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Face Numbers of Scarf Complexes*

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Abstract. Let A be a $(d+1) \times d$ real matrix whose row vectors positively span $\mathbb{R}^d$ and which is generic in the sense of Bárány and Scarf [BS1]. Such a matrix determines a certain infinite d-dimensional simplicial complex Σ, as described by Bárány et al. [BHS]. The group $\mathbb{Z}^d$ acts on Σ with finitely many orbits. Let f_i be the number of orbits of $(i+1)$-simplices of Σ. The sequence $f = (f_0, f_1, \dots, f_{d-1})$ is the f-vector of a certain triangulated $(d-1)$-ball T embedded in Σ. When A has integer entries it is also, as shown by the work of Peeva and Sturmfels [PS], the sequence of Betti numbers of the minimal free resolution of $k[x_1, \dots, x_{d+1}]/I$, where I is the lattice ideal determined by A.

In this paper we study relations among the numbers f_i. It is shown that $f_0, f_1, \dots, f_{\lfloor (d-3)/2 \rfloor}$ determine the other numbers via linear relations, and that there are additional nonlinear relations. In more precise (and more technical) terms, our analysis shows that f is linearly determined by a certain M-sequence $(g_0, g_1, \dots, g_{\lfloor (d-1)/2 \rfloor})$, namely, the g-vector of the $(d-2)$-sphere bounding T. Although T is in general not a cone over its boundary, it turns out that its f-vector behaves as if it were.

1. Introduction

A construction appearing in the work of Scarf and coauthors [Sc1], [BHS], [BS1] shows a way to associate with sufficiently generic real $(d+1) \times d$ matrices A a certain abstract simplicial complex Σ. This "big Scarf complex" is infinite, with $\mathbb{Z}^d$ as its set of vertices and with the group $\mathbb{Z}^d$ acting on it. It is however locally finite, and the choice of one of A's row vectors as priviliged determines a certain finite subcomplex T of the link of Σ at the origin. This "small Scarf complex" is a triangulation of a $(d-1)$-dimensional ball, and its faces are in bijection with the orbits of the $\mathbb{Z}^d$-action on Σ.

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Let f_i be the number of i-dimensional simplices of the complex T. It was empirically observed by Scarf that all the face numbers f_i seem to be determined by f_0 alone for $d = 3$ and $d = 4$, and by f_0 and f_1 when $d = 5$. This led him to ask [Sc2] whether it is true that in general $f_0, f_1, \dots, f_{\lfloor (d-3)/2 \rfloor}$ determine the other f_i-numbers. The main purpose of this paper is to prove that this is true. The proof will show that the face numbers of T are determined by the face numbers of its boundary. This boundary is a $(d-2)$-dimensional sphere, and relying on the Dehn–Sommerville relations among the face numbers of spheres we reach the conclusion. From a realization of the big Scarf complex Σ as a polyhedral surface, due to Bárány et al. [BHS], one can glean the information that the boundary of T is isomorphic to the boundary complex of a convex polytope. Via the work of Stanley [St1] this introduces further relations of an algebraic nature on the face numbers of T, namely, nonlinear inequalities of Macaulay type.

The construction of the Scarf complexes Σ and T is reviewed in Sections 2 and 3. Here for motivation we briefly mention the reasons for their study.

The complexes Σ were introduced by Scarf [Sc1] for purposes to do with integer programming. In fact, he defined such complexes for $n \times d$ real matrices A that are sufficiently generic. In this paper only the $n = d + 1$ case is considered. The relevance for integer programming is that the 1-skeleton of Σ provides a complete test set for integer programs of the form

$$\begin{cases} \text{minimize} & a_0 \cdot x \\ \text{subject to} & a_i \cdot x \le b_i, \qquad i = 1, \dots, d, \end{cases}$$

where $a_0, \dots, a_d$ are the row vectors of the matrix A. Namely, if a point $x_0 \in \mathbb{Z}^d$ is in the feasible region and if a local minimum is achieved at x_0 (meaning that no improvement of the objective function can be attained at any Σ-neighbor of x_0), then x_0 is a global minimum. Furthermore, the Σ-neighbors of x_0 in the direction of decreasing objective function are determined by the vertices of the small Scarf complex T. The higher-dimensional structure of Σ is very interesting mathematically; its meaning for integer programming is however more elusive, see, e.g., [SS] for a result in this direction.

Scarf complexes have recently become of interest also in commutative algebra, due to the work of Bayer, Peeva, and Sturmfels [BPS], [BS2], [PS] on free resolutions. Since the numbers f_i studied in this paper have algebraic meaning in that setting we want to outline the connection. This is done only for the case of a sufficiently generic $(d+1) \times d$ integer matrix A, although their work is more general.

For such a matrix A, let $\mathcal{L} \stackrel{\text{def}}{=} \{A \cdot y \mid y \in \mathbb{Z}^d\}$. Then $\mathcal{L}$ is a sublattice of $\mathbb{Z}^{d+1}$ with which we associate the ideal

$$I_{\mathcal{L}} \stackrel{\text{def}}{=} \langle \mathbf{x}^a - \mathbf{x}^b \mid a, b \in \mathbb{N}^{d+1} \text{ and } a - b \in \mathcal{L} \rangle$$

in the polynomial ring $S \stackrel{\text{def}}{=} k[\mathbf{x}] = k[x_1, \dots, x_{d+1}]$. In [PS] a minimal free resolution

$$0 \to S^{f_{d-1}} \to S^{f_{d-2}} \to \cdots \to S^{f_0} \to S \to S/I_{\mathcal{L}} \to 0$$

is constructed, where f_i denotes the face numbers of the small Scarf complex T, as previously discussed. Thus, the face numbers of T give the Betti numbers of the ring $S/I_{\mathcal{L}}$ in the generic case. In the nongeneric case it is shown in [PS] that f_i gives a lower bound for the ith Betti number of the ring.

2. The Big Complex Σ

In this section we review some definitions and general properties that are needed. Full details about this material can be found in [BHS] and [BS1].

Let A be a $(d+1)\times d$ real matrix. We require that A is of full rank, and that there exists a strictly positive vector $c \in \mathbb{R}^{d+1}$ such that $cA = 0$. For $b \in \mathbb{R}^{d+1}$ let

$$K_b \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d | Ax \le b\}.$$

If nonempty, K_b is either a point or a full-dimensional simplex in $\mathbb{R}^d$. K_b is said to be *lattice free* if $\operatorname{int} K_b \cap \mathbb{Z}^d = \emptyset$. A point $n \in \mathbb{R}^d$ is called a *neighbor* (or *neighbor of the origin*) if the smallest set of the form K_b containing n and 0 is lattice free. Let N denote the set of neighbors. It is proved in [BS1] that the set N is nonempty and finite.

Denote by $a_0, \dots, a_d$ the row vectors of A. The matrix A is said to be *generic* if

$$n \in N \quad \Rightarrow \quad a_i \cdot n \neq 0, \qquad \text{for all} \quad 0 \le i \le d.$$

We assume that A is generic. Also, since only the directions provided by the row vectors a_i are important, not their magnitudes, we can without loss of generality normalize the vector c to be the unit vector $\mathbf{1} = (1, \dots, 1)$. In summary, from now on we require of the matrix $A \in \mathbb{R}^{(d+1)\times d}$ that

(A1) A is of rank d,
(A2) $\mathbf{1} \cdot A = 0$,
(A3) A is generic.

Call a simplex K_b *maximal lattice free* if K_b is lattice free but every convex body strictly containing K_b has some point from $\mathbb{Z}^d$ in its interior. It can be shown that if K_b is maximal lattice free, then its boundary intersects $\mathbb{Z}^d$ in exactly $d+1$ points (one in the relative interior of each of its facets). The *Scarf complex* Σ (or Σ_A) is the abstract simplicial complex whose vertex set is $\mathbb{Z}^d$ and whose facets (maximal faces) are the intersections $K_b \cap \mathbb{Z}^d$, for all maximal lattice free K_b. Thus Σ is a pure d-dimensional complex. Its faces are the sets of the form $K_b \cap \mathbb{Z}^d$, for lattice free simplices K_b.

Although the convex hulls of Σ's facets in general intersect in complicated ways, it turns out that the geometric realization of Σ is homeomorphic to real d-space. This was shown by Bárány et al. [BHS]. Some details of their method of proof are needed later, so we review them here.

For a fixed positive real number t, consider the injective mapping $\varphi_t\colon \mathbb{R}^d \to \mathbb{R}^{d+1}$ defined by

$$\varphi_t\colon x \mapsto (e^{ta_0x}, \dots, e^{ta_dx}).$$

Let $M \stackrel{\text{def}}{=} \{(y_0, \ldots, y_d) \in \mathbb{R}^{d+1} | y_i > 0 \text{ for all } i, \text{ and } \prod_{i=0}^{d} y_i = 1\}$. Because of assumption (A2) we have that $\varphi_t(\mathbb{R}^d) \subseteq M$, and, in fact [BHS, Lemma 1], $\varphi_t(\mathbb{R}^d) = M$. Let $V \stackrel{\text{def}}{=} \varphi_t(\mathbb{Z}^d)$, and let C be the convex hull of V (which is a discrete set). Define $F \subseteq C$ to be a *face* of C if there exists a closed half-space H in $\mathbb{R}^{d+1}$ with bounding hyperplane H^0 such that $C \subseteq H$ and $C \cap H^0 = F$. The zero-dimensional faces are the points of V (and thus they correspond bijectively to the lattice points $\mathbb{Z}^d$). The higher-dimensional faces of C are described as follows, for large enough t.

Theorem 1 [BHS, Theorem 3]. *There exists t_0 such that for $t > t_0$:*

(i) *the maximal faces of C are d-simplices*;
(ii) *suppose $x^0, \ldots, x^d \in \mathbb{Z}^d$, then $\{x^0, \ldots, x^d\} \in \Sigma$ if and only if $\varphi_t(x^0), \ldots, \varphi_t(x^d)$ are the vertices of a maximal face of C.*

Remark. The proofs in [BHS] are based on a stricter definition of genericity than the one given here. However, they go through unchanged in the greater generality, see Remark 4 of [BS1].

Theorem 1 shows that the boundary complex of C provides a geometric realization of the Scarf complex Σ. In particular, since the boundary of C is obviously homeomorphic to $\mathbb{R}^d$ we obtain:

Corollary 2 [BHS]. $\|\Sigma\| \cong \mathbb{R}^d$.

3. The Small Complex *T*

It is clear from the definition that the group $\mathbb{Z}^d$ acts on Σ:

$$\sigma \in \Sigma \quad \text{and} \quad t \in \mathbb{Z}^d \quad \Rightarrow \quad \sigma + t \in \Sigma. \tag{1}$$

This action is transitive on vertices. Hence, to study the local structure of the complex it suffices to confine attention to the neighborhhood of the origin. Let

$$L \stackrel{\text{def}}{=} \{\sigma \in \Sigma \mid 0 \notin \sigma, \sigma \cup \{0\} \in \Sigma\}$$

define the *link* of Σ at the origin. The vertices of the subcomplex L are precisely the neighbors. Because of condition (A3) the set N splits into *negative neighbors*

$$N^- \stackrel{\text{def}}{=} \{n \in N \mid a_0 \cdot n < 0\}$$

and *positive neighbors* $N^+ \stackrel{\text{def}}{=} -N^-$. The subcomplex

$$T \stackrel{\text{def}}{=} \{\sigma \subseteq N^- \mid \sigma \cup \{0\} \in \Sigma\}$$

of L is called the *top* complex, a terminology suggested by Scarf [Sc2]. Its dependence on the choice of a_0, via the definition of N^-, is discussed at the end of this section.

The top complex T has been studied by Scarf for many years and its properties are discussed in a forthcoming treatise [Sc3]. To make this paper self-contained we give proofs of the key technical properties of T that are needed here. We refer to [Sc3] for a fuller treatment.

Proposition 3. *The mapping sending* $\{x^1, \ldots, x^j\}$ *to the orbit of* $\{0, x^1, \ldots, x^j\}$ *is a bijection between the* $(j-1)$*-faces of* T *and the* $\mathbb{Z}^d$*-orbits of* j*-faces of* Σ.

Proof. Let σ be a j-face of Σ. Choose $u \in \mathbb{R}$ such that the half-space $a_0 x \leq u$ contains σ while the plane $a_0 x = u$ intersects σ. Let x_0 be the unique intersection point (unique due to genericity). Then the translate $\sigma - x_0 = \{0, x^1, \ldots, x^j\}$ has the property that $a_0 x^i < 0$ for $i = 1, \ldots, j$, i.e., $\{x^1, \ldots, x^j\} \in T$. Hence, the mapping is surjective. Injectivity is clear. $\square$

Example 4. The matrix

$$A = \begin{pmatrix} -21 & -29 & 20 \\ 6 & -9 & 26 \\ -8 & 35 & -13 \\ 23 & 3 & -33 \end{pmatrix}$$

is generic and has 20 neighbors. Its top complex T is shown (unlabeled and up to combinatorial isomorphism) in Fig. 1. (Remark: This example was computed by a Maple program provided by B. Sturmfels.)

We say that a simplicial complex is a *polytopal* $(d-1)$*-sphere* if it is combinatorially isomorphic to the boundary complex of some convex d-polytope. See [G] or [Z] for notions relating to polytopes and convex geometry. A simplicial complex is called a *regular* d*-ball* if it is combinatorially isomorphic to some regular triangulation of a convex d-polytope. A triangulation Δ of a d-polytope is regular if it is the projection of the lower boundary of a convex $(d+1)$-polytope, see the discussion in Section 5.1 of [Z].

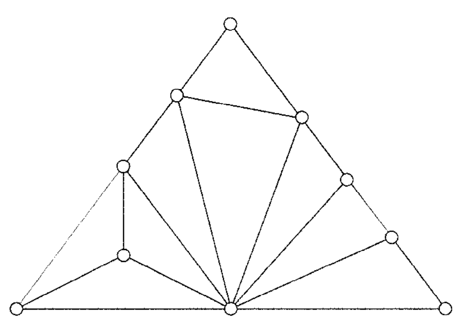

Fig. 1. The top complex of a 4×3 matrix.

The following result is important for our work with face numbers in the next section.

Theorem 5.

(i) *L is a polytopal $(d-1)$-sphere,*
(ii) *T is a regular $(d-1)$-ball,*
(iii) *∂T is a polytopal $(d-2)$-sphere.*

The proof follows a sequence of lemmas. The notation $[i, j] \overset{\text{def}}{=} \{z \in \mathbb{Z} \mid i \leq z \leq j\}$ is used.

Lemma 6. *Suppose $\{x^1, \dots, x^d\} \in L$. Then there exists a unique $k \in [0, d]$ such that $a_k x^j < 0$ for all $j \in [1, d]$.*

Proof. Since $\sigma = \{0, x^1, \dots, x^d\}$ is a facet of Σ there is a maximal lattice free simplex K_b such that $\sigma = K_b \cap \mathbb{Z}^d$. K_b has a unique facet F_0 containing 0 in its relative interior. Let $a_k x = 0$ be the equation of F_0's supporting hyperplane. Then, by definition of K_b, $a_k x^j < 0$ for all $j \in [1, d]$.

If $i \neq k$, then K_b has a supporting hyperplane of the form $a_i x = u$ containing some x^j in the interior of its intersection with K_b. Since $0 \in K_b$ it follows that $u > 0$, and hence $a_i x^j > 0$. □

For $x \in N$ let

$$I^+(x) \overset{\text{def}}{=} \{i \in [0, d] \mid a_i x > 0\},$$

$$I^-(x) \overset{\text{def}}{=} \{i \in [0, d] \mid a_i x < 0\}.$$

Both these sets are nonempty for all $x \in N$.

Lemma 7. *For every $j \in [0, d]$ there exists a unique $x \in N$ and a unique $x' \in N$ such that $I^+(x) = \{j\}$ and $I^-(x') = \{j\}$.*

Proof. Let $b = (0, \dots, 0, b_j, 0, \dots, 0)$ and choose $b_j > 0$ minimal such that K_b intersects $\mathbb{Z}^d$ in at least one point other than 0. This point x will be unique (due to genericity), it will be a neighbor, and it will satisfy $I^+(x) = \{j\}$. As a consequence, $-x \in N$ and $I^-(-x) = \{j\}$. □

Let $\{x^1, \dots, x^d\} \in L$. By Theorem 1 the set $\{\mathbf{1}, \varphi_t(x^1), \dots, \varphi_t(x^d)\}$ spans a facet of C. Let

$$\lambda(y) \overset{\text{def}}{=} \lambda_0(y_0 - 1) + \cdots + \lambda_d(y_d - 1) = 0$$

be the equation of the supporting hyperplane of this facet of C, oriented so that $\lambda(y) \geq 0$ for all $y \in C$. Then $\lambda_i > 0$ for all i, by Lemma 3 of [BHS].

Lemma 8. *Suppose* $\{x^1, \dots, x^d\} \in L$ *and choose "k" as in Lemma* 6. *Then, for sufficiently large* t,

$$\lambda_k > (d+1)\lambda_j, \qquad \textit{for all} \quad j \neq k.$$

Proof. Choose a number M such that $0 < M < a_i x^j$ for all $j \in [1, d]$ and $i \in I^+(x^j)$. Then choose $t > t_0$ large enough that $e^{Mt} - 1 > 2d$.

Choose $p \neq k$ such that $\lambda_p \geq \lambda_j$ for all $j \neq k$. Suppose (to reach a contradiction) that $\lambda_k \leq (d+1)\lambda_p$. By Lemma 6 (the uniqueness part) there exists $g \in [1, d]$ such that $p \in I^+(x^g)$. Using that $\lambda(\varphi_t(x^g)) = 0$ and the various inequalities we get

$$\begin{aligned}\sum_{i \in I^-(x^g)} \lambda_i &< 2d\lambda_p < (e^{Mt} - 1)\lambda_p \\ &\leq (e^{Mt} - 1) \sum_{i \in I^+(x^g)} \lambda_i \leq \sum_{i \in I^+(x^g)} (e^{ta_i x^g} - 1)\lambda_i \\ &= \sum_{i \in I^-(x^g)} (1 - e^{ta_i x^g})\lambda_i \leq \sum_{i \in I^-(x^g)} \lambda_i.\end{aligned}$$

This contradiction shows that $\lambda_k > (d+1)\lambda_p$, and since $\lambda_p \geq \lambda_j$ for all $j \neq k$ we are done. $\square$

Lemma 9. *Let* $\{x^1, \dots, x^d\} \in L$, *and let* $\lambda(y) = 0$ *be the equation of the supporting hyperplane of the corresponding facet of* C *as before. Then, for sufficiently large* t,

$$\lambda\left((d+2, 0, \dots, 0)\right) \textit{ is } \begin{cases} > 0, & \textit{if} \quad \{x^1, \dots, x^d\} \in T, \\ < 0, & \textit{otherwise}. \end{cases}$$

Proof. Let $\Phi \overset{\text{def}}{=} \lambda\left((d+2, 0, \dots, 0)\right) = (d+1)\lambda_0 - \sum_{j=1}^d \lambda_j$.

If $\{x^1, \dots, x^d\} \in T$, then $k = 0$ in Lemma 8 and we get $\sum_{j=1}^d \lambda_j < d(\lambda_0/(d+1)) < (d+1)\lambda_0$; hence $\Phi > 0$.

If $\{x^1, \dots, x^d\} \notin T$, then $k \neq 0$ in Lemma 8 and $(d+1)\lambda_0 < \lambda_k < \sum_{j=1}^d \lambda_j$; hence $\Phi < 0$. $\square$

Proof of Theorem 5. Choose t sufficiently large and let $P = C \cap H$, where H is the hyperplane $y_0 + \cdots + y_d = d + 2$ in $\mathbb{R}^{d+1}$. The vertex $\mathbf{1}$ of C is separated from its neighbors by H, so P is a bounded convex d-polytope whose boundary complex is isomorphic to L. This proves part (i).

Let $q = (d+2, 0, \dots, 0) \in \mathbb{R}^{d+1}$. Then $q \in H$, and Lemma 9 shows that (in the d-space H) q is beneath those facets of P that correspond to the facets of T, and beyond the remaining ones. (Being "beneath" means being on the same side as the interior of P, and the opposite for "beyond.") Let T' be the part of P's boundary complex that realizes T, and let $Q \overset{\text{def}}{=} \textit{conv}(P \cup \{q\})$. The facets of the polytope Q are the facets in T' and faces of the form $\textit{conv}(F \cup \{q\})$ for facets F of the boundary $\partial T'$. It follows that T' is a $(d-1)$-ball, and its boundary complex $\partial T'$ is isomorphic to the vertex figure of Q at the vertex q, which shows it is a polytopal $(d-2)$-sphere. See Section 5 of [BL1] for more about this kind of argument. Finally, a projective transformation that moves q (but no other vertex of Q) to infinity will move T' into a position showing that T is regular. $\square$

We return for a moment to the definition of the complex T and its dependence on the choice of a specific row of A. Generalizing our earlier definitions, let

$$N_i^- \stackrel{\text{def}}{=} \{n \in N \mid a_i \cdot n < 0\}$$

and

$$T_i \stackrel{\text{def}}{=} \{\sigma \subseteq N_i^- \mid \sigma \cup \{0\} \in \Sigma\},$$

for $i = 0, \ldots, d$. This creates $d + 1$ "top" complexes, all satisfying the enumerative property of Proposition 3, as well as all the other properties we have derived. Examples show, however, that these complexes T_i are in general not isomorphic. Using Theorem 5 and Lemma 6 we can conclude the following:

Proposition 10. *The $(d-1)$-balls $T_0, T_1, \ldots, T_d$ have pairwise disjoint interiors, and their union is the link L.*

4. Face Numbers of Scarf Complexes

We begin with a quick review of some definitions and results from the general theory of face numbers. For more about this topic, see, e.g., [G], [Z], or the survey [BB].

Let Δ be a $(d-1)$-dimensional simplicial complex, and let f_i be the number of i-dimensional faces of Δ. The sequence $f = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. We put $f_{-1} = 1$. The *h-vector* $h = (h_0, \ldots, h_d)$ of Δ is defined by the equation

$$\sum_{i=0}^{d} f_{i-1} x^{d-i} = \sum_{i=0}^{d} h_i (x+1)^{d-i}. \tag{2}$$

Note that $h_0 = 1$, $h_1 = n - d$, and $h_d = (-1)^{d-1} \tilde{\chi}(\Delta)$, where $\tilde{\chi}(\Delta)$ is the reduced Euler characteristic of Δ. In particular,

$$h_d = \begin{cases} 1, & \text{if } \Delta \text{ is a sphere,} \\ 0, & \text{if } \Delta \text{ is a ball,} \end{cases}$$

where the conditions are shorthand for saying that Δ's geometric realization is homeomorphic to a sphere, resp. a ball.

The following are called the *Dehn–Sommerville relations*:

$$\textit{If } \Delta \textit{ is a sphere, then } h_i = h_{d-i}, \textit{ for all } 0 \le i \le d. \tag{3}$$

Hence, for spheres all f-vector information is encoded in the much shorter *g-vector* $g = (g_0, \ldots, g_{\lfloor d/2 \rfloor})$, defined by $g_i = h_i - h_{i-1}$. We have that $g_0 = 1$, $g_1 = n - d - 1$.

A sequence of integers $(a_0, \dots, a_k)$ is called an *M-sequence* if there exists a nonempty family $\mathcal{M}$ of monomials in a_1 variables such that

(i) if m divides m' and $m' \in \mathcal{M}$, then $m \in \mathcal{M}$, and
(ii) $\mathcal{M}$ contains exactly a_i monomials of degree i, for all $0 \le i \le k$.

M-sequences have a number of algebraic and combinatorial characterizations, see, e.g., [St2] and [Z]. Their relevance for this paper is the following result due to Stanley [St1]:

$$\textit{If } \Delta \textit{ is a polytopal sphere, then } (g_0, \dots, g_{\lfloor d/2 \rfloor}) \textit{ is an } M\textit{-sequence.} \tag{4}$$

If Δ is a $(d-1)$-ball, its boundary complex $\partial\Delta$ is a $(d-2)$-sphere. Furthermore, $\partial\Delta$'s f-vector is determined by that of Δ, as shown by the following consequence of the Dehn–Sommerville relations, due to McMullen and Walkup [MW], see also Corollary 3.9 of [BL2]:

$$\textit{If } \Delta \textit{ is a ball with boundary } \partial\Delta, \textit{ then } h_i^{\Delta} - h_{d-i}^{\Delta} = g_i^{\partial\Delta}. \tag{5}$$

After this review, we now turn our attention back to Scarf complexes. The f-vectors of L, T, and ∂T are denoted f^L, f^T, and $f^{\partial T}$, and similarly for their h- and g-vectors.

Proposition 11.

(i) $f_i^L = (i+2) f_i^T$;
(ii) $h_i^L = (i+1)h_i^T + (d-i+1)h_{i-1}^T$.

Proof. Let σ be an i-simplex of T. Then $\sigma \cup \{0\}$ is an $(i+1)$-simplex of Σ. Each one of the $i+2$ vertices of $\sigma \cup \{0\}$ can be translated to the origin, and its $i+2$ maximal faces thus contribute distinct i-simplices to L. The proof of part (i) is concluded with the observation that every i-simplex of L is obtained from a unique i-simplex of T in this fashion.

To simplify notation, for the rest of this proof put $f \overset{\text{def}}{=} f^T$ and $h \overset{\text{def}}{=} h^T$. For part (ii) we begin by differentiating (2) and multiplying by $(x+1)$:

$$\sum_{i=0}^{d-1} (d-i) f_{i-1}(x+1)x^{d-i-1} = \sum_{i=0}^{d-1} (d-i) h_i (x+1)^{d-i}.$$

This relation gives

$$\begin{aligned} d\sum_{i=0}^{d} f_{i-1}x^{d-i} - \sum_{i=0}^{d} i f_{i-1}x^{d-i} + \sum_{i=0}^{d-1} (d-i) f_{i-1}x^{d-i-1} \\ = d\sum_{i=0}^{d} h_i (x+1)^{d-i} - \sum_{i=0}^{d} i h_i (x+1)^{d-i}. \end{aligned}$$

Using (2) and its derivative this simplifies to

$$\begin{aligned} \sum_{i=0}^{d} i f_{i-1}x^{d-i} &= \sum_{i=0}^{d} i h_i (x+1)^{d-i} + \sum_{i=0}^{d-1} (d-i) h_i (x+1)^{d-i-1} \\ &= \sum_{i=0}^{d} [i h_i + (d+1-i) h_{i-1}] (x+1)^{d-i}. \end{aligned}$$

Hence, using part (i) we get

$$\begin{aligned}\sum_{i=0}^{d} h_i^L (x+1)^{d-i} &= \sum_{i=0}^{d} (i+1)\, f_{i-1} x^{d-i}\\ &= \sum_{i=0}^{d} [i h_i + (d+1-i) h_{i-1}](x+1)^{d-i} + \sum_{i=0}^{d} f_{i-1} x^{d-i}\\ &= \sum_{i=0}^{d} [(i+1) h_i + (d+1-i) h_{i-1}](x+1)^{d-i},\end{aligned}$$

which proves part (ii). □

Proposition 12. *The h-vector of T satisfies*

$$h_i^T = h_{d-1-i}^T,$$

for all $0 \le i \le \lfloor (d-1)/2 \rfloor$.

Proof. The Dehn–Sommerville relations $h_i^L = h_{d-i}^L$ together with Proposition 11(ii) show that

$$(i+1)(h_i^T - h_{d-i-1}^T) = (d-i+1)(h_{d-i}^T - h_{i-1}^T). \tag{6}$$

We have that $h_d^T = 0$ and $h_d^L = 1$ (T being a ball and L a sphere), so $h_{d-1}^T = 1$ follows from Proposition 11(ii). Hence, since $h_0^T = 1$, (6) gives

$$h_0^T = h_{d-1}^T \quad \Rightarrow \quad h_1^T = h_{d-2}^T \quad \Rightarrow \quad h_2^T = h_{d-3}^T \quad \Rightarrow \quad \cdots.$$ □

The following is our main result.

Theorem 13. *Let ∂T be the boundary complex of T. Then*

$$h_i^T = h_i^{\partial T},$$

for all $0 \le i \le d-1$.

Proof. Using Proposition 12 and (5) we get that

$$h_i^T - h_{i-1}^T = h_i^T - h_{d-i}^T = g_i^{\partial T} = h_i^{\partial T} - h_{i-1}^{\partial T},$$

for all $1 \le i \le \lfloor (d-1)/2 \rfloor$. Since $h_0^T = h_0^{\partial T} = 1$ it follows that $h_i^T = h_i^{\partial T}$ for all $0 \le i \le \lfloor (d-1)/2 \rfloor$. This extends to all $0 \le i \le d-1$ via Proposition 12 and the Dehn–Sommerville relations $h_i^{\partial T} = h_{d-1-i}^{\partial T}$. □

Corollary 14. *T has the same f-vector as the cone over its boundary, namely,* $f_i^T = f_i^{\partial T} + f_{i-1}^{\partial T}$.

Proof. If $(h_0, \ldots, h_{d-1})$ is the h-vector of some $(d-2)$-complex Δ, straightforward computation (multiply (2) by $x+1$) shows that $(h_0, \ldots, h_{d-1}, 0)$ is the h-vector of the cone over Δ. □

One particular consequence, already known to Scarf [Sc2], [Sc3], is that T has a unique interior vertex, although (as illustrated by Fig. 1) T is in general *not* the cone over its boundary. The unique interior vertex of T is the neighbor x that in Lemma 7 was characterized by the property $I^-(x) = \{0\}$.

Corollary 15. *The f-vector of T determines, and is determined by, the M-sequence* $(1, g_1^{\partial T}, \ldots, g_{\lfloor (d-1)/2 \rfloor}^{\partial T})$.

Proof. The g-vector of ∂T is an M-sequence, since ∂T is polytopal (Theorem 5). The rest follows from the theorem. □

The theorem also implies a direct relationship between the g-vectors of the two spheres L and ∂T.

Corollary 16. $g_i^L = (i+1) g_i^{\partial T} + (d-i+2) g_{i-1}^{\partial T}$.

Proof. We have from Proposition 11 and the theorem that

$$\begin{aligned} g_i^L &= h_i^L - h_{i-1}^L = (i+1)h_i^T + (d-i+1)h_{i-1}^T - (i h_{i-1}^T + (d-i+2)h_{i-2}^T) \\ &= (i+1)(h_i^T - h_{i-1}^T) + (d-i+2)(h_{i-1}^T - h_{i-2}^T) \\ &= (i+1)g_i^{\partial T} + (d-i+2)g_{i-1}^{\partial T}. \end{aligned}$$ □

5. Remarks

1. The property of being a ball with a unique interior vertex does not by itself imply any special relationship between the f-vector of T and that of its boundary, such as that of Corollary 14. For example, take two tetrahedra glued together along one triangle and then perform a stellar subdivision of one of them, thus introducing an interior vertex. The resulting ball has five facets, whereas its boundary has six. One can also construct a unique-interior-vertex triangulation of the 3-ball with seven facets, whose boundary is the same six-facet 2-sphere.

2. Let $\mathcal{F}_d$ be the set of all f-vectors of Scarf top complexes T coming from generic $(d+1) \times d$-matrices. What is the dimension of the affine span of $\mathcal{F}_d$ in $\mathbb{R}^d$? We have shown that

$$\dim\ \mathrm{aff}\ \mathcal{F}_d \le \left\lfloor \frac{d-1}{2} \right\rfloor.$$

Is this upper bound sharp, or are other linear relations satisfied by these f-vectors ?

3. A more ambitious question is to ask which M-sequences $(1, g_1, \ldots, g_{\lfloor (d-1)/2 \rfloor})$ are "Scarf" in the sense of Corollary 15, i.e., correspond to the elements of $\mathcal{F}_d$.

4. The Stanley–Reisner rings of top complexes T may be interesting to investigate further from an algebraic point of view. For instance, Proposition 12 indicates that they might have a "Lefschetz element," similar to the ones provided by toric geometry in [St1].

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