Subspace Arrangements

ANDERS BJÖRNER

Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm, Sweden

Contents

1. Introduction
2. Basic definitions
3. Examples
4. The intersection semilattice
5. Operations on arrangements
6. Topology of the union and link
7. Topology of the complement
8. Consequences and examples
9. An application to computational complexity
10. More connections with complexity theory
11. Some ring-theoretic aspects
12. Cell complexes and matroids
13. Final remarks and open problems
References

1. Introduction

This paper will describe some recent developments in an area where combinatorics and complexity theory on one hand and geometry and topology on the other have interacted in several fruitful ways. By a *subspace arrangement* we mean a finite collection of affine subspaces in Euclidean space $\mathbb{R}^n$. There is a long tradition of work on ***hyperplane arrangements***, i.e., concerning subspaces of codimension 1. Here, however, the emphasis will be entirely on arrangements of subspaces of *arbitrary* dimensions, about which much less is known.

Dear Jesus,
Can you construct (univ.) Gröbner bases for all the orbit arrangements $\mathcal{A}_?$ (see §3.3 and 13.5) ?? Please keep me informed about further progress in this area. Best regards,
Anders Björner

For: Proceedings of 1st European Congress of Mathematics (Paris 1992)
Version of November, 1992

To motivate and prepare for the main topic, I will begin with a few comments about the study of hyperplane arrangements. There are two somewhat separate traditions here. One is combinatorial and studies mainly enumerative and structural properties of $\mathbb{R}$-arrangements (real hyperplanes in $\mathbb{R}^n$). The other is topological and mainly concerned with the topology of spaces associated with $\mathbb{C}$-arrangements (complex hyperplanes in $\mathbb{C}^n$). These two traditions were pursued more or less separately for a long time, although reflection arrangements (of finite Coxeter groups) always provided an area of interaction. A more unified view of the field has emerged in recent years, and much could be said about the vigour and breadth of current research on hyperplane arrangements. However, this has recently been done in two book-length expositions, see Björner, LasVergnas, Sturmfels, White & Ziegler [BLSWZ] and Orlik & Terao [OT1], of which the former mostly deals with the combinatorial and the latter with the topological aspects. So, my comments on hyperplane arrangements can be very brief.

An affine hyperplane cuts $\mathbb{R}^n$ into two connected regions. Several hyperplanes disconnect $\mathbb{R}^n$ into more regions, some of which may be bounded and others not. This simple fact makes the beginning of the combinatorial study of hyperplane arrangements. How many regions make up its complement? How many bounded regions are there? What can be said about the structure of these regions, their numbers of faces of various dimensions, their incidences, etc.? For instance, look at the line arrangement in Figure 1

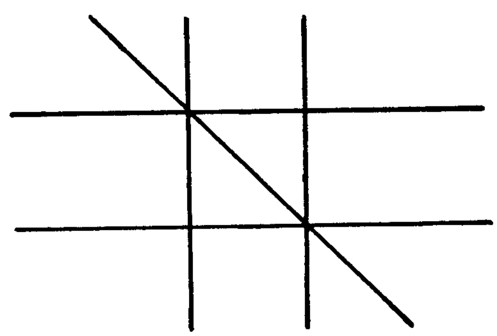


Figure 1.

It divides $\mathbb{R}^2$ into 12 regions, of which 2 are bounded. We can also observe that it contains simple intersection points (where only two lines meet) and triangular regions. These are all special cases of general facts, see [BLSWZ, Chapters 4 and 6].

The enumeration of regions of various kinds in the complement of a real hyperplane arrangement has a long history going back to the mid-1800's, and many nice formulas for special cases were discovered over the years. However, a satisfactory explanation and general formula was not achieved until 1975, when Zaslavsky published his enumerative theory [Za]. His main finding is that the number of regions of an arrangement $\mathcal{A}$ is determined by the intersection lattice $L_{\mathcal{A}}$ (defined in Section 2) in terms of its Möbius function μ.

Theorem 1.1 [Za]. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then*

$$\#\{\textit{regions}\} = \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)|$$

$$\#\{\textit{bounded regions}\} = \left| \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \right|.$$

This result gives a first indication of the important role played by intersection lattices in the theory of arrangements. We will later see many other examples of the amount of information encoded into these combinatorial objects. Figure 2 depicts the intersection lattice of the arrangement from Figure 1, with all values $\mu(0, x)$ of its Möbius function indicated.

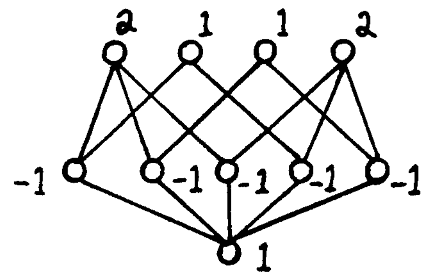


Figure 2.

A complex hyperplane in $\mathbb{C}^n$ also disconnects the space $\mathbb{C}^n \cong \mathbb{R}^{2n}$, but only in a higher-dimensional sense: its complement is 0-connected (i.e., arcwise connected) but not 1-connected (simply connected). This can be easily visulized only for the case $n = 1$, i.e., for the hyperplane $\{0\}$ in $\mathbb{C}$.

If $\mathcal{A}$ is an arrangement of complex hyperplanes in $\mathbb{C}^n$, then its complement $M_{\mathcal{A}}$ (i.e., those points of $\mathbb{C}^n$ that do not lie on any of the hyperplanes in $\mathcal{A}$) is a space with non-trivial topology. The relevant combinatorial invariants are the *Betti numbers* $\beta^i(M_{\mathcal{A}})$, i.e., the ranks of the singular cohomology groups $H^i(M_{\mathcal{A}})$. Recall that $\beta^0(T)$ is the number of connected components, for any space T, so that the Betti number sequence $\beta(M_{\mathcal{A}}) = (\beta^0, \beta^1, \ldots)$ is the natural generalization of the region-count that we met in connection with $\mathbb{R}$-arrangements. Note also that $\beta^i(M_{\mathcal{A}}) = 0$ for all $i > 0$ for an $\mathbb{R}$-arrangement $\mathcal{A}$ (all regions are contractible cones), so Theorem 1.1 in fact gives complete information about the Betti numbers in this case.

The sequences $\beta(M_{\mathcal{A}})$ were determined for the complexified braid arrangement (soon to be defined) by Arnol'd [Ar1] in 1969, and more generally for all complexified reflection arrangements by Brieskorn [Br] in 1971. Brieskorn also showed that the cohomology groups $H^i(M_{\mathcal{A}})$ are torsion-free for all complex hyperplane arrangements $\mathcal{A}$. The general rule for computing $\beta(M_{\mathcal{A}})$ was found by Orlik and Solomon [OS] in 1980. As in Theorem 1.1 it involves the intersection lattice $L_{\mathcal{A}}$ and its Möbius function μ.

Theorem 1.2 [OS]. *Let $\mathcal{A}$ be a complex hyperplane arrangement in $\mathbb{C}^n$ with complement $M_{\mathcal{A}}$. Then*

$$\beta^i(M_{\mathcal{A}}) = \sum_{\substack{x \in L_{\mathcal{A}} \\ \operatorname{codim}^{\mathbb{C}}(x)=i}} |\mu(\hat{0}, x)| \quad , \quad \textit{for all } i.$$

Here $\operatorname{codim}^{\mathbb{C}}(x) = n - \dim^{\mathbb{C}}(x)$, *where* $\dim^{\mathbb{C}}(x)$ *is the complex dimension of* x.

Theorems 1.1 and 1.2 are clearly related, and it is interesting to compare them in case $\mathcal{A}$ is an $\mathbb{R}$-arrangement and $\mathcal{A}^{\mathbb{C}}$ its *complexification.* By this we mean that the same real forms $\ell_i(x) = 0$ that define the hyperplanes of $\mathcal{A}$ in $\mathbb{R}^n$ are used to define the complex hyperplanes of $\mathcal{A}^{\mathbb{C}}$ in $\mathbb{C}^n$. It is easy to see that $\mathcal{A}$ and $\mathcal{A}^{\mathbb{C}}$ have isomorphic intersection lattices, so the following invariance of combinatorial properties under complexification can be deduced from Theorems 1.1 and 1.2:

$$\#\{\text{regions of } M_{\mathcal{A}}\} = \sum_{i \geq 0} \beta^i(M_{\mathcal{A}^{\mathbb{C}}})$$

(1.1) $$\#\{\text{bounded regions of } M_{\mathcal{A}}\} = |\chi(M_{\mathcal{A}^{\mathbb{C}}})|$$

For instance, the complexification of the line arrangement $\mathcal{A}$ in Figure 1 is an arrangement of 5 complex lines in $\mathbb{C}^2$ whose complement $M_{\mathcal{A}^{\mathbb{C}}}$ is a 4-manifold with Betti numbers $\beta(M_{\mathcal{A}^{\mathbb{C}}}) = (1, 5, 6, 0, 0)$ and Euler characteristic $\chi(M_{\mathcal{A}^{\mathbb{C}}}) = 2$, as can be quickly seen from Figure 2 and Theorem 1.2.

There is one particular family of hyperplane arrangements which has been the starting point for many investigations and which today occupies an in every sense central position in the theory. This is the family of *braid arrangements* $\mathcal{A}_{n,2}$ and their complexifications $\mathcal{A}_{n,2}^{\mathbb{C}}$ (the notation will be explained in Section 3.1), defined in terms of linear forms by $\mathcal{A}_{n,2} = \{x_i - x_j | 1 \leq i < j \leq n\}$. This is the reflection arrangement of the symmetric group S_n (acting on $\mathbb{R}^n$ by permuting coordinates), and it is easy to identify its intersection lattice with the partition lattice Π_n (i.e., all partitions of the set $\{1, \ldots, n\}$ ordered by refinement). It was shown in 1962 by Fadell, Fox and Neuwirth [FaN] [FoN] that the fundamental group of the complement of $\mathcal{A}_{n,2}^{\mathbb{C}}$ is the pure braid group, while all its other homotopy groups vanish. In particular, $\mathcal{A}_{n,2}^{\mathbb{C}}$ is a $K(\pi, 1)$ space. This result was in 1972 extended by Deligne [De] to all reflection arrangements (and a little beyond), but otherwise few general results are known about homotopy properties of complements of complex arrangements. See [OT1] for more about the braid arrangement and the topology of complex hyperplane arrangements generally.

So, what about general subspace arrangements? What are the relevant questions to ask, both from a combinatorial and from a topological point of view? I hope to have made it clear with the preceding discussion that the fundamental question, whose answer would indicate whether a useful general theory exists or not, is whether a combinatorial formula for the Betti numbers of the complement $M_{\mathcal{A}}$ of a general subspace arrangement $\mathcal{A}$ in terms of its intersection lattice $L_{\mathcal{A}}$ (or some similar combinatorial gadget) can be found.

Such a Betti number formula of striking simplicity and elegance was in the mid 1980's found by Goresky and MacPherson (published in 1988).

Theorem 1.3 [GM]. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$ with complement $M_{\mathcal{A}}$. Then*

$$\tilde{\beta}^i(M_{\mathcal{A}}) = \sum_{x \in L_{\mathcal{A}} - \{\hat{0}\}} \tilde{\beta}_{\text{codim}(x)-2-i}(\hat{0}, x).$$

In this formula $\tilde{\beta}_d(\hat{0}, x)$ is the rank of the d-dimensional reduced simplicial homology group of the order complex of the open interval $(\hat{0}, x)$ in $L_{\mathcal{A}}$. The meaning of these terms will be explained in Section 2.4, and a more complete discussion of the Goresky-MacPherson result (a formula for the actual cohomology groups) will be given in Section 7.

Theorem 1.3 contains the Betti number formulas of Theorems 1.1 and 1.2 as special cases. This is easy to see because of a special relation, due to Folkman [Fo], which for $\mathbb{R}$-arrangements takes the form:

$$(1.2) \qquad \tilde{\beta}_{\text{codim}(x)-2-i}(\hat{0}, x) = \begin{cases} (-1)^{\text{codim}(x)}\mu(\hat{0}, x) & , \text{ if } i = 0 \\ 0 & , \text{ if } i \neq 0 \end{cases}.$$

and for $\mathbb{C}$-arrangements:

$$(1.3) \quad \tilde{\beta}_{2\cdot\text{codim}^{\mathbb{C}}(x)-2-i}(\hat{0}, x) = \begin{cases} (-1)^{\text{codim}^{\mathbb{C}}(x)}\mu(\hat{0}, x) & , i = \text{codim}^{\mathbb{C}}(x) \\ 0 & , \text{otherwise} \end{cases}.$$

This kind of relation is quite special and depends on the fact that the intervals $[\hat{0}, x]$ are geometric lattices in the case of hyperplane arrangements (see Theorem 4.5.1). In general we find that $\tilde{\beta}_d(\hat{0}, x) \neq 0$ for several dimensions d, so the Goresky-MacPherson formula cannot be simplified to a form where only the Möbius function of $L_{\mathcal{A}}$ occurs. This may seem discouraging for potential applications, since a very rich combinatorial theory exists for the Möbius function, making explicit computations possible. However, there are also some combinatorial tools for computing Betti numbers of finite posets, and although more cumbersome to use than the Möbius function, such tools combined with Theorem 1.3 have produced explicit calculations in some cases. This will be exemplified in Section 8.

The breakthrough of Goresky and MacPherson not only opened up the area of subspace arrangements as a promising field of study, it has also provided a new perspective on complex hyperplane arrangements. The viewpoint of [GM] is to see these as just a special kind of arrangement of real subspaces of codimension 2 in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Continuing in this direction, one is led to single out a class called "c-arrangements" of arrangements of real subspaces of codimension c which preserve the essential features from the hyperplane theory. This is done by Goresky & MacPherson [GM], who observe that the crucial combinatorial fact that make "hyperplane-type" results true is that the intersetion lattice is a geometric lattice. This guarantees vanishing of homology as in (1.2) and (1.3). Thus, most general facts about $\mathbb{C}$-hyperplane arrangements can be generalized to 2-arrangements (and to c-arrangements), except for some fine details concerning the multiplicative structure of the cohomology algebra, as was shown by Ziegler [Z1]. Here both $\mathbb{R}$- and $\mathbb{C}$-hyperplane arrangements will be treated primarily as special cases of c-arrangements, see Section 4.2.

The material of this paper is organized in the following way. In Sections 2–5 I will discuss elementary combinatorial aspects of subspace arrangements. Several examples will be given, showing how subspace arrangements naturally arise in combinatorial situations, and a general look at intersection lattices will be taken.

The basic topological facts are presented in Sections 6–8. This includes a discussion not only of the complement of an arrangement, but also of the union of its hyperplanes. The two spaces are of course homologically related via Alexander duality.

Sections 9 and 10 are devoted to connections with complexity theory. The general idea is that measures of topological complexity, such as the covering number of a map, the Euler characteristic or the Betti numbers computable via Theorem 1.3, can somehow be converted to measures of computational complexity for computational problems with geometric content.

The final sections contain some brief mention of connections with various other topics, and in particular with ring theory. The latter stems from the fact that the union of a subspace arrangement is an algebraic variety, whose coordinate ring in some cases has interesting combinatorial properties.

Acknowledgements. I want to thank my coauthors L. Lovász, V. Welker and G. M. Ziegler, with whom I have recently collaborated on questions that involve subspace arrangements [BL2] [BW] [BZ1]. In the process of this joint work I have learned much of what is here reported. Also, the "Habilitations-Schrift" of Ziegler [Z4] has been an invaluable source of information, which I recommend as a reference. Useful comments and suggestions have been received from

2. Basic definitions

2.1. A *subspace arrangement* (or *affine subspace arrangement*) $\mathcal{A} = \{K_1, \ldots, K_t\}$ in $\mathbb{R}^n$ is a finite collection of affine proper subspaces K_i. It is called *central* if all K_i are linear subspaces, i.e., if $0 \in K_i$. Since all questions treated are invariant with respect to translations we can usually consider an arrangement to be central if $K_1 \cap \ldots \cap K_t \neq \emptyset$.

A d-dimensional subspace x of $\mathbb{R}^n$ is said to have *codimension* $n - d$: $\text{codim}(x) = n - \dim(x)$. A subspace arrangement $\mathcal{A} = \{K_1, \ldots, K_t\}$ will be called

(i) *simple*, if $K_i \subseteq K_j$ implies $i = j$ for all $1 \leq i, j \leq t$,

(ii) *pure*, if $\dim(K_i) = \dim(K_j)$, for all $1 \leq i, j \leq t$.

(iii) *d-dimensional*, if $d = \max\limits_{1 \leq i \leq t} \dim(K_i)$,

(iv) *c-codimensional*, if $c = \min\limits_{1 \leq i \leq t} \text{codim}(K_i)$.

Finally, $\mathcal{A}$ is a *c-arrangement* if $\mathcal{A}$ is central, pure c-codimensional and c divides $\text{codim}(x)$ for all intersections $x = K_{i_1} \cap \ldots \cap K_{i_p}$. See §4.2 for more about this concept.

Due to the space limitations I will only discuss affine and central subspace arrangements here. There are also other related concepts, such as *projective arrangements* and *spherical arrangements*, for which I must refer to the literature. See e.g. Goresky & MacPherson [GM] or Ziegler & Živaljević [ZŽ].

2.2. Two important spaces associated with an arrangement $\mathcal{A} = \{K_1, \ldots, K_t\}$ in $\mathbb{R}^n$ are:

$$V_{\mathcal{A}} = \bigcup_{i=1}^{t} K_i \quad \text{and} \quad M_{\mathcal{A}} = \mathbb{R}^n - V_{\mathcal{A}},$$

called the *union* and *complement*. Both are topological spaces as subspaces of $\mathbb{R}^n$ ($M_{\mathcal{A}}$ is an n-dimensional manifold), and $V_{\mathcal{A}}$ also has the structure of a real algebraic variety, being the union of spaces defined by systems of linear equations. We will by $\widehat{V}_{\mathcal{A}}$ denote the ***one-point compactification*** of $V_{\mathcal{A}}$, which is a subspace of the one-point compactification $\widehat{\mathbb{R}^n} \cong S^n$. Note that with this we achieve that $\widehat{V}_{\mathcal{A}}$ and $M_{\mathcal{A}}$ are complementary subspaces of S^n.

If $\mathcal{A}$ is central we define

$$V_{\mathcal{A}}^{\circ} = V_{\mathcal{A}} \cap S^{n-1} \quad , \quad M_{\mathcal{A}}^{\circ} = M_{\mathcal{A}} \cap S^{n-1} \quad ,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The union $V_{\mathcal{A}}$ is contractible in the central case, but the (*singularity*) ***link*** $V_{\mathcal{A}}^{\circ}$ carries an interesting topology. Note that in this case the compactification of the union equals the suspension of the link: $\widehat{V}_{\mathcal{A}} = \text{susp}\,(V_{\mathcal{A}}^{\circ})$.

2.3. The ***intersection semilattice*** $L_{\mathcal{A}}$ of an arrangement $\mathcal{A} = \{K_1, \ldots, K_t\}$ is the collection of all non-empty intersections $K_{i_1} \cap \ldots \cap K_{i_p}$, $1 \leq i_1 < \ldots < i_p \leq t$, ordered by reverse inclusion: $x \leq y \Leftrightarrow x \supseteq y$. This is a meet-semilattice, i.e., a partially ordered set such that a greatest lower bound (or *meet*) $x \wedge y$ exists for all $x, y \in L_{\mathcal{A}}$. There is a bottom element $\hat{0} = \mathbb{R}^n$ below all the others, but in general no top element. Figures 1 and 2 illustrate this concept, see also Section 4.

If $\mathcal{A}$ is *central*, then there is also a top element $\hat{1} = \cap\mathcal{A} = K_1 \cap \ldots \cap K_t$ and a least upper bound (or *join*) $x \vee y = x \cap y$ exists in $L_{\mathcal{A}}$ for all $x, y \in L_{\mathcal{A}}$. So, we may speak of the *intersection lattice* $L_{\mathcal{A}}$ in this case.

Given two arrangements $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^n$, let us say that $\mathcal{A}$ is ***embedded*** in $\mathcal{B}$ if $\mathcal{A} \subseteq L_{\mathcal{B}}$. In this case $L_{\mathcal{A}}$ has a join-preserving embedding in $L_{\mathcal{B}}$, meaning that $L_{\mathcal{A}} \subseteq L_{\mathcal{B}}$ and $x \vee_{\mathcal{A}} y = x \vee_{\mathcal{B}} y$ for all $x, y \in L_{\mathcal{A}}$ such that $x \vee y$ exists.

2.4. We will make frequent use of the order complex of a poset (finite partially ordered set), so the definitions and some basic facts will be reviewed here. See [Bj2] for more details and references, and also for a condensed review of the topological notions used.

For a poset P and elements $x, y \in P$, $x \leq y$, define

$$\begin{aligned}
(x, y) &= \{z \in P \mid x < z < y\}, \\
[x, y] &= \{z \in P \mid x \leq z \leq y\}, \\
P^{>x} &= \{z \in P \mid z > x\},
\end{aligned}$$

and similarly for $P^{\geq x}$, $P^{<x}$, $P^{\leq x}$. The ***order complex*** $\Delta(P)$ is the abstract simplicial complex whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \ldots < x_k$. $\widetilde{H}_i(P)$ denotes the reduced simplicial homology group of $\Delta(P)$ with integer coefficients. We will often consider the order complex of an open interval (x, y), and to simplify notation we will write $\Delta(x, y) = \Delta\,((x, y))$ and $\widetilde{H}_i(x, y) = \widetilde{H}_i\,((x, y))$, etc. When speaking about topological properties of a lattice L we will always have the complex

$\Delta(\hat{0},\hat{1})$ in mind.

The *Betti numbers* $\tilde{\beta}_i(P)$ are defined by $\tilde{\beta}_i(P) = \text{rank } \widetilde{H}_i(P)$. It is a basic fact, due to Ph. Hall in the 1930s (see [St2, p. 120]), that the Möbius function $\mu(x,y)$ is the reduced Euler characteristic of the order complex of the open interval (x,y):

$$\mu(x,y) = \sum_{i\geq -1} (-1)^i \, \tilde{\beta}_i(x,y). \tag{2.4.1}$$

3. Examples

Just as hyperplane arrangements naturally arise from graphs (represent each edge (i,j) by a hyperplane $x_i = x_j$), so subspace arrangements arise from hypergraphs. This and other examples of subspace arrangements (e.g., coming from reflection groups) will be mentioned in this section. Other examples arise from the constructions described in § 5. By a *hypergraph* $\mathcal{H} \subseteq 2^V$ we mean a finite ground set V (usually taken to be $[n] = \{1,\ldots,n\}$) together with a collection $\mathcal{H}$ of nonempty subsets (e.g., a simplicial complex).

3.1. For each subset $S = \{i_1,\ldots,i_s\} \subseteq [n], s \geq 2$, let $K_S = \{x \in \mathbb{R}^n \mid x_{i_1} = \ldots = x_{i_s}\}$. Then a hypergraph $\mathcal{H} \subseteq 2^{[n]}$ (without singletons) determines the subspace arrangement $\mathcal{A}_{\mathcal{H}} = \{K_S \mid S \in \mathcal{H}\}$. A special case of such *hypergraph arrangements* merits special mention, namely when $\mathcal{H}$ consists of *all* k-element subsets on $[n]$. This is called the *k-equal arrangement* and denoted $\mathcal{A}_{n,k}$. Note that for $k = 2$ this is the braid arrangement, whose intersection lattice is Π_n (the lattice of all partitions of the set $[n]$). Since every subspace K_S is the intersection of hyperplanes $x_i = x_j$, we see that every hypergraph arrangement $\mathcal{A}_{\mathcal{H}}$ is embedded in the braid arrangement $\mathcal{A}_{n,2}$. The intersection lattice of the k-equal arrangement $\mathcal{A}_{n,k}$ is the lattice $\Pi_{n,k}$ of partitions of $[n]$ with no blocks of sizes $2,3,\ldots,k-1$. Since the k-equal arrangements have been studied in considerable detail (in [BL2], [BLY], [BW]) they will be frequently used as examples in the following.

3.2. Let $\{b_1,\ldots,b_n\}$ be a basis of $\mathbb{F}^n$. For each subset $S = \{i_1,\ldots,i_s\} \subseteq [n]$ let $K'_S = \text{ span }\{b_{i_1},\ldots,b_{i_s}\}$. Then $\mathcal{H} \subseteq 2^{[n]}$ determines the subspace arrangement $\mathcal{B}_{\mathcal{H}} = \{K'_S \mid S \in \mathcal{H}\}$. We will call arrangements of this type *Boolean*; they are all embedded in the coordinate hyperplane arrangement spanned by the chosen basis, whose intersection lattice is the Boolean algebra of all subsets of $[n]$. This construction is particularly interesting for simplicial complexes Δ, since the union of a Boolean arrangement $\mathcal{B}_\Delta$ is an algebraic variety whose coordinate ring is the Stanley-Reisner ring of Δ (§ 11), and whose topology is closely related to that of Δ (§ 8.1).

Furthermore, the intersection lattice of $\mathcal{B}_\Delta$ is antiisomorphic to the face lattice of Δ.

3.3. If $\pi = (B_1, \ldots, B_p)$ is a nontrivial partition of the set $[n]$, then let $K_\pi = K_{B_1} \cap \ldots \cap K_{B_p} = \{x \in \mathbb{R}^n \mid i, j \in B_k \Rightarrow x_i = x_j$, for all $1 \leq i, j \leq n, 1 \leq k \leq p\}$. The *shape* of π is the sequence of block sizes $|B_i|$ arranged in non-increasing order; it is a partition of the number n. Note that every sublattice of Π_n is the intersection lattice of an arrangement of subspaces of type K_π.

Given a nontrivial number partition $\lambda \vdash n$, let

$$\mathcal{A}_\lambda = \{K_\pi \mid \pi \in \Pi_n \text{ and shape } (\pi) = \lambda\}.$$

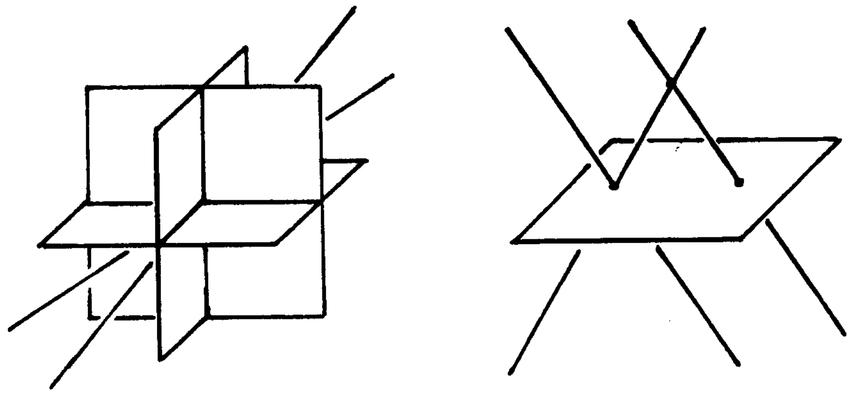
This is an *orbit arrangement* in the sense that $\mathcal{A}_\lambda$ is the orbit of any single subspace K_π under the natural action of S_n or $\mathbb{R}^n$ (permutation of coordinates). Note that all such orbit arrangements are embedded in the braid arrangement, which is the special case $\mathcal{A}_{(2,1,\ldots,1)}$. More generally, the k-equal arrangement $\mathcal{A}_{n,k}$ is the orbit arrangement $\mathcal{A}_{(k,1,\ldots,1)}$.

The intersection lattice of $\mathcal{A}_\lambda$ is the join-sublattice of Π_n that is join-generated by all set partitions of shape λ. For instance, for the rectangular shape $\lambda = (d, d, \ldots, d)$ the intersection lattice of A_λ is the lattice of "d-divisible" set partitions (for which all block sizes are divisible by d), studied by Calderbank, Hanlon & Robinson [CHR], Sagan [Sag] and Wachs [Wa].

3.4. Let G be a finite subgroup of $GL(\mathbb{R}^n)$ and K a proper subspace of $\mathbb{R}^n$. Then the orbit $\mathcal{A}_{G,K}$ of K under the action of G is a subspace arrangement. The most interesting case is when G is a finite reflection group (Coxeter group) and K is an intersection of reflecting hyperplanes. When specialized to the symmetric group S_n this gives precisely the class of orbit arrangements described in § 3.3.

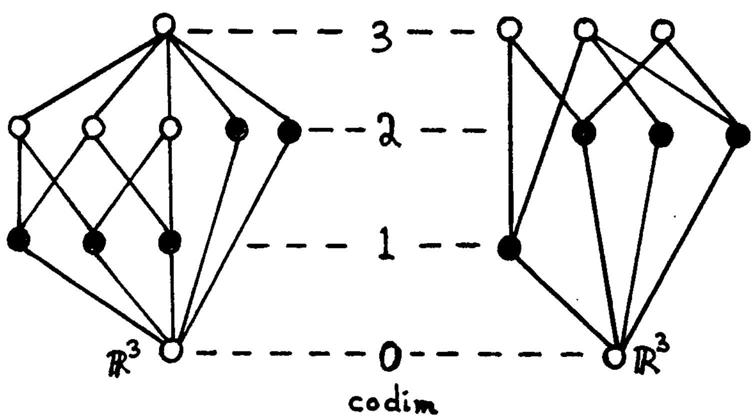
4. The intersection semilattice

4.1. The intersection semilattice $L_\mathcal{A}$ of an arrangement $\mathcal{A}$ was defined in § 2.3. One example was shown in Figure 2, and two more are given in Figure 4 (based on the $\mathbb{R}^3$-arrangements shown in Figure 3).



(a) (b)

Figure 3



(a) (b)

Figure 4

The semilattices are in Figure 4 drawn with their elements on different levels to emphasize that in addition to the order relation there is a ***rank function*** $r : L_{\mathcal{A}} \to \mathbb{N}$ given by $r(x) = \text{codim}(x)$. This rank function satisfies:

$$(4.1.1) \quad \begin{cases} (i) & r(\hat{0}) = 0 \\ (ii) & x < y \Rightarrow r(x) < r(y) \\ (iii) & r(x \vee y) + r(x \wedge y) \leq r(x) + r(y), \text{ if } x \vee y \text{ exists.} \end{cases}$$

The combinatorial information about $\mathcal{A}$ that is important resides in the *pair* $(L_{\mathcal{A}}, r)$ and not in the order structure of $L_{\mathcal{A}}$ alone.

It is natural to ask, given a finite semilattice $L = (L, \wedge)$ and a function $r : L \longrightarrow \mathbb{N}$ satisfying conditions (4.1.1), how can one know if $(L, r) \cong (L_{\mathcal{A}}, r)$ for some subspace arrangement $\mathcal{A}$? Questions of this type seem to have first been asked in this generality by A. M. Vershik [Ve1, Ve2]. There is no hope for a reasonable answer to the question of an effective characterization of rank-preserving representability, since it contains as a special case the question of representability over $\mathbb{R}$ of geometric lattices (or matroids), a problem which is known to be polynomially equivalent to the existential theory of the real numbers, and hence is NP-hard — see Bokowski & Sturmfels [BS] and Shor [Sh].

However, if we forget the rank function and ask only to represent a given semilattice L as the intersection semilattice $L_{\mathcal{A}}$ of some arrangement $\mathcal{A}$, then the situation improves: every L can be so represented by some arrangement $\mathcal{A}$ embedded in a braid arrangement $\mathcal{A}_{n,2}$ for suitable large n. This follows from a result of Pudlak & Tůma [PT] stating that every finite lattice can be lattice embedded into a finite partition lattice. See Rival & Stanford [RS] for a good survey of lattice results of this kind.

Since every semilattice L is representable as an intersection semilattice $L_{\mathcal{A}}$, one can go on and ask more about the representing arrangements $\mathcal{A}$. One such question that has been studied is: what is the minimal dimension of $\mathcal{A}$ such that $L \cong L_{\mathcal{A}}$? A result of Lovász (see (10.2.1)) implies that for every n there is a semilattice L of height 2 (maximal chains have 3 elements) which is not representable as $L_{\mathcal{A}}$ for any central subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$. Such an L is, for N sufficiently large, given by the subsets of $[2N]$ of sizes 0, 1, and 2, except for $\{i-1, i\}, i = 2, 4, \ldots, 2N$, ordered by inclusion. A non-constructive proof of the same result was given by Sapir & Scheinerman [SaS]. See also Ziegler [Z4] for a discussion of these questions.

4.2. A *geometric lattice* is a finite lattice that is semimodular and atomic. This is an important class of lattices and discussed in many places, see e.g. Birkhoff [Bi], Crapo & Rota [CR], Welsh [We] or White [W1, W2, W3]. A geometric lattice is essentially the same thing as a matroid (§ 12.2), see e.g. [CR] or [W1, Chapter 3] for the details of this correspondence.

A geometric lattice L has a *rank function* $r : L \longrightarrow \mathbb{N}$, where $r(x)$ denotes the common length of all maximal chains from $\hat{0}$ to x. The following fact was observed by Goresky & MacPherson [GM]:

Proposition 4.2.1. *If $\mathcal{A}$ is a c-arrangement then $L_{\mathcal{A}}$ is a geometric lattice, and* $\operatorname{codim}(x) = c \cdot r(x)$ *for all $x \in L_{\mathcal{A}}$.*

A 1-arrangement is the same thing as a central real hyperplane arrangement. Central complex hyperplane arrangements are examples of 2-arrangements, but not conversely: Goresky & MacPherson [GM] construct a 2-arrangement of nine 4-planes in $\mathbb{R}^6 \cong \mathbb{C}^3$ which if it were realizable by complex hyperplanes would violate the theorem of Pappus. A smaller example of a 2-arrangement without complex structure, consisting of four 2-planes in $\mathbb{R}^4 \cong \mathbb{C}^2$, was later given by Ziegler [Z1]. So, the class of c-arrangements strictly contains the real and complex central hyperplane arrangements and preserve the property that the intersection lattice is geometric. Note that in this case the rank function (=codimension) is deducible from the order structure alone.

The representation problem arises again: given a geometric lattice L (equivalently, a matroid) and a positive integer c, is L "*c-representable*", meaning is $L = L_{\mathcal{A}}$ for some c-arrangement $\mathcal{A}$? As was already mentioned, there is no hope for a good answer for the most restrictive case $c = 1$. However, it is easy to see using "c-plexification" (§ 5.2) that every t-representable geometric lattice is ct-representable for all $c \in \mathbb{Z}_+$, so one cannot a priori rule out the possibility that *every* geometric lattice is c-representable for some $c \geq 2$. However, this is not the case: L. Lovász has shown (personal communication) that e.g. the Vamos matroid is not c-representable for any c. His argument is that a certain rank-function inequality due to Ingleton [In] (also in Welsh [We], p. 158) must hold in every c-representable matroid. So, the question of c-representability of matroids is open, but probably hopeless.

4.3. Geometric lattices have a top element and therefore do not arise from non-central subspace arrangements. However, there is a more general notion of *geometric semilattice* which specializes to that of geometric lattice precisely when there is a top element. Due to the space constraints this is not the place to define or enter a discussion of geometric semilattices, see Wachs & Walker [WW] for this. Let it suffice to say that if $\mathcal{A}$ is an affine arrangement of real or complex hyperplanes then $L_{\mathcal{A}}$ is a geometric semilattice. The obvious-seeming generalization to "affine c-arrangements" is, however, problematic (see Ziegler [Z2]), so that geometric semilattices seem for the time being best suited for the study of hyperplane arrangements.

4.4. We will assume familiarity with the *Möbius function* $\mu(x,y)$ of a poset, see e.g. Stanley [St2] for an account of the basic theory. Whereas the Möbius function of a geometric lattice has some non-trivial properties (such as the sign property $(-1)^{h(x)-h(y)}\mu(x,y) \geq 0$ discovered by G.-C. Rota [Ro]) this is not the case for intersection semilattices $L_{\mathcal{A}}$ in general, for the simple reason that *every* finite semilattice is of this kind (§ 4.1).

Define the *characteristic polynomial* of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$ by

$$P_{\mathcal{A}}(t) = \sum_{x \in L_{\mathcal{A}}} \mu\left(\hat{0}, x\right) t^{\dim(x)}. \qquad (4.4.1)$$

Again, there is much to be said about such polynomials if $L_{\mathcal{A}}$ is a geometric lattice (see [W2, Chapter 7]), hardly any of which survives the generalization to arbitrary subspace arrangements (cf. § 8.2). However, computations in [BLY] and [BL2] for the k-equal arrangements show that the Möbius function and characteristic polynomial has interesting mathematical structure in special cases. I will quote the following results from [BL2]:

Theorem 4.4.1. *Let $\mu_{n,k}$ and $P_{n,k}(t)$ be the Möbius function and characteristic polynomial of the intersection lattice of $\mathcal{A}_{n,k}$. Furthermore, let $\alpha_1, \dots, \alpha_{k-1}$ be the roots of the polynomial $p_k(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{k-1}}{(k-1)!}$. Then*

(i) $\mu_{n,k}(\hat{0}, \hat{1}) = -(n-1)! \sum_{i=1}^{k-1} \alpha_i^{-n}$.

(ii) $P_{n,k}(-1) = n!(k-1)! \sum_{i=1}^{k-1} \alpha_i^{-n-k} = -\binom{n+k-1}{n}^{-1} \mu_{n+k,k}(\hat{0}, \hat{1})$.

(iii) $\sum_{n=0}^{\infty} P_{n,k}(t) \frac{x^n}{n!} = [p_k(x)]^t$.

Here is an explicitly computed example:

$$(4.4.2) \qquad \begin{cases} \mu_{6,3}(\hat{0}, \hat{1}) = 0 \\ P_{6,3}(t) = t^6 - 20t^4 + 45t^3 - 26t^2 \\ P_{6,3}(-1) = -90 = -\frac{1}{28}\mu_{9,3}(\hat{0}, \hat{1}) \end{cases}$$

4.5. What can be said about the topology of intersection lattices $L_{\mathcal{A}}$? Again, nothing much in general, since by § 3.2 the face lattice of every simplicial complex Δ is (up to order-reversal) the intersection lattice of a Boolean arrangement $\mathcal{B}_{\Delta}$. Let us however for later use record the fact that geometric (semi)lattices have good topological properties.

Theorem 4.5.1. *Let L be either a geometric lattice of rank r or a geometric semilattice of rank $r-1$ augmented with a top element $\hat{1}$.*

(i) *Then L has the homotopy type of a wedge of $|\mu(\hat{0}, \hat{1})|$ copies of the $(r-2)$-dimensional sphere.*

(ii) $\widetilde{H}_i(L) = \begin{cases} \mathbb{Z}^{|\mu(\hat{0}, \hat{1})|} & , \text{ if } i = r-2 \\ 0 & , \text{ otherwise} \end{cases}$.

For geometric lattices this result was proved by Folkman [Fo] (part (ii)) and Björner [Bj1] (part (i)), and the extension to geometric semilattices is due to Wachs & Walker [WW]. See also [Bj3] and [Z2].

The intersection lattices of some orbit arrangements $\mathcal{A}_\lambda$ (see § 3.3) also have well-behaved topological structure. Part (i) of the following result was first shown by M. Wachs, see Sagan [Sag] and Wachs [Wa]; part (ii) is due to Björner & Welker [BW].

Theorem 4.5.2. (i) *Let* $\lambda = (d, d, \ldots, d)$. *Then* $L_{\mathcal{A}_\lambda}$ *(the lattice of partitions with block sizes divisible by d) has the homotopy type of a wedge of spheres of dimension* $(\frac{n}{d} - 2)$.
(ii) *Let* $\lambda = (k, 1, \ldots, 1)$. *Then* $L_{\mathcal{A}_\lambda} = \Pi_{n,k}$ *(the lattice of partitions with no blocks of size* $2, 3, \ldots, k-1$*) has the homotopy type of a wedge of spheres of dimensions* $n - 3 - t(k-2)$, *for* $1 \leq k \leq \lfloor \frac{n}{k} \rfloor$.

So, in these cases all homology groups are torsion-free. Also, the same good topological behaviour is found in all lower intervals $[\hat{0}, x]$, which will later be of importance.

5. Operations on arrangements

The class of all subspace arrangements is closed under various simple constructions, and these constructions are combinatorially tractable in the sense that they behave well on the level of intersection lattices. This makes it easy to construct new arrangements from old ones, and the flexibility in this respect is of course greater than within the class of hyperplane arrangements. I will here list a few basic constructions, the proofs of claimed properties are immediate.

5.1. Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{F}^n$ and $L_{\mathcal{A}}$ its intersection lattice with rank function $r(x) = \text{codim}\,(x)$.

(i) The *contraction* to $K \in L_{\mathcal{A}}$ is the arrangement $\{K \cap K' \mid K' \in \mathcal{A} - \{K\}\}$ in $K \cong \mathbb{F}^d$. Its intersection lattice is the upper interval $L_{\mathcal{A}}^{\geq K}$ with rank function $r'(x) = r(x) - \textit{codim}\,(K)$.

(ii) The *deletion* of $K \in L_{\mathcal{A}}$ is the arrangement $\mathcal{A} - \{K\}$ with intersection lattice join-generated by the subset $\mathcal{A} - \{K\}$ in $L_{\mathcal{A}}$. The deletion of all subspaces of $\mathcal{A}$ that don't contain $x \in L_{\mathcal{A}}$ gives a subarrangement whose intersection lattice is the interval $[\hat{0}, x]$ in $L_{\mathcal{A}}$. In particular, by using contraction and deletion one sees that the class of ranked intersection lattices is closed under taking intervals $[y, x]$.

(iii) A *generic section* of $\mathcal{A}$ is defined as follows. Take a generic affine hyperplane H in $\mathbb{F}^n$ and let $\mathcal{A}' = \{H \cap K \mid K \in \mathcal{A}\}$. Then $L_{\mathcal{A}'} \cong \{x \in L_{\mathcal{A}} \mid r(x) < n\}$ with the same rank function. Thus, by repeated generic sections any *upper truncation* $\{x \in L_{\mathcal{A}} \mid r(x) \leq q\}$ can be realized as an intersection lattice.

(iv) The *p-truncation* of $\mathcal{A}, p \geq 1$, is the arrangement of all intersections of codimension p, i.e., $\{x \in L_{\mathcal{A}} \mid r(x) = p\}$. In some cases, e.g. for p-truncations of c-arrangements, the intersection lattice is then the *lower truncation* $\{x \in L_{\mathcal{A}} \mid r(x) = 0 \text{ or } r(x) \geq p\}$ of $L_{\mathcal{A}}$. By if necessary adding more elements from $L_{\mathcal{A}}$ to the p-truncations one sees that this lower truncation of $L_{\mathcal{A}}$ can always be realized as an intersection lattice. Thus the class of ranked intersection lattices is closed under both upper and lower truncation

As an example, the p-truncation of the braid arrangement is the family of all spaces K_π (defined in § 3.3) for partitions π with $n-p$ blocks. Note that the sequence of p-truncations for $p = 1, 2, 3, \dots$ provides a stratification of the union and a dual stratification of the complement.

5.2. The algebraic process of *complexification* $\mathcal{A} \to \mathcal{A}^{\mathbb{C}}$, i.e., turning a central arrangement $\mathcal{A}$ of real subspaces in $\mathbb{R}^n$ into the arrangement $\mathcal{A}^{\mathbb{C}}$ of complex subspaces in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ defined by the same real equations, has the following description on the combinatorial level. Since $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ we can place two copies of $\mathcal{A}$ into $\mathbb{C}^n$, one into the real $\mathbb{R}^n$-part and one into the imaginary $\mathbb{R}^n$-part. Then $\mathcal{A}^{\mathbb{C}}$ consists of all subspaces $K + iK$ of $\mathbb{C}^n$, for $K \in \mathcal{A}$.

This can be immediately generalized to "*c-plexification*", for $c \geq 2$, an operation that converts any central arrangement $\mathcal{A}$ in $\mathbb{R}^n$ into an arrangement $\mathcal{A}^c$ in $\mathbb{R}^{cn}$: just put $\mathbb{R}^{cn} = \mathbb{R}^n \oplus \dots \oplus \mathbb{R}^n$, place one copy of $\mathcal{A}$ in each of the c terms $\mathbb{R}^n$, and then take all subspaces $K + \dots + K$ of $\mathbb{R}^{cn}$ generated by $K \in \mathcal{A}$. It is easy to see that $L_{\mathcal{A}^c} \cong L_{\mathcal{A}}$, and that the rank (codimension) function has been scaled: $r_{\mathcal{A}^c}(x) = c \cdot r_{\mathcal{A}}(x)$ for all $x \in L_{\mathcal{A}}$. The c-plexification of a hyperplane arrangement is clearly a c-arrangement.

The process of c-plexification can be said to originate in the work of von Neumann [Ne]. He described how to construct lattice embeddings of the full subspace lattice of $\mathbb{R}^n$ (or any other field) into that of $\mathbb{R}^{cn}$ such that dimension is multiplied by c, which is exactly what we are talking about here. Constructions of such "stretch-embeddings" for other classes of geometric lattices were given by Björner & Lovász [BL1].

6. Topology of the union and link

As the study of subspace/hyperplane arrangements has developed, the focus has been primarily on the complement $M_{\mathcal{A}}$, as far as topology is concerned. In some recent work (e.g. [BZ1], [Fa], [JOS], [Va4], [ZŽ]), the idea has been to first work with the union, which has more combinatorial structure, and then pass to the complement via Alexander duality. Following this trend I will here treat the union first.

6.1. The following result says that the union of an arrangement has the same homotopy type as the order complex of its intersection lattice with bottom element removed.

Theorem 6.1.1. *For every affine arrangement:* $V_{\mathcal{A}} \simeq \Delta\left(L_{\mathcal{A}}^{>\hat{0}}\right)$.

A version of this appears in Goresky & MacPherson [GM, Section III.2.5]. Their formulation is a bit more involved, and (on p. 244) they say "this result is surprisingly difficult to verify". It was rediscovered by Björner, Lovász and Yao [BLY] in the simple form stated here and with an extremely simple proof based on the nerve theorem: the covering of $V_{\mathcal{A}}$ by the maximal subspaces $K_1, \ldots, K_m$ of $\mathcal{A}$ and the covering of $\Delta\left(L_{\mathcal{A}}^{>\hat{0}}\right)$ by the subcomplexes $\Delta\left(L_{\mathcal{A}}^{\geq K_i}\right), i = 1, \ldots, m$, have the same nerve, and all nonempty intersections are contractible.

The homotopy type of the compactification $\widehat{V}_{\mathcal{A}}$ can also be computed from combinatorial data, but this is considerably more difficult to prove. The following fundamental result is due to Ziegler & Živalević [ZŽ]. Here S^j denotes the j-dimensional sphere, "$*$" denotes join of spaces, and "$\simeq$" denotes homotopy equivalence.

Theorem 6.1.2. *For every affine arrangement:*

$$\widehat{V}_{\mathcal{A}} \simeq \underset{x \in L_{\mathcal{A}}^{>\hat{0}}}{wedge} \left(\Delta\left(0, x\right) * S^{\dim(x)}\right).$$

The proof of Ziegler & Živaljević uses homotopy limits of diagrams of spaces, a technique coming from semisimplicial topology. This method is used in [ZŽ] to prove several versions of their main result. For instance, if $\mathcal{A}$ is central, the previous formula can be "de-suspended" to the following formula for the link $V_{\mathcal{A}}^{\circ} = V_{\mathcal{A}} \cap S^{n-1}$.

Theorem 6.1.3. *For every central arrangement:*

$$V_{\mathcal{A}}^{\circ} \simeq \underset{x \in L_{\mathcal{A}}^{>\hat{0}}}{wedge} \left(\Delta\left(\hat{0}, x\right) * S^{\dim(x)-1}\right).$$

Theorems 6.1.2 and 6.1.3 are best possible in at least two ways: (1) the intersection lattice $L_{\mathcal{A}}$ alone (without the dimension function) does not determine the homotopy type of $\widehat{V}_{\mathcal{A}}$ and $V_{\mathcal{A}}^{\circ}$ (only of $V_{\mathcal{A}}$), and (2) the pair $(L_{\mathcal{A}}, \dim(x))$ does not determine $\widehat{V}_{\mathcal{A}}$ or $V_{\mathcal{A}}^{\circ}$ up to homeomorphism. The latter can be seen from arrangements of 6 planes in $\mathbb{R}^3$. Also, Ziegler [Z1] gives an example of non-homeomorphic links coming from two 2-arrangements of five 4-planes in $\mathbb{R}^6$ with identical ranked intersection lattices.

Theorem 6.1.2 implies the following formula on the level of homology groups:

Corollary 6.1.4. $\widetilde{H}_i\left(\widehat{V}_{\mathcal{A}}\right) \cong \bigoplus_{x > \hat{0}} \widetilde{H}_{i-\dim(x)-1}\left(\hat{0}, x\right)$.

There are results on the homotopy type of the union of a subspace arrangement or its compactification also by Nakamura [Na1] and Vassiliev [Va3, Va4]. The former treats simple homotopy type of spherical and projective arrangements and the latter proves stable homotopy type. The form of their results are in my opinion not as simple and explicit combinatorially as the formulas of Ziegler and Živaljević quoted above. I refer the reader directly to [Na1] and [Va3, Va4] for information about their work.

6.2. It follows from the preceding results that the spaces $V_{\mathcal{A}}, \widehat{V}_{\mathcal{A}}$ and $V^{\circ}_{\mathcal{A}}$ have the homotopy type of a wedge of spheres in many important cases. All that is needed is (for $V_{\mathcal{A}}$) that $L^{>\hat{0}}_{\mathcal{A}}$ has this homotopy type, or (for $\widehat{V}_{\mathcal{A}}$ and $V^{\circ}_{\mathcal{A}}$) that all lower intervals $[\hat{0}, x]$ in $L_{\mathcal{A}}$ have this homotopy type. This is true for geometric (semi)lattices (Theorem 4.5.1) and their truncations and for the intersection lattices of certain orbit arrangements $\mathcal{A}_\lambda$ (Theorem 4.5.2), so the following conclusions can be drawn.

Theorem 6.2.1. *The following spaces have the homotopy type of a wedge of spheres (of various dimensions):*

(i) $V_{\mathcal{A}}$ *and* $\widehat{V}_{\mathcal{A}}$, *for any truncation* $\mathcal{A}$ *of an affine hyperplane arrangement (over* $\mathbb{R}$ *or* $\mathbb{C}$*).*

(ii) $V^{\circ}_{\mathcal{A}}$, *for any truncation* $\mathcal{A}$ *of a c-arrangement.*

(iii) $V^{\circ}_{\mathcal{A}_\lambda}$, *for partitions* λ *of hook or rectangular shape.*

Note that the union of the 2-truncation of a real hyperplane arrangement $\mathcal{A}$ (resp. the 4-truncation of a complex hyperplane arrangement) is the singular locus of $V_{\mathcal{A}}$ considered as an algebraic variety, so such loci and their compactifications and links are also covered by this result. The "untruncated" version of part (ii) appeared with another proof in Björner & Ziegler [BZ1], the rest of Theorem 6.2.1 is from Ziegler & Živaljević [ZŽ], or is easily deducible from their results.

7. Topology of the complement

7.1. We now come to the result of Goresky & MacPherson [GM] cited in the introduction.

Theorem 7.1.1. *For every affine arrangement:*

$$\widetilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L^{>\hat{0}}_{\mathcal{A}}} \widetilde{H}_{\operatorname{codim}(x)-2-i}(\hat{0}, x).$$

This follows via Alexander duality in $\widehat{\mathbb{R}^n} \cong S^n$ from Corollary 6.1.4.

The original proof in [GM] uses stratified Morse theory. The result has also been proved using spectral sequence methods by Jewell-Orlik-Shapiro [JOS] and Vassiliev [Va4] and by induction and the Mayer-Vietoris sequence by Hu [Hu] and Ziegler [Z4]. The Hu-Ziegler approach makes it possible to relax the requirements on the sets $K \in \mathcal{A}$, they need not be flat affine subspaces — it suffices that they are topological balls in $\mathbb{R}^n$ and that the intersections are "nice". Ziegler & Živaljević [ZŽ] also prove a strengthening of Theorem 6.1.2 in that generality, thus via Alexander duality providing a different proof of Hu's generalization of Theorem 7.1.1.

Theorem 7.1.1 is best possible in the sense that $M_{\mathcal{A}}$ is definitely not determined up to homotopy type by the ranked intersection semilattice $(L_{\mathcal{A}}, codim)$. For instance, take the arrangement in Figure 3a and move one of the lines into another octant. This changes the homotopy type of $M_{\mathcal{A}}$, but not $L_{\mathcal{A}}$ or the homotopy type of $V_{\mathcal{A}}^{\circ}$. There is a long-standing conjecture due to P. Orlik that for complex hyperplane arrangements the homotopy type of $M_{\mathcal{A}}$ is determined by the geometric lattice $L_{\mathcal{A}}$. This now seems very doubtful in view of an example by Ziegler [Z1] of two 2-arrangements of four 2-planes in $\mathbb{R}^4$ with the same geometric intersection lattice but non-isomorphic cohomology algebras.

7.2. Here are some general facts about homotopy groups of complements of subspace arrangements of codimension greater than 2. The proof of part (i) is elementary, part (ii) uses Theorem 7.1.1 and the Hurewicz theorem, and part (iii) is based on a theorem of Serre. See Björner & Welker [BW] for the details.

Theorem 7.2.1. *Let $\mathcal{A}$ be c-codimensional, $c \geq 3$. Then*

(i) $\pi_i(M_{\mathcal{A}}) = 0$, *for* $i \leq c-2$,

(ii) $\pi_{c-1}(M_{\mathcal{A}}) \cong \mathbb{Z}^k$, *for some* $k \geq \#\{K \in \mathcal{A} \mid codim\,(K) = c\}$

(iii) $\pi_i(M_{\mathcal{A}}) \neq 0$ *(in fact, there is an element of infinite order or an element of order two), for infinitely many dimensions* i.

A famous result of Deligne [De] states that the complexification of a simplicial real hyperplane arrangement is a $K(\pi, 1)$ space. Recently three papers with proofs of Deligne's theorem have appeared: Cordovil [Co], Salvetti [Sa2] and Paris [Pa1]. The first two extend the result to a wider combinatorial setting (simplicial oriented matroids), while the third gives a particularly lucid analysis of the original theorem. Theorem 7.2.1 shows that no more Eilenberg-MacLane spaces are to be found among complements of subspace arrangements, unless the codimension is 2.

7.3. The following combinatorial formula for the Euler characteristic of $M_{\mathcal{A}}$ can be deduced from Theorem 7.1.1 using some basic properties of the Möbius function. See [BW] for the details. Here $P_{\mathcal{A}}(t)$ denotes the characteristic polynomial of $\mathcal{A}$ (§ 4.4).

Theorem 7.3.1. *Let $\mathcal{A}$ be an affine arrangement in $\mathbb{R}^n$. Then*
(i) $\chi(M_{\mathcal{A}}) = (-1)^n P_{\mathcal{A}}(-1) = \sum_{x \in L_{\mathcal{A}}} (-1)^{\text{codim}(x)} \mu(\hat{0}, x)$.
(ii) *If $\mathcal{A}$ is central, then also*

$$\chi(M_{\mathcal{A}}) = -2 \cdot \sum_{\substack{x \in L_{\mathcal{A}} \\ \text{codim}(x) \text{ odd}}} \mu(\hat{0}, x).$$

It follows that $\chi(M_{\mathcal{A}}) = 0$ if $\mathcal{A}$ is central and codim (x) is *even* for all $x \in L_{\mathcal{A}}$. For the case of complex subspace arrangements the vanishing of $\chi(M_{\mathcal{A}})$ is also a consequence of the fact that $M_{\mathcal{A}}$ is a fiber bundle with fiber $\mathbb{C}^*$, see Björner & Welker [BW]. If $\mathcal{A}$ is central and codim (x) is *odd* for all $x \in L_{\mathcal{A}}^{>\hat{0}}$, then $\chi(M_{\mathcal{A}}) = 2$.

7.4. Theorem 7.1.1 gives a description of the additive cohomology structure of $M_{\mathcal{A}}$. When $\mathcal{A}$ is a complex hyperplane arrangement there is also a well-known combinatorial description of the multiplicative structure, due to Orlik & Solomon [OS]. Their presentation of the cohomology algebra was extended to all 2-arrangements (except for the sign-pattern of the relations) by Björner & Ziegler [BZ1].

Also, a linear basis for $H^*(M_{\mathcal{A}})$ was constructed in [BZ1] for any c-arrangement $\mathcal{A}$. The elements of this basis are indexed by the so-called "broken circuit complex" of the underlying matroid, and the construction specializes to a well-known basis for the Orlik-Solomon algebra for $\mathbb{C}$-arrangements (see [W3, § 7.10]) and to a basis for the Varchenko-Gel'fand ring [VG] for $\mathbb{R}$-arrangements. The "broken circuit basis" of $H^*(M_{\mathcal{A}})$ constructed in [BZ1] consists of cohomology classes that are Alexander dual to the fundamental cycles of a system of explicitly constructed spheres that are embedded in $V_{\mathcal{A}}^{\circ}$, for any c-arrangement $\mathcal{A}$.

8. Consequences and examples

8.1. Let $\mathcal{B}_\Delta$ be the Boolean subspace arrangement of a simplicial complex Δ on n vertices. As was mentioned in § 3.2 the intersection lattice $L_{\mathcal{B}_\Delta}$ is (apart from $\hat{0}$) the face lattice of Δ upside-down, so if $x \in L_{\mathcal{B}_\Delta}$ corresponds to the face $\sigma \in \Delta$ then the open interval $(\hat{0}, x)$ is antiisomorphic to the face lattice of the *link* $lk(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}$. Hence Theorems 6.1.2 and 7.1.1 imply:

Theorem 8.1.1. (i) $\widehat{V_{\mathcal{B}_\Delta}} = \underset{\sigma\in\Delta}{wedge}\left(lk(\sigma) * S^{|\sigma|}\right)$,

(ii) $\widetilde{H}_i\left(\widehat{V_{\mathcal{B}_\Delta}}\right) \cong \widetilde{H}^{n-1-i}(M_{\mathcal{B}_\Delta}) \cong \bigoplus_{\sigma\in\Delta} \widetilde{H}_{i-|\sigma|-1}(lk(\sigma))$.

From these formulas it is easy to see that the topology of unions and complements of subspace arrangements can be almost arbitrarily bad. Take any simplicial complex Δ, then $V^{\circ}_{\mathcal{B}_\Delta}$ will contain $\Delta = lk(\emptyset)$ as a component in the wedge and $\widetilde{H}^*(M_{\mathcal{B}_\Delta})$ will have all homology groups of Δ as summands. In particular, homology of subspace arrangements needs not be torsion-free, as is the case for hyperplane arrangements. Using this construction, Ziegler and Živaljević [ZŽ] showed that every finitely presented group appears as the fundamental group of the link of some Boolean arrangement. Jewell, Orlik and Shapiro [JOS] also discuss the very general topological nature of subspace arrangements.

8.2. Having seen how bad things can get in general, let us now look at some examples with good topological properties. First, let $\mathcal{A}$ be a c-arrangement in $\mathbb{F}^n$ with characteristic polynomial $P_{\mathcal{A}}(t)$. Since $L_{\mathcal{A}}$ is a geometric lattice, known properties of such lattices (see [W2, Chapter 7]) show that $P_{\mathcal{A}}(t) = \sum_{i=0}^{r}(-1)^i w_i t^{n-ci}$, where w_i are nonnegative integers, $w_0 = 1$, and $r = \frac{1}{c}\cdot \text{codim}\,(\cap\mathcal{A})$. Goresky & MacPherson [GM] prove the following:

Theorem 8.2.1. *Let $\mathcal{A}$ be a c-arrangement, $c \geq 2$, and $P_{\mathcal{A}}(t) = \sum_{i=0}^{r}(-1)^i w_i t^{n-ci}$. Then*

(i) *all cohomology groups $H^i(M_{\mathcal{A}})$ are torsion-free,*

(ii) *$H^i(M_{\mathcal{A}}) \neq 0$ if and only if $i = t(c-1)$, $0 \leq t \leq r$,*

(iii) *$Poin\,(M_{\mathcal{A}}, t) = w_0 + w_1 t^{c-1} + w_2 t^{2(c-1)} + \ldots + w_r t^{r(c-1)}$.*

The good behaviour of $\Pi_{n,k}$ stated in Theorem 4.5.2 is also found in all lower intervals $[0, x]$ of $\Pi_{n,k}$. Using this and Theorem 7.1.1 the following is proved in Björner & Welker [BW].

Theorem 8.2.2. *Let $\mathcal{A} = \mathcal{A}_{n,k}$ be a k-equal arrangement. Then*

(i) *$H^i(M_{\mathcal{A}})$ is torsion-free, for all i,*

(ii) *$H^i(M_{\mathcal{A}}) \neq 0$ if and only if $i = t(k-2)$, $0 \leq t \leq \lfloor \frac{n}{k} \rfloor$.*

Several formulas for Betti numbers of $M_{\mathcal{A}_{n,k}}$ are given in [BW], but no closed formula for the Poincaré polynomial was found. Note however the formula for the Euler characteristic $\chi\left(M_{\mathcal{A}_{n,k}}\right) = (-1)^n P_{n,k}(-1)$ in terms of the roots of the truncated exponential series given in Theorem 4.4.1.

The Poincaré-polynomials of $\mathcal{A}_{6,3}$ and its complexification are

$$\operatorname{Poin}(M_{\mathcal{A}_{6,3}}, t) = 1 + 111t + 20t^2,$$
$$\operatorname{Poin}\left(M^{\mathbb{C}}_{\mathcal{A}_{6,3}}, t\right) = 1 + 20t^3 + 45t^4 + 36t^5 + 20t^6 + 10t^7,$$

which serves to illustrate the lack of relationship with the characteristic polynomial $P_{6,3}(t) = t^6 - 20t^4 + 45t^3 - 26t^2$ (except for the coincidence of values at $t = \pm 1$ that has already been explained). Actually, the three polynomials are put together by the same atomic parts, namely the Betti numbers of lower intervals $[\hat{0}, x]$ as explained by Theorem 7.1.1, but these parts are combined and distributed over the various dimensions in different ways. For instance, the only nonvanishing Betti numbers of $\Pi_{6,3}$ are $\beta_1 = \beta_2 = 10$, and these contribute to the homology of $\mathcal{A}_{6,3}$ (and $\mathcal{A}^{\mathbb{C}}_{6,3}$) in dimensions 1 and 2 (resp. 6 and 7), while their contribution to $P_{6,3}(t)$ is $\mu(\hat{0}, \hat{1}) = -\beta_1 + \beta_2 = 0$.

Part (i) of Theorem 4.5.2 implies the following for orbit arrangements $\mathcal{A}_\lambda$ with λ of rectangular shape.

Theorem 8.2.3. *Let $\lambda = (d, d, \ldots, d)$. Then $H^i(M_{\mathcal{A}_\lambda}) \neq 0$ if and only if $i = 0$ or $i = n - \frac{n}{d} - 1$, and all cohomology groups are torsion-free.*

8.3. Let $\mathcal{A}$ and $\mathcal{A}'$ be two subspace arrangements whose intersection lattices are isomorphic (as abstract posets without rank function). Then $\Sigma_{i\geq 0}\beta^i(M_{\mathcal{A}}) = \Sigma_{i\geq 0}\beta^i(M_{\mathcal{A}'})$, as is shown by Theorem 7.1.1. In particular, the sum of Betti numbers of the complement is unchanged by c-plexification. For $c = 2$ (complexification) this is called the *M-property*, a concept with interesting algebraic-geometric background, see Orlik & Terao [OT1] and Shapiro & Shapiro [ShS]. The work of Goresky & MacPherson [GM] shows that for the class of spaces studied here the M-property is an essentially combinatorial phenomenon.

8.4. If $\mathcal{A}$ is the p-truncation of a real hyperplane arrangement (§ 5.1) then $\widetilde{H}^i(M_{\mathcal{A}}) \neq 0$ if and only if $i = p - 1$. For instance, for p-truncations of the braid arrangement the following is computed in [BL2]:

$$\text{(8.4.1)} \qquad \operatorname{rank} \widetilde{H}^{p-1}\left(M_{\mathcal{A}^{(p)}_{n,2}}\right) = \sum_{r=1}^{n-p} r! r S(p + r - 1, r).$$

Here $S(m, r)$ is the Stirling number of the second kind, i.e., the number of partitions of an m-set into r blocks.

Furthermore, the p-truncation of a complex hyperplane arrangement (p even) in $\mathbb{C}^n$ has non-vanishing reduced cohomology precisely in dimensions $p-1, p, p+1, \ldots, \frac{1}{2}(p + \operatorname{codim}(\cap\mathcal{A})) - 1$, and similarly for c-arrangements.

8.5. Consider the space $\mathbb{R}^d[n,k]$ of all ordered n-tuples $(x_1,\dots,x_n)$ of points $x_i \in \mathbb{R}^d$ such that no point occurs k times. This is a submanifold of $\mathbb{R}^{dn}$, called an *ordered configuration space* by Vassiliev [Va4]. For $d=1$ we get the complement of the k-equal arrangement $\mathcal{A}_{n,k}$, and it is easy to see that $\mathbb{R}^d[n,k]$ in general is the complement of the d-plexification of $\mathcal{A}_{n,k}$. Therefore, the cohomology of $\mathbb{R}^d[n,k]$ is governed by its intersection lattice $\Pi_{n,k}$, and it follows from Theorem 4.5.2 that $H^*(\mathbb{R}^d[n,k])$ is torsion-free. Betti numbers can be computed from Theorem 7.1.1, and this was carried out by Björner & Welker [BW] for the cases $d=1$ and $d=2$ (complexification). It turns out that $\mathbb{R}^2[n,k] = M_{\mathcal{A}^{\mathbb{C}}_{n,k}}$ has non-vanishing cohomology in dimension d if and only if $d=0$ or there exist integers $1 \le m \le t \le \lfloor \frac{n}{k} \rfloor$ such that $tk \le d+m-t(k-2) \le n$. For $k=2$ (the space of n-tuples of distinct points in $\mathbb{R}^d$) non-zero cohomology occurs only in dimensions that are multiples of $d-1$, as shown by Theorem 8.2.1.

Let $\mathcal{P}^1_n$ be the space of real monic polynomials of degree n: $z^n + a_1 z^{n-1} + \dots + a_{n-1}z + a_n$, $a_i \in \mathbb{R}$. Similarly, let $\mathcal{P}^2_n$ be the space of complex monic polynomials of degree n. These are just spaces of sequences $(a_1,\dots,a_n)$, so $\mathcal{P}^1_n \cong \mathbb{R}^n$ and $\mathcal{P}^2_n \cong \mathbb{C}^n$. Let Σ^i_k be the subspace of all polynomials having some root of multiplicity k or higher. For $i=1,2$ there is a continuous map (surjective if $i=2$)

$$f^i_{n,k} : \mathbb{R}^i[n,k] \longrightarrow \mathcal{P}^i_n - \Sigma^i_k \tag{8.5.1}$$

which sends $x \in \mathbb{R}[n,k]$ resp. $x \in \mathbb{C}[n,k]$ to the monic polynomial with roots $x_1, x_2, \dots, x_n$. This is a polynomial map having the elementary symmetric functions as coordinates. Note that $f^i_{n,k}$ is invariant under the action of S_n on $\mathbb{R}^i[n,k]$ and this action is free on $\mathbb{R}^i[n,2]$ (all orbits are of size $n!$). Thus for $i=2$ there is an identification $\mathcal{P}^2_n - \Sigma^2_2 \cong \mathbb{R}^2[n,2]/S_n$, and this is even a diffeomorphism by a result of Arnol'd (see [Va4, p. 19]). The second space is the orbit space of the complement of the complexified braid arrangement modulo permutation of coordinates, i.e. what Vassiliev [Va4] calls the *unordered configuration space* of n distinct points in $\mathbb{R}^2 \cong \mathbb{C}$. It is known to be a $K(\pi,1)$ space with the braid group as fundamental group.

Spaces of the type $\mathcal{P}^i_n - \Sigma^i_k$, $i=1,2$, and other kinds of configuration spaces, have been intensively studied by V. I. Arnol'd and his school, see Vassiliev's book [Va4] for a general account. Let me here just quote one result of Arnol'd [Ar2] (see [Va4, p. 83]).

Theorem 8.5.1.

$$H^i(P^1_n - \Sigma^1_k) = \begin{cases} \mathbb{Z} & , \ \textit{if } i = t(k-2), 0 \le t \le \lfloor \frac{n}{k} \rfloor, \\ 0 & , \ \textit{otherwise.} \end{cases}$$

It is interesting to compare this with Theorem 8.2.2, which shows that the space $\mathbb{R}^1[n,k] = M_{\mathcal{A}_{n,k}}$ has non-vanishing cohomology in exactly the same dimensions. Is there a systematic explanation for this coincidence?

9. An application to computational complexity

9.1. Consider the following problem from theoretical computer science, called the "k-equal problem":

> *given n real numbers $x_1, x_2, \ldots, x_n$ and an integer $k \geq 2$, how many comparisons $x_i \geq x_j$ are needed to decide if some k of them are equal: $x_{i_1} = x_{i_2} = \ldots = x_{i_k}$?*

We are talking about the number of comparisons needed by the best algorithm in the worst case. Call this function $\gamma(n,k)$.

The answer for $k = 2$ (the "element distinctness problem") was given by Dobkin & Lipton [DL] in 1975:

$$\gamma(n,2) = \Theta(n \log n) \tag{9.1.1}$$

I will comment more on this in § 9.2. Let me now just remind the reader of the notational conventions used: $\gamma(n,k) = O(f(n,k))$ means that there exists a constant C such that $\gamma(n,k) \leq C \cdot f(n,k)$ for n sufficiently large (and all $2 \leq k \leq n$), $\gamma(n,k) = \Omega(f(n,k))$ means the same but with $\gamma(n,k) \geq C \cdot f(n,k)$, and "$\Theta$" means "both O and Ω".

The following solution to the k-equal problem was found by Björner, Lovász and Yao [BLY]:

Theorem 9.1.1. $\gamma(n,k) = \Theta\left(n \log \frac{2n}{k}\right)$.

The upper bound uses fairly standard sorting arguments, and will not concern us here. The lower bound, which is the difficult and more interesting part, uses the topology of subspace arrangements.

In a geometric reformation the k-equal problem concerns the complexity of deciding "$x \in V_{\mathcal{A}_{n,k}}$" for points $x \in \mathbb{R}^n$. Also, the comparisons "$x_i - x_j \geq 0$" are special cases of linear tests "$l(x) \geq 0$", for linear forms $l(x)$. Thus from a geometric point of view we are led to study the more general problem:

(9.1.2) *given a subspace arrangement $\mathcal{A}$, how many linear tests are needed (by the best algorithm in the worst case) to decide "$x \in V_{\mathcal{A}}$" for points $x \in \mathbb{R}^n$?*

It is only natural to expect that the topological complexity of $\mathcal{A}$, as measured by the cohomology of $M_{\mathcal{A}}$, should have some bearing on this algorithmic complexity.

9.2. The computational model for the kind of decision problems described in § 9.1 is that of a *linear decision tree*. This is a ternary tree with each interior node labelled by a linear form, the three outgoing edges labelled by "<", "=", ">", and each leaf (exterior node) labelled YES or NO. The inputs $x \in \mathbb{R}^n$ enter the tree at the root node, then travel down the tree, branching according to the tests performed, and finally reach a leaf where the answer is read off. Figure 5 shows an arrangement $\mathcal{A} = \{l_1, l_2, l_3\}$ of coplanar lines in $\mathbb{R}^3$. Let H_1, H_2 and H_3 be planes such that $H_i \cap H = l_i$. A linear decision tree for the problem "$x \in V_{\mathcal{A}}$?" is shown in Figure 6.

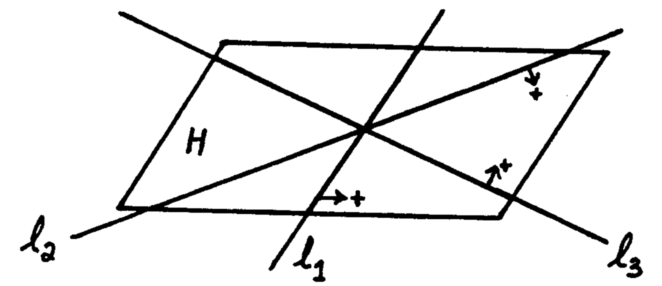


Figure 5.

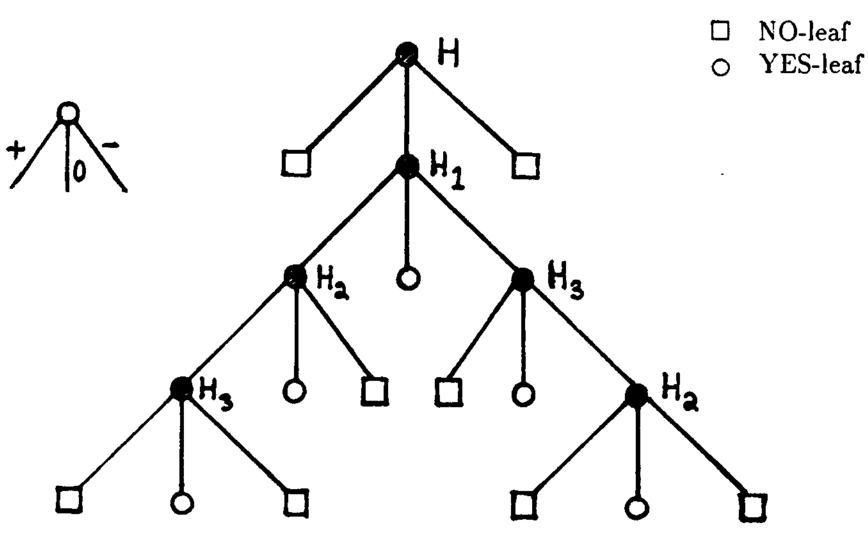


Figure 6.

Here is how Dobkin and Lipton proved the lower bound (9.1.1). Suppose that T is a linear decision tree for the 2-equal problem. For each NO-leaf w, let P_w be the set of inputs that arrive at w after traversing T. Clearly, P_w is a convex subset of $M_{\mathcal{A}_{n,2}}$, and the complement $M_{\mathcal{A}_{n,2}}$ is the disjoint union of all such sets P_w. In fact, *each connected component* of $M_{\mathcal{A}_{n,2}}$ must be a disjoint union of sets P_w, and therefore the number of components is less than or equal to the number of sets P_w; i.e., to the number of NO-leaves. The argument is of course perfectly general, and in topological notation we have proved for any tree testing for membership in an arrangement $\mathcal{A}$:

$$\text{(9.2.1)} \qquad \textit{number of NO-leaves} \geq \beta^0(M_{\mathcal{A}}).$$

Now, it is well-known and easy to see that the complement of the braid arrangement has $n!$ regions (Weyl chambers), so the depth of T must be at least $\log_3(n!) = \Omega(n \log n)$, which proves the lower bound (9.1.1).

The method (9.2.1) for obtaining lower bounds is known as the "component count method". It was extended to algebraic decision trees (where polynomial tests "$p(x) \geq 0$" are performed at the nodes) and algebraic computation trees (described in § 10.1) by Steele & Yao [SY] and Ben-Or [BO], and it has been successfully applied to several problems.

The component count method will clearly not work for the k-equal problem, $k > 2$, since $\beta^0(M_{\mathcal{A}}) = 1$ (the complement is connected). As was argued in the introduction, the higher Betti numbers are the relevant combinatorial (not only topological) invariants for general subspace arrangements, and a bound such as (9.2.1) should be sought in terms of these.

The following is proved in Björner & Lovász [BL2]. By the *dimension* of a leaf w is meant $\dim P_w$ (which is well-defined since P_w is an open convex polyhedron in its affine span).

Theorem 9.2.1. *Let T be a linear decision tree for an arrangement $\mathcal{A}$ in $\mathbb{R}^n$. Then, for all i:*

$$\textit{number of } i\textit{-dimensional NO-leaves} \geq \beta^{n-i}(M_{\mathcal{A}}).$$

For instance, in Figures 5 and 6 we have $\beta(M_{\mathcal{A}}) = (1, 5, 0)$ and there are two 3-dimensional NO-leaves and six 2-dimensional ones.

Corollary 9.2.2. *The number of tests needed in problem (9.1.2) is bounded below by either of:*

(i) $\log_3 \left(\sum_{i=0}^{n} \beta^i(M_{\mathcal{A}}) \right)$,

(ii) $\log_3 |\chi(M_{\mathcal{A}})|$,

(iii) $\log_3 |\mu_{L_{\mathcal{A}}}(\hat{0}, \hat{1})|$.

The lower bound (ii) and a weaker version of (iii) was proved in Björner, Lovász and Yao [BLY] by a cell decomposition method: the cells P_w, for all leaves w, "almost" form a CW decomposition of $\widehat{\mathbb{R}_n} = S_n$, having $V_{\mathcal{A}}$ as a subcomplex. The formula results by refining this decomposition, taking the expansion of the Euler characteristic as an alternating sum of numbers of cells and grouping terms. All three bounds result from Theorem 9.2.1, which is proved in [BL2] by induction on the size of the tree, using the Mayer-Vietoris sequence on the topological side. The Euler characteristic lower bound implied by Theorem 9.2.1 was extended to algebraic decision and computation trees by Yao [Ya].

Now back to the k-equal problem. The geometric analysis of the problem has shown that we should seek to prove that $\left|\chi\left(M_{\mathcal{A}_{n,k}}\right)\right| = |P_{n,k}(-1)|$ or $\left|\mu_{n,k}\left(\hat{0},\hat{1}\right)\right|$ is large enough. This is not always true, however; the formulas in Theorem 4.4.1 show that $P_{n,3}(-1) = \mu_{n+3,3}\left(\hat{0},\hat{1}\right) = 0$ for all $n \equiv 3 \pmod 4$. However, the same formulas can be used to prove that these numbers are "large enough, often enough", so that with the help of a monotonicity property of the function $\gamma(n,k)$ the lower bound of Theorem 9.1.1 can finally be established.

9.3. The following "k-unequal problem" is another variation on the same theme:

> *given n real numbers $x_1, x_2, \ldots, x_n$ and an integer $k \geq 2$, how many linear tests are needed to decide if some k of them are pairwise distinct?*

Let $\gamma'(n,k)$ denote this number. Note that this also contains the element distinctness problem as a special case.

Geometrically this problem concerns testing for membership in the $(n-k+1)$-truncation of the braid arrangement. Therefore the Betti number formula (8.4.1) is relevant. Indeed, the general method of § 9.2 applies, and leads to the following answer (see [BL2]):

Theorem 9.3.1. $\gamma'(n,k) = \Theta(n \log k)$.

10. More connections with complexity theory

10.1. How difficult is it to approximately (within ϵ) find the roots of a complex polynomial? This problem has been studied from many points of view, using different computation models and complexity measures. See Grigor'ev and Vorobjov [GV], Renegar [Re], Schönhage [Sc] and Smale [Sm1] for an overview and further references.

An interesting topological method for getting lower bounds was introduced by Smale [Sm2]. He uses algebraic computation trees as the model of computation, and the number of branch nodes as complexity measure. The result of Smale was later improved by Vassiliev [Va1] to essentially optimal form, by refinement of the same method. The topology of certain spaces closely related to the braid arrangement plays an intrisic role in this work.

I will describe here a few aspects of the Smale-Vassiliev work, as the space constraints permit. Apart from the statement of the results I hope to get across why the braid arrangement is relevant for this problem.

By an ***algebraic computation tree*** we will here understand the following. The interior nodes are of two kinds: ***computation nodes*** and ***branching nodes***, the former with one son and the latter with two. An input string of real numbers is fed into the tree at the root node and then the computation proceeds as a downward path through the tree. At each computation node a rational function is evaluated from arguments coming from the input string or values computed earlier along the path. At each branching node one of the rational functions already computed is compared to 0, and we pass to the left or to the right depending on the outcome. The process terminates when a leaf is reached, and the algorithm then presents some of the values computed along the path as output. Clearly, the number of leaves equals the number of branching nodes plus one.

A simple example of an algebraic computation tree is shown in Figure 7 to illustrate the definition. It computes the function $Im(z)^2 + |z \cdot \bar{z} - 2 \cdot Re(z)^2|$ for inputs $z = x + iy \in \mathbb{C}$.

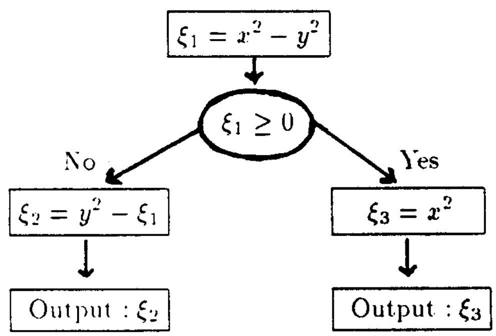


Figure 7.

Now consider the following problem:

Given a complex polynomial $p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ and $\epsilon > 0$, find $\xi_1, \ldots, \xi_n \in \mathbb{C}$ such that if $z_1, \ldots, z_n$ are the roots of $p(z)$ suitably ordered then $|\xi_i - z_i| < \epsilon$ for $1 \leq i \leq n$.

Define the *complexity* $\tau(n, \epsilon)$ to be the smallest number of branching nodes in any algebraic computation tree that accepts the string $(a_1, \ldots, a_n)$ as input and at each leaf outputs strings $(\xi_1, \ldots, \xi_n)$ of ϵ-approximate roots as required.

The result of Smale [Sm2] is that $\tau(n, \epsilon) > (\log_2 n)^{2/3}$, for sufficiently small $\epsilon > 0$. The following strengthening is due to Vassiliev [Va1, Va4].

Here $D_p(n)$ denotes the sum of the digits in the expansion of n in base p, and the minimum is taken over all prime numbers p.

Theorem 10.1.1. $n - \min_p D_p(n) \leq \tau(n,\epsilon) \leq n-1$, *for sufficiently small* $\epsilon > 0$.

Note that if n is a prime-power number then the two bounds coincide, so $\tau(n,\epsilon) = n-1$.

The method of Smale, also used by Vassiliev, hinges on the following link between complexity and topology. The *covering number* (or Schwartz genus) of a map $f : X \to Y$ of topological spaces is the size k of the smallest open covering $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of Y such that maps $g_i : \mathcal{O}_i \to X$ with $f o g_i = id_{\mathcal{O}_i}$ exist.

Theorem 10.1.2. *Let* $g(n)$ *be the covering number of the map (8.5.1)* $f^2_{n,2} : M_{\mathcal{A}^{\mathbb{C}}_{n,2}} \to \mathcal{P}^{\mathbb{C}}_n - \Sigma^{\mathbb{C}}_2$. *Then for all sufficiently small* $\epsilon > 0 : g(n) - 1 \leq \tau(n,\epsilon)$.

The idea, briefly, is to take an optimal tree T and for each leaf w look at the set P_w of all inputs that produce a computation path leading to w. This creates a partition of $\mathcal{P}^{\mathbb{C}}_n$ into $\tau(n,\epsilon)+1$ pieces P_w, which after a sequence of topological manipulations eventually is converted into an open covering $\mathcal{O}_1, \ldots, \mathcal{O}_{\tau(n,\epsilon)+1}$ of $\mathcal{P}^{\mathbb{C}}_n - \Sigma^{\mathbb{C}}_2$ having the required sections g_i with respect to the map $f^2_{n,2}$.

The work that remains to prove the lower bound in Theorem 10.1.1 is now to find a good estimate for the covering number of $f^2_{n,2}$. This part of the proof is entirely topological, and it is here that Vassiliev was able to improve on the estimate of Smale.

It should be mentioned that Vassiliev's results go much further. For instance, he proves that if n is a prime-power number then the complexity of the problem of finding just *one* root within ϵ is also equal to $n-1$. He has also extended these complexity results to the problem of finding approximate solutions to systems of polynomial equations in several variables [Va2]. See [Va4, Chapt. 2] for a general account.

10.2. Representations of graphs by subspace arrangements have been linked to questions in Boolean complexity theory in recent papers by Razborov [Ra] and Pudlák & Rödl [PR], and their work poses some intriguing combinatorial questions.

Let $\mathbf{k}$ be a field (not necessarily $\mathbb{R}$ now) and $G = ([t], E)$ a graph, $[t] = \{1, \ldots, t\}$. Then an affine subspace arrangement $\mathcal{A} = \{K_1, \ldots, K_t\}$ in $\mathbf{k}^n$ is said to provide an *affine representation* of G if $(i,j) \in E \Leftrightarrow K_i \cap K_j \neq \emptyset$, for all i, j. The minimal dimension n for which such a representation exists is called the *affine dimension* of G, denoted $adim_{\mathbf{k}}(G)$. Taking instead a central arrangement $\mathcal{A}$ and demanding that $(i,j) \in E \Leftrightarrow K_i \cap K_j \neq \{0\}$

we get the parallel notions of a ***projective representation*** of G, and of G's *projective dimension* $pdim_{\mathbf{k}}(G)$.

Ordering the vertices and edges of a graph G by inclusion, and adding the empty set, we get a semilattice $L(G)$ of height 2. A subspace representation of $L(G)$ in the sense of § 4.1 gives a subspace representation of G in the sense defined here, but not conversely (since distinct edges are not required to correspond to distinct subspaces here).

Why are graphs related to Boolean functions in the first place? The connection is this. Let $f : \{0,1\}^{2n} \rightarrow \{0,1\}$ be a Boolean function defined on strings of even length $2n$: $x = (x_1, \dots, x_n, y_1, \dots, y_n)$. Now, let $X = 2^{\{x_1,\dots,x_n\}}$ and $Y = 2^{\{y_1,\dots,y_n\}}$, where 2^S denotes the family of all subsets of the set S. Then $f^{-1}(1)$ can be viewed as a collection of incidence vectors describing pairs $(\mathbf{x},\mathbf{y}) \in X \times Y$. Conversely, every bipartite graph $E \subseteq X \times Y$ can be coded back to a Boolean function f.

Let G_f be the bipartite graph corresponding to a Boolean function f. Part (i) of the following result is due to Razborov [Ra], part (ii) to Pudlák & Rödl [PR].

Theorem 10.2.1. *For any field* $\mathbf{k}$ *and any Boolean function* f*:*

(i) $L(f) \geq adim_{\mathbf{k}}(G_f)$, *where* $L(f)$ *is the formula size of* f *in the basis* $\{-, \&, \vee\}$.

(ii) $L'(f) \geq pdim_{\mathbf{k}}(G_f) - 2$, *where* $L'(f)$ *is the minimal size of any branching program computing* f.

I refer the reader to the original papers and the references there for descriptions of these complexity measures and their background, but it should be clear why results about subspace representations of graphs have potential applicability to the important problem of proving lower bounds for the complexity of Boolean functions.

The best lower bound known for $pdim_{\mathbf{k}}(G)$ is the following inequality for the class of graphs G_N obtained by removing an N-to-N matching from the complete graph K_{2N}:

$$pdim_{\mathbf{k}}(G_N) = \Omega(\log N), \tag{10.2.1}$$

for every field $\mathbf{k}$ of characteristic other than 2. This result is due to L. Lovász, see Pudlák & Rödl [PR]. Note that a collection of N pairs of parallel lines in $\mathbb{R}^2$ shows that $adim_{\mathbb{R}}(G_N) = 2$ for all N.

What is needed for complexity theory is an explicit class of graphs G_t on t vertices (preferably bipartite) such that $pdim(G_t)$ or $adim(G_t)$ grows sufficiently fast with t.

11. Some ring-theoretic aspects

11.1. Let $\mathbf{k}$ be a field (not necessarily $\mathbb{R}$ or $\mathbb{C}$ in this section). If $\mathcal{A}$ is a subspace arrangement in $\mathbf{k}^n$ the union $V_{\mathcal{A}}$ is an affine algebraic variety, with vanishing ideal $I_{\mathcal{A}} = \{p \in \mathbf{k}[x_1, \dots, x_n] \mid p(x) = 0 \text{ for all } x \in K \in \mathcal{A}\}$ and coordinate ring $\mathbf{k}[\mathcal{A}] = \mathbf{k}[x_1, \dots, x_n]/I_{\mathcal{A}}$. This class of coordinate rings (and their seminormalization) has been studied by Yuzvinsky [Yu1, Yu2], particularly with respect to the Cohen-Macaulay property. Here I will mention a few facts about two particular cases that are of combinatorial interest.

The first case is the Boolean arrangement $\mathcal{B}_{\Delta}^{\mathbf{k}}$ in $\mathbf{k}^n$, corresponding to a simplicial complex Δ on vertex set $[n]$, cf. § 3.2. It is easy to see [Yu1] that the vanishing ideal I_{Δ} is generated by all square-free monomials $x_{i_1}x_{i_2}\dots x_{i_k}$ such that $\{i_1, i_2, \dots, i_k\} \notin \Delta$. Hence, the coordinate ring $\mathbf{k}[\Delta]$ of $\mathcal{B}_{\Delta}^{\mathbf{k}}$ is the *Stanley-Reisner ring* of Δ over $\mathbf{k}$, also called *face ring* [St1]. Some arrangement-theoretic aspects of such rings are discussed in § 11.2.

The second case is that of the k-equal arrangement $\mathcal{A}_{n,k}^{\mathbf{k}}$ in $\mathbf{k}^n$, cf. § 3.1. For each partition π of the set $[n]$, define the generalized Vandermonde polynomial

$$p_{\pi}(x_1, \dots, x_n) = \prod_{\substack{i>j \\ i \equiv j}} (x_i - x_j), \tag{11.1.1}$$

where $i \equiv j$ denotes that i and j belong to the same block of π. Let $b(\pi)$ be the number of blocks of π. The first part of the following result is due to Li & Li [LL], the second to DeLoera [DeL].

Theorem 11.1.1. *Let $I_{n,k}$ be the vanishing ideal of $V_{\mathcal{A}_{n,k}^{\mathbf{k}}}$. Then*
(i) $\{p_{\pi}(x) \mid b(\pi) = k-1\}$ *generates* $I_{n,k}$.
(ii) $\{p_{\pi}(x) \mid b(\pi) \le k-1\}$ *is a universal Gröbner basis for* $I_{n,k}$.

The ideal $I_{n,k}$ has an interesting connection to algorithmic graph theory. The *stability number* $\alpha(G)$ of a graph $G = ([n], E)$ is the maximum size of a set S of mutually non-adjacent nodes (i.e., such that $i, j \in S$ implies $(i,j) \notin E$). It is well-known that the computation of $\alpha(G)$ is NP-hard. The following ring-theoretic characterization of $\alpha(G)$ was discovered by Li & Li [LL]:

(11.1.2) $\alpha(G) < k$ *if and only if the polynomial* $\prod_{\substack{i>j \\ (i,j)\in E}} (x_i - x_j)$ *belongs to* $I_{n,k}$.

Thus $\alpha(G)$ can in principle be computed using Gröbner basis methods, but the algorithm is inefficient from the viewpoint of complexity theory. See

Lovász [Lo] for a discussion of this and other approaches to the computation of $\alpha(G)$.

Also the chromatic number $\chi(G)$ of a graph G has an ideal-theoretic characterization similar to (11.1.2), due to L. Lovász and D. Kleitman. The ideal in question here is also the vanishing ideal of a subspace arrangement (a truncation of the braid arrangement), and this ideal has a universal Gröbner basis consisting of certain polynomials of type (11.1.1). See Lovász [Lo] and DeLoera [DeL].

11.2. Let Δ be a simplicial complex on vertex set $[n]$, and let $\mathbf{k}$ be a field. There are some interesting connections between the Boolean arrangement $\mathcal{B}_\Delta^{\mathbf{k}}$ in $\mathbf{k}^n$ and its counterpart $\mathcal{B}_\Delta^{\mathbb{R}}$ in $\mathbb{R}^n$; namely, certain ring-theoretic invariants of the coordinate ring $\mathbf{k}[\Delta]$ can be computed from the topology of the real singularity link $V_\Delta^\circ = V_{\mathcal{B}_\Delta^{\mathbb{R}}} \cap S^{n-1}$. I will here assume familiarity with some basic facts about Stanley-Reisner rings and commutative algebra, see [St1] for definitions and explanations.

Let $\widehat{V}_\Delta$ denote the one-point compactification of the variety $V_{\mathcal{B}_\Delta^{\mathbb{R}}} = \{x \in \mathbb{R}^n \mid \text{supp}(x) \in \Delta\}$. Topologically $\widehat{V}_\Delta$ is the suspension of the link V_Δ°.

Theorem 11.2.1. *For any field* $\mathbf{k}$,
(i) $\dim \mathbf{k}[\Delta] = \max\{i \mid \widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right) \neq 0\}$,
(ii) $\text{depth}\ \mathbf{k}[\Delta] = \min\{i \mid \widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right) \neq 0\}$.

Proof. The following formula for reduced singular homology with coefficients in $\mathbf{k}$ can be deduced from Theorem 8.1.1 via the Universal Coefficient Theorem:

$$\widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right) \cong \bigoplus_{\sigma\in\Delta} \widetilde{H}_{i-|\sigma|-1}(lk\sigma;\mathbf{k}). \tag{11.2.1}$$

It is known [St1, p. 63] that the Krull dimension of $\mathbf{k}[\Delta]$ equals $d = \max_{\sigma\in\Delta}|\sigma|$. Formula (11.2.1) shows that each σ of maximal size contributes a copy of $\mathbf{k}$ to $\widetilde{H}_d\left(\widehat{V}_\Delta;\mathbf{k}\right)$, since $lk\sigma = \emptyset$ for such σ. Also, for dimensional reasons $\widetilde{H}_i = 0$ if $i > d$. This proves part (i).

A result of M. Hochster [St1, p. 70] implies that

$$\text{depth}\ \mathbf{k}[\Delta] = \min\{i \mid \widetilde{H}_{i-|\sigma|-1}(lk\sigma;\mathbf{k}) \neq 0 \text{ for some } \sigma \in \Delta\}. \tag{11.2.2}$$

Together with (11.2.1) this gives part (ii). □

This result shows that singular homology of the space $\widehat{V}_\Delta$ with coefficients in $\mathbf{k}$ has strong formal similarity with local cohomology of the ring $\mathbf{k}[\Delta]$. The two concepts are sensitive to depth and dimension in the same way. In the case $\mathbf{k} = \mathbb{F}$ this phenomenon seems related to the original geometric motivation for Grothendieck's definition of local cohomology [Ha].

The singular homology $\widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right)$ and the local cohomology $H^i(\mathbf{k}[\Delta])$ are usually not isomorphic (local cohomology is in most cases not even finitely generated). However, they coincide in the following case. If Δ is *Buchsbaum* (i.e., Δ is connected and $lk\sigma$ is $CM/\mathbf{k}$ (soon to be defined) for all $\sigma \in \Delta - \{\emptyset\}$), then for all $i \leq \dim \Delta$:

$$\widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right) \cong \widetilde{H}_{i-1}(\Delta;\mathbf{k}) \cong H^i(\mathbf{k}[\Delta]). \tag{11.2.3}$$

The first isomorphism is a consequence of (11.2.1), the second is due to Schenzel [Sc].

Let us call a d-dimensional compact topological space X *Cohen-Macaulay over* $\mathbf{k}$ (written "$CM/\mathbf{k}$") if for all $x \in X$ and all $i < d$

$$\widetilde{H}_i(X;\mathbf{k}) = H_i(X, X - x;\mathbf{k}) = 0. \tag{11.2.4}$$

It is known from theorems of G. Reisner and J. Munkres [Sta1, pp. 70–71] that $\mathbf{k}[\Delta]$ is a Cohen-Macaulay ring (i.e., dim = depth) if and only if the geometric realization of Δ is $CM/\mathbf{k}$ as a space. Here are a few more characterizations of Cohen-Macaulayness in terms of the real Boolean arrangement $\mathcal{B}_\Delta$.

Theorem 11.2.2. *The following are equivalent:*
(i) $\mathbf{k}[\Delta]$ *is Cohen-Macaulay.*
(ii) $\widetilde{H}_i(V^\circ_\Delta;\mathbf{k}) = 0$ *for all* $i < \dim \Delta$.
(iii) V°_Δ *is* $CM/\mathbf{k}$.
(iv) $\widetilde{H}_i\left(\widehat{V}_\Delta;\mathbf{k}\right) = 0$ *for all* $i \leq \dim \Delta$.
(v) $\widehat{V}_\Delta$ *is* $CM/\mathbf{k}$.

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) follow from $\widehat{V}_\Delta = \operatorname{susp} V^\circ_\Delta$ and the definition (11.2.4). Theorem 11.2.1 shows that (iv) $\Rightarrow$ (i). So it remains to verify (i) $\Rightarrow$ (iii).

Construct from Δ a new simplicial complex $\tilde{\Delta}$ on the vertex set $[\tilde{n}] = \{-n, \ldots, -1, 1, \ldots, n\}$, defined by

$$\tilde{\sigma} \in \tilde{\Delta} \quad \Leftrightarrow \quad \{i, -i\} \not\subseteq \tilde{\sigma} \text{ and } \{|i| \mid i \in \tilde{\sigma}\} \in \Delta.$$

An equivalent geometric description is that $\tilde{\Delta}$ is the symmetrized simplicial complex generated from Δ (geometrically realized as a subcomplex of the standard spherical simplex $\mathbb{R}^n_+ \cap S^{n-1}$) by reflections in the coordinate hyperplanes. This picture also shows that $\tilde{\Delta}$ triangulates the space V°_{Δ}. The complex $\tilde{\Delta}$ is obtained from Δ by repeated "doubling of points"; the n vertices of Δ may be doubled in arbitrary order. It was shown by Baclawski [Ba, Theorem 7.3] that the $CM/\mathbf{k}$ property is preserved both by doubling and un-doubling of points. Hence, Δ is $CM/\mathbf{k}$ if and only if $\tilde{\Delta}$ is $CM/\mathbf{k}$, which proves that (i) $\Leftrightarrow$ (iii). □

12. Cell complexes and matroids

12.1. To study the topology of $M_{\mathcal{A}}$ and $\widehat{V}_{\mathcal{A}}$ for a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$ it can be useful to have an encoding of these spaces in terms of a finite cell complex. Examples of such use of cell complexes are so far limited to the case of complex hyperplane arrangements and c-arrangements (see below), but the basic constructions are completely general.

The first construction of a cell complex for the complement is due to Salvetti [Sa1]. His construction, which is quite intricate, assumes that $\mathcal{A}$ is the complexification of a real hyperplane arrangement $\mathcal{A}'$ and describes $M_{\mathcal{A}}$ up to homotopy type in terms of the combinatorics of the real arrangement $\mathcal{A}'$. Salvetti's work was inspired by Deligne [De]. See Paris [Pa1, Pa2] for a discussion of Deligne's work from this point of view.

In 1990 Björner and Ziegler [BZ1] had the idea of a very simple and general construction of cell complexes for the complement. This idea was communicated to P. Orlik, and a version of it was used by Orlik [Or] and Orlik & Terao [OT1]. Another construction of cell complexes appears in Nakamura [Na2], who also considers infinite locally finite subspace arrangements.

The idea of [BZ1, Sections 3 and 9] is the following. Let $\mathcal{A} = \{K_1, \dots, K_t\}$ be an arrangement of subspaces in $\mathbb{R}^n$, which for simplicity we take to be central and essential ($\cap\mathcal{A} = \{0\}$). Construct a regular CW decomposition Γ of S^{n-1} which contains $V^{\circ}_{\mathcal{A}}$ as a subcomplex, and whose barycentric subdivision is a PL sphere (more about this in a moment). Let $\widetilde{\Gamma}_{\mathcal{A}} = \{\sigma \in \Gamma \mid \sigma \not\subseteq V^{\circ}_{\mathcal{A}}\}$. Then a quite elementary argument shows that the set $\widetilde{\Gamma}_{\mathcal{A}}$ ordered by reverse inclusion describes the cellular incidence structure of a regular CW complex $\Gamma_{\mathcal{A}}$ having the homotopy type of $M^{\circ}_{\mathcal{A}}$, and hence of $M_{\mathcal{A}}$.

There are two main ways of constructing the auxiliary complex Γ.

(1) For each $K \in \mathcal{A}$ choose a flag of subspaces $K = K_0 \subset K_1 \subset \dots \subset K_e = \mathbb{R}^n$, and put $K_{-1} = \emptyset$. Say that two points $x, y \in S^{n-1}$ are equivalent if for all $K \in \mathcal{A}$: x and y belong to the same connected component of $K_i - K_{i-1}$ for some $i \geq 0$. Then the equivalence classes are the open cells of a regular cell decomposition of S^{n-1} that we may use as Γ.

Salvetti's complex [Sa1] is equivalent to a special case of this construction.

(2) For each $K \in \mathcal{A}$ choose $\epsilon = \operatorname{codim}(K)$ hyperplanes $H_1, \ldots, H_\epsilon$ such that $K = \bigcap_{i=1}^{\epsilon} H_i$, and let $\mathcal{A}'$ be the collection of all hyperplanes so chosen for all $K \in \mathcal{A}$. Say that $x, y \in S^{n-1}$ are equivalent if for all $H \in \mathcal{A}'$: x and y both belong to H or else to the same connected component of $S^{n-1} - H$. Then the equivalence classes are the open cells of a complex that can be used as Γ. This is the method used by Orlik [Or] and Orlik & Terao [OT1].

Salvetti [Sa1] used his cell complex to give a presentation of the fundamental group $\pi_1(M_{\mathcal{A}})$ for the complexification of a real hyperplane arrangement. Björner & Ziegler [BZ1] used their cell complex in the case of a c-arrangement to compute the homology of $V^\circ_{\mathcal{A}}$ and the cohomology of $M_{\mathcal{A}}$ in terms of explicit bases, which are matched by Alexander duality, and to determine the homotopy type of $V^\circ_{\mathcal{A}}$ (cf. Theorem 6.2.1 (ii)). In the case of a complex hyperplane arrangement the "cellular" method leads to a quite elementary proof (completely avoiding differential topology) of the Brieskorn-Orlik-Solomon theorem on the structure of the cohomology algebra of $M_{\mathcal{A}}$ [BZ1]. The fact that the cell complex $\Gamma_{\mathcal{A}}$ for $M^\circ_{\mathcal{A}}$ and the cell complex $\Gamma - \widetilde{\Gamma}_{\mathcal{A}}$ for $V^\circ_{\mathcal{A}}$ are "combinatorially dual" to each other, as complementary subsets of cells in the spherical complex Γ, makes this class of cell complexes particularly useful in combination with Alexander duality.

12.2. The combinatorial study of hyperplane arrangements is closely linked to the theory of matroids. Matroids come in several versions, each of which can be defined in a multitude of ways that I cannot discuss now, see e.g. Aigner [Ai], Björner, Las Vergnas, Sturmfels, White & Ziegler [BLSWZ], Welsh [We], or White [W1, W2, W3]. Here is one approach that is particularly useful from the arrangements point of view.

Let $\mathcal{A} = \{H_1, \ldots, H_t\}$ be a central arrangement of hyperplanes in $\mathbf{k}^n$. Suppose given a finite set of symbols (or "signs") Σ and a function $\mathrm{s} : \mathbf{k}^n \times \mathcal{A} \to \Sigma$ which in some sense measures the "position" $\mathrm{s}(x, H)$ of a point x with respect to the hyperplane H. Then the position of x relative to the arrangement $\mathcal{A}$ is indicated by a "sign vector"

$$\mathrm{s}(x) = (\mathrm{s}(x, H_1), \ldots, s(x, H_t)) \in \Sigma^t,$$

and the collection of all such sign vectors

$$M_{\mathrm{s}}(\mathcal{A}) = \{\mathrm{s}(x) \mid x \in \mathbf{k}^n\} \subseteq \Sigma^t \qquad (12.2.1)$$

is the associated "matroid".

Now, suppose that $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$ and let $l_1, \ldots, l_t$ be linear forms such that $H_i = Ker\, l_i$.

(1) If $\Sigma = \{0, 1\}$ and

$$s^0(x, H_i) = \begin{cases} 0 \ , & \text{if } l_i(x) = 0 \\ 1 \ , & \text{if } l_i(x) \neq 0 \end{cases}$$

then we get the (ordinary) *matroid* of $\mathcal{A}$, in terms of its closed sets.

(2) If $\mathbf{k} = \mathbb{R}$, $\Sigma = \{0, +, -\}$ and

$$s^{\mathbb{R}}(x, H_i) = \begin{cases} 0 \ , & \text{if } l_i(x) = 0 \\ + \ , & \text{if } l_i(x) > 0 \\ - \ , & \text{if } l_i(x) < 0 \end{cases}$$

then we get the *oriented matroid* of $\mathcal{A}$, in terms of its covectors.

(3) If $\mathbf{k} = \mathbb{C}$, $\Sigma = \{0, +, -, i, j\}$ and

$$s^{\mathbb{C}}(x, H_i) = \begin{cases} 0 \ , & \text{if } l_i(x) = 0 \\ + \ , & \text{if } Im(l_i(x)) = 0, Re(l_i(x)) > 0 \\ - \ , & \text{if } Im(l_i(x)) = 0, Re(l_i(x)) < 0 \\ i \ , & \text{if } Im(l_i(x)) > 0 \\ j \ , & \text{if } Im(l_i(x)) < 0 \end{cases}$$

then we get the *complex matroid* of $\mathcal{A}$. (In (2) and (3) this depends, to be accurate, on the choice of forms l_i.)

The precise definition of these 3 kinds of matroids (which I will not give) is in each case an axiomatization abstracting essential properties of such sign vector systems $M_{\mathbf{s}}(\mathcal{A}) \subseteq \Sigma^t$. Note that the set $M_{\mathbf{s}^0}(\mathcal{A}) \subseteq \{0, 1\}^t$, i.e., the ordinary matroid, ordered by reverse inclusion of supports, is isomorphic to the intersection lattice $L_{\mathcal{A}}$. Matroids and oriented matroids have been studied since many years and have a quite developed theory, see the already cited sources. Complex matroids, on the other hand, are a very recent addition; the notion was initiated by Björner & Ziegler [BZ1], then made precise and further studied by Ziegler [Z3]. It should be said that not all matroids in either of the three classes arise from hyperplane arrangements. However, it can be shown that every oriented matroid and complex matroid corresponds to a topologically deformed hyperplane arrangement.

The general definition (12.2.1) applies equally well to affine hyperplane arrangements, but little work has been done on the affine case as such. The sign vector systems $M_{\mathbf{s}^{\mathbb{R}}}(\mathcal{A})$ coming from real affine arrangements have been studied by Karlander [Ka].

Other recent extensions and variations of the matroid concept are the *matroids with coefficients* of Dress [Dr, DW], the *WP-matroids* of Gel'fand & Serganova [GS1, GS2], and the *greedoids* of Korte & Lovász [KLS, BZ2]. The motivations for these concepts are mainly algebraic and algorithmic, and no clear connection with the theory of subspace arrangements can be seen at this point.

The sign vectors $M_{s\mathbb{R}}(\mathcal{A})$ of an oriented matroid coming from a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ correspond to the cells of a regular cell decomposition of S^{n-1} having $V^{\circ}_{\mathcal{A}}$ as a subcomplex (corresponding to those sign vectors that have at least one "0" entry). The sign vectors $M_{s\mathbb{C}}(\mathcal{A})$ of a complex matroid coming from a hyperplane arrangement in $\mathbb{C}^n$ describe a cell decomposition of S^{2n-1} in a similar way. Thus these two kinds of matroids contain descriptions of cell complexes for the link $V^{\circ}_{\mathcal{A}}$ and the complement $M_{\mathcal{A}}$, by the construction described in § 12.1. Having the same oriented or complex matroid structure is therefore a combinatorial equivalence relation on hyperplane arrangements with powerful topological consequences. Here is an example of a result obtained by elaborating this connection [BEZ, BZ1]: *If for two real central hyperplane arrangements there is a bijection between their sets of regions that preserves adjacency in both directions, then the complements (and links) of their complexifications are homeomorphic.*

Topological uses of the matroid concept along completely different lines appear in recent work of Gel'fand & MacPherson [GeM] and MacPherson [M]. There oriented matroids are used to define a class of manifolds intermediate between PL manifolds and differentiable manifolds, one purpose being to obtain combinatorial formulas for Pontryagin classes. The work of Mnëv [Mn1, Mn2] should also be mentioned, and in particular his "Universality Theorem" showing that every semialgebraic set in $\mathbb{R}^n$ is stably equivalent (and hence homotopy equivalent) to the realization space of some oriented matroid. See also [BLSWZ] for a discussion of realization spaces.

From the viewpoint of this paper it is relevant to ask: *Is there any useful notion of "matroid" for general subspace arrangements?* One immediate observation is that a matroid-like structure is provided by the (ranked) intersection semilattice. This semilattice is extremely useful for the general theory and specializes to the ordinary matroid in the hyperplane case (as we have seen). One could attempt to axiomatize this concept as pairs (L, r) consisting of a $\wedge$-semilattice L and a function $r : L \to \mathbb{N}$ satisfying conditions (4.1.1). However, it is doubtful whether this would give rise to a useful combinatorial theory. Also, there is little hope for a good general notion on the level of oriented and complex matroids, due to the great topological complexity that general subspace arrangements can have. One miniscule step in this direction is the class of "c-matroids" defined in [BZ1], which corresponds to the topologically well-behaved class of c-arrangements. In conclusion, I think that the available information indicates that matroid theory is a phenomenon which is essentially confined to the setting of hyperplane arrangements.

13. Final remarks and open problems

13.1. There have been several recent papers on what might be called low-dimensional subspace arrangements. This concerns mainly arrangements of lines in affine or projective 3-space. Tools are developed to distinguish different equivalence classes of line arrangements (isotopy classification), and to distinguish their 2-dimensional projections within the class of planar line diagrams with above/below information at crossing points (weaving diagrams). An interesting point is that an arrangement of skew lines in $\mathbb{R}P^3$ is the same thing as a 2-arrangement in $\mathbb{R}^4$, and (after intersection with the unit sphere) therefore the same thing as an arrangement of disjoint great circles in S^3, i.e., a certain kind of link in the sense of knot theory. Therefore link invariants play an important role.

See Crapo & Penne [CP], Mazurovskii [Ma1, Ma2, Ma3], Pach, Pollack & Welzl [PPW], Penne [Pe], Richter-Gebert [RG], Viro [Vi], Viro & Drobotuchina [VD], and Ziegler [Z1].

13.2. We have seen in § 8.2 several examples of c-codimensional arrangements $\mathcal{A}$ for which $H^i(M_{\mathcal{A}}) \neq 0$ only if i is a multiple of $c-1$, and for which $V^o_{\mathcal{A}}$ is homotopically a wedge of spheres (§ 6.2): namely, c-arrangements and orbit arrangements $\mathcal{A}_\lambda$ for partitions λ of hook and rectangular shape. Also Theorem 8.5.1 has this flavor. Is there a common underlying reason for such periodicity of cohomology and such topologically well behaved links?

Perhaps what is needed is to find a good combinatorially defined class of intersection lattices that can play for these well-behaved examples within the general theory the role that geometric lattices play in the theory of hyperplane arrangements. Such lattices should have the homotopy type of a wedge of spheres whose distribution in numbers and dimensions should be signalled by combinatorial data. The class should be closed under taking (lower) intervals, and should include the mentioned examples.

13.3. Let $\mathcal{A}_\lambda$ be the orbit arrangement (§ 3.3) corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 1$, $\lambda_1 + \ldots + \lambda_p = n$. Define numbers $s \geq 0$ and $k \geq 2$ by $\lambda_p = \lambda_{p-1} = \lambda_{p-s+1} = 1$ and $\lambda_{p-s} = k > 1$. Let Π_λ be the intersection lattice of $\mathcal{A}_\lambda$, i.e., the subposet of Π_n consisting of all partitions of $[n]$ that can be obtained as joins of partitions of shape λ.

Conjecture 13.3.1. $\widetilde{H}_*(\Pi_\lambda)$ is torsion-free, and

$$\widetilde{H}_i(\Pi_\lambda) \neq 0 \quad \Leftrightarrow \quad i = p - 2 - t(k-2),\ 0 \leq t \leq \lfloor \tfrac{s}{k} \rfloor.$$

Conjecture 13.3.2. $H^*(M_{\mathcal{A}_\lambda})$ is torsion-free, and

$$\widetilde{H}^i(M_{\mathcal{A}_\lambda}) \neq 0 \quad \Leftrightarrow \quad i = n - p - 1 + t(k-2),\ 0 \leq t \leq \lfloor \tfrac{s}{k} \rfloor.$$

Conjecture 13.3.3. Π_λ and $V^\circ_{\mathcal{A}_\lambda}$ have the homotopy type of a wedge of spheres.

These statements are true for $\lambda = (k, 1, \dots, 1)$ and $\lambda = (k, k, \dots, k)$.

13.4. Suppose that an arrangement $\mathcal{A}$ is stable under the action of some finite group G on $\mathbb{R}^n$. Then G acts also on the intersection lattice $L_{\mathcal{A}}$ and on the spaces $V_{\mathcal{A}}$ and $M_{\mathcal{A}}$, and there will be induced representations on the homology of these spaces. This setup has been studied in several cases for hyperplane arrangements (see [OT1, Chapter 6]), but not much seems to have been done beyond that. For instance, one can ask [BW]: what are the homology representations arising from the action of S_n on $\mathcal{A}_{n,k}$?

13.5. Find (universal) Gröbner bases or other useful combinatorial descriptions of the vanishing ideals of orbit arrangements for finite Coxeter groups (§ 3.3–4). Apparently only the case $\mathcal{A}_{n,k}$ has been dealt with (Theorem 11.1.1).

13.6. With few exceptions all arrangements considered in this paper are defined over $\mathbb{R}$. However, much of what has been discussed makes sense for subspace arrangements defined over other fields **k**, certainly all the algebraic and combinatorial aspects (e.g. intersection lattices) carry over. The problem with a general treatment (over arbitrary **k** other than $\mathbb{R}$ or $\mathbb{C}$) comes from the topological aspects. Nevertheless, whenever some (co)homology theory exists for subsets of $\mathbf{k}^n$ one can attempt to use it as a measure of complexity for such sets. This might be of use e.g. as a complexity measure for computation or decision algorithms over finite fields. A relevant question seems to be: Is there a Goresky-MacPherson formula (Theorem 7.1.1) for subspace arrangements over finite fields?

13.7. Let $\mathcal{A}$ be a central arrangement of $(n-2)$-dimensional subspaces in $\mathbb{R}^n$. No general effective procedure seem to be known for computing the fundamental group of the complement $M_{\mathcal{A}}$. For $n \leq 3$ the problem is easy, and for $n = 4$ special methods are provided by knot theory, since $M^\circ_{\mathcal{A}} \simeq M_{\mathcal{A}}$ is the complement in S^3 of a "link", cf. § 13.1. The case of complex hyperplane arrangements is discussed in [OT1, Chapter 5]. What is $\pi_1(M_{\mathcal{A}_{n,3}})$? Is $M_{\mathcal{A}_{n,3}}$ a $K(\pi, 1)$-space?

13.8. How much information about the topological structure of an arrangement $\mathcal{A}$ is contained in the ranked intersection semilattice $(L_{\mathcal{A}}, r)$, where $r(x) = \operatorname{codim}(x)$? We have seen in Sections 6 and 7 that it determines $\widehat{V}_{\mathcal{A}}$ up to homotopy type and $M_{\mathcal{A}}$ up to the additive structure of cohomology, and in general these results seem to be best possible. For the special case of complex hyperplane arrangements the multiplicative structure of $H^*(M_{\mathcal{A}})$ is also combinatorially determined, as was shown by Orlik

& Solomon [OS], and it is a long-standing conjecture of Orlik that $M_{\mathcal{A}}$ is in fact determined by $L_{\mathcal{A}}$ up to homotopy type in this case. Ziegler [Z1] has shown that multiplication in $H^*(M_{\mathcal{A}})$ is *not* determined by $L_{\mathcal{A}}$ for 2-arrangements, so the dividing lines in this area are subtle. See Chapter 1 of Ziegler [Z4] for a comprehensive and up-to-date discussion of this question.

13.9. Let $\mathcal{A} = \{K_1, \ldots, K_t\}$ be a central subspace arrangement in $\mathbb{R}^n$. Define

$$T(\mathcal{A}) = \{p = (p_1, \ldots, p_n) \in (\mathbb{R}[x_1, \ldots, x_n])^n \mid p(K_i) \subseteq K_i, 1 \leq i \leq t\}.$$

This is a module over $\mathbb{R}[x_1, \ldots, x_n]$. For the case of hyperplane arrangements this module was introduced and studied as a module of derivations by H. Terao, see [OT1, Chapter 4]. A basic result of Terao [T] states that if $T(\mathcal{A})$ is a free module and $\mathcal{A}$ consists of hyperplanes, then the characteristic polynomial $P_{\mathcal{A}}(t)$ (§ 4.4) splits into linear factors over the integers. It has been shown by S. Yuzvinsky (personal communication) that $T(\mathcal{A})$ is never free if $\operatorname{codim}(\mathcal{A}) \geq 2$, so it seems uncertain whether Terao's theory has any nontrivial extension beyond the hyperplane case.

13.10. There are several intriguing problems having to do with subspace representations of graphs (§ 10.2) and of semilattices (§ 4.1). It would be particularly useful to have an understanding of the combinatorial obstructions to subspace representations in a space of given dimension.

References.

[Ai] M. Aigner, *Combinatorial Theory*, Springer-Verlag, Grundlehren Series **234**, 1979.

[Ar1] V.I. Arnol'd, *The cohomology ring of the colored braid group*, Mat. Zametki 5 (1969), 227–231. (Transl.: Mathematical Notes **5** (1969), 138-140.)

[Ar2] V.I. Arnol'd, *The spaces of functions with mild singularities*, Funkts. Anal. i Prilozh. **23** (1989), 1-10. (Transl.: Functional Anal. Appl. **23** (1989).)

[Ba] K. Baclawski, *Cohen-Macaulay ordered sets*, J. Algebra **63** (1980), 226–258.

[BO] M. Ben-Or, *Lower bounds for algebraic computation trees*, in: Proc. 15th Ann. ACM Symp. on Theory of Computing, ACM Press, New York, 1983, pp. 80–86.

[Bi] G. Birkhoff, *Lattice Theory*, 3rd ed., Amer. Math. Soc., Providence, R.I., 1967.

[Bj1] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Transactions Amer. Math. Soc. **260** (1980), 159-183.

[Bj2] A. Björner, *Topological Methods*, in: Handbook of Combinatorics (eds. R. Graham, M. Grötschel, L. Lovász), North-Holland, to appear. (Preprint, 1989.)

[Bj3] A. Björner, *The homology and shellability of matroids and geometric lattices*, in: Matroid Applications (ed. N. White), Cambridge University Press, 1992, pp. 226–283.

[BEZ] A. Björner, P. Edelman & G. M. Ziegler, *Hyperplane arrangements with a lattice of regions*, Discr. Comp. Geometry **5** (1990), 263–288.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White & G. M. Ziegler, *Oriented Matroids*, Cambridge University Press, 1992.

[BL1] A. Björner & L. Lovász, *Pseudomodular lattices and continuous matroids*, Acta Sci. Math. Szeged **51** (1987), 295–308.

[BL2] A. Björner & L. Lovász, *Linear decision trees, subspace arrangements and Möbius functions*, preprint, 1992.

[BLY] A. Björner, L. Lovász & A. Yao, *Linear decision trees: volume estimates and topological bounds*, in: Proc. 24th Ann. ACM Symp. on Theory of Computing, ACM Press, New York, 1992, pp. 170–177.

[BW] A. Björner & V. Welker, *The homology of "k-equal" manifolds and related partition lattices*, Advances in Math., to appear. (Preprint, 1992.)

[BZ1] A. Björner & G. M. Ziegler, *Combinatorial stratification of complex arrangements*, Journal Amer. Math. Soc. **5** (1992), 105-149.

[BZ2] A. Björner & G. M. Ziegler, *Introduction to greedoids*, in: Matroid Applications (ed. N. White), Cambridge Univ. Press, 1992, pp. 284–357.

[BS] J. Bokowski & B. Sturmfels, *Computational Synthetic Geometry*, Lecture Notes in Mathematics **1355**, Springer, 1989.

[Br] E. Brieskorn, *Sur les groupes de tresses (d'après V. I. Arnol'd)*, Séminaire Bourbaki 24e année 1971/72, Lecture Notes in Mathematics **317**, Springer 1973, 21-44.

[CHR] A. R. Calderbank, P. Hanlon & R. W. Robinson, *Partitions into even and odd block size and some unusual characters of the symmetric groups*, Proc. London Math. Soc. (3) **53** (1986), 288–320.

[Co] R. Cordovil, *On the homotopy of the Salvetti complexes determined by simplicial arrangements*, preprint, 1991.

[CP] H. Crapo & R. Penne, *Chirality and the isotopy classification of skew lines in projective 3-space*, Institut Mittag-Leffler, preprint, 1992.

[CR] H. Crapo & G.-C. Rota, *Combinatorial Geometries*, MIT Press, 1970.

[De] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Inventiones math. **17** (1972), 273-302.

[DeL] J. A. DeLoera, *Gröbner bases for arrangements of linear subspaces related to graphs*, preprint, Cornell Univ., 1992.

[DL] D. Dobkin & R. Lipton, *On the complexity of computations under varying sets of primitives*, in: Automata Theory and Formal Languages (ed. H. Bradhage), Lecture Notes in Comp. Sci. **33**, Springer-Verlag, 1975, pp. 110–117.

[Dr] A. W. M. Dress, *Duality theory for finite and infinite matroids with coefficients*, Advances in Math. **59** (1986), 97–123.

[DW] A. W. M. Dress & W. Wenzel, *Grassmann-Plücker relations and matroids with coefficients*, Advances in Math. **86** (1991), 68–110.

[FaN] E. Fadell & L. Neuwirth, *Configuration spaces*, Math. Scandinavica **10** (1961), 111-118.

[Fa] M. J. Falk, *A geometric duality for order complexes and hyperplane complements*, European J. Combin. **13** (1992), 351–355.

[Fo] J. Folkman, *The homology groups of a lattice*, J. Math. Mech. **15** (1966), 631–636.

[FoN] R. Fox & L. Neuwirth, *The braid groups*, Math. Scandinavica **10** (1961), 119-126.

[GeM] I. M. Gel'fand & R. D. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Math. Soc. **26** (1992), 304–309.

[GR] I. M. Gel'fand & G. L. Rybnikov, *Algebraic and topological invariants of oriented matroids*, Soviet Math. Doklady **40** (1990), 148-152.

[GS1] I. M. Gel'fand & V. V. Serganova, *On the general definition of a matroid and a greedoid*, Soviet Math. Doklady **35** (1987), 6–10.

[GS2] I. M. Gel'fand & V. V. Serganova, *Combinatorial geometries and torus strata on homogeneous compact manifolds*, Russian Math. Surveys **42** (1987), 133–168.

[GM] M. Goresky & R. D. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 14, Springer Verlag 1988.

[GV] D. Y. Grigor'ev & N. N. Vorobjov, *Solving systems of polynomial equations in subexponential time*, J. Symbolic Computation **5** (1988), 37–64.

[Ha] R. Hartshorne, *Local cohomology*, Lecture Notes in Math. **41**, Springer-Verlag, 1967.

[Hu] Y. Hu, *Homology of complements of subspaces*, preprint, 1992.

[In] A. W. Ingleton, *Representation of matroids*, in: Combinatorial Math. and its Applic. (ed. D. J. A. Welsh), Academic Press, 1971, pp. 149–167.

[JOS] K. Jewell, P. Orlik & B. Z. Shapiro, *On the complement of affine subspace arrangements*, preprint, 1991.

[Ka] J. Karlander, *A characterization of affine sign vector systems*, preprint, KTH, Stockholm, 1992.

[KLS] B. Korte, L. Lovász & R. Schrader, *Greedoids*, Springer-Verlag, Berlin, 1991.

[LL] S. R. Li & W. W. Li, *Independence number of graphs and generators of ideals*, Combinatorica **1** (1981), 55–61.

[Lo] L. Lovász, *Stable sets and polynomials*, Discrete Math., to appear. (Preprint, 1990.)

[M] R. D. MacPherson, *Combinatorial differential manifolds*, preprint, 1991.

[Ma1] V. F. Mazurovskii, *Configurations of six skew lines*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Institut Steklov (LOMI) **167** (1988), 121–134. (Transl.: J. Soviet Math. **52** (1990), 2825–2832.)

[Ma2] V. F. Mazurovskii, *Kauffmann polynomials of non-singular configurations of projective lines*, Russian Math. Surveys **44** (1989), 212-213.

[Ma3] V. F. Mazurovskii, *Rigid isotopies of the real projective configurations*, preprint, 1991.

[Mn1] N. E. Mnëv, *On manifolds of combinatorial types of projective configurations and convex polyhedra*, Soviet Math. Doklady **32** (1985), 335–337.

[Mn2] N. E. Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytope varieties*, in: Topology and Geometry – Rohlin Seminar (ed. O. Ya. Viro), Lecture Notes in Math. **1346**, Springer, 1988, 527–544.

[Na1] T. Nakamura, *The topology of the configuration of projective subspaces in a projective space I*, Sci. Papers Coll. Arts Sci. Univ. Tokyo **37** (1987), 13–35.

[Na2] T. Nakamura, *The topology of the configuration of projective subspaces in a projective space II*, Sci. Papers Coll. Arts Sci. Univ. Tokyo **41** (1991), 59–81.

[Ne] J. von Neumann, *Examples of continuous geometries*, Proc. Math. Acad. Sci. USA **22** (1936), 101–108.

[Or] P. Orlik, *Complements of subspace arrangements*, J. Algebraic Geometry **1** (1992), 147-156.

[OS] P. Orlik & L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Inventiones math. **56** (1980), 167–189.

[OT1] P. Orlik & H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, 1992.

[OT2] P. Orlik & H. Terao, *Nerves of arrangements*, preprint, 1992.

[PPW] J. Pach, R. Pollack & E. Welzl, *Weaving patterns of lines and line segments in space*, preprint, Courant Institute, NYU, 1991.

[Pa1] L. Paris, *Universal cover of Salvetti's complex and topology of simplicial arrangements of hyperplanes*, Trans. Amer. Math. Soc., to appear. (Preprint, 1991.)

[Pa2] L. Paris, *The Deligne complex of a real arrangement of hyperplanes*, preprint, 1991.

[Pe] R. Penne, *Lines in 3-Space, Isotopy, Chirality and Weavings.* Ph.D. Dissertation, University of Antwerp, 1992.

[PR] P. Pudlák & V. Rödl, *A combinatorial approach to complexity*, Combinatorica **12** (1992), 221–226.

[PT] P. Pudlák & J. Tůma, *Every finite lattice can be embedded in a finite partition lattice*, Algebra Universalis **10** (1980), 74–95.

[Ra] A. A. Razborov, *Applications of matrix methods to the theory of lower bounds in computational complexity*, Combinatorica **10** (1990), 81–93.

[Re] J. Renegar, *Computational complexity of solving real algebraic formulae*, in: Proc. Intern. Congress of Math. 1990 (Kyoto), Springer-Verlag, 1991, pp. 1594–1606.

[RG] J. Richter-Gebert, *Combinatorial obstructions to the lifting of weaving diagrams*, Discr. Comp. Geometry, to appear. (Preprint 1992.)

[RS] I. Rival & M. Stanford, *Algebraic aspects of partition lattices*, in [W3], 1992, pp. 106–122.

[Ro] G.-C. Rota, *On the foundations of combinatorial theory: I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.

[Sag] B. E. Sagan, *Shellability of exponential structures*, Order **3** (1986), 47–54.

[Sa1] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$, Inventiones math. **88** (1987), 603–618.

[Sa2] M. Salvetti, *On the homotopy theory of the complexes associated to metrical hemisphere complexes*, Discrete Math., to appear. (Preprint 1990.)

[SaS] M. V. Sapir & E. R. Scheinerman, *Irrepresentability of short semilattices by Euclidean subspaces*, preprint, 1992.

[Sc] P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Zeitschrift **178** (1981), 125–142.

[Sch] A. Schönhage, *Equation solving in terms of computational complexity*, in: Proc. Intern. Congress of Math. 1986 (Berkeley), Amer. Math. Soc., 1987, pp. 131–153.

[ShS] B. Z. Shapiro & M. Z. Shapiro, *The M-property of flag varieties*, Topology and its Appl. **43** (1992), 65–81.

[Sh] P. Shor, *Stretchability of pseudolines is NP-hard*, in: Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift (eds. P. Gritzmann, B. Sturmfels), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc. **4** (1991), 531–554.

[Sm1] S. Smale, *Algorithms for solving equations*, in: Proc. Intern. Congress of Math. 1986 (Berkeley), Amer. Math. Soc., 1987, pp. 87–121.

[Sm2] S. Smale, *On the topology of algorithms, I*, J. Complexity **3** (1987), 81–89.

[St1] R. P. Stanley, *Combinatorics and Commutative Algebra*, Progress in Mathematics **41**, Birkhäuser, 1983.

[St2] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Wadsworth, 1986.

[SY] M. Steele & A. Yao, *Lower bounds for algebraic decision trees*, J. Algorithms **3** (1982), 1–8.

[T] H. Terao, *Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula*, Inventiones Math. **63** (1981), 159–179.

[VG] A. N. Varchenko & I. M. Gel'fand, *Heaviside functions of a configuration of hyperplanes*, Funkts. Anal. i Prilozh. **21** (1987), 1–18. (Transl.: Functional Anal. Appl. **21** (1988), 255–270.)

[Va1] V. A. Vassiliev, *Cohomology of braid groups and the complexity of algorithms*, Funkts. Anal. i Prilozh. **22** (1988), 15–24. (Transl.: Functional Anal. Appl. **22** (1988).)

[Va2] V. A. Vassiliev, *Topological complexity of algorithms for approximate solution of systems of polynomial equations*, Algebra i Analiz **1** (1989), 98–113. (Transl.: Leningrad Math. J. **1** (1990), 1401–1417.)

[Va3] V. A. Vassiliev, *Stable homotopy type of the complement to affine plane arrangement*, Mathematical Express, to appear. (Preprint, 1991.)

[Va4] V. A. Vassiliev, *Complements of discriminants of smooth maps: Topology and applications*, Transl. of Math. Monographs Vol. 98, Amer. Math. Soc., Providence, R.I., 1992.

[Ve1] A. M. Vershik, *Topology of the convex polytopes' manifolds, the manifold of the projective configurations of a given combinatorial type and representations of lattices*, in: Topology and Geometry – Rohlin Seminar (ed. O. Ya. Viro), Lecture Notes in Math. **1346**, Springer 1988, 557-581.

[Ve2] A. M. Vershik, *A geometric approach to representations of partially ordered sets*, Vestnik Leningrad Univ. Math. **21** (1988), 11–15.

[Vi] O. Ya. Viro, *Topological problems concerning lines and points of three-dimensional space*, Soviet Math. Dokl. **32** (1985), 528-531.

[VD] O. Ya. Viro & Yu. V. Drobotukhina, *Configurations of skew lines*, Leningrad J. Math. **1** (1990), 1027-1050.

[Wa] M. L. Wachs, *A basis for the homology of d-divisible partition lattices*, preprint, 1992.

[WW] M. L.Wachs & J. W. Walker, *On geometric semilattices*, Order **2** (1986), 367-385.

[We] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

[W1] N. White (ed.), *Theory of Matroids*, Cambridge Univ. Press, 1986.

[W2] N. White (ed.), *Combinatorial Geometries*, Cambridge Univ. Press, 1987.

[W3] N. White (ed.), *Matroid Applications*, Cambridge Univ. Press, 1992.

[Ya] A. Yao, *Algebraic decision trees and Euler characteristics*, preprint, 1992.

[Yu1] S. Yuzvinsky: *Flasque sheaves on posets and Cohen-Macaulay unions of regular varieties*, Advances in Math. **73** (1989), 24-42.

[Yu2] S. Yuzvinsky, *Cohen-Macaulay seminormalizations of unions of linear subspaces*, J. Algebra **132** (1990), 431–445.

[Za] T. Zaslavsky, *Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes*, Memoirs Amer. Math. Soc. No. 154, **1** (1975).

[Z1] G. M. Ziegler, *On the difference between real and complex arrangements*, Math. Zeitschrift, to appear. (Preprint, 1991.)

[Z2] G. M. Ziegler, *Matroid shellability, β-systems and affine hyperplane arrangements*, J. Algebraic Combinatorics **1** (1992).

[Z3] G. M. Ziegler, "*What is a complex matroid?*" Discr. Comp. Geometry, to appear. (Preprint 1992.)

[Z4] G. M. Ziegler, *Combinatorial models for subspace arrangements*, Habilitations-Schrift, Techn. Univ., Berlin, 1992.

[ZŽ] G. M. Ziegler & R. T. Živaljević, *Homotopy types of arrangements via diagrams of spaces*, Math. Annalen, to appear. (Preprint, 1991.)