# The Homology of "$k$-equal" Manifolds and Related Partition Lattices

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Contents



# 1 Introduction

For $2 \leq k \leq n$, let $V^{\mathbf{R}}_{n,k}$ denote the set of points $\underline{\mathbf{x}} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ such that $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for some $k$-set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Define $V^{\mathbf{C}}_{n,k}$ similarly for $\underline{\mathbf{x}} \in \mathbf{C}^n$. The main purpose of this paper is to obtain topological and combinatorial information about these subspace arrangements and the manifolds $M^{\mathbf{R}}_{n,k} = \mathbf{R}^n - V^{\mathbf{R}}_{n,k}$ and $M^{\mathbf{C}}_{n,k} = \mathbf{C}^n - V^{\mathbf{C}}_{n,k}$.

These objects have been much studied in the $k = 2$ case. Namely, $V^{\mathbf{R}}_{n,2}$ is the union of the reflecting hyperplanes of the Coxeter arrangement of type $A_{n-1}$ (corresponding to the symmetric group $S_n$), and $M^{\mathbf{R}}_{n,2}$ consists of $n!$ disjoint simplicial cones (times a copy of $\mathbf{R}$). Hence, the cohomology of $M^{\mathbf{R}}_{n,2}$ is free with $\beta^0(M^{\mathbf{R}}_{n,2}) = n!$ as its only non-vanishing Betti number. Furthermore, $M^{\mathbf{C}}_{n,2}$ is known as the "pure braid space". It is a $K(\pi, 1)$ space with the pure braid group as fundamental group [FaN62], [FoN62]. Arnol'd [Arn69] showed that the cohomology of $M^{\mathbf{C}}_{n,2}$ is free and he also computed its Betti numbers. See [OT92] for more information about the $k = 2$ case.

The interest in obtaining information about the cohomology of $M^{\mathbf{R}}_{n,k}$, also for $k \geq 3$, arose in connection with a problem from computer science in [BLY92], [BL92]. Namely, the Betti numbers of $M^{\mathbf{R}}_{n,k}$ are there shown to be the essential ingredient in a lower bound for the complexity of deciding membership in $V^{\mathbf{R}}_{n,k}$ using the linear decision tree model of computation.

The following are the main results of this paper.

**Theorem 1.1** *The cohomology groups of $M^{\mathbf{R}}_{n,k}$ are free. Furthermore*

(a) *$M^{\mathbf{R}}_{n,k}$ is $(k-3)$-connected.*

(b) *$H^d(M^{\mathbf{R}}_{n,k}) \neq 0$ if and only if $d = t \cdot (k-2)$, for some integer $t$ such that $0 \leq t \leq \lfloor \frac{n}{k} \rfloor$.*

(c) *$rank H^{k-2}(M^{\mathbf{R}}_{n,k}) = \sum_{i=k}^{n} \binom{n}{i} \cdot \binom{i-1}{k-1}$ if $k \geq 3$.*

**Theorem 1.2** *The cohomology groups of $M^{\mathbf{C}}_{n,k}$ are free. Furthermore,*

(a) *$M^{\mathbf{C}}_{n,k}$ is $(2k-4)$-connected.*

(b) *$H^d(M^{\mathbf{C}}_{n,k}) \neq 0$ if and only if $d = 0$ or there exist integers $1 \leq m \leq t \leq \lfloor \frac{n}{k} \rfloor$ such that*

$$t \cdot (k-2) - m + t \cdot k \leq d \leq t \cdot (k-2) - m + n.$$

(c) *$rank H^{2k-3}(M^{\mathbf{C}}_{n,k}) = \binom{n}{k}$.*

(d) *$\chi(M^{\mathbf{C}}_{n,k}) = 0$.*

**Theorem 1.3** *The spaces $V^{\mathbf{R}}_{n,k} \cap S^{n-1}$ and $V^{\mathbf{C}}_{n,k} \cap S^{2n-1}$ have the homotopy type of a wedge of spheres.*

More precise information about the size and the distribution of non-vanishing Betti numbers will be given in Section 5, where also the proofs of Theorems 1.1-1.3 appear. These theorems make a result of Serre [Ser53] applicable, from which the following conclusions can be drawn about the higher homotopy groups.

**Corollary 1.4**

(a) *If $k \geq 4$ then $\pi_d(M^{\mathbf{R}}_{n,k}) \neq 0$ for infinitely many $d \geq k-2$.*

(b) *If $k \geq 3$ then $\pi_d(M^{\mathbf{C}}_{n,k}) \neq 0$ for infinitely many $d \geq 2k-3$.*

Our proofs rely on a combination of two techniques. The first stems from recent progress in the study of subspace arrangements, more precisely the cohomology formula of Goresky and MacPherson [GM88] and the related formula for homotopy type of Ziegler and Živaljević [ZŽ91]. These results reduce questions of the type we study here (at least in principle) to questions about the combinatorics of certain finite lattices. The second technique consists in combinatorial methods for computing the homotopy type of partially ordered sets. In Section 2 the relevant background from these two areas will be reviewed.

The lattices which are of interest for our work have the following combinatorial characterization. For $2 \le n$, $2 \le k$, and $0 \le l$, let $\Pi_{n,k}(l)$ be the family of all partitions $\pi$ of the set $\{1, 2, \ldots, n\}$ such that each block $B$ of $\pi$ satisfies at least one of the following requirements :

(i) $|B| = 1$,

(ii) $k \le |B| \le n$,

(iii) $B \cap \{1, 2, \ldots, l\} \ne \emptyset$.

Ordered by refinement, $\Pi_{n,k}(l)$ is a lattice, in fact a join-sublattice of the lattice of all partitions of $\{1, 2, \ldots, n\}$. The lattice $\Pi_{n,k} = \Pi_{n,k}(0)$ appeared in [BLY92] (there denoted $\Pi_{n,k-1}$) where its Möbius function was computed. It is the intersection lattice of the subspace arrangements determining $M^{\mathbf{R}}_{n,k}$ and $M^{\mathbf{C}}_{n,k}$. The sense in which we speak about the topology of a lattice will be explained in Sections 2 and 4.

**Theorem 1.5** *Assume that* $2 \le k \le n$.

*(a) The lattice* $\Pi_{n,k}$ *has the homotopy type of a wedge of spheres (possibly of different dimensions). Therefore its homology groups are free.*

*(b)* $\widetilde{H}_d(\Pi_{n,k}) \ne 0$ *if and only if* $d = n - 3 - t \cdot (k-2)$, *for some integer* $t$ *such that* $1 \le t \le \lfloor \frac{n}{k} \rfloor$.

*(c)* $rank \widetilde{H}_d(\Pi_{n,k})$ *is divisible by* $\binom{n-1}{k-1}$, *for all* $d$.

*(d)* $rank \widetilde{H}_{n-k-1}(\Pi_{n,k}) = \binom{n-1}{k-1}$, *if* $k \ge 3$.

This will be proved in Section 4. More general information valid for $\Pi_{n,k}(l)$, all $l \ge 0$, is given there, and also a formula for $rank \widetilde{H}_d(\Pi_{n,k})$ of which (b), (c) and (d) are special cases, see Theorem 4.5. Part (a) is strengthened to $S_{n-1}$-equivariant homotopy equivalence in Remark 7.7, where the representations of symmetric groups induced on the homology of $\Pi_{n,k}$, $M^{\mathbf{R}}_{n,k}$, and $M^{\mathbf{C}}_{n,k}$ are briefly discussed.

The lattices $\Pi_{n,k}$ and their generalizations $\Pi_{n_1,\ldots,n_r;k}$ defined in Section 6 are closely related to certain complexes of disconnected $k$-graphs, which have been considered by Fock, Nekrasov, Rosly and Selivanov [FNRS91] and V.A.Vassiliev in connection with questions in quantum theory and the homotopy classification of links. We comment on such complexes and answer some questions asked by these authors in Remark 7.8.

The various recursions and formulae for computing the Betti numbers of $M^{\mathbf{R}}_{n,k}$, $M^{\mathbf{C}}_{n,k}$ and $\Pi_{n,k}$ make computer calculations possible. At the end of the paper we present some tables of computer generated Betti numbers.

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# 2 Preliminaries

The purpose of this section is to gather some tools that will be needed, and to establish notation.

A finite collection $\mathcal{A} = \{K_1, \ldots, K_t\}$ of linear proper subspaces in $\mathbf{R}^n$ is called a SUBSPACE ARRANGEMENT. Most of what is said in this section and the following one is true also for arrangements of affine subspaces, but since that generality will not be needed we will for simplicity assume that the spaces $K_i$ are linear, *i.e.*, contain the origin. We may without loss of generality assume that there are no containments $K_i \subseteq K_j$, $i \neq j$.

Let $V_{\mathcal{A}} = K_1 \cup \cdots \cup K_t$ and $M_{\mathcal{A}} = \mathbf{R}^n - V_{\mathcal{A}}$. These spaces are called the UNION and COMPLEMENT of $\mathcal{A}$, respectively. The complement $M_{\mathcal{A}}$ is an $n$-dimensional manifold. The INTERSECTION LATTICE $L_{\mathcal{A}}$ is the collection of all intersections $\{ K_{i_1} \cap \cdots \cap K_{i_j} \mid 1 \leq i_1 < \cdots < i_j \leq t \}$ ordered by reverse inclusion : $x \leq y$ if and only if $x \supseteq y$. This makes $L_{\mathcal{A}}$ into a finite lattice with bottom element $\hat{0} = \mathbf{R}^n$ and top element $\hat{1} = K_1 \cap \cdots \cap K_t$. The notation $L_{\mathcal{A}}^{>\hat{0}} = L_{\mathcal{A}} - \{\hat{0}\}$ will be convenient.

Let $P$ be a poset (finite partially ordered set) and for $x, y \in P$, $x < y$, let $\Delta(x,y)$ denote the ORDER COMPLEX of the open interval $(x,y) = \{ z \in P \mid x < z < y \}$, i.e., the simplicial complex of all chains $x_0 < x_1 < \cdots < x_k$ in $(x,y)$. We will write $\widetilde{H}_i(x,y) = \widetilde{H}_i(\Delta(x,y), \mathbf{Z})$ for the $i$-th reduced simplicial homology group of $\Delta(x,y)$, and $\tilde{\beta}_i(x,y) = rank \widetilde{H}_i(x,y)$ for the corresponding Betti numbers. For the closed interval $(x,y) \cup \{x,y\}$ we will use the notation $[x,y]$. Further, $\widetilde{H}^i(T)$ will denote the reduced singular cohomology of a space $T$. All homology and cohomology groups appearing in this paper are taken to have coefficients in $\mathbf{Z}$.

The following two results establish the fundamental link between the topology of the arrangement $\mathcal{A}$ and the combinatorics of its intersection lattice $L_{\mathcal{A}}$.

**Proposition 2.1** *(Goresky and MacPherson [GM88]) For every subspace arrangement $\mathcal{A}$ and all dimensions $i$ :*

$$\widetilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}}^{>\hat{0}}} \widetilde{H}_{codim(x)-2-i}(\hat{0}, x)$$

**Proposition 2.2** *(Ziegler and Živaljević [ZŽ91]) For every arrangement $\mathcal{A}$ in $\mathbf{R}^n$ there is a homotopy equivalence*

$$V_{\mathcal{A}} \cap S^{n-1} \simeq \underset{x \in L_{\mathcal{A}}^{>\hat{0}}}{wedge} \Big( \Delta(\hat{0}, x) * S^{dim(x)-1} \Big)$$

In these formulas, $codim(x) = n - dim(x)$, $S^{j-1}$ denotes the unit sphere in $\mathbf{R}^j$, and "$*$" denotes the join of spaces. Note that the Ziegler-Živaljević result implies the Goresky-MacPherson formula via Alexander duality in $S^{n-1}$. A different proof of Proposition 2.1, and a result similar to Proposition 2.2, appear in Vassiliev [Vas92, Theorems 6.2 and 6.4].

Two elements $x$ and $y$ of a lattice $L$ are said to be COMPLEMENTS if $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Denote by $\mathcal{CO}(x)$ the set of all complements of $x$. Here we write $\vee$ for the join operation (supremum) and $\wedge$ for the meet operation (infimum) in the lattice $L$. A subset $A \subseteq L$ is called an ANTICHAIN if no two distinct elements are comparable, i.e., if $x \leq y$ implies $x = y$ for all $x, y \in A$.

**Proposition 2.3** *(Björner and Walker [BW83]) For a finite lattice $L$, and every element $x \neq \hat{0}, \hat{1}$ such that $\mathcal{CO}(x)$ is an antichain, there is a homotopy equivalence :*

$$\Delta(\hat{0}, \hat{1}) \simeq \underset{y \in \mathcal{CO}(x)}{wedge}\ susp\Big(\Delta(\hat{0}, y) * \Delta(y, \hat{1})\Big)$$

In the preceding proposition we denote by "*susp*" the suspension of topological spaces. The DIRECT PRODUCT $P \times Q$ of two posets $P$ and $Q$ is the set of all ordered pairs $< x, y >$, $x \in P$, $y \in Q$, ordered by $< x, y > \leq < x', y' >$ if and only if $x \leq x'$ in $P$ and $y \leq y'$ in $Q$.

**Lemma 2.4** *(Quillen [Qui78], Walker [Wal88]) If $x < x'$ and $y < y'$, then there is a homeomorphism :*

$$\Delta(<x,y>,<x',y'>) \cong susp\Big(\Delta(x,x') * \Delta(y,y')\Big)$$

To be able to use this homeomorphism, we need the following information about topological properties of the join operation.

**Lemma 2.5** *Suppose that $\Delta_1$ and $\Delta_2$ are finite simplicial complexes.*

*(i) If $\widetilde{H}_i(\Delta_1)$ and $\widetilde{H}_i(\Delta_2)$ are free for all $i$, then*

$$\widetilde{H}_{i+1}(\Delta_1 * \Delta_2) \cong \bigoplus_{p+q=i} \Big(\widetilde{H}_p(\Delta_1) \otimes \widetilde{H}_q(\Delta_2)\Big).$$

*(ii) If $\Delta_1$ and $\Delta_2$ both have the homotopy type of a wedge of spheres, then so does $\Delta_1 * \Delta_2$.*

*(iii) If $\Delta$ has the homotopy type of a wedge of spheres, then so does $susp(\Delta)$.*

**Proof:** Part (i) follows from the formula

$$\widetilde{H}_i(\Delta_1 \times \Delta_2) \cong \widetilde{H}_{i+1}(\Delta_1 * \Delta_2) \oplus \widetilde{H}_i(\Delta_1) \oplus \widetilde{H}_i(\Delta_2),$$

see Munkres [Mun84, p. 373], together with the Künneth formula [Mun84, p. 351]. Alternatively, it can be shown via direct arguments that isomorphism of homology is induced by the map of chain complexes $\mathcal{C}(\Delta_1) \otimes \mathcal{C}(\Delta_2) \to \mathcal{C}(\Delta_1 * \Delta_2)$ defined on generators by

$$(x_0, \ldots, x_p) \otimes (y_0, \ldots, y_q) \mapsto (x_0, \ldots, x_p, y_0, \ldots, y_q).$$

Part (ii) follows from these two facts :

(iv) If $T_1 \simeq T_1'$ and $T_2 \simeq T_2'$, then $T_1 * T_2 \simeq T_1' * T_2'$,

(v) $(\underset{i}{wedge}\, S^{a_i}) * (\underset{j}{wedge}\, S^{b_j}) \simeq \underset{i,j}{wedge}\, S^{a_i+b_j+1}$.

Note that part (iii) is the special case $\Delta_1 = S^0$ of part (ii).

Statement (iv) can be proven directly from the definitions: take homotopy inverses $f_i : T_i \to T_i'$ and $g_i : T_i' \to T_i$, $i = 1, 2$, and combine them to homotopy inverses $f_1 * f_2 : T_1 * T_2 \to T_1' * T_2'$ and $g_1 * g_2 : T_1' * T_2' \to T_1 * T_2$.

For statement (v) we note that there are natural inclusions

$$S^{a_i} * S^{b_j} \hookrightarrow (\underset{i}{wedge}\, S^{a_i}) * (\underset{j}{wedge}\, S^{b_j})$$

such that

$$(\underset{i}{wedge}\, S^{a_i}) * (\underset{j}{wedge}\, S^{b_j}) = \bigcup_{i,j} (S^{a_i} * S^{b_j}).$$

Let $p$ be the wedge point of $\underset{i}{wedge}\, S^{a_i}$. Then :

$$(\underset{i}{wedge}\, S^{a_i}) * (\underset{j}{wedge}\, S^{b_j}) \simeq \Big(\bigcup_{i,j}(S^{a_i} * S^{b_j})\Big) / \Big(p * (\underset{j}{wedge}\, S^{b_j})\Big) =$$

$$= \bigcup_{i,j} \Big((S^{a_i} * S^{b_j}) / (p * S^{b_j})\Big) =$$

$$= \operatorname*{wedge}_{i,j}\Big((S^{a_i} * S^{b_j})/(p * S^{b_j})\Big) \simeq$$

$$\simeq \operatorname*{wedge}_{i,j} S^{a_i} * S^{b_j}$$

Here the first and the last homotopy equivalences use that the homotopy type is unaffected by smashing a contractible subcomplex (see e.g. [BW83, p.12]), and the last equivalence also uses that the wedge operation is well-defined on homotopy classes of spaces (see [BW83, p. 16]). Finally, there is a homeomorphism $S^a * S^b \cong S^{a+b+1}$, as can be deduced from [Mun84, pp. 370-371]. ∎

The following facts about intervals in product posets $P \times Q$ are directly implied by the two preceding lemmas.

**Proposition 2.6** *Suppose that $x < x'$ in $P$ and $y < y'$ in $Q$.*

*(i) If $\widetilde{H}_i(x, x')$ and $\widetilde{H}_i(y, y')$ are free for all $i$, then*

$$\widetilde{H}_{i+2}(< x, y >, < x', y' >) \cong \bigoplus_{p+q=i} \Big(\widetilde{H}_p(x, x') \otimes \widetilde{H}_q(y, y')\Big).$$

*(ii) If $\Delta(x, x')$ and $\Delta(y, y')$ are both homotopy equivalent to wedges of spheres, then so is also $\Delta(< x, y >, < x', y' >)$.*

# 3 General subspace arrangements

Some of the arguments used to prove the theorems stated in Section 1 do not rely on the specific structure found in the "$k$-equal" arrangements. We have gathered such general results in this section.

**Theorem 3.1** *Let $\mathcal{A}$ be a subspace arrangement and $c = \min_{K \in \mathcal{A}} codim(K)$. Then*

*(i) $M_{\mathcal{A}}$ is $(c-2)$-connected,*

*(ii) $H^{c-1}(M_{\mathcal{A}})$ is a free Abelian group,*

*(iii) $rank(H^{c-1}(M_{\mathcal{A}})) \geq |\mathcal{A}^c|$, $\mathcal{A}^c = \{ K \in \mathcal{A} \mid codim(K) = c \}$, with equality if and only if $codim(K \cap K') \neq c+1$ for all $K, K' \in \mathcal{A}^c$.*

**Proof:** The statements are trivially true for $c = 1$, so we may assume that $c \geq 2$. Suppose that $x > \hat{0}$ in $L_{\mathcal{A}}$. We have that $codim(y) \notin \{1, \ldots, c-1\}$ for all $\hat{0} < y < x$ in $L_{\mathcal{A}}$, hence $dim\Delta(\hat{0}, x) \leq codim(x) - 2 - (c-1)$. It follows that $\widetilde{H}_{codim(x)-2-(c-1)}(\hat{0}, x)$ is necessarily free, since no boundaries are present in the maximal dimension. By Proposition 2.1,

$$H^{c-1}(M_{\mathcal{A}}) \cong \bigoplus_{x > \hat{0}} \widetilde{H}_{codim(x)-2-(c-1)}(\hat{0}, x),$$

and part (ii) follows.

Let us look more closely at the various contributions to $H^{c-1}(M_{\mathcal{A}})$ generated by $x \in L_{\mathcal{A}}^{>\hat{0}}$. First of all, each $K \in \mathcal{A}^c$ contributes a copy of $\widetilde{H}_{-1}(\hat{0}, K) = \widetilde{H}_{-1}(\emptyset) = \mathbf{Z}$.

Suppose now that $codim(K \cap K') \neq c+1$ for all $K, K' \in \mathcal{A}^c$. If $x \in L_{\mathcal{A}}^{>\hat{0}} - \mathcal{A}^c$ then no chain in $\Delta(\hat{0}, x)$ can contain elements of both codimensions $c$ and $c+1$, hence $dim\Delta(\hat{0}, x) \leq codim(x) - 2 - c$.

For dimensional reasons there are then no non-zero contributions coming from $x \notin \mathcal{A}^c$, and hence $H^{c-1}(M_{\mathcal{A}}) \cong \mathbf{Z}^{|\mathcal{A}^c|}$.

Suppose instead that $codim(K \cap K') = c+1$ for some $K, K' \in \mathcal{A}^c$. Let $x = K \cap K'$. Then $\Delta(\hat{0}, x)$ is a 0-dimensional complex with at least two points $K$ and $K'$, so $\widetilde{H}_0(\hat{0}, x) \neq 0$. This contribution shows that $rank H^{c-1}(M_{\mathcal{A}}) > |\mathcal{A}^c|$, and part (iii) has been proved.

The dimension arguments that we have so far used show also that $\widetilde{H}^i(M_{\mathcal{A}}) = 0$ for all $i \leq c-2$. To prove the stronger property that homotopy groups vanish up to dimension $c-2$, i.e. to prove part (i), we will proceed differently.

Take any regular CW-decomposition of the unit sphere $S^{n-1}$ in $\mathbf{R}^n$ that contains $V_{\mathcal{A}} \cap S^{n-1}$ as a subcomplex, and whose barycentric subdivision is a PL-sphere. Two different constructions of such decompositions, called the "$\mathbf{s}^{(1)}$- and $\mathbf{s}^{(2)}$-stratifications", are described in Björner and Ziegler [BZ92, Sections 2 and 9]. Let $P$ be the face poset of such a cell complex, and let $P_0$ be the subset of cells whose union is the $(n-c-1)$-dimensional subcomplex $V_{\mathcal{A}} \cap S^{n-1}$. Then as shown in [BZ92, Proposition 3.1] the opposite poset $P^{op}$ is the face poset of a regular CW-decomposition of $S^{n-1}$, and the subposet $(P - P_0)^{op}$ determines a subcomplex having the homotopy type of $S^{n-1} - (V_{\mathcal{A}} \cap S^{n-1}) = M_{\mathcal{A}} \cap S^{n-1}$. Clearly, $M_{\mathcal{A}} \cap S^{n-1}$ is a strong deformation retract of $M_{\mathcal{A}}$. Every cell $\sigma \in P_0$ satisfies $dim_P(\sigma) \leq n-c-1$, hence $dim_{P^{op}}(\sigma) \geq (n-1)-(n-c-1) = c$. It follows that the full $(c-1)$-skeleton of $P^{op}$ is contained in the subcomplex $(P-P_0)^{op}$, and since $P^{op} \cong S^{n-1}$ is $(c-2)$-connected so is therefore also $(P-P_o)^{op} \simeq M_{\mathcal{A}}$. (In the last step we have used the well-known fact that a complex is $k$-connected if and only if its $(k+1)$-skeleton is $k$-connected.) ∎

For $c = 2$ the theorem says only that $M_{\mathcal{A}}$ is arc-wise connected and that $H^1(M_{\mathcal{A}}) \cong \mathbf{Z}^k$ for some $k \geq |\mathcal{A}^c|$. For $c \geq 3$ it implies together with results of Hurewicz and Serre some information about the homotopy groups.

**Corollary 3.2** *Suppose that $c \geq 3$. Then*

*(i)* $\pi_i(M_{\mathcal{A}}) = 0$, *for* $i \leq c-2$,

*(ii)* $\pi_{c-1}(M_{\mathcal{A}}) \cong \mathbf{Z}^k$, *for some* $k \geq |\mathcal{A}^c|$,

*(iii)* $\pi_i(M_{\mathcal{A}}) \neq 0$ *(in fact, there is an element of infinite order or an element of order two), for infinitely many values of* $i$.

**Proof:** Part (i) is a restatement of $(c-2)$-connectedness. It implies via the Hurewicz Theorem and the Universal Coefficient Theorem that $\pi_{c-1}(M_{\mathcal{A}}) \cong H_{c-1}(M_{\mathcal{A}}) \cong H^{c-1}(M_{\mathcal{A}})$, and gives part (ii).

It has been shown by Serre [Ser53, p. 217] that if $T$ is a topological space having the homotopy type of a finite CW-complex, and if $T$ is 1-connected and $H_i(T, \mathbf{Z}_2) \neq 0$ for at least one $i > 0$, then there are infinitely many dimensions $i$ such that the homotopy group $\pi_i(T)$ contains a subgroup isomorphic to $\mathbf{Z}$ or to $\mathbf{Z}_2$. This implies part (iii). ∎

We will now show that the Euler characteristic $\chi(M_{\mathcal{A}})$ equals zero in many interesting cases. See also Remark 7.6.

**Theorem 3.3** *Assume that $codim(x)$ is even, for all $x \in L_{\mathcal{A}}$. Then $\chi(M_{\mathcal{A}}) = 0$.*

**Proof:** We will use Proposition 2.1 and the fact that the Möbius function $\mu(\hat{0}, x)$ is the reduced Euler characteristic of $\Delta(\hat{0}, x)$, see e.g. [Sta86, p. 120].

$$\chi(M_{\mathcal{A}}) - 1 = \sum_{i \geq 0} (-1)^i \cdot \widetilde{\beta}^i(M_{\mathcal{A}}) =$$

$$= \sum_{i\geq 0}(-1)^i \cdot \sum_{x>\hat{0}} \tilde{\beta}_{codim(x)-2-i}(\hat{0},x) =$$

$$= \sum_{x>\hat{0}}\sum_{i\geq 0}(-1)^i \cdot \tilde{\beta}_{codim(x)-2-i}(\hat{0},x) =$$

$$= \sum_{x>\hat{0}}\sum_{i\geq -1}(-1)^i \cdot \tilde{\beta}_i(\hat{0},x) =$$

$$= \sum_{x>\hat{0}} \mu(\hat{0},x) = -1. \qquad \blacksquare$$

**Corollary 3.4** *If $\mathcal{A}$ is an arrangement of complex subspaces in $\mathbf{C}^n \cong \mathbf{R}^{2n}$, then $\chi(M_{\mathcal{A}}) = 0$.*

# 4 The $\Pi_{n,k}(l)$ partition lattices

In this section the homology groups and the homotopy type of the lattices $\Pi_{n,k}(l)$ will be described. The description includes the proof of Theorem 1.5, with several added details. When we speak about topological properties of a lattice $L$ we always have the order complex of $L - \{\hat{0},\hat{1}\}$ in mind. So, in terms of the notation introduced in Section 2 we have $\Delta(L) = \Delta(\hat{0},\hat{1})$, $\widetilde{H}_i(L) = \widetilde{H}_i(\hat{0},\hat{1})$, etc. .

It is well known that the lattice $\Pi_n$ of all partitions of the set $\{1,\ldots,n\}$ ordered by refinement is a geometric lattice of rank $n-1$ with Möbius function $\mu(\hat{0},\hat{1}) = (-1)^{n-1}\cdot(n-1)!$, see [Sta86, p. 128]. The following is therefore implied by [BW83, Theorem 5.1].

**Proposition 4.1** *The lattice $\Pi_n$ has the homotopy type of a wedge of $(n-1)!$ copies of the sphere $S^{n-3}$.*

Let $2 \leq n$, $2 \leq k$ and $0 \leq l$, and consider the partition lattice $\Pi_{n,k}(l) \subseteq \Pi_n$ defined in Section 1. The join operation of $\Pi_{n,k}(l)$ is the same as that of $\Pi_n$, whereas the meet operation of $\Pi_{n,k}(l)$ consists in first taking the meet in $\Pi_n$ and then breaking up all blocks $B$ such that $|B| < k$ and $B \cap \{1,\ldots,l\} = \emptyset$ into singletons.

The case $l = 0$ is the most interesting one for our purposes and we will make a few observations about it. If $3 \leq k \leq \frac{n}{2}$ then the lattice $\Pi_{n,k}(0)$ is not pure (i.e., there exist maximal chains of different lengths). For example, the partition (1 2 3)(4 5 6) covers the partition (1 2 3)(4)(5)(6) in $\Pi_{6,3}(0)$ but (1 2 3 4 5)(6) does not. On the other hand, if $\frac{n}{2} < k \leq n$ then the lattice $\Pi_{n,k}(0)$ is pure. In fact, it is easy to see that in this case $\Pi_{n,k}(0)$ is isomorphic to the inclusion lattice of all subsets $A \subseteq \{1,\ldots,n\}$ such that $|A| = 0$ or $|A| \geq k$. This is a rank-selected boolean lattice, and one can deduce from standard facts about rank-selected shellable posets (see e.g. [Bjö89, (11.13)]) that $\Pi_{n,k}(0)$ has the homotopy type of a wedge of $\binom{n-1}{k-1}$ copies of the $(n-k-1)$-dimensional sphere. This can also be derived via the observation that $\Pi_{n,k}(0)$ is anti-isomorphic to the $(n-k-1)$-skeleton of an $(n-1)$-simplex, and it will be proved again as a special case of the results in this section. It will turn out that $k = 2$ and $\frac{n}{2} < k \leq n$ are the only cases for which $\Pi_{n,k}(0)$ is homotopically a wedge of *equidimensional* spheres.

If $n < k$ the lattice $\Pi_{n,k}(0)$ consists of only the partition $(1)(2)\cdots(n)$. To guarantee non-degenerate lattices (with $\hat{0} \neq \hat{1}$) we will always require that either $k \leq n$ or $0 < l$. Therefore let us call $(n,k,l)$ an ADMISSIBLE TRIPLE if $n$, $k$ and $l$ are integers such that $2 \leq n$, $2 \leq k$, $0 \leq l$, and either $k \leq n$ or $0 < l$.

In the following let $\widetilde{B}^d_{n,k}(l)$ denote the rank of the reduced homology group $\widetilde{H}_d(\Pi_{n,k}(l))$, and $\widetilde{B}^d_{n,k} = \widetilde{B}^d_{n,k}(0)$.

**Lemma 4.2** *Let $(n,k,l)$ be an admissible triple with $0 < l$. If $n-1 \leq l$ or $n \leq k$, then $\Pi_{n,k}(l)$ has the homotopy type of a wedge of $(n-3)$-dimensional spheres. All homology groups are free, and the following formulas hold for their ranks :*

*(i) If $n-1 \leq l$, then*

$$\tilde{B}^d_{n,k}(l) = \begin{cases} (n-1)! & , \text{ if } d = n-3 \\ 0 & , \text{ otherwise} \end{cases}$$

*(ii) If $0 < l < n \leq k$, then $\tilde{B}^d_{n,k}(l) = l \cdot \tilde{B}^{d-1}_{n-1,k}(l)$, and hence*

$$\tilde{B}^d_{n,k}(l) = \begin{cases} l^{n-l-1} \cdot l! & , \text{ if } d = n-3 \\ 0 & , \text{ otherwise} \end{cases}$$

**Proof:** In case (i) we have that $\Pi_{n,k}(l) = \Pi_n$, so all claims follow from Proposition 4.1.

For the case (ii) we first observe that $n \geq 3$, and that $(1 \cdots n-1)(n) \in \Pi_{n,k}(l)$ since $l > 0$. We will investigate the complements of $(1 \cdots n-1)(n)$ in $\Pi_{n,k}(l)$. These correspond to partitions with $n-2$ blocks of size 1 and a single block $(j\ n)$ of size 2, where $j \in \{1,\ldots,l\}$. There are $l$ partitions of that kind. All of them are minimal in $\Pi_{n,k}(l) - \{\hat{0}\}$, and hence they form an antichain. For every complement $\pi$ of $(1 \cdots n-1)(n)$ the closed interval $[\pi, \hat{1}]$ in $\Pi_{n,k}(l)$ is isomorphic to $\Pi_{n-1,k}(l)$. So we infer from Proposition 2.3 that there is a homotopy equivalence

$$\Pi_{n,k}(l) \simeq \underset{l}{wedge}\, susp\big(\Pi_{n-1,k}(l)\big). \tag{4.1}$$

In particular, $\tilde{B}^d_{n,k}(l) = l \cdot \tilde{B}^{d-1}_{n-1,k}(l)$. Now by an $(n-l-1)$-fold application of the reduction (4.1) we arrive at the lattice $\Pi_{l+1,k}(l)$. The assertions then follow from the formula for $\tilde{B}^d_{l+1,k}(l)$ and the homotopy type information given in part (i). ∎

Now we are in a position to state and prove the basic theorem on the topological properties of the lattices $\Pi_{n,k}(l)$.

**Theorem 4.3** *Let $(n,k,l)$ be an admissible triple.*

*(i) The lattice $\Pi_{n,k}(l)$ has the homotopy type of a wedge of spheres (possibly of different dimensions). Therefore the homology groups are free.*

*(ii) For $0 \leq l \leq n-1$, $2 \leq k \leq n-1$, and all $d$, the following recursion formula holds for the rank $\tilde{B}^d_{n,k}(l)$ of the d-th reduced homology group (equivalently, the number of d-spheres in the wedge) :*

$$\tilde{B}^d_{n,k}(l) = \binom{n-1-l}{k-1} \cdot \tilde{B}^{d-1}_{n-k+1,k}(l+1) + l \cdot \tilde{B}^{d-1}_{n-1,k}(l).$$

**Proof:** We will use induction on $n \geq 2$. The degenerate case $n = 2$ is trivially correct, since $\Delta(\Pi_{2,k}(l)) = \{\emptyset\}$ which is the $(-1)$-dimensional sphere and $\widetilde{H}_{-1}(\{\emptyset\}) = \mathbf{Z}$. Assume that $n \geq 3$. If $n \leq l$, or if $n \leq k$ and $0 < l$, then correctness follows from Lemma 4.2. The case $n = k$ and $l = 0$ is also correct, since $\Delta(\Pi_{n,n}(0)) = \{\emptyset\}$. So let us assume that $0 \leq l \leq n-1$ and $2 \leq k \leq n-1$.

Our assumptions imply that the partition $(1 \cdots n-1)(n)$ is an element of $\Pi_{n,k}(l)$. Let $\pi$ be a complement of $(1 \cdots n-1)(n)$ in $\Pi_{n,k}(l)$. In this situation there can be no non-trivial block in $\pi$ which is contained in $\{1,\ldots,n-1\}$. Otherwise the meet of $\pi$ and $(1 \cdots n-1)(n)$ would be non-trivial. Therefore $\pi$ contains only one block of size greater than 1. Now assume that $B_1$ is the unique non-trivial block of $\pi$. Then by definition of $\Pi_{n,k}(l)$ either $|B_1| \geq k$ and $B_1 \cap \{1,\ldots,l\} = \emptyset$, or $B_1$ intersects $\{1,\ldots,l\}$ non-trivially. In the first case we immediately conclude that $|B_1| = k$ and $n \in B_1$ from the fact that $\pi$ complements $(1\cdots n-1)(n)$. In the second case it follows by the same reasoning that $|B_1| = 2$ and $B_1 = (j\ n)$ for an integer $j$ between 1 and $l$. These partitions are all the complements of $(1\cdots n-1)(n)$, and it is easily seen that they from an antichain in $\Pi_{n,k}(l)$. Therefore we can apply Proposition 2.3. All complements $\pi$ are minimal in $\Pi_{n,k}(l) - \{\hat{0}\}$, so our considerations are reduced to the intervals $[\pi, \hat{1}]$. We have to distinguish between the two different types of complements :

(a) The unique non-trivial block $B_1$ in $\pi$ is a block of size $k$ such that $B_1 \cap \{1,\ldots,l\} = \emptyset$. Regarding this block as a single point shows that in the interval $[\pi, \hat{1}]$ all blocks containing this point are allowed to occur. For the other blocks the same restrictions hold as before. After a suitable renumbering we obtain $[\pi, \hat{1}] \cong \Pi_{n-k+1,k}(l+1)$.

(b) The unique non-trivial block in $\pi$ is a block $(j\ n)$, where $j \in \{1,\ldots,l\}$. Again we contract this block to a single point. Analogously to case (a) we infer that in $[\pi, \hat{1}]$ all blocks containing this point are allowed. Again the remaining blocks have to obey the inherited restrictions. Now suitable renumbering shows that $[\pi, \hat{1}] \cong \Pi_{n-1,k}(l)$.

There are $\binom{n-1-l}{k-1}$ complements which satisfy condition (a) and $l$ complements which satisfy condition (b). Since the sets of complements described by (a) and (b) are disjoint the decomposition provided by Proposition 2.3 proves the recursion formula of part (ii) and, with the induction assumption, also the claim of part (i). ∎

The recursion given in part (ii) provides an algorithm for computing $\widetilde{B}^d_{n,k}(l)$ for any admissible triple $(n,k,l)$ and any $d \in \mathbf{Z}$. If $n \leq l$ or $n \leq k$ the answer is already provided by Lemma 4.2. Otherwise repeated use of the recursion will lead to parameter values where Lemma 4.2 is applicable. In actual computations one can often stop branches of the recursion before Lemma 4.2 becomes applicable, since $\widetilde{B}^d_{n,k}(l)$ may be zero for dimensional reasons. Clearly,

$$\widetilde{B}^d_{n,k}(l) = 0, \text{ if } d > n-3 \text{ or } d < -1.$$

The following computation of $\widetilde{B}^0_{n,k}(l)$ provides a stopping rule which somewhat improves on $\widetilde{B}^{-2}_{n,k}(l) = 0$. We omit the proof, which is via case-by-case checking.

**Lemma 4.4** *For all admissible triples $z = (n,k,l)$*

$$\widetilde{B}^0_{n,k}(l) = \begin{cases} n-1 & , \text{ if } z = (n, n-1, 0) \quad , \quad n \geq 3 \\ 1 & , \text{ if } z = (n, n-1, 1) \quad , \quad n \geq 4 \\ 1 & , \text{ if } z = (3, k, 1) \quad , \quad k \geq 3 \\ 2 & , \text{ if } z = (3, k, l) \quad , \quad k \geq 2, l \geq 2 \\ 0 & , \text{ in all other cases.} \end{cases}$$

Having established the basic recursion formulas it is natural to ask which homology groups vanish and which do not. We will investigate this in detail and provide a complete answer for the case $l = 0$. The case $l \neq 0$ can be treated similarly. Since the case $k = 2$ is just the case of the full partition lattice, which is completely understood (see Proposition 4.1), we may assume that $k > 2$.

What has to be done is to trace the recursion given in Theorem 4.3. For a single step in the recursion we will refer to the left term $\widetilde{B}^{d-1}_{n-k+1,k}$ in Theorem 4.3 as a branch of type $L$ and to the right term $\widetilde{B}^{d-1}_{n-1,k}$ as a branch of type $R$. One immediately observes that the parameter $k$ is constant. Hence we may omit the reference to $k$ in the further derivations.

Assume that we start our recursion with the parameters $(n, d, l)$ where $l = 0$. Assume further that we meet the conditions of Lemma 4.2 (i) or (ii) after a suitable number of recursion steps with the parameters $(n', d', l')$. Let $t_L$ be the number of recursion steps of type $L$ and $t_R$ the number of steps of type $R$ which had to be taken. Then the parameters satisfy :

$$(4.2) \qquad n' = n - (k-1) \cdot t_L - t_R, \quad d' = d - t_L - t_R \quad \text{and} \quad l' = t_L.$$

From Lemma 4.2 we deduce that we get a contribution to homology only in case $n' - 3 = d'$, or equivalently $n - (k-1) \cdot t_L - t_R - 3 = d - t_L - t_R$. Adding $t_L + t_R$ to both sides we see that the dimension $d$ to which the contribution is made actually depends only on $t_L$, and the defining equation for $d = d(t_L)$ is

$$(4.3) \qquad d(t_L) = n - (k-2) \cdot t_L - 3.$$

Therefore, the dimensions $d$ for which there are non-vanishing homology groups $\widetilde{H}_d(\Pi_{n,k}(0))$ are determined by the parameters $t_L$ which can occur in a complete recursion path starting with the parameters $n$, $2 < k < n$ and $l = 0$ and involving exactly $t_L$ branchings of type $L$. The parameter $l$ is 0 at the beginning, so the first step in the recursion must be of type $L$ (since we want to count only non-zero contributions to the rank of the homology group). Hence,

$$t_L \geq 1$$

and the ranks of all homology groups are divisible by $\binom{n-1}{k-1}$. Since one can always meet the condition of Lemma 4.2 (ii) by branching a suitable number of times into branch $R$ it remains to find tight upper and lower bounds for the parameter $t_L$.

The lower bound is given by $t_L = 1$. This bound is achieved by going into branch $L$ once and then branching to the $R$ with the parameter $l = 1$ until the parameters meet the conditions of Lemma 4.2. The multiplication by $l$ does not change the value and the factor given by Lemma 4.2 (i) or (ii) is always 1 in this case. Therefore the maximal-dimensional non-vanishing homology group is given by :

$$(4.4) \qquad d_{max} = n - k - 1, \qquad \widetilde{B}_{n,k}^{d_{max}} = \binom{n-1}{k-1}.$$

The formula for the rank follows from the fact that the parameters $t_L$ and $t_R$ and the path in the recursion are uniquely determined in this case. This simple observation is an immediate consequence of the maximality of $d_{max}$.

Let us now investigate the general case. Here the combinatorial analysis of the recursion paths gets a bit more involved. Assume that $\widetilde{B}_{n,k}^d(0) \neq 0$, and let $\rho$ be a recursion path that contributes a non-zero summand to $\widetilde{B}_{n,k}^d(0)$. For instance, we might have $\rho = LRLLLRRL$, in terms of the left and right branchings. Let us assume that $\rho$ ends with a left branch $L$, if not then truncate $\rho$ so that it does (recall that $\rho$ must start with $L$). Write $\rho = \rho' L$. As before we let $t_L$ be the number of $L$-branchings and $t_R$ the number of $R$-branchings in $\rho$, so $t_L \geq 1$ and $t_R \geq 0$. When applied after the branching sequence $\rho'$, the recursion formula (Theorem 4.3) has the form (see (4.2) for the parameter values) :

$$\widetilde{B}^{d-(t_L-1)-t_R}_{n-(k-1)\cdot(t_L-1)-t_R,k}(t_L - 1) =$$

$$= \binom{n - kt_L + k - 1 - t_R}{k-1} \cdot \widetilde{B}^{d-t_L-t_R}_{n-(k-1)\cdot t_L - t_R,k}(t_L) + (t_L - 1) \cdot \widetilde{B}^{d-(t_L-1)-t_R-1}_{n-(k-1)\cdot(t_L-1)-t_R-1,k}(t_L - 1)$$

By assumption $\rho = \rho' L$ makes a non-zero contribution, hence

$$n - kt_L + k - 1 - t_R \geq k - 1,$$

which for $(t_L, t_R) \geq (1, 0)$ is equivalent to

$$\text{(4.5)} \qquad \begin{cases} 1 \leq t_L \leq \lfloor \frac{n}{k} \rfloor \\ 0 \leq t_R \leq n - t_L k. \end{cases}$$

So, (4.5) gives necessary conditions that an $L$-ending recursion path contributes to non-zero homology, and the contribution then is to dimension $d(t_L)$, as shown by (4.3). We will now show that this condition is sufficient, in the sense that if $1 \leq t \leq \lfloor \frac{n}{k} \rfloor$, then $\tilde{B}^{d(t)}_{n,k}(0) \neq 0$. For this we will work with complete recursion paths, and it will be useful to picture these as paths in a rectangular integer lattice.

Two recursion paths with the same distribution of $L$-branches and $R$-branches must end at the same Betti number $\tilde{B}^{d'}_{n',k}(l')$. This follows from the fact that the effect on the parameters of a sequence $LR$ is the same as the effect of the sequence $RL$. Therefore we obtain the following description.

Let $n \geq k > 2$ and $1 \leq t \leq \lfloor \frac{n}{k} \rfloor$ be natural numbers. We set $s = n - t \cdot k$. Let $\mathcal{G}(n, k, t)$ denote the integer lattice $\{0, \ldots, s\} \times \{1, \ldots, t\} \cup \{(0, 0)\}$. We regard $\mathcal{G}(n, k, t)$ as a directed graph whose edges are $((i, j), (i+1, j))$ and $((i, j), (i, j+1))$. Hence every maximal path starts in $(0, 0)$ and ends in $(s, t)$. Now we label the edge $((i, j), (i+1, j))$ by $j$ and the edge $((i, j), (i, j+1))$ by $\binom{n-jk-i-1}{k-1}$.

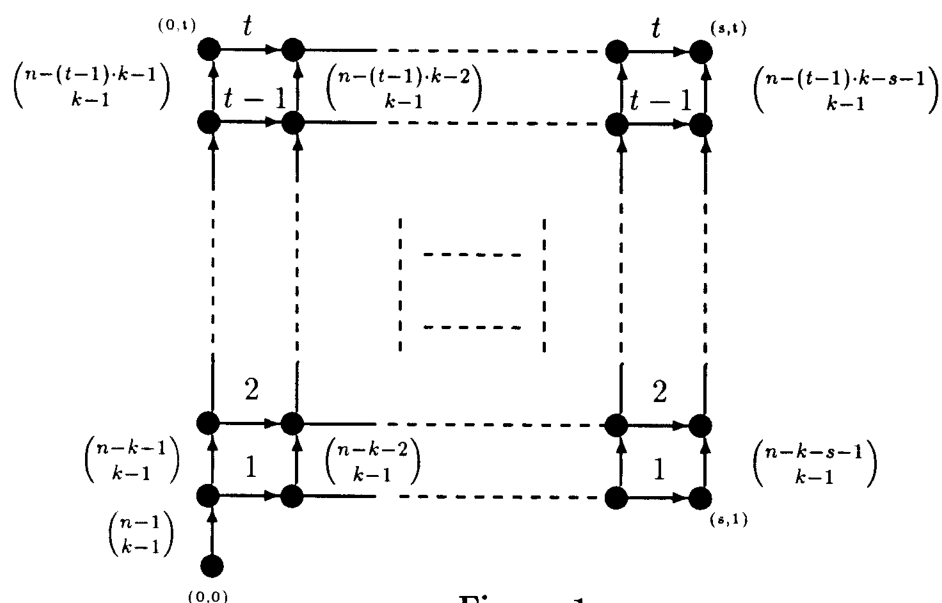


**Figure 1**

Finally we label each node $(i, j)$ by $\tilde{B}^{d(t)-j-i}_{n-j\cdot(k-1)-i,k}(j)$. The labels are chosen to correspond to the Betti numbers (nodes) and coefficients (edges) that will be encountered in a recursion process starting from $\tilde{B}^{d(t)}_{n,k}(0)$.

We claim that every recursion path that ends in the sink node $(s, t)$ will give a non-zero contribution to $\tilde{B}^{d(t)}_{n,k}(0)$, and conversely every such contributing path can be extended or truncated to one ending in $(s, t)$. For the first claim it suffices to observe that the node $(s, t)$ is labeled by $\tilde{B}^{d(t)-t-s}_{n-t\cdot(k-1)-s,k}(t) = \tilde{B}^{t-3}_{t,k}(t)$, which by Lemma 4.2 (i) equals $(t-1)! \neq 0$, and that all edge-labels in $\mathcal{G}(n, k, t)$ are non-zero. For the second claim we use the fact (4.5) that if a path $\rho = \rho' RR \cdots R$ makes a non-zero contribution to $\tilde{B}^{d(t)}_{n,k}(0)$ and if $\rho'$ which ends with $L$ has step distribution $(t_L, t_R)$, then $t_L = t$ and

$0 \le t_R \le n - tk = s$. Hence, such a path $\rho$ meets the upper border (the $y = t$ line) of the graph $\mathcal{G}(n, k, t)$ at the point $(t_R, t)$ to the left of the sink node $(s, t)$, and then (possibly) takes some steps to the right. By adding or deleting such $R$-steps, the path $\rho$ can be made to end in $(s, t)$. Figure 2 shows a close-up view of the upper right-hand corner of the labeled graph $\mathcal{G}(n, k, t)$, which is helpful for verifications.

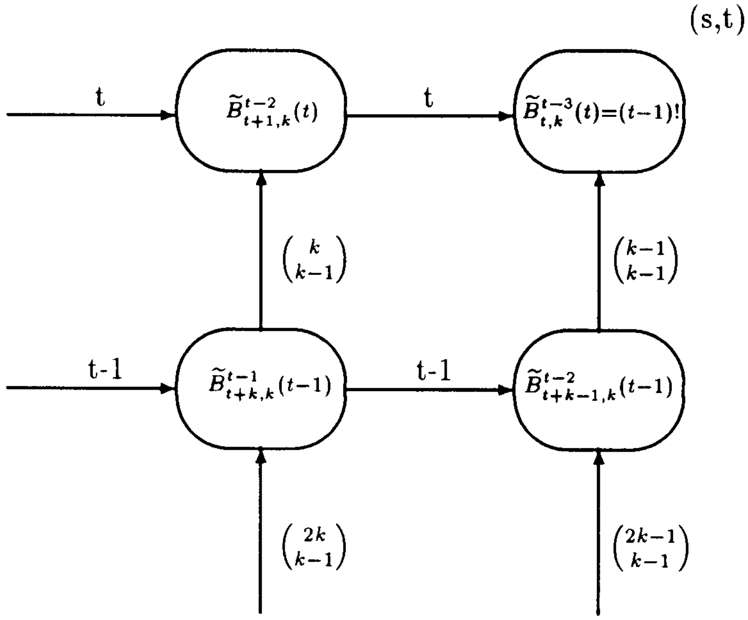


**Figure 2**

Let us assign to each maximal path $\rho$ in $\mathcal{G}(n, k, t)$ the product of the labels of its edges, and call this number $w(\rho)$ the WEIGHT of $\rho$. The preceding discussion has shown that

$$\widetilde{B}^{d(t)}_{n,k}(0) = (t-1)! \cdot \sum_{\rho} w(\rho). \tag{4.6}$$

Each path $\rho$ can be encoded by the sequence $0 \le i_1 \le i_2 \le \cdots \le i_{t-1} \le s$, given by the edges $(i_j, j) \to (i_j, j+1)$ that belong to $\rho$. This leads to a more efficient reformulation of (4.6), which is stated in the following theorem together with the other major conclusions that have been reached.

**Theorem 4.5** *Assume that* $2 < k \le n$, *and let* $\widetilde{B}^d_{n,k} = rank \widetilde{H}_d(\Pi_{n,k})$. *Then*

*(i)* $\widetilde{B}^d_{n,k} \ne 0$ *if and only if* $d = n - 3 - t \cdot (k-2)$ *for some integer* $t$ *such that* $1 \le t \le \lfloor \frac{n}{k} \rfloor$.

*(ii) If* $1 \le t \le \lfloor \frac{n}{k} \rfloor$, *then*

$$\widetilde{B}^{n-3-t\cdot(k-2)}_{n,k} = (t-1)! \cdot \binom{n-1}{k-1} \cdot \sum_{0=i_0 \le i_1 \le \cdots \le i_{t-1} \le i_t = n-tk} (*),$$

*where* $(*) =$

$$\binom{n-k-1-i_1}{k-1} \cdot \binom{n-2k-1-i_2}{k-1} \cdots \binom{n-(t-1)\cdot k-1-i_{t-1}}{k-1} \cdot 1^{i_1-i_0} \cdot 2^{i_2-i_1} \cdots t^{i_t-i_{t-1}}.$$

This unwieldy expression specializes to a more manageable form for the two highest-dimensional non-vanishing Betti numbers, and when $k$ divides $n$ also for the lowest-dimensional one.

**Corollary 4.6**

*(i)* $\tilde{B}_{n,k}^{n-k-1} = \binom{n-1}{k-1}$

*(ii)* $\tilde{B}_{n,k}^{n-2k+1} = \binom{n-1}{k-1} \cdot \sum_{j=0}^{n-2k} \binom{k-1+j}{k-1} \cdot 2^j$, *if* $n \geq 2k$.

*(iii) If* $\frac{n}{k} = q \in \mathbf{Z}$, *then* $\tilde{B}_{n,k}^{n-3-q(k-2)} = (q-1)! \cdot \prod_{j=0}^{q-1} \binom{n-1-jk}{k-1}$.

For example, we compute

$$\tilde{B}_{16,4}^{11} = \binom{15}{3} = 455$$

$$\tilde{B}_{16,4}^{9} = \binom{15}{3} \cdot \sum_{j=0}^{8} \binom{3+j}{3} \cdot 2^j = 29\ 819\ 335$$

$$\tilde{B}_{16,4}^{5} = 3! \cdot \binom{15}{3} \cdot \binom{11}{3} \cdot \binom{7}{3} \cdot \binom{3}{3} = 15\ 765\ 750.$$

Also, for all $k \geq 3$ :

$$\tilde{B}_{2k,k}^{d} = \begin{cases} \binom{2k-1}{k-1} & , \quad \text{if } d = 1 \text{ or } d = k-1 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{4.7}$$

It is interesting to compare the following known formula for the Euler characteristic of $\Pi_{n,k}$ with the expressions for the individual Betti numbers in Theorem 4.5.

**Proposition 4.7** *[BLY92, Theorem 5.6] Let* $\alpha_1, \ldots, \alpha_{k-1}$ *be the roots of the polynomial* $\sum_{i=0}^{k-1} \frac{x^i}{i!}$, $k \geq 2$. *Then*

$$\sum_{d \geq 0} (-1)^d \cdot \tilde{B}_{n,k}^{d} = -(n-1)! \cdot \sum_{i=1}^{k-1} \alpha_i^{-n}.$$

We will end this section by generalizing the preceding results to lower intervals

$$[\hat{0}, \pi] = \{\pi' \in \Pi_{n,k} \mid \hat{0} \leq \pi' \leq \pi\} \text{ in } \Pi_{n,k} = \Pi_{n,k}(0).$$

**Theorem 4.8** *Let* $\pi \in \Pi_{n,k}$, $2 < k \leq n$, *be a partition with* $m$ *non-singleton blocks of sizes* $a_1, \ldots, a_m$. *Then*

*(i)* $[\hat{0}, \pi] \cong \Pi_{a_1,k} \times \Pi_{a_2,k} \times \cdots \times \Pi_{a_m,k}$.

*(ii) The interval* $[\hat{0}, \pi]$ *has the homotopy type of a wedge of spheres. Its homology groups are therefore free.*

*(iii)* $\tilde{H}_d(\hat{0}, \pi) \neq 0$ *if and only if* $d = \sum_{i=1}^{m} a_i - m - 2 - t \cdot (k-2)$, *for some integer* $t$ *satisfying* $m \leq t \leq \sum_{i=1}^{m} \lfloor \frac{a_i}{k} \rfloor$.

(iv) $rank\widetilde{H}_d(\hat{0},\pi) = \sum_{p_1+\cdots+p_m=d-2(m-1)} \widetilde{B}^{p_1}_{a_1,k} \cdot \widetilde{B}^{p_2}_{a_2,k} \cdots \widetilde{B}^{p_m}_{a_m,k}$.

(v) $rank\widetilde{H}_{d_{max}}(\hat{0},\pi) = \prod_{i=1}^{m} \binom{a_i-1}{k-1}$, for $d_{max} = \sum_{i=1}^{m} a_i - m - 2 - m\cdot(k-2)$.

(vi) $rank\widetilde{H}_d(\hat{0},\pi)$ is divisible by $\prod_{i=1}^{m} \binom{a_i-1}{k-1}$, for all $d$.

**Proof:** The direct product decomposition (i) is obvious from the fact that the non-singleton blocks can be independently refined. Theorem 4.3 and $(m-1)$-fold use of Proposition 2.6 prove part (ii) and yield the formula

$$\widetilde{H}_d(\hat{0},\pi) = \bigoplus_{p_1+\cdots+p_m=d-2(m-1)} \left(\widetilde{H}_{p_1}(\Pi_{a_1,k}) \otimes \cdots \otimes \widetilde{H}_{p_m}(\Pi_{a_m,k})\right),$$

which gives part (iv). The rest can be easily deduced from the information given by Theorem 4.5. ∎

# 5 The "$k$-equal" arrangements and their manifolds

Fix $2 \le k \le n$, and for each $k$-subset $i_1 < i_2 < \cdots < i_k$ of $\{1,\ldots,n\}$ let $K^{\mathbf{R}}_{i_1,\ldots,i_k} = \{\underline{x} \in \mathbf{R}^n \mid x_{i_1} = x_{i_2} = \cdots = x_{i_k}\}$. The collection $\mathcal{A}^{\mathbf{R}}_{n,k} = \{ K^{\mathbf{R}}_{i_1,\ldots,i_k} \mid 1 \le i_1 < \cdots < i_k \le n \}$ of $(n-k+1)$-dimensional subspaces is the real "$k$-equal" arrangement. Replacing $\mathbf{R}$ by $\mathbf{C}$ we get the complex $k$-equal arrangement $\mathcal{A}^{\mathbf{C}}_{n,k}$, which can be regarded as an arrangement of $2(n-k+1)$-dimensional subspaces in $\mathbf{R}^{2n} \cong \mathbf{C}^n$. The spaces $M^{\mathbf{R}}_{n,k}$ and $M^{\mathbf{C}}_{n,k}$ considered in Section 1 are the complements of these arrangements.

**Lemma 5.1** *The intersection lattice of $\mathcal{A}^{\mathbf{R}}_{n,k}$, and of $\mathcal{A}^{\mathbf{C}}_{n,k}$, is isomorphic to $\Pi_{n,k}$. Suppose that $\pi \in \Pi_{n,k}$ is a partition with $m$ non-singleton blocks of sizes $a_1,\ldots,a_m$, respectively. Then, viewed as a subspace of $\mathbf{R}^n$ (resp. of $\mathbf{C}^n$) via such an isomorphism, $\pi$ has the following real codimension :*

(i) $codim^{\mathcal{A}^{\mathbf{R}}}(\pi) = \sum_{i=1}^{m} a_i - m$,

(ii) $codim^{\mathcal{A}^{\mathbf{C}}}(\pi) = 2\cdot\left(\sum_{i=1}^{m} a_i - m\right)$.

**Proof:** Associate with each partition $\pi \in \Pi_n$ the subspace

$$K^{\mathbf{R}}_{\pi} = \{ \underline{x} \in \mathbf{R}^n \mid i,j \in B \Rightarrow x_i = x_j, \text{ for all } i,j \in \{1,\ldots,n\} \text{ and every block } B \text{ of } \pi \}.$$

The dimension of $K_\pi$ is clearly equal to the number of blocks of $\pi$, which is $n - \sum_{i=1}^{m}(a_i-1)$. If $\pi$ has only one non-singleton block $\{i_1,\ldots,i_k\}$ then $K_\pi = K_{i_1,\ldots,i_k}$. This identifies the minimal elements of $\Pi_{n,k} - \{\hat{0}\}$ with the subspaces in $\mathcal{A}^{\mathbf{R}}_{n,k}$, and it is easy to verify that $K_\pi \cap K_{\pi'} = K_{\pi\vee\pi'}$, from which the identification of $\Pi_{n,k}$ with the intersection lattice of $\mathcal{A}^{\mathbf{R}}_{n,k}$ then follows.

For the complex case, make the replacement $\mathbf{R} \to \mathbf{C}$ and double the dimensions. ∎

We will now deal with the real and complex cases separately. Define

$$\widetilde{\beta}^{\mathbf{R},d}_{n,k} = rank\widetilde{H}^d(M^{\mathbf{R}}_{n,k}) \text{ and } \widetilde{\beta}^{\mathbf{C},d}_{n,k} = rank\widetilde{H}^d(M^{\mathbf{C}}_{n,k}).$$

**Theorem 5.2** *For each partition $\pi \neq \hat{0}$, let $m$ be the number of non-singleton blocks and $a_1,\ldots,a_m$ their sizes. Then*

$$\widetilde{\beta}^{\mathbf{R},d}_{n,k} = \sum_{\pi\in\Pi^{>\hat{0}}_{n,k}} \sum_{q_1+\cdots+q_m=d} \widetilde{B}^{a_1-3-q_1}_{a_1,k} \cdots \widetilde{B}^{a_m-3-q_m}_{a_m,k}.$$

**Proof:** Proposition 2.1 gives

$$\widetilde{H}^d(M^{\mathbf{R}}_{n,k}) \cong \bigoplus_{\pi\in\Pi^{>\hat{0}}_{n,k}} \widetilde{H}_{codim(\pi)-2-d}(\hat{0},\pi). \tag{5.1}$$

Passing to ranks, and using Theorem 4.8 (iv) and Lemma 5.1, we obtain

$$\widetilde{\beta}^{\mathbf{R},d}_{n,k} = \sum_{\pi>\hat{0}} \sum_{(*)} \widetilde{B}^{p_1}_{a_1,k} \cdots \widetilde{B}^{p_m}_{a_m,k},$$

with summation $(*)$ over all $m$-tuples $(p_1,\ldots,p_m)$ such that

$$p_1 + \cdots + p_m = \sum_{i=1}^{m} a_i - m - 2 - d - 2(m-1) = \sum_{i=1}^{m} a_i - 3m - d.$$

The substitutions $p_i = a_i - 3 - q_i$ turn this formula into final form. ∎

All the supporting steps have now been taken, so that the main results of Section 1 can be proven.

**Proof of Theorem 1.1:** Formula (5.1) together with Theorem 4.8 (ii) show that the cohomology of $M^{\mathbf{R}}_{n,k}$ is free.

(a) This part follows from Theorem 3.1 (i).

(b) For this we must combine the information from Theorems 4.5 and 5.2. The formula for $\widetilde{\beta}^{\mathbf{R},d}_{n,k}$ shows that $\widetilde{\beta}^{\mathbf{R},d}_{n,k} = 0$ if $(k-2) \nmid d$, since then also $(k-2) \nmid q_i$ for some $i$ which shows that all terms are equal to zero. A general partition $\pi > \hat{0}$ makes non-zero contributions to cohomology in dimensions $t_1(k-2) + \cdots + t_m(k-2)$, $1 \le t_i \le \lfloor \frac{a_i}{k} \rfloor$, $1 \le i \le m$, i.e., precisely to the dimensions

$$t \cdot (k-2), \quad \text{for} \quad m \le t \le \sum_{i=1}^{m} \lfloor \frac{a_i}{k} \rfloor \le \lfloor \frac{n}{k} \rfloor. \tag{5.2}$$

In particular, $\widetilde{\beta}^{\mathbf{R},t(k-2)}_{n,k}$ gets non-zero contributions from $\pi = \hat{1}$ (for which $m = 1$ and $a_1 = n$) for all $1 \le t \le \lfloor \frac{n}{k} \rfloor$, and $\widetilde{\beta}^{\mathbf{R},t(k-2)}_{n,k} = 0$ for all $t$ outside this range.

(c) The preceding analysis (5.2) shows that $\widetilde{\beta}^{\mathbf{R},k-2}_{n,k}$ gets a non-zero contribution only from partitions $\pi > \hat{0}$ with $m = 1$, i.e., with exactly one non-singleton block. There are $\binom{n}{i}$ such partitions with a block of size $i$, and they each contribute $\binom{i-1}{k-1}$ by Corollary 4.6 (i). ∎

The analysis of $\tilde{\beta}^{\mathbf{R},t(k-2)}_{n,k}$ just made shows that while in principle a closed formula valid for all $1 \le t \le \lfloor \frac{n}{k} \rfloor$ could be produced by inserting the formula of Theorem 4.5 into that of Theorem 5.2, such an expression would be quite awkward. The superposition of contributions of different form coming from the various $\pi \in \Pi_{n,k}$ makes "nice" formulas for the $t \ge 2$ cases in general seem improbable. Let us note two very special exceptions ($k > 2$):

$$(5.3) \qquad \begin{aligned} \tilde{\beta}^{\mathbf{R},q(k-2)}_{qk,k} &= \binom{qk}{k,\ldots,k} \\ \tilde{\beta}^{\mathbf{R},q(k-2)}_{qk+1,k} &= \binom{qk+1}{k,\ldots,k,k+1} \cdot (qk+k+1) \end{aligned}$$

We will now continue with the complex case.

**Theorem 5.3** *Define $m$ and $a_i$ (depending on $\pi$) as in Theorem 5.2. Then*

$$\tilde{\beta}^{\mathbf{C},d}_{n,k} = \sum_{\pi \in \Pi^{>\hat{0}}_{n,k}} \; \sum_{q_1+\cdots+q_m = d+m-\sum_{i=1}^{m} a_i} \tilde{B}^{a_1-3-q_i}_{a_1,k} \cdots \tilde{B}^{a_m-3-q_m}_{a_m,k}.$$

**Proof:** Argue as for Theorem 5.2, but with the codimension function of $\mathcal{A}^{\mathbf{C}}$ replacing that of $\mathcal{A}^{\mathbf{R}}$ (see Lemma 5.1). ∎

**Proof of Theorem 1.2 :** The freeness of cohomology follows from Proposition 2.1 and Theorem 4.8 (ii).

(a) See Theorem 3.1 (i).

(b) Theorems 4.5 and 5.3 show that $\tilde{\beta}^{\mathbf{C},d}_{n,k} \neq 0$ if and only if

$$d = \sum_{i=1}^{m} a_i - m + t_1 \cdot (k-2) + \cdots + t_m \cdot (k-2),$$

for some integers $m, a_1, \ldots, a_m, t_1, \ldots, t_m$ such that $1 \le m \le \lfloor \frac{n}{k} \rfloor$, $k \le a_i \le n$, $\sum_{i=1}^{m} a_i \le n$, and $1 \le t_i \le \lfloor \frac{a_i}{k} \rfloor$. Putting $t = t_1 + \cdots + t_m$ it is easy to see that this condition on $d$ is equivalent to saying that

$$t \cdot (k-2) - m + tk \le d \le t \cdot (k-2) - m + n$$

for some integers $1 \le m \le t \le \lfloor \frac{n}{k} \rfloor$.

(c) See Theorem 3.1 (iii).

(d) See Corollary 3.4. ∎

Theorem 1.2 (b) shows that the range of positive dimensions for which $M_{n,k}^{\mathbf{C}}$ has non-vanishing cohomology is a union of integer intervals $[t \cdot (k-2) - m + tk, t \cdot (k-2) - m + n]$ indexed by the parameters $(m,t)$, $1 \leq m \leq t \leq \lfloor \frac{n}{k} \rfloor$. These intervals frequently overlap, but their union is not necessarily connected. Clearly, the least and the greatest such dimensions $d_{min}$ and $d_{max}$, are found for the choices $(m,t) = (1,1)$ and $(m,t) = (1, \lfloor \frac{n}{k} \rfloor)$. This gives the explicit expressions

$$d_{min} = 2k - 3 \quad \text{and} \quad d_{max} = n - 1 + \lfloor \frac{n}{k} \rfloor \cdot (k-2). \tag{5.4}$$

The rank $\widetilde{\beta}_{n,k}^{\mathbf{C},d_{min}} = \binom{n}{k}$ was already computed. We observe that

$$\widetilde{\beta}_{n,k}^{\mathbf{C},d_{max}} = \widetilde{B}_{n,k}^{n-3-\lfloor \frac{n}{k} \rfloor \cdot (k-2)}, \tag{5.5}$$

as can be deduced from Theorem 5.3 (the only contribution in this case comes from $\pi = \hat{1}$).

Note that for $k > \frac{n}{2}$ the choice $(m,t) = (1,1)$ is unique, and the cohomology of $M_{n,k}^{\mathbf{C}}$ is found precisely in the interval of dimensions $[2k-3, n+k-3]$.

The case $k = \frac{n}{2}$ is interesting. Here there are 3 choices for $(m,t)$ : $(1,1), (1,2)$ and $(2,2)$, and the two latter produce single element intervals. Thus the total range of dimensions for non-vanishing cohomology is in this case :

$$[2k-3, 3k-3] \cup \{4k-6, 4k-5\}.$$

See e.g. the case $(n,k) = (16,8)$ in the tables 8.2. Another curious feature here is that for $k \geq 4$ :

$$\widetilde{\beta}_{2k,k}^{\mathbf{C},3k-3} = \widetilde{\beta}_{2k,k}^{\mathbf{C},4k-6} = \widetilde{\beta}_{2k,k}^{\mathbf{C},4k-5} = \binom{2k-1}{k-1}.$$

Here we have $\widetilde{\beta}_{2k,k}^{\mathbf{C},3k-3} = \widetilde{B}_{2k,k}^{k-1}$ and $\widetilde{\beta}_{2k,k}^{\mathbf{C},4k-5} = \widetilde{B}_{2k,k}^{1}$, see (5.5) and (4.7), and $\widetilde{\beta}_{2k,k}^{\mathbf{C},4k-6}$ equals the number of partitions of $\{1,\ldots,2k\}$ into 2 blocks of size $k$ (which each contributes 1 to the rank of cohomology in this dimension). Thus there are always two "isolated" isomorphic cohomology groups of very high dimension in this case.

**Proof of Theorem 1.3 :** This follows from Proposition 2.2 and Theorem 4.8 (ii), see also Lemma 2.5. ∎

**Proof of Corollary 1.4 :** See Corollary 3.2. ∎

# 6 Generalized "$k$-equal" arrangements

In this section we will generalize the notion of $k$-equal arrangements and show that several of the main facts remain true. This generalization was suggested by a question from V.A. Vassiliev, see Remark 7.8. The arguments here are completely parallel to the ones used in Sections 4 and 5, so we will omit many details. We will first treat the intersection lattices of these arrangements using their combinatorial characterization, and then deal with the arrangements and their complements.

Let $2 \leq k \leq r$ and $n_1 \geq 1, \ldots, n_r \geq 1$, $l \geq 0$ be integers. Put $n = n_1 + \cdots + n_r + l$, and fix a partition $\{1,\ldots,n\} = \{1,\ldots,l\} \cup E_1 \cup \cdots \cup E_r$ such that $|E_i| = n_i$ for all $1 \leq i \leq r$. Define $\Pi_{n_1,\ldots,n_r;k}(l)$ to be the family of all partitions $\pi \in \Pi_n$ such that each non-singleton block $B \in \pi$ satisfies one of the following conditions :

(i) $card\{ \; i \mid E_i \cap B \neq \emptyset \; \} \geq k$,

(ii) $B \cap \{1,\ldots,l\} \neq \emptyset$.

Then $\Pi_{n_1,\ldots,n_r;k}(l)$ is a join-sublattice of $\Pi_n$, and the minimal elements of $\Pi_{n_1,\ldots,n_r;k}(l) - \{\hat{0}\}$ are the partitions of $\{1,\ldots,n\}$ with exactly one non-singleton block $B_1$ such that either $B_1$ is a $k$-block that intersects $k$ different classes $E_i$ or else $B_1$ is a 2-block that touches the set $\{1,\ldots,l\}$. Note that the partition lattices treated in Section 4 are the special cases $n_1 = n_2 = \cdots n_r = 1$ :

$$\Pi_{1,\ldots,1;k}(l) = \Pi_{r+l,k}(l).$$

**Theorem 6.1**

*(i) The lattice $\Pi_{n_1,\ldots,n_r;k}(l)$ has the homotopy type of a wedge of spheres. Consequently, its homology groups are free.*

*(ii) If $\widetilde{H}_d(\Pi_{n_1,\ldots,n_r;k}(l)) \neq 0$, then $d = n_1 + \cdots + n_r + l - 3 - t \cdot (k-2)$ for some $t \geq 0$.*

*(iii) If $l = 0$, then $t \geq 1$ in part (ii).*

**Proof:** Theorems 4.3 and 4.5 show that the statements here are true for $n_1 = \cdots = n_r = 1$. Part (ii) is actually not covered by Theorem 4.5 as stated when $l > 0$, but it is easy to check that the reasoning around formulas (4.2) and (4.3) is valid also when $l > 0$, and this is all that is needed. Let $\widetilde{B}^d_{n_1,\ldots,n_r;k}(l) = rank \widetilde{H}_d(\Pi_{n_1,\ldots,n_r;k}(l))$. We will use induction on $n_1 + \cdots + n_r - r \geq 0$, based on the fact that the result is true when this quantity equals zero (i.e., when $n_1 = \cdots = n_r = 1$).

Suppose that $n_r \geq 2$, and let $x \in E_r$. Consider the complements in $\Pi_{n_1,\ldots,n_r;k}(l)$ of the partition $(\{1,\ldots,n\} - \{x\})(x)$. These complements $\pi$ are minimal in $\Pi_{n_1,\ldots,n_r;k}(l) - \{\hat{0}\}$, and their unique non-singleton block $B_1$ must either be a $k$-block containing $x$ and touching $k-1$ of the classes $E_1,\ldots,E_{r-1}$, or else $B_1$ must be a 2-block $(j\ x)$ with $1 \leq j \leq l$. Arguing as in the proof of Theorem 4.3 about the structure of the upper intervals $[\pi, \hat{1}]$, and using Proposition 2.3, we are led to the recursion formula :

$$(6.1) \qquad \widetilde{B}^d_{n_1,\ldots,n_r;k}(l) = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq r-1} n_{i_1} \cdots n_{i_{k-1}} \cdot \widetilde{B}^{d-1}_{n'_1,\ldots,n'_{r-1},n_r-1;k}(l+1) +$$

$$+ l \cdot \widetilde{B}^{d-1}_{n_1,\ldots,n_{r-1},n_r-1;k}(l).$$

Here the string $(n'_1,\ldots,n'_{r-1})$ is obtained from $(n_1,\ldots,n_{r-1})$ by subtracting 1 in the positions $i_1,\ldots,i_{k-1}$.

The result is by induction true for the lattices $\Pi_{n'_1,\ldots,n'_{r-1},n_r-1;k}(l+1)$ and $\Pi_{n_1,\ldots,n_{r-1},n_r-1;k}(l)$, and hence for all intervals $[\pi, \hat{1}]$ above complements $\pi$ to $(\{1,\ldots,n\} - \{x\})(x)$. Part (i) then follows directly from Proposition 2.3. For part (ii) one must also check that the condition for non-vanishing homology is correctly transferred via relation (6.1). ∎

In the preceding proof we assumed that some $n_i \geq 2$ (or after relabeling : $n_r \geq 2$) in order that the recursion formula (6.1) would take us back to the previously treated case $n_1 = \cdots n_r = 1$. However, formula (6.1) is valid also if $n_r = 1$, in fact for $n_1 = \cdots = n_r = 1$ it specializes to the recursion formula in Theorem 4.3 (ii).

We will now define the generalized "$k$-equal" arrangements. Let $2 \leq k \leq r$ and $n_1 \geq 1, \ldots, n_r \geq 1$ be integers. Fix a partition $E_1 \cup \cdots \cup E_r$ of $\{1, \ldots, n\}$, such that $|E_i| = n_i$ and $n = n_1 + \cdots + n_r$. For instance, we can take $E_1$ to be the first $n_1$ positive integers, $E_2$ the next $n_2$ ones, and so on. Let $\mathcal{A}^{\mathbf{R}}_{n_1,\ldots,n_r;k}$ be the arrangement of subspaces in $\mathbf{R}^n$ given by all equations $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ such that $|\{i_1, i_2, \ldots, i_k\} \cap E_j| \leq 1$ for all $1 \leq j \leq r$. Let $M^{\mathbf{R}}_{n_1,\ldots,n_r;k}$ denote its complement in $\mathbf{R}^n$, and similarly $M^{\mathbf{C}}_{n_1,\ldots,n_r;k}$ the complexified complement. Note that $M^{\mathbf{R}}_{1,\ldots,1;k} = M^{\mathbf{R}}_{r,k}$ are the $k$-equal manifolds discussed in Sections 1 and 5. Also, $\mathcal{A}^{\mathbf{R}}_{n_1,n_2;2}$ is the "graphic" hyperplane arrangement corresponding to the complete bipartite graph $K_{n_1,n_2}$, and given by the set of equations

$$\{ \, x_i - y_j \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2 \, \} \text{ in } \mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}.$$

**Theorem 6.2** *The cohomology groups of $M^{\mathbf{R}}_{n_1,\ldots,n_r;k}$ and $M^{\mathbf{C}}_{n_1,\ldots,n_r;k}$ are free. Furthermore, if $H^d(M^{\mathbf{R}}_{n_1,\ldots,n_r;k}) \neq 0$ then $d = t \cdot (k-2)$ for some integer $t \geq 0$.*

**Proof:** The intersection lattice of $\mathcal{A}^{\mathbf{R}}_{n_1,\ldots,n_r;k}$ and of $\mathcal{A}^{\mathbf{C}}_{n_1,\ldots,n_r;k}$ is isomorphic to $\Pi_{n_1,\ldots,n_r;k}(0)$. This is easily seen as in the proof of Lemma 5.1. Also, the formulas for codimension from Lemma 5.1 are valid. Therefore Proposition 2.1 shows that all the stated properties of cohomology are transferred from the corresponding homology properties of lower interval $[\hat{0}, \pi]$ in $\Pi_{n_1,\ldots,n_r;k}(0)$. These intervals $[\hat{0}, \pi]$ have the homotopy type of a wedge of spheres. This follows from Theorem 6.1 (i) by arguing as in Theorem 4.8 (i) and (ii). Hence cohomology is free. The necessary condition for the dimensions of non-vanishing cohomology follows by reasoning parallel to that used in proving Theorems 4.8 (iv), 5.2 and 1.1 (b). We omit the details. ∎

# 7 Final remarks

### 7.1 Generalization to other Coxeter groups

Results of the type obtained in this paper might exist in the wider setting of Coxeter groups, and we would like to raise this question. Let $W$ be a finite Coxeter group acting as a reflection group on $\mathbf{R}^n$, and let $K$ be some subspace which is an intersection of reflecting hyperplanes. Then the orbit $\mathcal{A}^{\mathbf{R}}_{W,K} = \{ w(K) \mid w \in W \}$ is a subspace arrangement, and we can consider its complement $M^{\mathbf{R}}_{W,K}$ and the complexification $M^{\mathbf{C}}_{W,K}$. Is cohomology always torsion-free for such spaces ? Do the dimensions of non-zero cohomology exhibit periodicity ?

### 7.2 Fundamental groups

What can be said about the fundamental group of $M^{\mathbf{R}}_{n,3}$ ? Is $M^{\mathbf{R}}_{n,3}$ a $K(\pi, 1)$ space ? Note that the fact that $M^{\mathbf{C}}_{n,k}$ is a $K(\pi, 2k-3)$ space for $k = 2$ does not generalize to $k \geq 3$, as shown by Corollary 1.4.

### 7.3 Algebra structure of the cohomology rings

Can the multiplicative structure of the free graded modules $H^*(M^{\mathbf{R}}_{n,k})$ and $H^*(M^{\mathbf{C}}_{n,k})$ be described ? This was done in the $k = 2$ case by Arnol'd [Arn69], see also Orlik and Solomon [OS80] and [OT92].

### 7.4 CW-complexes

Is there any geometrically motivated CW-complex $\mathcal{C}$, such that $\mathcal{C}$ and $M^{\mathbf{R}}_{n,k}$ are homotopy equivalent and $\mathcal{C}$ has cells only in dimensions $t \cdot (k-2)$, for $0 \leq t \leq \lfloor \frac{n}{k} \rfloor$ ?

### 7.5 Recursions for Betti numbers

Let $b(n,k) = \sum_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1}$, i.e., $b(n,k) = rankH^{k-2}(M^{\mathbf{R}}_{n,k})$ for $n \geq k \geq 3$. The Zeilberger algorithm [Zei91] proves that there is no "closed" formula for $b(n,k)$ (i.e. the quotient $\frac{b(n,k)}{b(n+1,k)}$ is not a rational function in $n$). But the following two recursions hold :

(i) $(n-k+2)\cdot b(n+2,k) - (3n-k+4)\cdot b(n+1,k) + (2n+2)\cdot b(n,k) = 0$,

(ii) $b(n,k+1) + b(n,k) = \binom{n}{k}\cdot 2^{n-k}$.

The first recursion has been computed by the Zeilberger algorithm and the second is due to V. Strehl.

### 7.6 Fiber bundles

Let $\mathcal{A}$ be an arrangement of complex subspaces in $\mathbf{C}^n = \mathbf{R}^{2n}$, with minimal occurring complex codimension equal to $c$. Is it true that

$$M_{\mathcal{A}} \cong B \times (\mathbf{C}^c - \{\underline{0}\})$$

for some space $B$ ? (I.e., does $M_{\mathcal{A}}$ have the structure of a trivial fiber bundle?) This is true for complex hyperplane arrangements, see [OT92, Proposition 5.1.1]. If true in general it would explain on the topological level why $\chi(M_{\mathcal{A}}) = 0$, since $\chi(\mathbf{C}^c - \{\underline{0}\}) = \chi(S^{2c-1}) = 0$. Therefore Theorem 3.3 suggests that a similar decomposition might exist for real subspace arrangements satisfying the even codimension condition.

### 7.7 The induced $S_n$ representations

All homology and cohomology groups computed in Section 4 for the lattice $\Pi_{n,k}$ and in Section 5 for the spaces $M^{\mathbf{R}}_{n,k}$ and $M^{\mathbf{C}}_{n,k}$ are $S_n$-modules for the symmetric group $S_n$. The $S_n$ action on homology and cohomology is induced by the natural action of $S_n$ on $\mathbf{R}^n$ (by permuting the coordinates) which makes $S_n$ a group of automorphisms of the arrangements $\mathcal{A}^{\mathbf{R}}_{n,k}$ and $\mathcal{A}^{\mathbf{C}}_{n,k}$. In order to have a proper setting for classic representation theory we will assume in this remark that all homology and cohomology groups are taken with coefficients in the field $\mathbf{C}$ of complex numbers.

The $k = 2$ cases are already known. The character $\pi^{n-3}_{n,2}$ of $S_n$ on the unique non-vanishing reduced homology group of $\Pi_{n,2} = \Pi_n$ has been computed by Stanley [Sta82, Theorem 7.3]. The representation of $S_n$ on the cohomology of $M^{\mathbf{C}}_{n,2}$ has been determined by Lehrer and Solomon [LS86], see also Orlik and Solomon [OS80]. The action of $S_n$ on $M^{\mathbf{R}}_{n,2}$ amounts to faithfully permuting the $n!$ simplicial cones, as is easy to see, so this gives the regular representation.

We denote by $\pi^d_{n,k}$ the representation of $S_n$ on the reduced homology group $\widetilde{H}_d(\Pi_{n,k})$. Stanley observes [Sta82, Corollary 7.6] that the restricted character $\pi^{n-3}_{n,2}\downarrow_{S_{n-1}}$ is a permutation character, namely the character of the regular representation. We will show that an analogous result holds for all the characters $\pi^d_{n,k}$. We will not give the full details of the proof since it runs parallel to the proofs of Lemma 4.2 and Theorem 4.3 in an equivariant setting. Here $S_{n-1}$ will be regarded as the subgroup of $S_n$ fixing $n$.

**Proposition 7.1** *Let $2 \leq k \leq n$ be integers. Then the character $\pi_{n,k}^{n-3-t\cdot(k-2)} \downarrow_{S_{n-1}}$ is a permutation character for $0 \leq t \leq \lfloor \frac{n}{k} \rfloor$.*

**Proof:** In order to be able to use the recursive technique of Lemma 4.2 and Theorem 4.3 we will for every admissible triple $(n,k,l)$ denote by $\pi^d_{n,k}(l)$ the character of the group of automorphisms of $\Pi_{n,k}(l)$ acting on the reduced homology group $\widetilde{H}_d(\Pi_{n,k}(l))$. Note that for $0 \leq l \leq n-2$ the group of automorphisms of $\Pi_{n,k}(l)$ is $S_l \times S_{n-l-1}$, the direct product of the symmetric group $S_l$ on $\{1,\ldots,l\}$ and the symmetric group $S_{n-l}$ on $\{l+1,\ldots,n\}$ . For $l \geq n-1$ we have $\Pi_{n,k}(l) = \Pi_n$ and in this case the full symmetric group $S_n$ is the automorphism group of $\Pi_{n,k}(l)$. Now the more general claim is :

$$(7.1) \qquad \text{The characters} \begin{cases} \pi^d_{n,k}(l) \downarrow_{S_l \times S_{n-l-1}} & , \quad \text{if } 0 \leq l \leq n-2, \\ \pi^d_{n,k}(l) \downarrow_{S_{n-1}} & , \quad \text{if } l \geq n-1, \end{cases} \quad \text{are permutation characters.}$$

This will be deduced from the following topological fact:

(7.2) $\Pi_{n,k}(l)$ has the $G$-homotopy type of a wedge of spheres which are permuted under the action of $G$, where $G = S_l \times S_{n-l-1}$ if $0 \leq l \leq n-2$ and $G = S_{n-1}$ if $l \geq n-1$.

Clearly,the action of $G$ on the $d$-spheres gives rise to a permutation representation of $G$ on $\widetilde{H}_d(\Pi_{n,k}(l))$, which proves (7.1). Proposition 7.1 is the $l = 0$ case.

To prove (7.2) we will use an equivariant version of Proposition 2.3 [Wel90, Proposition 2.5] which says the following : If for an element $x$ of a lattice $L$ which is invariant under a group $G$ of lattice automorphisms the set of complements $\mathcal{CO}(x)$ is an antichain, then $L$ is $G$-homotopy equivalent to the space given in Proposition 2.3 and $G$ acts on it by fixing the wedge point and permuting the complexes in the wedge according to its operation on the lattice. For an inductive procedure one has to proceed as follows. If $G$ is the group of automorphisms of the lattice $L$ which fixes the element $x$ then one has to determine the $G_y$-homotopy type of $\Delta(\hat{0}, y)$ and $\Delta(y, \hat{1})$ for the stabilizer $G_y$ of each element $y \in \mathcal{CO}(x)$. Applying this equivariant version of Proposition 2.3 in the case $l \geq n-1$ to the $S_{n-1}$-action on $\Pi_{n,k}(l) = \Pi_n$ and letting $x = (1 \cdots n-1)(n)$, we easily verify (7.2) in this case, which is the case treated in Lemma 4.2 (i). Continuing with this induction technique, the method of the proofs of Lemma 4.2 (ii) and Theorem 4.3 leads to a proof of (7.2). ∎

The determination of the homology characters $\pi^d_{n,k}$ themselves seems more difficult, and we know the solution only for the top dimension $d = n-k-1$. As mentioned, this was done by Stanley [Sta82] for $k = 2$, and we will now show that these characters are irreducible when $k \geq 3$.

**Proposition 7.2** *The homology character $\pi^{n-k-1}_{n,k}$, for $2 < k \leq n$, is the irreducible character of $S_n$ corresponding to the "hook" partition $(k,1,\ldots,1)$.*

**Proof:** Let $B_{n,k}$ denote the rank-selected boolean lattice $B_n$, where rank-levels $1,\ldots,k-1$ have been removed. There is an obvious embedding $\varphi : B_{n,k} \hookrightarrow \Pi_{n,k}$ which sends a subset $E \in B_{n,k}$ to the partition of $\{1,\ldots,n\}$ with unique non-trivial block $E$. The order complexes of $B_{n,k}$ and of $\Pi_{n,k}$ are both $(n-k-1)$-dimensional, so the embedding $\varphi$ induces an injection $\tilde{\varphi} : \widetilde{H}_{n-k-1}(B_{n,k}) \hookrightarrow \widetilde{H}_{n-k-1}(\Pi_{n,k})$. In fact, $\tilde{\varphi}$ is an isomorphism, since $rank\widetilde{H}_{n-k-1}(B_{n,k}) = \binom{n-1}{k-1} = rank\widetilde{H}_{n-k-1}(\Pi_{n,k})$ and we are working over a field. From the equivariance of the inclusion map $\varphi$ we deduce that $\tilde{\varphi}$ is actually an isomorphism of $S_n$-modules.

Now we use a result of Stanley [Sta82, Theorem 4.3] which says that the irreducible $S_n$-character corresponding to $\lambda \vdash n$ occurs in the $S_n$-module $\widetilde{H}_{n-k-1}(B_{n,k})$ as many times as the number of standard Young tableaux of shape $\lambda$ and descent set $\{k, k+1, \ldots, n-1\}$. But there is clearly only one standard Young tableau with descent set $\{k, k+1, \ldots, n-1\}$, and its shape is $(k, 1, \ldots, 1)$. ∎

It follows, of course, that $\pi_{n,k}^{n-k-1}$ is not a permutation character, since it does not contain the trivial character (except in the degenerate case $k = n$). It would be interesting to have more information about the representations of $S_n$ on the (co)homology of $\Pi_{n,k}$, $M_{n,k}^{\mathbf{R}}$ and $M_{n,k}^{\mathbf{C}}$, for $k \geq 3$.

### 7.8 Complexes of disconnected $k$-graphs

By a $k$-GRAPH we will mean a collection of $k$-element subsets of $\{1, \ldots, n\}$. A $k$-graph $\mathcal{G}$ is CONNECTED if for every pair $i, i' \in \{1, \ldots, n\}$ there exist a sequence of elements $i = i_0, i_1, \ldots, i_p = i'$ and a sequence of $k$-edges $B_1, \ldots, B_p$ in $\mathcal{G}$ such that $\{i_{j-1}, i_j\} \subseteq B_j$ for $j = 1, \ldots, p$.

The following question was asked by Fock, Nekrasov, Rosly and Selivanov [FNRS91]. Let the $k$-element subsets of $\{1, \ldots, n\}$ be the vertices of a $(\binom{n}{k} - 1)$-simplex, and consider the subcomplex $\Delta_{n,k}$ of its boundary consisting of all disconnected $k$-graphs. What is the homology of $\Delta_{n,k}$ ?

A more precise question was later asked by V.A. Vassiliev [private communication, July 1992]. Namely, let $2 \leq k \leq r$ and $n_i \geq 1$, $i = 1, \ldots, r$, be positive integers. Take pairwise disjoint sets $E_1, \ldots, E_r$, such that $|E_i| = n_i$ for all $i$. Call a $k$-element subset $B \subseteq E_1 \cup \cdots \cup E_r$ BALANCED if $|B \cap E_i| \leq 1$ for $i = 1, \ldots, r$. Now, let the balanced $k$-element subsets of $E_1 \cup \cdots \cup E_r$ be the vertices of a simplex, and consider the subcomplex $\Delta_{n_1, \ldots, n_r; k}$ consisting of all disconnected $k$-graphs. What is the homology of $\Delta_{n_1, \ldots, n_r; k}$ ?

These questions (particularly the first one) can be answered using the Crosscut Theorem (see below) and the results of this paper. Namely, $\Delta_{n,k}$ is the crosscut complex of the lattice $\Pi_{n,k}$, so $\Delta_{n,k}$ and $\Pi_{n,k}$ are homotopy equivalent. Hence, complete information about the homology and the homotopy type of $\Delta_{n,k}$ can be obtained from Theorems 1.5 and 4.5.

In a similar way, $\Delta_{n_1, \ldots, n_r; k}$ is the crosscut complex of the lattice $\Pi_{n_1, \ldots, n_r; k}(0)$, hence these complexes are homotopy equivalent and partial information about the homology of $\Delta_{n_1, \ldots, n_r; k}$ can be found in Theorem 6.1.

The Crosscut Theorem says the following. Let $L$ be a finite lattice and let $A$ be the set of minimal elements of $L - \{\hat{0}\}$. Define a simplicial complex $\Gamma(L, A)$ by taking $A$ as the vertex set and as simplices those subsets $A_0 \subseteq A$ such that $\bigvee_{\pi \in A_0} \pi \neq \hat{1}$. Then $L$ and the crosscut complex $\Gamma(L, A)$ are homotopy equivalent. See [Bjö89, (10.8)] for a proof and further references.

# 8 Tables of Betti numbers for $\Pi_{n,k}$, $M^{\mathbf{R}}_{n,k}$ and $M^{\mathbf{C}}_{n,k}$

## 8.1 Tables of the cohomology of $M^{\mathbf{R}}_{n,k}$ :

| n | k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 2 | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 3 | 1 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 4 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 2 | 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 3 | 1 | 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 4 | 1 | 0 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 5 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 2 | 720 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 3 | 1 | 111 | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 4 | 1 | 0 | 49 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 5 | 1 | 0 | 0 | 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 6 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 2 | 5040 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 3 | 1 | 351 | 350 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 4 | 1 | 0 | 209 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 5 | 1 | 0 | 0 | 71 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 6 | 1 | 0 | 0 | 0 | 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 7 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 2 | 40320 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 3 | 1 | 1023 | 3542 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 4 | 1 | 0 | 769 | 0 | 70 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 5 | 1 | 0 | 0 | 351 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 6 | 1 | 0 | 0 | 0 | 97 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 7 | 1 | 0 | 0 | 0 | 0 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 2 | 362880 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 3 | 1 | 2815 | 27174 | 1680 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 4 | 1 | 0 | 2561 | 0 | 1638 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 5 | 1 | 0 | 0 | 1471 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 6 | 1 | 0 | 0 | 0 | 545 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 7 | 1 | 0 | 0 | 0 | 0 | 127 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 17 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 10 | 2 | 3628800 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 3 | 1 | 7423 | 175422 | 54600 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 4 | 1 | 0 | 7937 | 0 | 21462 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 5 | 1 | 0 | 0 | 5503 | 0 | 0 | 252 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 6 | 1 | 0 | 0 | 0 | 2561 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 7 | 1 | 0 | 0 | 0 | 0 | 799 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 161 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 19 | 0 | 0 | 0 | 0 | 0 |
| 10 | 10 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 11 | 2 | 39916800 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 3 | 1 | 18943 | 1005312 | 986370 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 4 | 1 | 0 | 23297 | 0 | 207702 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 5 | 1 | 0 | 0 | 18943 | 0 | 0 | 7392 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 6 | 1 | 0 | 0 | 0 | 10625 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 7 | 1 | 0 | 0 | 0 | 0 | 4159 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 1121 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 199 | 0 | 0 | 0 | 0 | 0 |
| 11 | 10 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 21 | 0 | 0 | 0 | 0 |
| 11 | 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 12 | 2 | 479001600 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 3 | 1 | 47103 | 5279252 | 13086150 | 369600 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 4 | 1 | 0 | 65537 | 0 | 1655412 | 0 | 34650 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 5 | 1 | 0 | 0 | 61183 | 0 | 0 | 118932 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 6 | 1 | 0 | 0 | 0 | 40193 | 0 | 0 | 0 | 924 | 0 | 0 | 0 | 0 |
| 12 | 7 | 1 | 0 | 0 | 0 | 0 | 18943 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 6401 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1519 | 0 | 0 | 0 | 0 | 0 |
| 12 | 10 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 241 | 0 | 0 | 0 | 0 |
| 12 | 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 23 | 0 | 0 | 0 |
| 12 | 12 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 13 | 2 | 6227020800 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 3 | 1 | 114687 | 25929332 | 142331046 | 19219200 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 4 | 1 | 0 | 178177 | 0 | 11503492 | 0 | 1531530 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 5 | 1 | 0 | 0 | 187903 | 0 | 0 | 1389102 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 6 | 1 | 0 | 0 | 0 | 141569 | 0 | 0 | 0 | 32604 | 0 | 0 | 0 | 0 |
| 13 | 7 | 1 | 0 | 0 | 0 | 0 | 78079 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 31745 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 9439 | 0 | 0 | 0 | 0 | 0 |
| 13 | 10 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2001 | 0 | 0 | 0 | 0 |
| 13 | 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 287 | 0 | 0 | 0 |
| 13 | 12 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 0 | 0 |
| 13 | 13 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 14 | 2 | 87178291200 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 3 | 1 | 274431 | 120796832 | 1344785442 | 543182640 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 4 | 1 | 0 | 471041 | 0 | 72135492 | 0 | 36702666 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 5 | 1 | 0 | 0 | 553983 | 0 | 0 | 13166582 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 6 | 1 | 0 | 0 | 0 | 471041 | 0 | 0 | 0 | 622050 | 0 | 0 | 0 | 0 |
| 14 | 7 | 1 | 0 | 0 | 0 | 0 | 297727 | 0 | 0 | 0 | 0 | 3432 | 0 | 0 |
| 14 | 8 | 1 | 0 | 0 | 0 | 0 | 0 | 141569 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 9 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 50623 | 0 | 0 | 0 | 0 | 0 |
| 14 | 10 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 13441 | 0 | 0 | 0 | 0 |
| 14 | 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2575 | 0 | 0 | 0 |
| 14 | 12 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 337 | 0 | 0 |
| 14 | 13 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  | 0 | 0 | 27 | 0 |
| 14 | 14 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

## 8.1 Tables of the cohomology of $M^{\mathbf{R}}_{n,k}$ (continued) :

| n | k | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 15 | 2 | 1307674368000 | 0 | 0 | 0 | 0 | 0 |
| 15 | 3 | 1 | 647167 | 539181162 | 11431616196 | 11061250200 | 168168000 |
| 15 | 4 | 1 | 0 | 1216513 | 0 | 417527812 | 0 |
| 15 | 5 | 1 | 0 | 0 | 1579007 | 0 | 0 |
| 15 | 6 | 1 | 0 | 0 | 0 | 1496065 | 0 |
| 15 | 7 | 1 | 0 | 0 | 0 | 0 | 1066495 |
| 15 | 8 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 9 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 10 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 11 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 12 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 13 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 14 | 1 | 0 | 0 | 0 | 0 | 0 |
| 15 | 15 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 2 | 20922789888000 | 0 | 0 | 0 | 0 | 0 |
| 16 | 3 | 1 | 1507327 | 2323102602 | 89478384996 | 181667205720 | 12780768000 |
| 16 | 4 | 1 | 0 | 3080193 | 0 | 2266142762 | 0 |
| 16 | 5 | 1 | 0 | 0 | 4374527 | 0 | 0 |
| 16 | 6 | 1 | 0 | 0 | 0 | 4571137 | 0 |
| 16 | 7 | 1 | 0 | 0 | 0 | 0 | 3629055 |
| 16 | 8 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 9 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 10 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 11 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 12 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 13 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 14 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 15 | 1 | 0 | 0 | 0 | 0 | 0 |
| 16 | 16 | 1 | 0 | 0 | 0 | 0 | 0 |

| n | k | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 4 | 632305674 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 5 | 107440762 | 0 | 0 | 756756 | 0 | 0 | 0 | 0 | 0 |
| 15 | 6 | 0 | 0 | 8509930 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 7 | 0 | 0 | 0 | 0 | 141570 | 0 | 0 | 0 | 0 |
| 15 | 8 | 580865 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 9 | 0 | 242815 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 10 | 0 | 0 | 77505 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 11 | 0 | 0 | 0 | 18591 | 0 | 0 | 0 | 0 | 0 |
| 15 | 12 | 0 | 0 | 0 | 0 | 3249 | 0 | 0 | 0 | 0 |
| 15 | 13 | 0 | 0 | 0 | 0 | 0 | 391 | 0 | 0 | 0 |
| 15 | 14 | 0 | 0 | 0 | 0 | 0 | 0 | 29 | 0 | 0 |
| 15 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 16 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 4 | 8766802044 | 0 | 63063000 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 5 | 782487862 | 0 | 0 | 42378336 | 0 | 0 | 0 | 0 | 0 |
| 16 | 6 | 0 | 0 | 93470806 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 7 | 0 | 0 | 0 | 0 | 3124550 | 0 | 0 | 0 | 0 |
| 16 | 8 | 2228225 | 0 | 0 | 0 | 0 | 0 | 12870 | 0 | 0 |
| 16 | 9 | 0 | 1066495 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 10 | 0 | 0 | 397825 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 11 | 0 | 0 | 0 | 114687 | 0 | 0 | 0 | 0 | 0 |
| 16 | 12 | 0 | 0 | 0 | 0 | 25089 | 0 | 0 | 0 | 0 |
| 16 | 13 | 0 | 0 | 0 | 0 | 0 | 4031 | 0 | 0 | 0 |
| 16 | 14 | 0 | 0 | 0 | 0 | 0 | 0 | 449 | 0 | 0 |
| 16 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 31 | 0 |
| 16 | 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

## 8.2 Tables of the cohomology of $M^{\mathbf{C}}_{n,k}$

| n | k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 2 | 6 | 11 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 3 | 0 | 0 | 4 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 4 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 2 | 10 | 35 | 50 | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 3 | 0 | 0 | 10 | 15 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 4 | 0 | 0 | 0 | 0 | 5 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 2 | 15 | 85 | 225 | 274 | 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 3 | 0 | 0 | 20 | 45 | 36 | 20 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 4 | 0 | 0 | 0 | 0 | 15 | 24 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 2 | 21 | 175 | 735 | 1624 | 1764 | 720 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 3 | 0 | 0 | 35 | 105 | 126 | 140 | 190 | 105 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 4 | 0 | 0 | 0 | 0 | 35 | 84 | 70 | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 21 | 35 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 2 | 28 | 322 | 1960 | 6769 | 13132 | 13068 | 5040 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 3 | 0 | 0 | 56 | 210 | 336 | 560 | 1240 | 1512 | 651 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 4 | 0 | 0 | 0 | 0 | 70 | 224 | 280 | 160 | 35 | 35 | 35 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 56 | 140 | 120 | 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 28 | 48 | 21 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 7 | 0 | 0 | 0 | 0 | 0 |
| 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 9 | 2 | 36 | 546 | 4536 | 22449 | 67284 | 118124 | 109584 | 40320 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 3 | 0 | 0 | 84 | 378 | 756 | 1680 | 5160 | 9828 | 9275 | 3948 | 560 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 4 | 0 | 0 | 0 | 0 | 126 | 504 | 840 | 720 | 315 | 371 | 819 | 504 | 0 | 0 | 0 | 0 | 0 |
| 9 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 126 | 420 | 540 | 315 | 70 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 84 | 216 | 189 | 56 | 0 | 0 | 0 | 0 | 0 |
| 9 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 36 | 63 | 28 | 0 | 0 | 0 | 0 |
| 9 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 8 | 0 | 0 | 0 |
| 9 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 10 | 2 | 45 | 870 | 9450 | 63273 | 269325 | 723680 | 1172700 | 1026576 | 362880 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 3 | 0 | 0 | 120 | 630 | 1512 | 4200 | 16500 | 42840 | 63455 | 58452 | 37136 | 12600 | 0 | 0 | 0 | 0 | 0 |
| 10 | 4 | 0 | 0 | 0 | 0 | 210 | 1008 | 2100 | 2400 | 1575 | 2135 | 6699 | 9156 | 4116 | 0 | 0 | 0 | 0 |
| 10 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 252 | 1050 | 1800 | 1575 | 700 | 126 | 0 | 126 | 126 | 0 | 0 |
| 10 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 210 | 720 | 945 | 560 | 126 | 0 | 0 | 0 | 0 |
| 10 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 120 | 315 | 280 | 84 | 0 | 0 | 0 |
| 10 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 45 | 80 | 36 | 0 | 0 |
| 10 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 9 | 0 |
| 10 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

| n | k | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 8 | 12870 | 91520 | 288288 | 524160 | 600600 | 443520 | 205920 | 54912 | 6435 | 0 | 0 | 0 | 0 | 6435 | 6435 | 0 |

## 8.3 Tables of the reduced homology of $\Pi_{n,k}$ :

| n | k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 2 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 2 | 0 | 0 | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 3 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 2 | 0 | 0 | 0 | 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 3 | 0 | 10 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 4 | 0 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 2 | 0 | 0 | 0 | 0 | 720 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 3 | 0 | 0 | 105 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 4 | 0 | 0 | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 5 | 0 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 6 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 2 | 0 | 0 | 0 | 0 | 0 | 5040 | 0 | 0 | 0 | 0 | 0 |
| 8 | 3 | 0 | 0 | 0 | 651 | 21 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 4 | 0 | 35 | 0 | 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 5 | 0 | 0 | 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 6 | 0 | 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 7 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 40320 | 0 | 0 | 0 | 0 |
| 9 | 3 | 0 | 0 | 0 | 560 | 3108 | 28 | 0 | 0 | 0 | 0 | 0 |
| 9 | 4 | 0 | 0 | 504 | 0 | 56 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 5 | 0 | 0 | 0 | 70 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 6 | 0 | 0 | 56 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 7 | 0 | 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 362880 | 0 | 0 | 0 |
| 10 | 3 | 0 | 0 | 0 | 0 | 12600 | 12636 | 36 | 0 | 0 | 0 | 0 |
| 10 | 4 | 0 | 0 | 0 | 4116 | 0 | 84 | 0 | 0 | 0 | 0 | 0 |
| 10 | 5 | 0 | 126 | 0 | 0 | 126 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 6 | 0 | 0 | 0 | 126 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 7 | 0 | 0 | 84 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 8 | 0 | 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 9 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 33633600 | 1190839650 | 480411932 | 4286373 |
| 15 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 75603528 | 0 | 8480108 | 0 | 364 |
| 15 | 5 | 0 | 0 | 0 | 252252 | 0 | 0 | 5508503 | 0 | 0 | 1001 | 0 |
| 15 | 6 | 0 | 0 | 0 | 0 | 1091090 | 0 | 0 | 0 | 2002 | 0 | 0 |
| 15 | 7 | 0 | 0 | 45045 | 0 | 0 | 0 | 0 | 3003 | 0 | 0 | 0 |
| 15 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 3432 | 0 | 0 | 0 | 0 |
| 15 | 9 | 0 | 0 | 0 | 0 | 0 | 3003 | 0 | 0 | 0 | 0 | 0 |
| 15 | 10 | 0 | 0 | 0 | 0 | 2002 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 11 | 0 | 0 | 0 | 1001 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 12 | 0 | 0 | 364 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 13 | 0 | 91 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 14 | 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2018016000 | 14944759830 | 2722579860 |
| 16 | 4 | 0 | 0 | 0 | 0 | 0 | 15765750 | 0 | 753212460 | 0 | 29819335 | 0 |
| 16 | 5 | 0 | 0 | 0 | 0 | 10090080 | 0 | 0 | 25857195 | 0 | 0 | 1365 |
| 16 | 6 | 0 | 0 | 0 | 0 | 0 | 7690683 | 0 | 0 | 0 | 3003 | 0 |
| 16 | 7 | 0 | 0 | 0 | 635635 | 0 | 0 | 0 | 0 | 5005 | 0 | 0 |
| 16 | 8 | 0 | 6435 | 0 | 0 | 0 | 0 | 0 | 6435 | 0 | 0 | 0 |
| 16 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 6435 | 0 | 0 | 0 | 0 |
| 16 | 10 | 0 | 0 | 0 | 0 | 0 | 5005 | 0 | 0 | 0 | 0 | 0 |
| 16 | 11 | 0 | 0 | 0 | 0 | 3003 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 12 | 0 | 0 | 0 | 1365 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 13 | 0 | 0 | 455 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 14 | 0 | 105 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 15 | 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| n | k | 11 | 12 | 13 | | n | k | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 2 | 0 | 87178291200 | 0 | | 16 | 2 | 0 | 0 | 1307674368000 |
| 15 | 3 | 91 | 0 | 0 | | 16 | 3 | 12042135 | 105 | 0 |
| 15 | 4 | 0 | 0 | 0 | | 16 | 4 | 455 | 0 | 0 |

# References


[Arn69] V.I. Arnol'd. The cohomology ring of the colored braid group. *Math. Notes*, 5:138–140, 1969.

[Bjö89] A. Björner. Topological methods. In : R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. North-Holland, to appear. (Preprint 1989).

[BL92] A. Björner and L. Lovász. Work in progress. 1992.

[BLY92] A. Björner, L. Lovász, and A. Yao. Linear decision trees: volume estimates and topological bounds. In *Proc. 24th ACM Symp. on Theory of Computing*, pages 170–177, 1992.

[BW83] A. Björner and J.W. Walker. A homotopy complementation formula for partially ordered sets. *European Journal of Combinatorics*, 4:11–19, 1983.

[BZ92] A. Björner and G. M. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5:105–149, 1992.

[FaN62] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.

[FoN62] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.

[FNRS91] V.V. Fock, N.A. Nekrasov, A.A. Rosly, and K.G. Selivanov. What we think about the higher-dimensional Chern-Simons theories. Technical Report 70-91, Institute of Theoretical and Experimental Physics, Moscow, 1991.

[GM88] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer, Berlin, Heidelberg, New York, 1988.

[LS86] G.I. Lehrer and L. Solomon. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. *J. Algebra*, 104:410–424, 1986.

[Mun84] J.R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Menlo Park, CA, 1984.

[OS80] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Inventiones Math.*, 56:167–189, 1980.

[OT92] P. Orlik and H. Terao. *Arrangements of Hyperplanes.* Springer, Berlin, Heidelberg, New York, 1992.

[Qui78] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Advances in Mathematics*, 28:101–128, 1978.

[Ser53] J.P. Serre. Cohomologie mod 2 des complexes d'Eilenberg-MacLane. *Comm. Math. Helv.*, 27:198–232, 1953.

[Sta82] R.P. Stanley. Some aspects of groups acting on finite posets. *Journal of Combinatorial Theory (A)*, 32:132–161, 1982.

[Sta86] R. P. Stanley. *Enumerative Combinatorics I.* Wadsworth & Brooks/Cole, Monterey, CA, 1986.

[Vas92] V.A. Vassiliev. *Complements of discriminants of smooth maps : Topology and Applications.* Volume 98 of *Transl. of Math. Monographs.* Amer. Math. Soc., Providence, RI, 1992.

[Wal88] J.W. Walker. Canonical homeomorphisms of posets. *Europ. J. of Combinatorics*, 9:97–107, 1988.

[Wel90] V. Welker. Equivariant homotopy of posets and some applications to subgroup lattices. (Preprint 1990).

[Zei91] D. Zeilberger. The method of creative telescoping. *Journal of Symb. Comp.*, 11(3):195–204, 1991.

[ZŽ91] G. M. Ziegler and R. Živaljević. Homotopy types of subspace arrangements via diagrams of spaces. *Math. Annalen*, to appear. (Preprint 1991).