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ESSENTIAL CHAINS AND HOMOTOPY TYPE OF POSETS

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ABSTRACT. A short proof of a result of Billera, Kapranov, and Sturmfels (*Cellular strings on polytopes*, preprint, 1991), verifying a homotopy type conjecture of Baues (*Geometry of loop spaces and the cobar construction*, Mem. Amer. Math. Soc., no. 230, 1980), is given.

1. INTRODUCTION

The topology of a poset (partially ordered set) is that of the simplicial complex of its chains. Here we observe that up to homotopy type it is sufficient to consider only certain chains, which we call essential.

The motivation for this note comes from a recent paper of Billera, Kapranov, and Sturmfels [2], in which they settle a conjecture of Baues [1]. This conjecture, which in turn is motivated by questions in the theory of loop spaces, states (in a somewhat different language) that the poset of essential chains in the weak Bruhat order of S_n (or n-permutohedron) is homotopy equivalent to the $(n-3)$-sphere. Our approach leads to a proof of Baues's conjecture, which is quite different from that of Billera, Kapranov, and Sturmfels.

Actually, a much stronger result valid for all polytopes is proved in [2]. The method of this paper applies only to the subclass of zonotopes, but, on the other hand, it is applicable to other classes of posets that have nothing to do with polytopes. See §3 for more precise comments.

2. ESSENTIAL CHAINS IN A POSET

For any poset P, let $\Delta(P)$ denote its *order complex*, i.e., the simplicial complex whose simplices are the chains (totally ordered subsets) $x_0 < x_1 < \cdots < x_k$ in P. When referring to the topology of P we will always have $\Delta(P)$ in mind. For any simplicial complex Δ, let $\mathscr{F}(\Delta)$ denote its *face poset*, i.e., the poset of simplices of Δ ordered by inclusion. Then $\Delta(\mathscr{F}(\Delta))$ is the first barycentric subdivision of Δ, and $\|\Delta\| \cong \|\Delta(\mathscr{F}(\Delta))\|$. We will assume some familiarity with poset topology, particularly for questions about homotopy type. For background see, for example, [4].

Let P be a *bounded* poset (i.e., P has top and bottom elements, denoted $\hat{1}$ and $\hat{0}$, respectively). A chain $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ will be called

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essential if $k \geq 2$ and every open interval (x_{i-1}, x_i) has noncontractible order complex $\Delta(x_{i-1}, x_i)$, $1 \leq i \leq k$. In particular, every maximal chain is essential (the open intervals being empty). Let $\mathrm{Ess}(P)$ be the poset of essential chains ordered by inclusion, and let $\overline{P} = P - \{\hat{0}, \hat{1}\}$.

Theorem 1. *For any bounded poset P of finite length, the posets $\overline{P}$ and $\mathrm{Ess}(P)$ have the same homotopy type.*

The proof will rely on the following lemma, which follows from the Fiber Theorem of Quillen [7]. See [4, Theorem 10.5 and Lemma 11.12].

Lemma. *Let Q be a poset of finite length and A a subset. Assume that $\Delta(Q_{>x})$ is contractible for all $x \in Q - A$. Then the inclusion $A \to Q$ induces homotopy equivalence $\Delta(A) \cong \Delta(Q)$.*

Proof of Theorem 1. Let $Q = \mathscr{F}(\Delta(\overline{P}))$ and $A = \mathrm{Ess}(P)$. Since $\Delta(Q) = \Delta(\mathscr{F}(\Delta(\overline{P}))) \cong \Delta(\overline{P})$, it suffices to check that $Q_{>\sigma}$ is contractible for all nonessential chains σ. Let $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ be a nonessential chain, $k \geq 2$. Then one open interval (x_{i-1}, x_i), say (x_0, x_1), is contractible. Now

$$Q_{>\sigma} \cong \mathscr{F}(\Delta(x_0, x_1) * \Delta(x_1, x_2) * \cdots * \Delta(x_{k-1}, x_k)),$$

where $*$ denotes simplicial join. Being a subcomplex of this join, $\Delta(x_0, x_1)$ can be contracted to a point without changing the global homotopy type (see [4, Lemma 10.2]). But this turns the join into a cone, and it follows that $Q_{>\sigma}$ is contractible. □

Essential chains can be combinatorially characterized in terms of the Möbius function μ for many interesting classes of posets. Namely, it is often (but not always) the case that

$$\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1} \text{ is essential}$$
$$\Leftrightarrow \mu(x_{i-1}, x_i) \neq 0 \text{ for all } 1 \leq i \leq k.$$

This is by the results of [3, 5, 6] true for the posets considered in §3, namely, weak Bruhat order of Coxeter groups and tope posets of oriented matroids. It is also true for all shellable posets (including semimodular lattices and Bruhat order on Coxeter groups and various subsets), for subword order and factor order on free monoids, and for dominance order on the partitions of an integer. Hence, in all these (and other) cases, a chain is essential if and only if it avoids steps of zero Möbius value.

Theorem 1 has a version for simplicial complexes that is proved the same way. Say that a simplex σ is essential in a complex Δ if $\mathrm{link}_\Delta(\sigma)$ is not contractible. Then for any simplicial complex Δ of finite dimension, the poset of essential simplices (ordered by inclusion) is homotopy equivalent to Δ.

3. Examples

The conjecture of Baues [1] is concerned with weak Bruhat order on the symmetric group S_n. This poset $WB(S_n)$ is defined as follows: $\pi \leq \sigma$ if there exist permutations $\pi = \pi_0, \pi_1, \ldots, \pi_k = \sigma$ such that π_i differs from π_{i-1} by an adjacent transposition that increases (by one) the number of inversions. Say that permutations π and σ differ by an *N-move* if σ is obtained from π by reversing a collection of k disjoint increasing substrings of lengths $n_1, \ldots, n_k$

such that $N = n_1 + \cdots + n_k - k$. Equivalently, in Coxeter group terms, this is the case if $\pi^{-1}\sigma$ is the maximal length element $w_0(J)$ of some parabolic subgroup W_J with $|J| = N$ and $\text{inv}(\sigma) = \text{inv}(\pi) + \text{inv}(w_0(J))$. The 1-moves are precisely the covering relations of $WB(S_n)$.

Theorem 6 of [3], when specialized to symmetric groups, gives the following description of the topology of intervals in $WB(S_n)$:

(i) $\Delta(\pi, \sigma)$ is contractible if $\pi < \sigma$ is not a move,
(ii) $\Delta(\pi, \sigma)$ is homotopy equivalent to the $(N-2)$-sphere if $\pi < \sigma$ is an N-move.

In particular, $WB(S_n) - \{e, w_0\}$ has the homotopy type of an $(n-3)$-sphere (since $e < w_0$ is an $(n-1)$-move), and therefore by Theorem 1 so does also $\text{Ess}(WB(S_n))$, as conjectured by Baues.

Essential chains in $WB(S_n)$ have been studied under the name of "allowable sequences" in several papers of Goodman and Pollack. Ungar has shown that every essential chain has length at least $(n-1)$ (or n, if n is even). For a discussion of these facts see [5, §§1.10, 1.11, and 6.4].

Weak Bruhat order on a *finite* Coxeter group is a special case of a tope poset of an oriented matroid. Using the description of the topology of intervals in tope posets given in [6; 5, Theorem 4.4.2], we can in the same way as for $WB(S_n)$ deduce the following.

Theorem 2. *Let $\mathcal{T}(\mathcal{L}, B)$ be the tope poset of an oriented matroid of rank r. Then $\text{Ess}(\mathcal{T}(\mathcal{L}, B))$ has the homotopy type of an $(r-2)$-sphere.*

In the realizable case, i.e., for zonotopes, this is subsumed by the results of Billera, Kapranov, and Sturmfels [2]. These authors prove a stronger result, also conjectured by Baues in the $WB(S_n)$ case; namely, that the order complex $\Delta = \Delta(\text{Ess}(\mathcal{T}(\mathcal{L}, B)))$ for a realizable oriented matroid $\mathcal{L}$ contains a subcomplex that is *homeomorphic* to S^{r-2} and to which Δ retracts. This is undoubtedly true also in general, but such a strengthening of Theorem 2 would require other methods.

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