Geometriae Dedicata **52**: 129–142, 1994.

Ordinary 3-Polytopes

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(Received: 26 February 1993; revised version: 2 November 1993)

Abstract. We introduce a class of three-dimensional polytopes P with the property that there is a total ordering of the vertices of P that determines completely the facial structure of P. This class contains the cyclic 3-polytopes.

Mathematics Subject Classifications (1991): 52A15, 52B10, 52B12.

Let $C(v, d)$ denote a cyclic d-polytope with v vertices in E^d, $d \geq 3$. We recall that $C(v, d)$ is combinatorially equivalent to the convex hull of v points on the moment curve, or on any curve of order d, in E^d. The importance of $C(v, d)$ is well known and it is due to the fact that there is a vertex array (a total ordering of vertices) of $C(v, d)$ that determines completely the facial structure of $C(v, d)$. It is our belief that there are other classes of a-polytopes, induced by curves in E^d, with a vertex array that is instrumental in determining their facial structure.

Presently, we verify this conjecture for $d = 3$.

As the first step in the introduction of this new class of 3-polytopes, we present an overview of our motivations, definitions and main results.

In Section 1, we describe the class of oriented ordinary spherical space curves (cf. Figure 1) and show that if we choose vertices on such a curve in a particular manner then the facets of the resultant 3-polytope satisfy a global and a local condition ((01) and (02)) that can be expressed solely in terms of the order of appearance of the vertices on the curve. With this observation in mind, we define an ordinary 3-polytope as one with a vertex array such that its facets satisfy (01) and (02). Except for the notations at the beginning and the definition at the end, the reader may choose to skip this section.

The central concept in understanding and describing an ordinary 3-polytope P with the vertex array $x_0 < x_1 < \cdots < x_n$, $n \geq 3$, is its characteristic. Specifically, the characteristic of P is an integer $k(k = \text{char } P)$, where $3 \leq k \leq n$, and x_0 and x_i determine an edge of P iff $1 \leq i \leq k$ iff x_n and x_{n-i} determine an edge of P. The introduction of char P is the subject of Lemmas 6, 7 and 8, and requires the description of the vertex figures of P at x_0 (Lemma 7) and x_n (Lemma 8), and a set of facets of P which do not contain x_0 or x_n. In Lemmas 9 and 10, we determine which of the facets above may be equal and which must be distinct. These are the last results required to describe P.

Theorem 11 states that, in fact, we have all the facets of P and that the number $f_2(P)$ of facets of P increases as $k = \text{char } P$ increases. In particular, P is cyclic

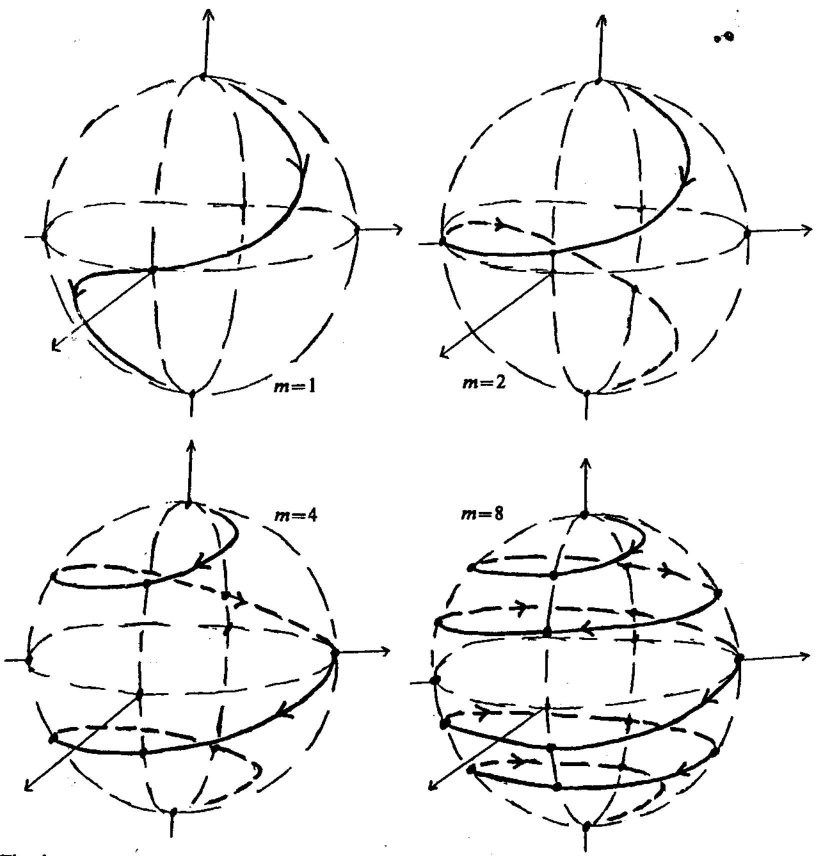


Fig. 1.

if $k = n$ (Theorem 13), and a P with maximum $f_2(P)$ 'looks' more cyclic as k approaches n (cf. Theorem 12 and Figures 4 and 5).

Finally, we determine a lower bound for $f_2(P)$ in Theorem 14, and show that if the characteristic of P is minimum ($k = 3$) then there is a P with the least number of facets of any 3-polytope with $n+1$ vertices (cf. Theorem 15 and Figures 2, 3, 4 and 7). Thus, we introduce a class of 3-polytopes with the unexpected property that for a fixed number of vertices, the polytope with the maximum number of facets and a polytope with the minimum number of facets are in the class.

1. The Curves

Let $Y \subset E^3$. Then conv Y and aff Y denote, respectively, the convex hull and the affine hull of Y. If $Y = \{y_1, \ldots, y_n\}$, we set

$$[y_1, \ldots, y_n] = \text{conv } Y \quad \text{and} \quad \langle y_1, \ldots, y_n \rangle = \text{aff } Y.$$

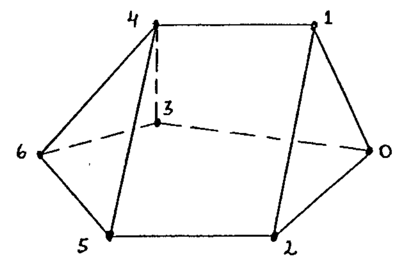


Fig. 2. (n=6)

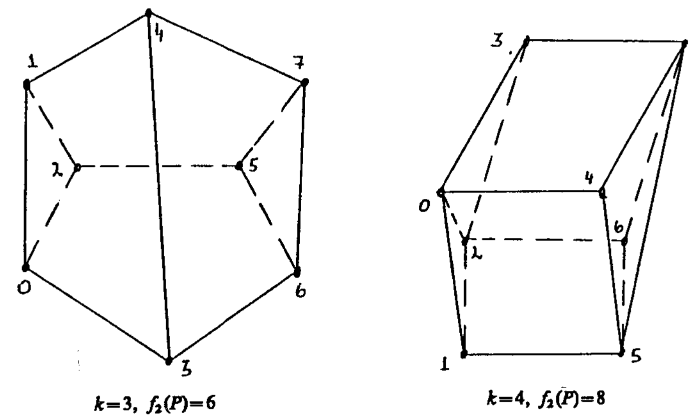


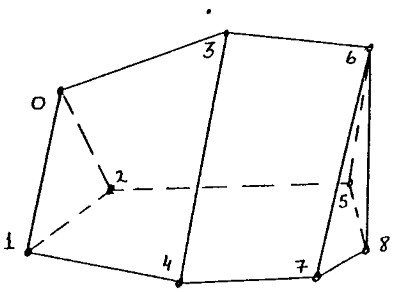
Fig. 3. (n=7)

Thus, as usual, $[y_1, y_2]$ is the closed segment with endpoints y_1 and y_2. We set $(y_1, y_2) = [y_1, y_2] \backslash \{y_1, y_2\}$.

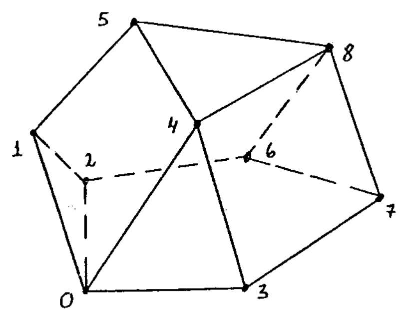
Let $I \subset E^1$ be an open interval and let $S \subset E^3$ be a sphere of positive radius. Let $\Gamma: I \to S$ be a simple finite C^∞ curve; that is, Γ is injective and any plane intersects $\Gamma(I)$ at a finite number of points. For convenience, we identify Γ and $\Gamma(I)$. For $r < t$ in I, we set $\Gamma[r, t] = \Gamma([r, t])$ and $\Gamma(r, t) = \Gamma((r, t))$.

Let $s \in I$ and $U \subset I$ be an open neighbourhood of s. We say that $\Gamma(U)$ is of *order* k if k is the maximum number of coplanar points of $\Gamma(U)$. Clearly, $k \geq 3$. We say that $\Gamma(s)$ is *ordinary* if there is an open neighbourhood $U \subset I$ of s such that $\Gamma(U)$ is of order three, and that Γ is *ordinary* if each of its points is ordinary. Finally, let H be a plane through $\Gamma(s)$. Then $|H \cap \Gamma| < \infty$ implies that either there is an open neighbourhood $U \subset I$ of s such that $\Gamma(U)$ lies on one side of H or not. In case of the former [latter], we say that H *supports* [*cuts*] Γ at $\Gamma(s)$.

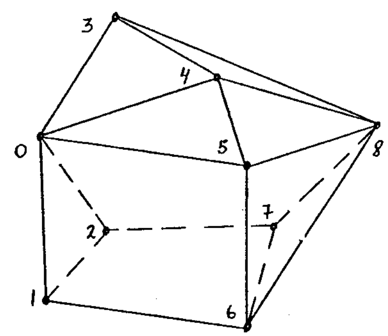
Henceforth, we assume that $\Gamma: I \to S$ is a simple, finite, ordinary C^∞ curve. From p. 169 of [1], we cite the property of such a Γ that we require for this study.



$k=3,\ f_2(P)=7$



$k=4,\ f_2(P)=8$



$k=5,\ f_2(P)=10$

Fig. 4. $(n=8)$

LEMMA 1. *Let* $r < s < t$ *in* I. *Then* $\langle \Gamma(r), \Gamma(s), \Gamma(t)\rangle$ *is a plane that cuts* Γ *at* $\Gamma(s)$.

LEMMA 2. *Let* $r < s < t < u$ *in* I *such that* $H = \langle \Gamma(r), \Gamma(s), \Gamma(t), \Gamma(u)\rangle$ *is a plane and* $H \cap \Gamma(s, t) = \emptyset$. *Then* $(\Gamma(r), \Gamma(u)) \cap (\Gamma(s), \Gamma(t)) \neq \emptyset$.

Proof. Let $A = [\Gamma(r), \Gamma(s), \Gamma(t), \Gamma(u)]$. Since Γ is spherical, A is a convex 4-gon.

If $(\Gamma(r), \Gamma(u)) \cap (\Gamma(s), \Gamma(t)) = \emptyset$ then $[\Gamma(r), \Gamma(u)]$ is an edge of A, and $\Gamma(s)$ and $\Gamma(t)$ are on the same side of $\langle \Gamma(r), \Gamma(u)\rangle$ in H. Since $H \cap \Gamma[s, t] = \{\Gamma(s), \Gamma(t)\}$, H supports $B = \mathrm{conv}(\Gamma[s, t])$. Since $\Gamma(s)$ and $\Gamma(t)$ are on the same side of $\langle \Gamma(r), \Gamma(u)\rangle$, $\langle \Gamma(r), \Gamma(u)\rangle \cap B = \emptyset$. Thus there is a plane $H' \neq H$ through $\langle \Gamma(r), \Gamma(u)\rangle$ that supports B. Since H' necessarily supports Γ at $\Gamma(s')$ for some $s < s' < t$, we have a contradiction by 1. □

LEMMA 3. *Let* $Y = \{y_1, \ldots, y_m\} \subset \Gamma$ *such that* $y_i = \Gamma(r_i)$, $r_1 < r_2 \ldots < r_m$ *in* I *and* $m \geq 4$. *If* $H = \langle y_1, \ldots, y_m\rangle$ *is a plane and* $H \cap \Gamma[r_1, \ldots, r_m] = Y$ *then* $[y_1, \ldots, y_m]$ *is a convex m-gon with the edges* $[y_1, y_2]$, $[y_{m-1}, y_m]$ *and* $[y_i, y_{i+2}]$, $i = 1, \ldots, m-2$.

Proof. For $i = 1, \ldots, m-3$, we apply 2 with $r_i < r_{i+1} < r_{i+2} < r_{i+3}$. □

Let $n \geq 3$, $s_0 < s_1 < \cdots < s_n$ in I, $z_i = \Gamma(s_i)$, $V = \{z_0, \ldots, z_n\}$ and $Q = \mathrm{conv}\ V$. Since Γ is simple and spherical, Q is a 3-polytope and $V = \mathrm{ext}\ Q$. We set $z_i < z_j$ if $s_i < s_j$ in I, and call $z_0 < z_1 < \cdots < z_n$ *a vertex array* of Q. If we reverse this ordering on V then $z_n < z_{n-1} < \cdots < z_0$ is a reverse vertex array of Q.

We note that if Γ is of order three then (cf. [2] and [3]) Q is a cyclic 3-polytope and the vertex array $z_0 < \cdots < z_n$ satisfies *Gale's Evenness Condition*: A set V' of three points of V determines a facet of Q if and only if every two points of $V \backslash V'$ are separated in the vertex array by an even number of points of V'. Thus

$$\{[z_0, z_i, z_{i+1}] | i = 1, \ldots, n-1\} \cup \{[z_j, z_{j+1}, z_n] | j = 0, 1, \ldots, n-2\}$$

is the set of facets of Q.

If Γ is not of order 3 then, of course, we do not expect that $z_0 < \cdots < z_n$ satisfies Gale's Evenness Condition. We do, however, obtain the necessary part of the condition for a certain type of facet of Q.

LEMMA 4. *Let* F *be a facet of* Q *such that* $(\mathrm{aff}\ F) \cap \Gamma[s_0, s_n] = F \cap V$ *and* $\mathrm{aff}\ F$ *cuts* $\Gamma(s_0, s_n)$ *at each point of intersection. Then every two points of* $V \backslash F$ *are separated in* $z_0 < \cdots < z_n$ *by an even number of points of* $F \cap V$.

Proof. Let $y = \Gamma(r) \neq \Gamma(t) = w$ in $V \backslash F$, $y < w$. Since $H = \mathrm{aff}\ F$ supports Q, y and w lie in the same open half-space determined by H. Since $H \cap \Gamma(r, t) \subset F \cap V$ and H cuts $\Gamma(r, t)$ at each point of intersection, it follows that H cuts, and meets, $\Gamma(r, t)$ at an even number of points. □

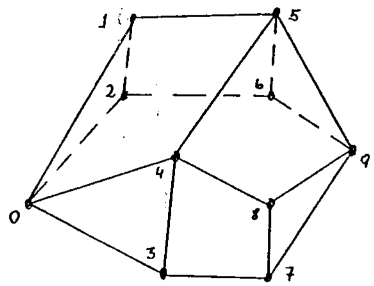
We note that if, in Lemma 4, $(\mathrm{aff}\ F) \cap \Gamma[s_0, s_n] = \{y_1, \ldots, y_m\}$ where $y_1 < y_2 < \cdots < y_m$, then $\mathrm{aff}\ F$ cuts Γ at y_i for $i = 2, \ldots, m-1$ by 1.

In summary, if Q has the property that for each facet F of Q, $(\mathrm{aff}\ F) \cap \Gamma[s_0, s_n] = F \cap V$ and $\mathrm{aff}\ F$ cuts $\Gamma(s_0, s_n)$ at each point of intersection then Q with the vertex array $z_0 < \cdots < z_n$ satisfies 3 and 4.

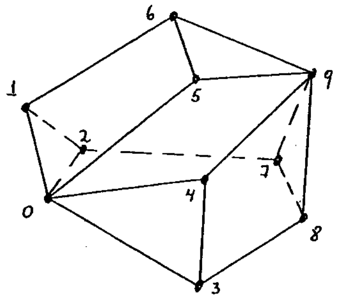
Let P be a 3-polytope with $V = \mathrm{ext}\ P = \{x_0, x_1, \ldots, x_n\}$, $n \geq 3$. We say that P is *ordinary* if there is a vertex array, say, $x_0 < \cdots < x_n$ such that for each facet F of P:

(01) every two points of $V \backslash F$ are separated in $x_0 < \cdots < x_n$ by an even number of points of $F \cap V$, and

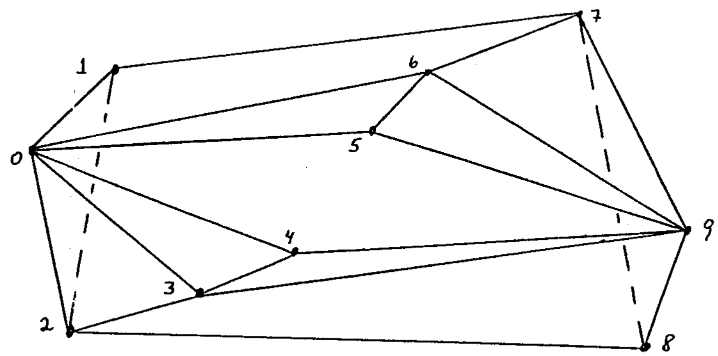
(02) if $F \cap V = \{y_1, \ldots, y_m\}$ where $y_1 < y_2 < \cdots < y_m$ then F is a convex m-gon with the edges $[y_1, y_2]$, $[y_{m-1}, y_m]$ and $[y_i, y_{i+2}]$; $i = 1, \ldots, m-2$.



$k=4,\ f_2(P)=9$



$k=5,\ f_2(P)=9$



$k=6,\ f_2(P)=12$

Fig. 5. ($n=9$)

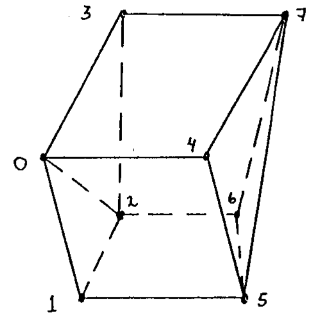
We note that if P is ordinary with $x_0 < \cdots < x_n$ then it is also ordinary with $x_n < \cdots < x_0$.

2. The Polytopes

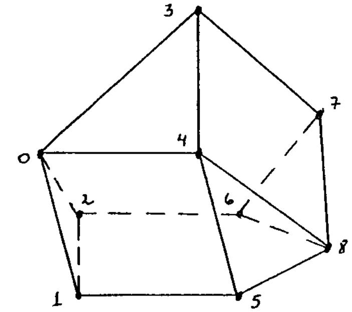
In this section, we assume that P is an ordinary 3-polytope with $V = \operatorname{ext} P = \{x_0, x_1, \ldots, x_n\}$, $n \geq 3$, and the vertex array $x_0 < \cdots < x_n$ satisfying (01) and (02).

We denote by $\mathcal{E}$ or $\mathcal{E}(P)[\mathcal{F}$ or $\mathcal{F}(P)]$, the set of edges [facets] of P. As usual, $f_1(P) = |\mathcal{E}(P)|$ and $f_2(P) = |\mathcal{F}(P)|$. Next, we say that $x_i \neq x_j$ in V are *adjacent* if $[x_i, x_j] \in \mathcal{E}$. For $i = 0, 1, \ldots, n$, we set

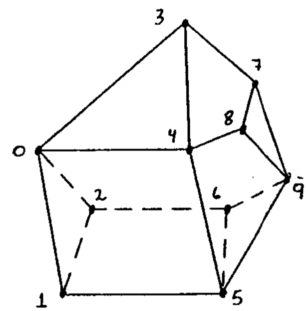
$$\mathcal{F}_i = \{F \in \mathcal{F} | x_i \in F\} \quad \text{and} \quad V_i = \{x_j \in V | [x_i, x_j] \in \mathcal{E}\}.$$



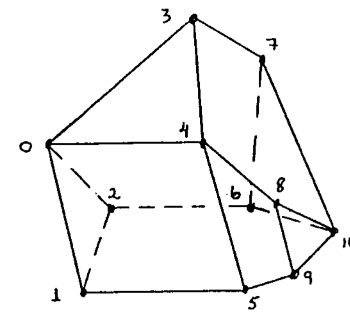
$k=7,\ f_2(P)=8$



$n=8,\ f_2(P)=8$



$n=9,\ f_2(P)=9$



$n=10,\ f_2(P)=9$

Fig. 6. ($k=4$)

Finally, for $i = 0, 1, \ldots, n-1$, we set $L_i = [x_i, x_{i+1}]$. We recall that $[x_i, x_j] \in \mathcal{E}$ if and only if $|\mathcal{F}_i \cap \mathcal{F}_j| = 2$.

LEMMA 5.1 *If* $F \in \mathcal{F}_i$ *and* $1 \leq i \leq n-1$ *then* $F \cap \{x_{i-1}, x_{i+1}\} \neq \emptyset$.

2. *If* $F \in \mathcal{F}$ *contains* $\{x_0, x_1, x_2\}$ *or* $\{x_{n-2}, x_{n-1}, x_n\}$ *then* $|F \cap V| = 3$.

3. *If* $L_i \notin \mathcal{E}$ *and* $1 \leq i \leq n-2$ *then* $\{L_{i-1}, L_{i+1}\} \subset \mathcal{E}$.

Proof. The first assertion follows from (01).

Let $\{x_0, x_1, x_2\} \subset F \in \mathcal{F}$. If there is a smallest $i > 2$ such that $x_i \in F$ then $[x_1, x_i] \in \mathcal{E}$ by (02). Let $\mathcal{F}_1 \cap \mathcal{F}_i = \{F, G\}$. Then $F \cap G = [x_1, x_i]$ and $G \cap \{x_0, x_2\} = \emptyset$; a contradiction by 5.1. We argue similarly if $\{x_{n-2}, x_{n-1}, x_n\} \subset F$.

Let $L_i \notin \mathcal{E}$ for some $1 \leq i \leq n-2$. Then $|\mathcal{F}_i \cap \mathcal{F}_{i+1}| \leq 1$. Thus $|\mathcal{F}_i| \geq 3$ and 5.1 yield that $|\mathcal{F}_i \cap \mathcal{F}_{i-1}| \geq 2$ and hence, $|\mathcal{F}_i \cap \mathcal{F}_{i-1}| = 2$. Similarly, $|\mathcal{F}_{i+1} \cap \mathcal{F}_{i+2}| = 2$. □

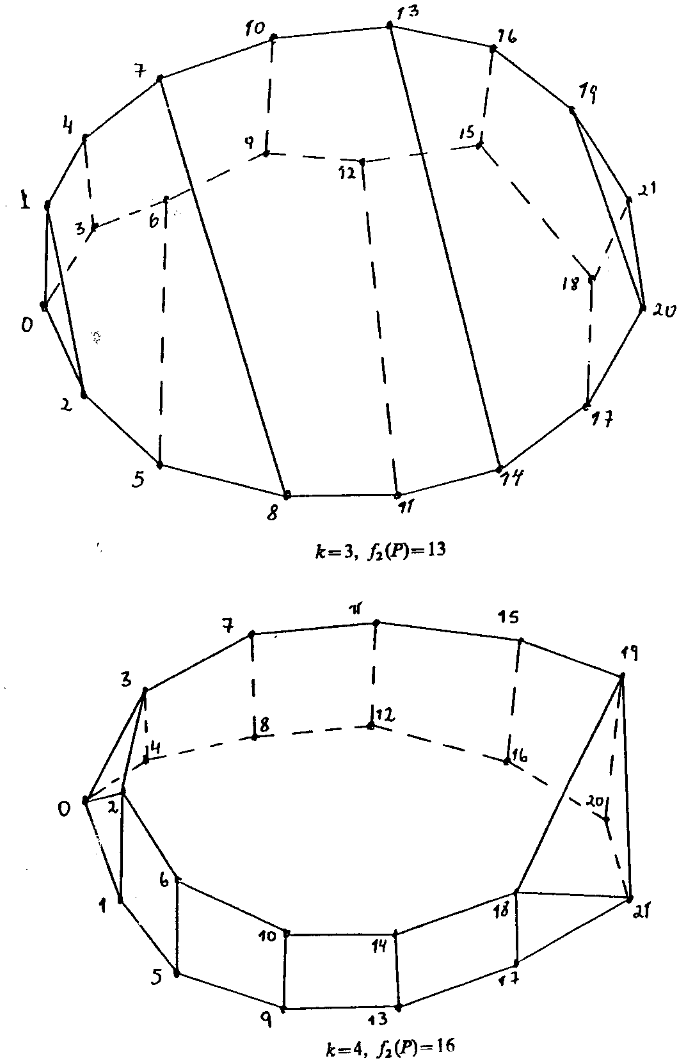


Fig. 7. (n=21)

LEMMA 6. *There are integers k and m such that $3 \leq k,\ m \leq n$, $V_0 = \{x_1, x_2, \ldots, x_k\}$ and $V_n = \{x_{n-m}, \ldots, x_{n-2}, x_{n-1}\}$.*

Proof. We verify that $V_0 = \{x_1, x_2, \ldots, x_k\}$ for some $k \geq 3$ by showing that if $i \geq 2$ and $x_i \in V_0$ then $x_{i-1} \in V_0$.

Clearly, $|V| \geq 3$ and $x_i \in V_0$ for some $i \geq 2$. Then $[x_0, x_i] \in \mathcal{E}$ and $[x_0, x_i] = F \cap G$ for some F and G in $\mathcal{F}$. By (02), this is possible only if

$$|F \cap \{x_0, x_1, \ldots, x_i\}| \leq 3 \quad \text{and} \quad |G \cap \{x_0, x_1, \ldots, x_i\}| \leq 3.$$

If $i = n$ then $|F \cap V| = |G \cap V| = 3$ by the above and $\{F, G\} = \{[x_0, x_1, x_n], [x_0, x_{n-1}, x_n]\}$ by (01). Thus $[x_0, x_{n-1}] \in \mathcal{E}$. Let $i \leq n-1$. Then $F \cap G \cap V = \{x_0, x_i\}$ and 5.1 yield that, say, $x_{i-1} \in F$ and $x_{i+1} \in G$. Thus $F \cap \{x_0, x_1, \ldots, x_i\} = \{x_0, x_{i-1}, x_i\}$ by the above, and $[x_0, x_{i-1}] \in \mathcal{E}$ by (02).

A similar argument with the reverse vertex array yields:

$$V_n = \{x_{n-m}, \ldots, x_{n-2}, x_{n-1}\} \quad \text{for some } m \geq 3. \qquad \square$$

LEMMA 7. *Let $V_0 = \{x_1, x_2, \ldots, x_k\}$, $3 \leq k \leq n$.*

1. *$|\mathcal{F}_0| = k$ and $\mathcal{F}_0 = \{F_1^0, \ldots, F_k^0\}$ where $[x_0, x_i, x_{i+1}] \subseteq F_i^o$ for $i = 1, \ldots, k-1$, $[x_0, x_1, x_k] \subseteq F_k^0$, $F_1^0 = [x_0, x_1, x_2]$ and either $k = n$ and $F_n^0 = [x_0, x_1, x_n]$ or $[x_0, x_1, x_k, x_{k+1}] \subseteq F_k^0$.*
2. *If $k \leq n-1$ then for $j = 0, 1, \ldots, n-1-k$, $[x_j, x_{j+k}]$ and $[x_{j+1}, x_{j+1+k}]$ are edges of a facet G_j of P.*

Proof. We note first that each $F \in \mathcal{F}_0$ contains exactly two edges through x_0 and thus, $|F \cap V_0| = 2$.

Since there are k edges of P containing x_0; the vertex figure of P at x_0 is a convex k-gon. The k-gon has exactly k edges and thus, $|\mathcal{F}_0| = k$.

Let $F \in \mathcal{F}_0$ and $F \cap V_0 = \{x_i, x_j\}$ for some $1 \leq i < j \leq k$. If $i > 1$ then $x_{i-1} \notin F$ and 5.1 yield that $x_j = x_{i+1}$. Let $i = 1$. Then $2 < j < k$ is not possible by 5.1 and so, either $x_j = x_2$ or $x_j = x_k$. If $F \cap V_0 = \{x_i, x_{i+1}\}$ for some $1 \leq i \leq k-1$, we denote F by F_i^0. If $F \cap V_0 = \{x_1, x_k\}$, we denote F by F_k^0. Since $|\mathcal{F}_0| = k$, it follows that $\mathcal{F}_0 = \{F_1^0, \ldots, F_k^0\}$. We note that $F_1^0 = [x_0, x_1, x_2]$ by 5.2, and that if $k = n$ then $F_n^0 \cap V = \{x_0, x_1, x_n\}$.

Let $k \leq n-1$. Since $F_k^0 \cap \{x_0, x_1, \ldots, x_k\} = \{x_0, x_1, x_k\}$ and $k \geq 3$, $x_{k+1} \in F_k^0$ by 5.1. Then $x_0 < x_1 < x_k < x_{k+1}$, and (02) yield that $[x_0, x_k]$ and $[x_1, x_{k+1}]$ are edges of F_k^0. Let $1 \leq j \leq n-1-k$ and assume that there is an $F \in \mathcal{F}$ with edges $[x_{j-1}, x_{j-1+k}]$ and $[x_j, x_{j+k}]$. Let $G \in \mathcal{F}$ such that $[x_j, x_{j+k}] = F \cap G$. Then $1 \leq j, j+k \leq n-1$, $G \cap \{x_{j-1}, x_{j+k-1}\} = \emptyset$ and 5.1 imply that $\{x_{j+1}, x_{j+k+1}\} \subset G$. Since $[x_j, x_{j+k}] \in \mathcal{E}$, it follows by (02) that

$$G \cap \{x_j, x_{j+1}, \ldots, x_{j+k}\} = \{x_j, x_{j+1}, x_{j+k}\}.$$

Thus, $x_{j+k+1} \in G$ and (02) yield that $[x_{j+1}, x_{j+k+1}] \in \mathcal{E}$. $\square$

LEMMA 8. *Let $V_0 = \{x_1, x_2, \ldots, x_k\}$, $3 \leq k \leq n$. Then $V_n = \{x_{n-k}, \ldots, x_{n-2}, x_{n-1}\}$, $|\mathcal{F}_n| = k$ and $\mathcal{F}_n = \{F_1^*, \ldots, F_k^*\}$ where $[x_{n-i-1}, x_{n-i}, x_n] \subseteq F_i^*$*

for $i = 1, \dots, k-1$, $[x_{n-k}, x_{n-1}, x_n] \subseteq F_k^*$, $F_1^* = [x_{n-2}, x_{n-1}, x_n]$ *and either* $k = n$ *and* $F_n^* = [x_0, x_{n-1}, x_n]$ *or* $[x_{n-k-1}, x_{n-k}, x_{n-1}, x_n] \subseteq F_k^*$.

Proof. By 6, $V_n = \{x_{n-m}, \dots, x_{n-2}, x_{n-1}\}$ for some $3 \leq m \leq n$. Clearly, we need only to show that $m = k$.

If $k = n$ then $[x_0, x_n] \in \mathcal{E}$ and $m = n$. Let $k \leq n-1$. From 7.2, $[x_{n-k}, x_n] = [x_{(n-1-k)+1}, x_{(n-1-k)+1+k}] \in \mathcal{E}$ and thus, $n-k \geq n-m$ and $k \leq m$. Now, with the reverse vertex array, $m \leq n-1$ implies that $m \leq k$. □

In view of 8, we say that P has *characteristic* k (char $P = k$) if $|V_0| = |V_n| = k$, $3 \leq k \leq n$.

Let char $P = k$. Then the following are (not necessarily distinct) facets of P: $F_1^0, \dots, F_k^0$, $F_1^*, \dots, F_k^*$ and when $k \leq n-1$, $G_0, G_1, \dots, G_{n-k-1}$. For consistency of notation, we set $\mathcal{F}^0 = \mathcal{F}_0$ and $\mathcal{F}^* = \mathcal{F}_n$. It is clear that $|\mathcal{F}^0 \cap \mathcal{F}^*| \leq 2$. If $k \leq n-1$ then

$$[x_0, x_1, x_k, x_{k+1}] \subseteq G_0 \cap F_k^0$$

and

$$[x_{n-k-1}, x_{n-k}, x_{n-1}, x_n] \subseteq G_{n-k-1} \cap F_k^*.$$

Thus $G_0 = F_k^*$ and $G_{n-k-1} = F_k^*$. When $k \leq n-3$, we set

$$\mathcal{G} = \{G_1, \dots, G_{n-k-2}\}.$$

We recall that $[x_j, x_{j+k}]$ and $[x_{j+1}, x_{k+j+1}]$ are edges of $G_j \in \mathcal{G}$. Clearly, if L_j and L_{j+k} are also edges of G_j then G_j is distinct from each facet in $\mathcal{F}^0 \cup \mathcal{F}^* \cup (\mathcal{G} \backslash \{G_j\})$. Accordingly, we determine when $L_i \in \mathcal{E}$ for $i = 0, 1, \dots, n-1$. From 7 and 8, $\{L_0, L_1, L_{n-2}, L_{n-1}\} \subset \mathcal{E}$. We set

$$\mathcal{L} = \{L_2, \dots, L_{n-3}\}.$$

LEMMA 9. *Let* char $P = k$ *and* $L_i \in \mathcal{L} \cup \{L_1, L_{n-2}\}$.

1. *If* $i \leq \min\{k, n-k-2\}$ *then* $L_i \subseteq F_i^0 \cap G_i$.
2. *If* $k+1 \leq i \leq n-k-2$ *then* $L_i \subseteq G_{i-k} \cap G_i$.
3. *If* $n-k-1 \leq i \leq k$ *then* $L_i \subseteq F_i^0 \cap F_{n-i-1}^*$.
4. *If* $i \geq \max\{k+1, n-k-1\}$ *then* $L_i \subseteq G_{i-k} \cap F_{n-i-1}^*$.

Furthermore, $L_i \in \mathcal{E}$ *if and only if the two denoted facets are distinct.*

Proof. We note that

$$L_i = [x_{(i-k)+k}, x_{(i-k)+k+1}] = [x_{n-(n-i-1)-1}, x_{n-(n-i-1)}]$$

and thus, 9.1 to 9.4 readily follow from 7 and 8. Next, if $L_i \notin \mathcal{E}$ then L_i is contained in at most one facet of P.

Finally, let $F \in \mathcal{F}$ contain $L_i \in \mathcal{E}$. Then by (02), either $F \cap V \subseteq \{x_0, \dots, x_{i-1}\}$ or $F \cap V \subseteq \{x_i, \dots, x_n\}$. It is now easy to check that, in each of 9.1 to 9.4, the two denoted facets are distinct when $L_i \in \mathcal{E}$. □

From 9, we obtain that if $L_i \in \mathcal{L} \backslash \mathcal{E}$ then certain facets in $\mathcal{F}^0 \cup \mathcal{F}^* \cup \mathcal{G}$ are equal. We now investigate which facets in $\mathcal{F}^0 \cup \mathcal{F}^* \cup \mathcal{G}$ may be equal and which are necessarily distinct.

LEMMA 10. *Let* char $P = k$, $3 \leq k \leq n$.

1. *If* $G_i = G_j$ *then* $j \equiv i \pmod{k}$; *moreover, if* $j = i + lk$ *and* $l \geq 1$ *then* $G_i = G_{i+k} = \dots = G_{i+lk} = G_j$.
2. *If* $F_i^0 = F_j^*$ *then* $j + i \equiv n-1 \pmod{k}$; *moreover, if* $n-1 = j+i+lk$ *and* $l \geq 1$ *then* $F_i^0 = G_i = \dots = G_{i+(l-1)k} = F_j^*$.

Proof. Let $G = G_i = G_j$, $1 \leq i < j \leq n-k-2$. Then

$$\{x_i, x_{i+1}, x_{i+k}, x_{i+k+1}, x_j, x_{j+1}, x_{j+k}, x_{j+k+1}\} \subset G$$

by 7.2. Since $x_i < x_j < x_{j+1} < x_{j+k}$, it follows from (02) that $L_j \notin \mathcal{E}$.

Let $j \leq k$. Then $L_j \notin \mathcal{E}$ and 9.1 imply that $G = F_j^0$ and $x_0 \in G$. Since $x_0 < x_i < x_{i+1} < x_{i+k}$, we also obtain that $L_i \notin \mathcal{E}$ and hence, $G = F_i^0$. Since $F_i^0 = F_j^0$ and $|\{F_1^0, \dots, F_k^0\}| = k$, $i = j$. This is a contradiction and so, $j \geq k+1$. Then $L_j \notin \mathcal{E}$ and 9.2 imply that $G = G_{j-k}$. Now arguing as above, we obtain that either $j-k \leq k$ and $j-k = i$ or $j-k \geq k+1$ and $G = G_{j-k} = G_{j-2k}$. 13.1 now follows readily.

Let $F = F_i^0 = F_j^*$, $2 \leq i, j \leq k$. Let $k = n$. Then $[x_0, x_n] \in \mathcal{E}$ and $|\mathcal{F}^0 \cap \mathcal{F}^*| = 2$. From 7 and 8,

$$[x_0, x_{n-1}, x_n] \subseteq F_{n-1}^0 \cap F_n^* \quad \text{and} \quad [x_0, x_1, x_n] \subseteq F_n^0 \cap F_{n-1}^*.$$

Thus (i, j) is either $(n-1, n)$ or $(n, n-1)$. Let $k = n-1$. Then $|\mathcal{F}^0 \cap \mathcal{F}^*| \leq 1$ and from 7 and 8,

$$[x_0, x_1, x_{n-1}, x_n] \subseteq F_{n-1}^0 \cap F_{n-1}^*.$$

Thus $F_{n-1}^0 = F_{n-1}^*$ and $(i, j) = (n-1, n-1)$. Let $k \leq n-2$. Then

$$\{x_0, x_i, x_{i+1}, x_{n-j-1}, x_{n-j}, x_n\} \subset F$$

and by (02), $\{L_i, L_{n-j-1}\} \cap \mathcal{E} = \emptyset$.

If $i \geq n-k-1$ then $L_i \notin \mathcal{E}$ and 9.3 imply that $F_i^0 = F_{n-i-1}^*$. Thus $F_j^* = F_{n-i-1}^*$ and $j = n-i-1$. We note that $n-k-1 \leq n-j-1$. Hence if $n-j-1 \leq k$ then $L_{n-j-1} \notin \mathcal{E}$ and 9.3 imply that $F_{n-j-1}^0 = F_{n-(n-j-1)-1}^* = F_j^*$. Thus $F_i^0 = F_{n-j-1}^0$ and again $i = n-j-1$.

Let $i \leq n-k-2$ and $n-j-1 \geq k+1$. Then $F_i^0 = G_i$ by 9.1, and $G_{(n-j-1)-k} = F^*_{n-(n-j-1)-1} = F_j^*$ by 9.4. Since $G_i = G_{(n-j-1)-k}$, it follows from 10.1 that $i \equiv n-j-1 (\bmod k)$, $n-j-1 = i + lk$ for some $l \geq 1$ and

$$F_i^0 = G_i = \cdots = G_{i+(l-1)k} = G_{(n-j-1)-k} = F_j^*. \qquad \square$$

We are now ready to describe ordinary 3-polytopes. We continue with the introduced terminology and remark that for a real number b, $[b]$ denotes the largest integer equal to or less than b.

THEOREM 11. *Let P be an ordinary 3-polytope with the vertex array $x_0 < \cdots < x_n$ and the characteristic k. Then*

1. $\mathcal{F}(P) = \{F_1^0, \ldots, F_k^0, F_k^*, \ldots, F_1^*\}$ *for* $k \geq n-2$,
2. $\mathcal{F}(P) = \{F_1^0, \ldots, F_k^0, G_1, \ldots, G_{n-k-2}, F_k^*, \ldots, F_1^*\}$ *for* $k \leq n-3$,
3. $f_2(P) \leq n+k-2$ *and*
4. $f_2(P) = n+k-2$ *for* $k \geq n-1$.

Proof. Let $F \in \mathcal{F} \backslash (\mathcal{F}^0 \cup \mathcal{F}^*)$. Then there is a smallest i, $1 \leq i \leq n-2$, such that $x_i \in F$. We note that $x_{i+1} \in F$ by 5.1, and $L_i \in \mathcal{E}$ by (02). If $i \leq n-k-2$ then $k \leq n-3$ and $F \in \{G_{i-k}, G_i\}$ by 9.1 or 9.2. If $i \geq n-k-1$ then $i \geq k+1$ by 9.3. Thus $n-3 \geq k$ and $F = G_{i-k}$ by 9.4.

We recall from the proof of 10 that if $k = n[n-1]$ then $|\mathcal{F}^0 \cap \mathcal{F}^*| = 2[1]$. Now, 11.3 and 11.4 readily follow from 11.1 and 11.2. $\square$

THEOREM 12. *Let P be an ordinary 3-polytope with the vertex array $x_0 < \cdots < x_n$, the characteristic k and $f_2(P) = n+k-2$. Then*

1. $F_i^0 = [x_0, x_i, x_{i+1}]$ *and* $F_i^* = [x_{n-i-1}, x_{n-i}, x_n]$ *for* $i = 1, \ldots, k-1$,
2. $F_k^0 = [x_0, x_i, x_k, x_{k+1}]$ *and* $F_k^* = [x_{n-k-1}, x_{n-k}, x_{n-1}, x_n]$ *when* $k \leq n-1$ *and*
3. $G_j = [x_j, x_{j+1}, x_{j+k}, x_{j+k+1}]$ *when* $k \leq n-3$ *and* $j = 1, \ldots, n-k-2$.

Proof. We note that $f_2(P) = n+k-2$ and 9 yield that $\mathcal{L} \subset \mathcal{E}$, and that the descriptions of the facets readily follow from 7, 8 and $\mathcal{L} \subset \mathcal{E}$. $\square$

With the remark that if char $P = n$ then $f_2(P) = 2n-2$, $F_n^0 = F_{n-1}^* = [x_0, x_1, x_n]$ and $F_n^* = F_{n-1}^0 = [x_0, x_{n-1}, x_n]$, and if char $P = n-1$ then $f_2(P) = 2n-3$ and $F_{n-1}^0 = F_{n-1}^* = [x_0, x_1, x_{n-1}, x_n]$, we have from 12 a complete description of ordinary 3-polytopes with maximal number of facets.

Next, we recall that the starting point for the development of this theory was the cyclic 3-polytope. We now elaborate on this relationship.

THEOREM 13. *Let P be a 3-polytope with vertices $x_0, x_1, \ldots, x_n$. Then P is cyclic with $x_0 < x_1 < \cdots < x_n$ if and only if P is ordinary with $x_0 < x_1 < \cdots < x_n$ and* char $P = n$.

Proof. If P is cyclic with $x_0 < x_1 < \cdots < x_n$ then

$$\mathcal{F}(P) = \{[x_0, x_i, x_{i+1}] | i = 1, \ldots, n-1\} \cup \{[x_i, x_{i+1}, x_n] | i = 0, \ldots, n-2\}$$

by Gale's Evenness Condition. Clearly, P satisfies (01) and (02), and $[x_0, x_n]$ is an edge of P.

If P is ordinary with $x_0 < x_1 < \cdots < x_n$ and char $P = n$ then

$$\mathcal{F}(P) = \{F_i^0 = [x_o, x_i, x_{i+1}] | i = 1, \ldots, n-1\} \cup \{F_i^* = [x_{n-i-1}, x_{n-i}, x_n] | i = 1, \ldots, n-1\}$$

by 11 and 12, and Gale's Evenness Condition is satisfied. $\square$

Next, we consider P with the vertex array $x_0 < \cdots < x_n$, the characteristic $k \leq n-2$ and $f_2(P) < n+k-2$.

If $k = n-2$ then $f_2(P) = 2n-5$ by 11.1 and $|\mathcal{F}^0 \cap \mathcal{F}^*| \leq 1$, and $F_i^0 = F_{n-i-1}^*$ for some $2 \leq i \leq n-2$ by 10.2.

Let $k \leq n-3$. Then

$$\mathcal{F}(P) = \{F_1^0, \ldots, F_k^0, G_1, \ldots, G_{n-k-2}, F_k^0, \ldots, F_1^*\}$$

and some of the facets are equal under the restrictions of $F_1^0 = [x_0, x_1, x_2]$, $F_1^* = [x_{n-2}, x_{n-1}, x_n]$, $|\mathcal{F}^0| = |\mathcal{F}^*| = k$ and $|\mathcal{F}^0 \cap \mathcal{F}^*| \leq 1$. It is now a matter of applying 10, 9 and 5.3 to determine which facets may be equal, and of applying 12 to describe the identified facets in terms of the vertices. For small n, it is an easy exercise to determine all ordinary 3-polytopes with $n+1$ vertices. In the general case, we restrict our attention to P with a maximum number of identified facets.

THEOREM 14. *Let P be an ordinary 3-polytope with the vertex array $x_0 < \cdots < x_n$ and this characteristic $k \leq n-3$. Then $f_2(P) \geq k + [n/2]$.*

Proof. We recall that $\mathcal{L} = \{L_2, \ldots, L_{n-3}\}$. By 9 and 10, all possible identifications of facets are related to $\mathcal{L} \backslash \mathcal{E}$; that is,

$$f_2(P) = (n+k-2) - |\mathcal{L} \backslash \mathcal{E}|.$$

Since $\{L_1, L_{n-2}\} \subset \mathcal{E}$, it follows from 5.3 that $|\mathcal{L} \backslash \mathcal{E}| \leq [(n-3)/2]$. Thus,

$$f_2(P) \geq (n+k-2) - \left[\frac{n-3}{2}\right] = k + \left[\frac{n}{2}\right]. \qquad \square$$

We note that if $k = 3$ then $3 + [n/2] = [((n+1)+5)/2]$ is the greatest lower bound for the number of facets of any 3-polytope with $n+1$ vertices; cf. p. 184 of [3]. In the article, we present some examples of P with $k = 3$ and $f_2(P) = 3 + [n/2]$.

THEOREM 15. *Let P be an ordinary 3-polytope with the vertex array $x_0 < \cdots < x_n$, the characteristic 3 and $\mathcal{L}\backslash\mathcal{E} = \{L_{2i} | i = 1, \ldots, m\}$ where $m = [(n-3)/2] \geq 3$. Then $f_2(P) = 3 + [n/2]$ with the following identifications: $F_2^0 = G_2$, $G_{2i-3} = G_{2i}$ for $i = 2, \ldots, m-1$ and $G_{2m-3} = F^*_{n-2m-1}$.*

Proof. Apply 9.1, 9.2 and 9.4. □

From 13, 15 and the examples, it follows that in the class of 3-polytopes with $n+1 \geq 6$ vertices, there is an ordinary one with the maximum number of facets, and an ordinary one with the minimum number of facets.

Finally, we observe that if k is large compared to n then $k + [n/2]$ is certainly not the greatest lower bound for $f_2(P)$. In particular, we can show that if $n = k+4 \geq 9$ or $k+5 \leq n \leq 2k-1$ (n even) then $f_2(P) \geq 2k-1$, and if $n = k+3 \geq 7$ or $7 \leq n = k+4 \leq 8$ or $k+5 \leq n \leq 2k-1$ (n odd) then $f_2(P) \geq 2k$.

3. Remarks and Examples

Our rationale for the definition of an ordinary 3-polytope is based upon some properties of a simple, finite, ordinary C^∞ curve $\Gamma: I \to S \subset E^3$. We note that in fact it is sufficient to assume that $\Gamma: I \to E^3$ is a simple, finite regular C^∞ curve which is *convex*, namely, $|L \cap \Gamma| \leq 2$ for any line $L \subset E^3$ and $\Gamma \subset \mathrm{bd}(\mathrm{conv}\,\Gamma)$. Clearly, it is easy to visualize spherical curves, and regular polytopes have been already defined.

Let $I = (o, \pi)$, $m \in \mathbf{Z}^+$ and $\Gamma^m: I \to E^3$ be defined by

$$\Gamma^m(t) = (\cos(mt)\sin(t),\ \sin(mt)\sin(t),\ \cos(t)).$$

Then Γ^m is a simple, finite C^∞ spherical curve. It is tedious but not too difficult to check that Γ^m is ordinary. In Figure 1, we depict Γ^m for $m = 1, 2, 4$ and 8.

Next, we recall that unlike the definition of a cyclic polytope, the definition of an ordinary 3-polytope is not in terms of an ordinary curve Γ. This approach permits us to avoid the difficult task of verifying that for large n, there exist $s_0 < \cdots < s_n$ in I such that for each facet F of $Q = \mathrm{conv}\{\Gamma(s_0), \ldots, \Gamma(s_n)\}$,

$$(\mathrm{aff}\ F) \cap \Gamma[s_0, s_n] = F \cap \{\Gamma(s_0), \ldots, \Gamma(s_n)\}$$

and aff F cuts $\Gamma(s_0, s_n)$ at each point of intersection.

Finally, in Figures 2 to 7, we present examples of ordinary 3-polytopes with the vertex array $x_0 < \cdots < x_n$ and the characteristic $k \leq n-3$. In each case, $f_2(P)$ is minimum for the type of polytope depicted. It is easy to check that P need not always be combinatorially unique.

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Geometriae Dedicata **52:** 143–153, 1994.

Semiregular Surfaces with a Single Triple-Point

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(Received: 1 March 1993)

Abstract. We exhibit 18 surfaces that can be mapped generically into 3-space with a single triple-point. The family should be used as a source of examples and counter-examples.

Mathematics Subject Classifications (1991): 57R45, 52B70.

There are two well-known surfaces which have a single triple-point. Steiner's surface is the image of a semiregular map and has six non-immersive points or crosscaps. Its domain is the projective plane $\mathbf{P}^2$. When given as the zero-set of the function

$$f(x, y, x) = x^2y^2 + y^2z^2 + z^2x^2 + xyz$$

the surface has tetrahedral symmetry and the singular set consists of an interval on each of the three axes. These intervals are lines of double-points which intersect in a triple-point at the origin and are terminated at each end by a crosscap. Boy's surface is an immersion of $\mathbf{P}^2$. Its singular set is a bouquet of three circles and we can assume that these also contain intervals of the coordinate axes connected by 270° circular arcs.

Here we exhibit a family of 18 surfaces with a single triple-point which includes the Steiner and Boy surfaces; one of the others (which we have designated 2_B) has been investigated by the second author and David Mond ([M], [M-M]). They are built from the basic ingredients in the Steiner and Boy surfaces: the singular set consists of an interval in each axis, and these three intervals are either terminated by crosscaps or connected to one another by arcs. The four cases are shown in Figure 1. Although none of our surfaces is given as the image of a map we shall refer to them as *semiregular* surfaces and to their abstract topological type as their *domain*. We also discuss transitions between the surfaces that can be achieved by anihilating pairs of crosscaps by a surgery. There is a unique path from the Steiner to the Boy surface. François Apéry described a one-parameter family of mappings of $\mathbf{P}^2$ in $\mathbf{R}^3$ which transforms the Steiner to the Boy surface ([A, p. 80]). The three hyperbolic confluences occur simultaneously as the crosscaps disappear.

1. Surfaces with Six Crosscaps

Since all our surfaces must contain a triple-point, we begin by constructing this. The vertices $\{a, b, c, d, e, f\}$ of a regular octahedron determine three squares that