the vertex array $z_0 < z_1 < \dots < z_k$. Also, if $d = 2m$ then

$$|\mathcal{F}^0| = 2p(m-1, k-1) = 2\binom{k-m}{m-1},$$

and if $d = 2m+1$ then

$$\begin{aligned}|\mathcal{F}^0| &= p(m,k) + p(m-1, k-2) \\ &= \binom{k-m}{m} + \binom{k-m-1}{m-1} = \frac{k}{k-m}\binom{k-m}{m}.\end{aligned}$$

Next, let

$$V^* = \{x_i \in V \,|[x_{n-i}, x_n] \in \mathcal{F}_1\} \text{ and } \mathcal{F}^* = \{F \in \mathcal{F}|\, x_n \in F\}.$$

By reversing the vertex array, we obtain that there is an l such that $d \le l \le n$ and $V^* = \{x_{n-l}, \dots, x_{n-1}\}$, and the analogues of 3, 5, 6 and 7.

9. LEMMA: *Let* $V^* = \{x_{n-l}, \dots, x_{n-1}\}$

9.1 *If* $d = 2m$ *then*

$$\begin{aligned}\mathcal{F}^* =&\{F_{n-1,n}(S_{d-2}) \,|S_{d-2} \subset \{x_{n-l}, \dots, x_{n-2}\}\} \\ &\cup\{F_n^{n-l}(S_{d-2})|\, S_{d-2} \subset \{x_{n-l+1}, \dots, x_{n-1}\}\}\end{aligned}$$

where

$$F_{n-1,n}(S_{d-2}) \cap \{x_{n-l}, \dots, x_n\} = S_{d-2} \cup \{x_{n-1}, x_n\}$$

and

$$F_n^{n-l}(S_{d-2}) \cap \{x_{n-l}, \dots, x_n\} = \{x_{n-l}\} \cup \; S_{d-2} \cup \{x_n\}.$$

9.2 *If* $d = 2m + 1$ *then*

$$\begin{aligned}\mathcal{F}^* =&\{F_n(S_{d-1}) \,|S_{d-1} \subset \{x_{n-l}, \dots, x_{n-1}\}\} \\ &\cup\{F_{n-1,n}^{n-l}(S_{d-3})|\, S_{d-3} \subset \{x_{n-l+1}, \dots, x_{n-2}\}\}\end{aligned}$$

where

$$F_n(S_{d-1}) \cap \{x_{n-l}, \dots, x_n\} = S_{d-1} \cup \{x_n\}$$

and

$$F_{n-1,n}^{n-l}(S_{d-3}) \cap \{x_{n-l}, \dots, x_n\} = \{x_{n-l}\} \cup \; S_{d-3} \cup \{x_{n-1}, x_n\}.$$

We are now ready to exclude the case $d = 2m$ from our considerations.

THEOREM A: *Let* P *be an ordinary* d*-polytope with the vertex array* $x_0 < x_1 < \dots < x_n$, $d = 2m \ge 4$. *Then* P *is cyclic.*

Proof:

(i) $[x_0, x_n] \in \mathcal{F}_1$:

We suppose that $V^0 = \{x_1, \dots, x_k\}$, $d \le k < n$, and seek a contradiction. By 8.1, there is an $F \in \mathcal{F}^0$ with the vertex array $y_0 < y_1 < \dots < y_s$ such that $\{y_0, \ldots, y_{d-1}\} = \{x_0, x_1, \ldots, x_{d-2}, x_k\}$. Since $d \le k < n$, $x_{k-1} \notin F$ and $x_{k+1} \in F$. By (02),

$$G = [y_0, y_2, \ldots, y_{d-1}] = [x_0, x_2, \ldots, x_{d-2}, x_k] \in \mathcal{F}_{d-2}.$$

Let $F' \in \mathcal{F}$ such that $F' \cap F = G$. Then $F' \cap \{x_1, x_{k+1}\} = \emptyset$ and $x_{k-1} \in F'$. By 1.4,

$$F' \cap \{x_0, \ldots, x_k\} = \{x_0, x_2, \ldots, x_{d-2}, x_{k-1}, x_k\}.$$

Hence, x_1 and x_{k+1} are separated by an odd number $d-1$ of vertices of F', a contradiction. Thus, $[x_0, x_n] \in \mathcal{F}_1$ and $k = n = l$.

(ii) P IS SIMPLICAL:

We suppose that

$$\mathcal{F}' = \{F \in \mathcal{F} \mid f_0(F) \ge d+1\}$$

is not empty and seek a contradiction.

Since $k = n = l$, $\mathcal{F}' \cap (\mathcal{F}^0 \cup \mathcal{F}^*) = \emptyset$ by 3. Let $F \in \mathcal{F}'$ with the vertex array $y_0 < y_1 < \cdots < y_s$. Then $F \cap \{x_0, x_n\} = \emptyset$ implies that $\{y_0, y_1, \ldots, y_s\}$ is a paired set and $s \ge d+1$. Let

$$\{y_0, y_1, y_d, y_{d+1}\} = \{x_i, x_{i+1}, x_v, x_{v+1}\}$$

for some suitable i and v. We note that $i \ge 1$. Without loss of generality, we may assume that if $\tilde{F} \in \mathcal{F}'$ then $\tilde{F} \cap V \subset \{x_i, \ldots, x_{n-1}\}$.

We observe that

$$G = [y_0, y_1, y_3, \ldots, y_d] \in \mathcal{F}_{d-2}$$

by (02), and there is an $F' \in \mathcal{F}'$ such that $F' \cap F = G$. Since $f_0(G) = d$,

$$\left| F' \cap \{x_j \mid y_0 \le x_j \le y_d\} \right| = d+1$$

by 1.4. Since $F' \cap \{x_{i-1}, x_{v+1}\} = \emptyset$ and $d+1$ is odd, the set above is not paired; a contradiction.

(iii) FOR EACH $S_d \subset \{x_1, \ldots, x_{n-1}\}$, THERE IS AN $F \in \mathcal{F}$ SUCH THAT $F \cap V = S_d$:

Let $\{y_0, \ldots, y_{d-1}\} \subset V$ be a paired set with $x_0 < y_0 < \cdots < y_{d-1} < x_n$. Then $\{y_0, y_1\} = \{x_r, x_{r+1}\}$ for some $r \ge 1$, and $y_2 = x_t$ for some $t \ge r+2$. Since $S_{d-2} = \{y_2, \ldots, y_{d-1}\} \subset \{x_{r+2}, \ldots, x_{n-1}\}$, it follows from 8.1 and $k = n$ that

$$[x_0, x_1, y_2, \ldots, y_{d-1}] \in \mathcal{F}.$$

Then $G = [x_1, y_2, \dots, y_{d-1}] \in \mathcal{F}_{d-2}$ by (02). Since $y_2 = x_t \geq x_{r+2} \geq x_3$ and P is simplicial, it is clear that

$$[x_1, x_2, y_2, \dots, y_{d-1}]$$

is the other facet of P containing G. Reiteration of this argument yields that

$$[x_i, x_{i+1}, y_2, \dots, y_{d-1}] \in \mathcal{F}$$

for $i = 1, \dots, t-2$, and hence for $i = r$.

(iv) P IS CYCLIC: By 8., 9., the preceding and (01), we have that P is simplicial and satisfies Gale's Evenness Condition. ∎

3. Ordinary $(2m+1)$-polytopes

In this section, we assume that $d = 2m+1 \geq 5$. From 8.2 and 9.2, we have the facets of P passing through x_0 or x_n. We proceed now with the task of finding the remaining facets of P.

10. LEMMA: *Let $V^0 = \{x_1, \dots, x_k\}$, $d \leq k \leq n-2$ and $1 \leq i \leq n-k-1$. Let j be an odd integer, $1 \leq j \leq d-2$, $S_{d-j-2} \subseteq \{x_{i+2}, \dots, x_{i+k-j-1}\}$ and $F \in \mathcal{F}$ such that*

$$F \cap \{x_{i-1}, \dots, x_{i+k}\} = \{x_{i-1}, x_i\} \cup S_{d-j-2} \cup \{x_{i+k-j}, \dots, x_{i+k}\}.$$

Then there is an $F' \in \mathcal{F}$ such that

$$F' \cap \{x_i, \dots, x_{i+k+1}\} = \{x_i, x_{i+1}\} \cup S_{d-j-2} \cup \{x_{i+k-j+1}, \dots, x_{i+k+1}\}.$$

Proof: Let $y_0 < y_1 < \cdots < y_s$ be the vertex array of F. Then

$$F \cap \{x_{i-1}, \dots, x_{i+k}\} = \{y_{r-2}, \dots, y_{r-2+d}\}$$

for some $2 \leq r \leq s-d+2$. We note that $\{y_{r-2}, y_{r-1}\} = \{x_{i-1}, x_i\}$ and $y_r \geq x_{i+2}$. Hence, $s \leq r+d-2$ by 2.1; that is, $s = r+d-2$ and

$$F \cap \{x_{i-1}, \dots, x_{i+k}\} = \{y_{s-d}, \dots, y_s\}.$$

From 1.6, $[x_{i-1}, x_{i+k}] = [y_{s-d}, y_s] \notin \mathcal{F}_1$. Since

$$G = [y_{s-j-d+2}, \dots, y_{s-j-1}, y_{s-j+1}, \dots, y_{s-j+d-2}] \in \mathcal{F}_{d-2}$$

for $1 \leq j \leq d-2$ and $x_{i+k-j} = y_{s-j}$, we have that

$$G = [y_{s-j-d+2}, \ldots, y_{s-j-1}, x_{i+k-j+1}, \ldots, x_{i+k}].$$

Let $F' \in \mathcal{F}$ such that $F' \cap F = G$. Then $x_{i+k-j} \notin F'$, $i+k < n$ and $|\{x_{i+k-j+1}, \ldots, x_{i+k}\}| = j$ (odd) yield that $x_{i+k+1} \in F'$.

If $j = 1$ then $y_{s-j-d+2} = y_{s-d+1} = y_{r-1} = x_i$ and $x_{i-1} \notin G$. Thus, $x_{i-1} \notin F'$ and $x_{i-1} \in F'$. Let $j \geq 3$. Then $y_{s-j-d+2} < y_{s-d} = x_{i-1}$ and

$$G \cap \{x_{i-1}, \ldots, x_n\} = \{x_{i-1}, x_i\} \cup S_{d-j-2} \cup \{x_{i+k-j+1}, \ldots, x_{i+k}\}.$$

Since $[x_{i-1}, x_{i+k}] \notin \mathcal{F}_1$, it follows from 1.4 and 1.5 that there is exactly one vertex x of F' such that $x \notin G$ and $x_{i-1} \leq x \leq x_{i+k}$. Then $x_{i+k-j} \notin F'$ and (01) clearly yield that $x = x_{i+1}$. ∎

11. Lemma: *Let $V^0 = \{x_1, \ldots, x_k\}$, $d \leq k \leq n-1$ and $0 \leq i \leq n-k-1$. For each $S_{d-3} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}$, there is a facet $F_i(S_{d-3})$ of P such that*

$$F_i(S_{d-3}) \cap \{x_i, \ldots, x_{i+k+1}\} = \{x_i, x_{i+1}\} \cup S_{d-3} \cup \{x_{i+k}, x_{i+k+1}\}.$$

Proof: We note that by 8.2, the assertion is true for $i = 0$. (Since $k < n$, $x_{k+1} \in F^k_{0,1}(S_{d-3})$.) Let $1 \leq i \leq n-k-1$ and assume that the assertion is true for $i-1$.

Let $S_{d-3} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}$. If $x_{i+k-1} \notin S_{d-3}$ then $F_{i-1}(S_{d-3})$ exists by the induction hypothesis. Since

$$F_{i-1}(S_{d-3}) \cap \{x_{i-1}, \ldots, x_{i+k}\} = \{x_{i-1}, x_i\} \cup S_{d-3} \cup \{x_{i+k-1}, x_{i+k}\},$$

the existence of $F_i(S_{d-3})$ follows from 10. with $j = 1$. Let $x_{i+k-1} \in S_{d-3}$. Since S_{d-3} is paired, there is a largest odd integer j such that $3 \leq j \leq d-2$ and $x_{i+k-j} \notin S_{d-3}$. Then

$$S_{d-3} = S_{d-j-2} \cup \{x_{i+k-j+1}, \ldots, x_{i+k-1}\}$$

with

$$S_{d-j-2} = S_{d-3} \cap \{x_{i+2}, \ldots, x_{i+k-j-1}\},$$

and

$$S' = S_{d-j-2} \cup \{x_{i+k-j}, \ldots, x_{i+k-2}\}$$

is a paired set of cardinality $d-3$. Now, $F_{i-1}(S')$ exists by induction, and $F_i(S_{d-3})$ exists by 10. ∎

12. COROLLARY: *Let $V^0 = \{x_1, \dots, x_k\}$, $d \le k \le n-1$. Then*

$$[x_i, x_{i+1}, x_{i+k}, x_{i+k+1}] \in \mathcal{F}_2 \quad \text{for } i = 0, \dots, n-k-1.$$

Proof: Let $0 \le i \le n-k-1$, $S_{d-3} \subset \{x_{i+2}, \dots, x_{i+k-1}\}$ and $y_0 < y_1 < \dots < y_s$ be the vertex array of $F_i(S_{d-3})$. Then $F_i(S_{d-3}) \cap \{x_i, \dots, x_{i+k+1}\} = \{y_j, \dots, y_{j+d}\}$ for some $0 \le j \le s-d$, and

$$[x_i, x_{i+1}, x_{i+k}, x_{i+k+1}] = [y_j, y_{j+1}, y_{j+d-1}, y_{j+d}] \in \mathcal{F}_2$$

by 1.6. ∎

13. COROLLARY: *Let $V^0 = \{x_1, \dots, x_k\}$, $d \le k \le n$. Then*

$$V^* = \{x_{n-k}, \dots, x_{n-1}\}.$$

Proof: As we have already noted, $V^* = \{x_{n-l}, \dots, x_{n-1}\}$ for some $d \le l \le n$. If $k = n$ then $[x_0, x_n] \in \mathcal{F}_1$, and $n = l$.

Let $k \le n-1$ and consider $S_{d-3} = \{x_{n-d+2}, \dots, x_{n-2}\} \subset \{x_{n-k+1}, \dots, x_{n-2}\}$. By 11., $F_{n-k-1}(S_{d-3})$ exists and

$$F_{n-k-1}(S_{d-3}) \cap \{x_{n-k-1}, \dots, x_n\} = \{x_{n-k-1}, x_{n-k}\} \cup S_{d-3} \cup \{x_{n-1}, x_n\}.$$

By 1.5, $[x_{n-k}, x_n] \in \mathcal{F}_1$. Thus $n - l \le n - k$ and $k \le l$. Now by reversing the vertex array, $l \le k$. ∎

Since $|V^0| = |V^*| = k$ for some $d \le k \le n$, we call k the **characteristic** of P and write $k = \operatorname{char} P$.

For $i = 0, \dots, n-d+1$, let

$$\mathcal{F}^i = \{F \in \mathcal{F} \mid x_i \in F \cap V \subseteq \{x_i, \dots, x_n\}\}.$$

Since $|F \cap V| \ge d$ for any $F \in \mathcal{F}$, we have that $\mathcal{F} = \bigcup_{i=0}^{n-d+1} \mathcal{F}^i$. Finally, let

$$\tilde{\mathcal{F}} = \{F_i(S_{d-3}) \mid S_{d-3} \subset \{x_{i+2}, \dots, x_{i+k-1}\} \text{ and } i = 0, \dots, n-k-1\}$$

when $k \le n-1$, and set $\tilde{\mathcal{F}} = \emptyset$ otherwise.

As noted in the introduction, Lemma 11 will yield all the facets of P not containing x_0 or x_n. This next Lemma will enable us to prove it.

14. LEMMA: *Let $k = \operatorname{char} P$, $F \in \mathcal{F}$ with the vertex array $y_0 < y_1 < \cdots < y_s$ and $\{y_0, y_1\} = \{x_i, x_{i+1}\}$. Then $y_{d-3} \le x_{i+k-2}$.*

Proof: If $i = 0$ then $y_{d-1} \le x_k$ by 8.2, and the assertion follows. Let $i \ge 1$ and assume that if $\tilde{F} \in \mathcal{F}$ with the vertex array $z_0 < z_1 < \cdots < z_t$ and $\{z_0, z_1\} = \{x_{i-1}, x_i\}$ then $z_{d-3} \le x_{i+k-3}$.

Since $y_0 \ne x_0$ and $F \cap V$ is a Gale set, we have that $\{y_0, \ldots, y_{d-2}\}$ is a paired set and either $y_{d-1} = x_n$ or $s \ge d$ and $\{y_{d-1}, y_d\}$ is a paired set.

If $y_{d-1} = x_n$ then $F \in \mathcal{F}^*$. Now 9.2 implies that $S_{d-1} = \{x_i, x_{i+1}, y_0, \ldots, y_{d-2}\} \subset \{x_{n-k}, \ldots, x_{n-1}\}$ and $F = F_n(S_{d-1})$. Hence, $x_{n-k} \le x_i$ and $y_{d-3} \le x_{n-2} = x_{(n-k)+(k-2)} \le x_{i+k-2}$. Let $s \ge d$ and, say,

$$\{y_{d-3}, y_{d-2}, y_{d-1}, y_d\} = \{x_j, x_{j+1}, x_l, x_{l+1}\}.$$

We note that

$$G' = [y_0, y_2, \ldots, y_{d-1}] = [x_i, y_2, \ldots, y_{d-2}, x_l] \in \mathcal{F}_{d-2}.$$

Let $F' \in \mathcal{F}$ with the vertex array $w_0 < w_1 < \cdots < w_r$ such that $F' \cap F = G'$. Since $F' \cap \{x_{i+1}, x_{l+1}\} = \emptyset$, we have that $\{x_{i-1}, x_{l-1}\} \subset F'$. Then $f_0(G') = d-1 \le 2d-5$ and 1.4 yield that $x_l = y_{d-1} = w_r$, and so $x_{l-1} = w_{r-1}$. From $f_0(G') = d-1$ and 1.3,

$$\text{either } G' = [w_{r-d+2}, \ldots, w_r] \quad \text{or } G' = [w_{r-d+1}, \ldots, w_{r-2}, w_r].$$

In case of the former, y_1 and y_d are separated by the $d-2$ (odd) vertices $y_2, \ldots, y_{d-1}$ of F'. Hence,

$$[y_0, y_2, \ldots, y_{d-1}] = [w_{r-d+1}, \ldots, w_{r-2}, w_r]$$

and

$$\{w_{r-d}, \ldots, w_r\} = \{x_{i-1}, x_i, y_2, \ldots, y_{d-2}, x_{l-1}, x_l\}.$$

Accordingly,

$$\begin{aligned}\tilde{G} &= [w_{r-d}, \ldots, w_{r-3}, w_{r-1}, w_r] \\ &= [x_{i-1}, x_i, y_2, \ldots, y_{d-3}, x_{l-1}, x_l] \in \mathcal{F}_{d-2}.\end{aligned}$$

Let $\tilde{F} \in \mathcal{F}$ with the vertex array $z_0 < z_1 < \cdots < z_t$ such that $\tilde{F} \cap F' = \tilde{G}$. Since $f_0(\tilde{G}) = d \le 2d-5$, it follows from 1.4 that there is exactly one vertex z of $\tilde{F}$ such that $z \notin \tilde{G}$ and $x_{i-1} \le z \le x_l$, and $z_0 = x_{i-1}$ or $z_t = x_l$. Since

$$\{x_{i-1}, x_i, y_2, \ldots, y_{d-4}, x_{l-1}, x_l\}$$

is a paired set, $x_j = y_{d-3}$ and $x_{j+1} = y_{d-2} \notin \tilde{F}$, it follows that $x_{j-1} \in \tilde{F}$ and $x_i < z \leq x_{j-1}$.

If $z_0 = x_{i-1}$ then $\{x_{l-1}, x_l\} = \{z_{d-1}, z_d\}$. Thus $z \leq x_{j-1}$ implies that $x_j = y_{d-3} = z_{d-2}$, and $x_{j-1} = z_{d-3}$. By the induction, $x_{j-1} \leq x_{i+k-3}$ and so $x_j \leq x_{i+k-2}$.

Let $z_t = x_l$. Then

$$\{x_{i-1}, x_i\} = \{z_{t-d}, z_{t-d+1}\} \quad \text{and} \quad \{x_{j-1}, x_j, x_{l-1}\} = \{z_{t-3}, z_{t-2}, z_{t-1}\}.$$

We note that

$$\tilde{G} = [z_{u-d+2}, \ldots, z_{u-1}, z_{u+1}, \ldots, z_{u+d-2}]$$

for some $t-d+2 \leq u \leq t-3$. Now $u \leq t-3$ implies that $u-d+2 < t-d$. Therefore $z_{u-d+2} < z_{t-d}$ and our convention yield that $z_{t-d} = z_0$. Since $z_0 = x_{i-1}$, $y_{d-3} = x_j \leq x_{i+k-2}$ from above. ∎

15. LEMMA: *Let P be an ordinary d-polytope with the vertex array $x_0 < x_1 < \cdots < x_n$ and the characteristic k, $d = 2m+1 \geq 5$. Then*

$$\mathcal{F} = F^0 \cup \tilde{\mathcal{F}} \cup \mathcal{F}^*.$$

Proof: Let $F \in \mathcal{F}$ with the vertex array $y_0 < y_1 < \cdots < y_s$. We may assume that $x_0 < y_0$. Then

$$S^0_{d-1} = \{y_0, \ldots, y_{d-2}\}$$

is a paired set with, say, $\{y_0, y_1\} = \{x_i, x_{i+1}\}$ and $\{y_{d-3}, y_{d-2}\} = \{x_j, x_{j+1}\}$. By 14., $j \leq i+k-2$; that is,

$$S^0_{d-1} \subseteq \{x_i, \ldots, x_{i+k-1}\} \cap \{x_{(j-k)+2}, \ldots, x_{j+1}\}.$$

Then

$$S^1_{d-3} = \{y_0, \ldots, y_{d-4}\} \subset \{x_{(j-k)+2}, \ldots, x_{j-1}\}$$

and

$$S^2_{d-3} = \{y_2, \ldots, y_{d-2}\} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}.$$

We note that $x_{j+1} = y_{d-2} \leq x_{n-1}$ and $G = [y_0, \ldots, y_{d-2}] \in \mathcal{F}_{d-2}$.

If $i \geq n-k$ then $S^0_{d-1} \subset \{x_{n-k}, \ldots, x_{n-1}\}$ and 9.2 imply that $G \subset F_n(S^0_{d-1})$. If $i \leq n-k-1$ then $S^2_{d-3} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}$ and 11. yield that $G \subset F_i(S^2_{d-3})$.

If $j \leq k-1$ then $S^0_{d-1} \subset \{x_1, \ldots, x_k\}$ and 8.2 imply that $G \subset F_0(S^0_{d-1})$. If $j \geq k$ then $S^1_{d-3} \subset \{x_{(j-k)+2}, \ldots, x_{(j-k)+k-1}\}$, $0 \leq j-k \leq n-k-2$ and 11. yield that $G \subset F_{j-k}(S^1_{d-3})$.

Since G is the intersection of exactly two facets of P, it follows that $F \in \tilde{\mathcal{F}} \cup \mathcal{F}^*$.

∎

We can now list all the facets of P and it remains only to describe them in terms of their vertices. To that end, we use the decomposition

$$\mathcal{F} = \bigcup_{i=0}^{n-d+1} \mathcal{F}^i.$$

THEOREM B: *Let P be an ordinary d-polytope with the vertex array $x_0 < x_1 < \cdots < x_n$ and the characteristic k, $d = 2m+1 \geq 5$. Then*

$$f_{d-1}(P) = 2\binom{k-m}{m} + (n-k)\binom{k-m-2}{m-1}$$

and, with $\{y_{i+1}, \ldots, y_{i+j}\}$ denoting a paired set of cardinality j, the following are the facets of P.

B1. *For $j = d-2, \ldots, k-2$ and $\{y_1, \ldots, y_{d-3}\} \subset \{x_1, \ldots, x_{j-1}\}$,*

$$[x_0, y_1, \ldots, y_{d-3}, x_j, x_{j+1}].$$

B2. *For $r = 0, \ldots, m-2$ and $\{y_{2r+1}, \ldots, y_{d-3}\} \subset \{x_{2r+2}, \ldots, x_{k-2}\}$,*

$$[x_0, \ldots, x_{2r}, y_{2r+1}, \ldots, y_{d-3}, x_{k-1}, \ldots, x_{k+2r}]$$

and

$$[x_0, \ldots, x_{d-3}, x_{k-1}, \ldots, x_{k+d-3}].$$

B3. *For $i = 0, \ldots, n-k-1$, $r = 0, \ldots, m-2$,*

$$\{y_{2r+2}, \ldots, y_{d-2}\} \subset \{x_{i+2r+3}, \ldots, x_{i+k-1}\}$$

and $y_{d-2} \neq x_{k+i-1}$ for $i > 0$,

$$[x_i, \ldots, x_{i+2r+1}, y_{2r+2}, \ldots, y_{d-2}, x_{k+i}, \ldots, x_{k+i+2r+1}]$$

and

$$[x_i, \ldots, x_{i+d-2}, x_{k+i}, \ldots, x_{k+i+d-2}].$$

B4 *For* $\{y_1, \dots, y_{d-3}\} \subset \{x_{n-k+2}, \dots, x_{n-1}\}$ *and,* $y_{d-3} \neq x_{n-1}$ *if* $k < n$,

$$[x_{n-k}, x_{n-k+1}, y_1, \dots, y_{d-3}, x_n].$$

B5 *For* $j = d-2, \dots, k-2$ *and* $\{y_1, \dots, y_{d-3}\} \subset \{x_{n-j+1}, \dots, x_{n-1}\}$,

$$[x_{n-j-1}, x_{n-j}, y_1, \dots, y_{d-3}, x_n].$$

Proof B1: Let $d-2 \leq j \leq k-2$ and $S_{d-3} = \{y_1, \dots, y_{d-3}\} \subset \{x_1, \dots, x_{j-1}\}$. Set

$$S_{d-1} = S_{d-3} \cup \{x_j, x_{j+1}\} \text{ and } S^*_{d-3} = S_{d-1} \backslash \{y_1, y_2\}.$$

From $S_{d-1} \subset \{x_1, \dots, x_{k-1}\}$ and 8.2,

$$F_0(S_{d-1}) \cap \{x_0, \dots, x_k\} = \{x_0\} \cup S_{d-1} = \{x_0, y_1, \dots, y_{d-3}, x_j, x_{j+1}\}.$$

Let $x_0 < y_1 < \cdots < y_s$ be the vertex array of $F_0(S_{d-1})$. Then $y_{d-1} = x_{j+1}$, and we need to show that $s = d-1$. By 3., we may assume that $k < n$. Since 2.1 (with $r = 1$) implies that if $y_1 \neq x_1$ then $s < 1 + d - 1$, we may assume also that $y_1 = x_1$. Then $y_2 = x_2$ and $S^*_{d-3} \subset \{x_3, \dots, x_k\}$. If $k < n-1$ then from 11.,

$$F_1(S^*_{d-3}) \cap \{x_1, \dots, x_{k+2}\} = \{x_1, x_2\} \cup S^*_{d-3} \cup \{x_{k+1}, x_{k+2}\} = S_{d-1} \cup \{x_{k+1}, x_{k+2}\}.$$

Let $z_0 < z_1 < \cdots < z_t$ be the vertex array of $F_1(S^*_{d-3})$. We recall that $[x_0, x_1, x_k, x_{k+1}] \in \mathcal{F}_2$ by 12. Therefore, $x_k \notin F_1(S^*_{d-3})$ implies that $x_0 \notin F_1(S^*_{d-3})$, and $\{z_0, \dots, z_{d-2}\} = S_{d-1}$. Since

$$G = [z_0, \dots, z_{d-2}] = [y_1, y_2, \dots, y_{d-3}, x_j, x_{j+1}]$$

is a $(d-2)$-face of P such that $G \subset F_0(S_{d-1})$, $f_0(G) = d-1$ and $x_0 \notin G$, it follows from 1.3 that $y_s \in G$; that is, $y_{d-1} = x_{j+1} = y_s$.

If $k = n-1$ then we need only that $x_n \notin F_0(S_{d-1})$. This is immediate since $[x_0, x_1, x_{n-1}, x_n] \in \mathcal{F}_2$, $\{x_0, x_1\} \subset F_0(S_{d-1})$ and $x_{n-1} \notin F_0(S_{d-1})$.

B2. Let $0 \leq r \leq m-2$, $\{y_{2r+1}, \dots, y_{d-3}\} \subset \{x_{2r+2}, \dots, x_{k-2}\}$ and $S^r_{d-1} \subset \{x_1, \dots, x_k\}$ such that

$$\{x_0\} \cup S^r_{d-1} = \{x_0, \dots, x_{2r}, y_{2r+1}, \dots, y_{d-3}, x_{k-1}, x_k\}.$$

From 8.2,

$$F_0(S^r_{d-1}) \cap \{x_0, \dots, x_k\} = \{x_0\} \cup S^r_{d-1}.$$

We may assume that $k < n$. Then $\{x_0, x_k\} \subset F_0(S^r_{d-1})$ and 12. yield that for $j = 1, \ldots, n-k-1$,

$$x_j \in F_0(S^r_{d-1}) \text{ if and only if } x_{j+k} \in F_0(S^r_{d-1}).$$

Thus, $\{x_0, \ldots, x_{2r}\} \subset F_0(S^r_{d-1})$ implies that

$$F_0(S^r_{d-1}) \cap \{x_0, \ldots, x_{k+2r}\} = \{x_0, \ldots, x_{2r}, y_{2r+1}, \ldots, y_{d-3}, x_{k-1}, \ldots, x_{k+2r}\}.$$

Let $y_0 < y_1 < \cdots < y_s$ be the vertex array of $F_0(S^r_{d-1})$. Then $x_{k+2r} = y_{d+2r-1}$, and $y_{2r+1} \neq x_{2r+1}$ and 2.1 imply that $s < 2r+1+d-1 = d+2r$.

We observe that with $S_{d-1} = \{x_1, \ldots, x_{d-3}, x_{k-1}, x_k\}$, 8.2 yields that

$$F_0(S_{d-1}) \cap \{x_0, \ldots, x_k\} = \{x_0, \ldots, x_{d-3}, x_{k-1}, x_k\}.$$

Now, we argue as above and obtain that

$$F_0(S_{d-1}) = [x_0, \ldots, x_{d-3}, x_{k-1}, \ldots, x_{k+d-3}].$$

We may think of this facet as the $r = m-1$ case.

B3. Let $0 \leq i \leq n-k-1$, $0 \leq r \leq m-2$,

$$\{y_{2r+2}, \ldots, y_{d-2}\} \subset \{x_{i+2r+3}, \ldots, x_{i+k-1}\}$$

and $S^r_{d-3} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}$ such that

$$\{x_i, x_{i+1}\} \cup S^r_{d-3} = \{x_i, \ldots, x_{i+2r+1}, y_{2r+2}, \ldots, y_{d-2}\}.$$

From 11.,

$$F_i(S^r_{d-3}) \cap \{x_i, \ldots, x_{i+k+1}\} = \{x_i, \ldots, x_{i+2r+1}, y_{2r+2}, \ldots, y_{d-2}, x_{k+i}, x_{k+i+1}\}.$$

Since $\{x_i, \ldots, x_{i+2r+1}, x_{k+i}\} \subset F_i(S^r_{d-3})$ and $x_{i+2r+2} \notin F_i(S^r_{d-3})$, we apply 12. and 2.1 as above and obtain that

$$F_i(S^r_{d-3}) \cap \{x_i, \ldots, x_n\} = \{x_i, \ldots, x_{i+2r+1}, y_{2r+2}, \ldots, y_{d-2}, x_{k+i}, \ldots, x_{k+i+2r+1}\}.$$

We may now assume that $i > 0$. As we are describing here the facets with the initial vertex x_i, it is an easy consequence of 2.2 and 12. that $y_{d-2} \neq x_{k+i-1}$ if and only if

$$F_i(S^r_{d-3}) = [x_i, \ldots, x_{i+2r+1}, y_{2r+2}, \ldots, y_{d-2}, x_{k+i}, \ldots, x_{k+i+2r+1}].$$

With $S_{d-3} = \{x_{i+2}, \ldots, x_{i+d-2}\}$, we have that

$$F_i(S_{d-3}) \cap \{x_i, \ldots, x_{i+k+1}\} = \{x_i, \ldots, x_{i+d-2}, x_{k+i}, x_{k+i+1}\}$$

for $0 \leq i \leq n-k-1$. Noting that $x_{i+d-2} \neq x_{k+i-1}$ and arguing as above, we obtain that

$$F_i(S_{d-3}) = [x_i, \ldots, x_{i+d-2}, x_{k+i}, \ldots, x_{k+i+d-2}].$$

Again, we may think of this as the $r = m-1$ case.

B4. Let $S_{d-3} = \{y_1, \ldots, y_{d-3}\} \subset \{x_{n-k+2}, \ldots, x_{n-1}\}$. Then

$$S_{d-1} = \{x_{n-k}, x_{n-k+1}\} \cup S_{d-3} \subset \{x_{n-k}, \ldots, x_{n-1}\},$$

and from 9.2,

$$F_n(S_{d-1}) \cap \{x_{n-k}, \ldots, x_n\} = \{x_{n-k}, x_{n-k+1}, y_1, \ldots, y_{d-3}, x_n\}.$$

Now if $k < n$, we obtain from 2.2 and 12. that

$$F_n(S_{d-1}) = [x_{n-k}, x_{n-k+1}, y_1, \ldots, y_{d-3}, x_n]$$

if and only if $y_{d-3} \neq x_{n-1}$.

B5. Apply B1 with the reverse vertex array.

Now, let $F \in \mathcal{F}^i$, $0 \leq i \leq n-d+1$, have the vertex array $y_0 < y_1 < \cdots < y_s$. If $i = 0$ then by 8.2, either $\{y_1, \ldots, y_{d-1}\}$ is a paired subset of $\{x_1, \ldots, x_k\}$ [type B1 or B2] or $\{y_0, y_1, y_{d-1}\} = \{x_0, x_1, x_k\}$ and $\{y_2, \ldots, y_{d-2}\}$ is a paired subset of $\{x_2, \ldots, x_{k-1}\}$ [type B3 ($k < n$) or B4 ($k = n$)]. If $1 \leq i \leq n-k-1$ then $\{y_0, \ldots, y_{d-2}\}$ is a paired set and by 14., $\{y_2, \ldots, y_{d-2}\} \subset \{x_{i+2}, \ldots, x_{i+k-1}\}$ [type B3]. If $0 < n-k \leq i \leq n-d+1$ then 9.2 yields that F is type B4 or B5.

Finally, we note that $d-3 = 2(m-1)$ and recall that

$$\sum_{i=u}^{v} \binom{i}{u} = \binom{v+1}{u+1}.$$

Clearly, there are

$$2 \sum_{j=d-2}^{k-2} p(m-1, j-1) = 2 \sum_{j=d-2}^{k-2} \binom{j-m}{m-1}$$

facets in B1 and B5. Since each facet in B2 is determined by an $S_{d-3} \subset \{x_1, \ldots, x_{k-2}\}$, there are $p(m-1, k-2) = \binom{k-m-1}{m-1}$ of them.

Let $k = n$. Then each facet in B4 is determined by an $S_{d-3} \subset \{x_2, \ldots, x_{n-1}\}$ and there are $p(m-1, n-2) = \binom{n-m-1}{m-1}$ of them. Thus, in this case,

$$f_{d-1}(P) = 2\left(\sum_{j=d-2}^{n-2}\binom{j-m}{m-1} + \binom{n-m-1}{m-1}\right) = 2\sum_{j=d-2}^{n-1}\binom{j-m}{m-1}$$
$$= 2\sum_{i=d-2-m}^{n-m-1}\binom{i}{m-1} = 2\sum_{i=m-1}^{n-m-1}\binom{i}{m-1}$$
$$= 2\binom{n-m}{m}.$$

Let $k < n$. Considering B3, each facet in $\mathcal{F}^0(\mathcal{F}^i,\ 1 \le i \le n-k-1)$ is determined by an $S_{d-3} \subset \{x_2, \ldots, x_{k-1}\}(\{x_{i+2}, \ldots, x_{i+k-2}\})$, and there are

$$p(m-1, k-2) + (n-k-1)p(m-1, k-3) = \binom{k-m-1}{m-1} + (n-k-1)\binom{k-m-2}{m-1}$$

of them. In B4, each facet is determined by an $S_{d-3} \subset \{x_{n-k+2}, \ldots, x_{n-2}\}$ and there are $p(m-1, k-3) = \binom{k-m-2}{m-1}$ of them. Therefore,

$$f_{d-1}(P) = 2\left(\sum_{j=d-2}^{k-2}\binom{j-m}{m-1} + \binom{k-m-1}{m-1}\right) + (n-k)\binom{k-m-2}{m-1}$$
$$= 2\binom{k-m}{m} + (n-k)\binom{k-m-2}{m-1}. \quad \blacksquare$$

16. Corollary: *Let P be an ordinary d-polytope with the vertex array $x_0 < x_1 < \cdots < x_n$ and the characteristic k, $d = 2m+1 \ge 5$.*

16.1 *If $k = n$ then P is cyclic with the same vertex array.*

16.2 *If $k = d$ then $f_{d-1}(P) = n+1$ and the $(d-1)$-faces of P are $[x_0, x_1, \ldots, x_{d-1}]$, $[x_{n-d+1}, \ldots, x_{n-1}, x_n]$ and $[x_{i-d+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+d-1}]$ for $i = 1, \ldots, n-1$.*

Proof: 1. It is immediate that if $k = n$ then $\mathcal{F} = F^0 \cup F^*$ and P is simplicial. Theorem B or Lemmas 8.2 and 9.2 now yield that any d element Gale set of V is the set of vertices of a facet of P.

2. Let $k = d$. The assertion is trivial if $n = d$, and

$$f_{d-1}(P) = 2\binom{m+1}{m} + (n-d)\binom{m-1}{m-1} = 2m + 2 + n - d = n + 1.$$

Let $F_i = [x_{i-d+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+d-1}]$ for $i = 1, \ldots, n-1$, and assume that $d < n$.

From B1 and B5, we obtain $[x_0, \ldots, x_{d-1}]$ and $[x_{n-d+1}, \ldots, x_1]$, respectively. B2 yields F_i for odd $i = 1, \ldots, d-2$. B3 yields F_i for even $i = 2, \ldots, d-1$, and $i = d, \ldots, n-2$. Finally, B4 yields F_{n-1}. ∎

4. Remarks and examples

It is clear that although we can describe ordinary $(2m+1)$-polytopes, further study is needed to really understand them. For example, 16.2 is a surprising result that hints of something special about ordinary $(2m+1)$-polytopes with characteristic $2m+1$, $m \geq 2$. Also, while the present definition of an ordinary d-polytope is a reasonable one because it recognizes the parity of d, it does not indicate in any way how to obtain non-cyclic ordinary $2m$-polytopes. Is there a better definition of ordinary $(2m+1)$-polytopes? This relates of course to the problem of a second definition of an ordinary d-polytope that yields cyclic $(2m+1)$-polytopes and non-trivial $2m$-polytopes.

Next, the difference between the theory of ordinary 3-polytopes and that of those of higher dimension. From [1], we note that if P is an ordinary 3-polytope with $f_0(P) = n+1$ and char $P = k$ then

$$\left[\frac{n}{2}\right] + k \leq f_2(P) \leq n + k - 2 = 2\binom{k-1}{1} + (n-k)\binom{k-3}{0}.$$

Thus, P is not combinatorially unique. It is somewhat surprising that already an ordinary 5-polytope with $n+1$ vertices and characteristic k is combinatorially unique.

Finally, we refer to [1] for examples of ordinary 3-polytopes. Below, we present two examples of higher dimensional ones. In each case, the polytope is d-dimensional with the vertex array $x_0 < x_1 < \cdots < x_n$ and the characteristic k, $d = 2m+1$. We specify the polytope by (n, k, d) and denote the facets using the subscripts of the x_i's. We list the facets via Theorem B.

Example 1: $(n, k, d) = (7, 6, 5)$ *and* $f_4 = 14$.

B1: $[0,1,2,3,4], [0,1,2,4,5], [0,2,3,4,5]$;

B2: $[0,2,3,5,6], [0,3,4,5,6], [0,1,2,5,6,7]$;

B3: $[0,1,3,4,6,7], [0,1,4,5,6,7], [0,1,2,3,6,7]$;

B4: $[1,2,3,4,7], [1,2,4,5,7]$;

B5: $[2,3,4,5,7], [2,3,5,6,7], [3,4,5,6,7]$.

Example 2: $(n, k, d) = (10, 8, 7)$ *and* $f_6 = 26$.

B1: $[0,1,2,3,4,5,6], [0,1,2,3,4,6,7], [0,1,2,4,5,6,7], [0,2,3,4,5,6,7]$;

B2: $[0,2,3,4,5,7,8], [0,2,3,5,6,7,8], [0,3,4,5,6,7,8]$,
$[0,1,2,4,5,7,8,9,10], [0,1,2,5,6,7,8,9,10], [0,1,2,3,4,7,8,9,10]$;

B3: $[0,1,3,4,5,6,8,9], [0,1,3,4,6,7,8,9], [0,1,4,5,6,7,8,9]$,
$[0,1,2,3,5,6,8,9,10], [0,1,2,3,6,7,8,9,10]$,
$[1,2,4,5,6,7,9,10], [1,2,3,4,6,7,9,10]$,
$[0,1,2,3,4,5,8,9,10], [1,2,3,4,5,6,9,10]$;

B4: $[2,3,4,5,6,7,10], [2,3,4,5,7,8,10], [2,3,5,6,7,8,10]$;

B5: $[3,4,5,6,7,8,10], [3,4,5,6,8,9,10], [3,4,6,7,8,9,10], [4,5,6,7,8,9,10]$.

References

[1] T. Bisztriczky, *Ordinary 3-polytopes*, Geometriae Dedicata **52** (1994), 129–142.

[2] D. Gale, *Neighborly and cyclic polytopes*, Proceedings of Symposia in Pure Mathematics **7** (convexity) (1963), 225–232.

[3] H.W. Gould, *Combinatorial Identities*, Morgantown Printing, W. Virginia, 1972.

[4] B. Grünbaum, *Convex Polytopes*, Interscience, New York, 1967.

[5] G. C. Shephard, *A theorem on cyclic polytopes*, Israel Journal of Mathematics **6** (1968), 368–372.