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On the homology of geometric lattices

ANDERS BJÖRNER

Introduction

It has been known since the work of J. Folkman [8] that reduced homology vanishes in all dimensions other than $r-2$ for a geometric lattice L of rank r. In this paper we study the non-vanishing homology group $\tilde{H}_{r-2}(L)$ of such a lattice. In particular, we try to relate the algebraic structure of $\tilde{H}_{r-2}(L)$ to the geometric structure of L. The key component in establishing such interconnection is the observation that the geometric bases of L can be represented as cycles in $\tilde{H}_{r-2}(L)$. We identify certain families of bases of L such that the corresponding cycles form an algebraic basis for the free Abelian group $\tilde{H}_{r-2}(L)$. This is true, for instance, for the family consisting of those bases of L which contain no broken circuit under a given ordering of the points. Thus there are deeper structural reasons for the enumerative result of G.-C. Rota and H. Whitney on broken circuits of finite geometric lattices [10].

This paper is organized into 7 sections having the following main contents. §1: Preliminary material. §2: A presentation of the order homology of a geometric lattice. The theorem of J. Folkman is proved, and a method for blowing up cycles in order homology is presented and then used to derive a rank inequality. §3: Certain families of bases of a geometric lattice, called neat base-families, are presented and the connection with broken circuits explained. §4: It is shown that the algebraic cycles coming from a neat base-family form a basis for order homology. §5: Whitney homology is studied in a similar spirit. A formula relating Whitney homology to order homology is proved, and K. Baclawski's result [1] about the Betti numbers of Whitney homology is deduced. Furthermore, it is shown that the members of the broken circuit complex give rise to cycles whose classes form a basis for Whitney homology. §6: A structure of combinatorial geometry is naturally induced on the set of bases of a geometric lattice by representation in order homology. This geometry is shown to be 2-partitionable. §7: Let w_k be the kth Whitney number of the first kind and I_k the number of

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k-element independent sets in a finite geometric lattice. We prove that

$$\frac{m_k}{k} w_k \leq I_k \leq \frac{M_k+1}{2} w_k,$$

where m_k is the minimum cardinality of a k-flat and M_k is the maximum Möbius invariant of a k-flat. Finally, there are some remarks about the Whitney numbers of infinite geometric lattices.

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1. Preliminaries

A *geometric lattice* is a semimodular point-lattice of finite length. Such a lattice is complete and relatively complemented and every lattice-element x has a well-defined rank equal to the length of any maximal chain from the minimum element 0 to x. The *rank* of a geometric lattice L is the rank of its maximum element 1: rank L = rank 1. When $x<y$ in a geometric lattice L the interval $[x, y]=\{z \in L \mid x \leq z \leq y\}$ is also a geometric lattice under the inherited order. References [3] and [6] contain proofs of these and other properties basic to the theory of geometric lattices.

For a geometric latice L let $L_i=\{x \in L \mid \text{rank } x = i\}$ for $0 \leq i \leq r = \text{rank } L$. The elements of L_1 are called *points* and the elements of L_{r-1} are called *hyperplanes* of L. The cardinality of a set A will be denoted by $|A|$.

PROPOSITION 1. *Let L be an infinite geometric lattice of rank r. Then* $|L_1|=|L_2|=\cdots=|L_{r-1}|=|L|$.

Proof. Since L is a point-lattice (i.e., every element of L is a join of points) L_1 must be infinite. Let $1 \leq i \leq (r-2)$ and assume that L_i is infinite. Since L is relatively complemented it is possible for every element x of L_i to find two elements y and z of L_{i+1} such that $x = y \wedge z$. Making such a choice for every $x \in L_i$ we obtain an injection $L_i \rightarrow L_{i+1}^2$. Hence, L_{i+1} is infinite and $|L_i| \leq |L_{i+1}^2| = |L_{i+1}|$. Reversing our argument we find that $|L_{i+1}| \leq |L_i^2| = |L_i|$. Consequently, $|L_i|=|L_{i+1}|$ for $i=1, 2, \ldots, r-2$. The last equality of the proposition is a consequence of $L-\{0, 1\}$ being the disjoint union of the sets L_i, $i=1, 2, \ldots, r-1$.

The Möbius function μ of a finite lattice L is that function of two lattice-

variables which for all $x, y \in L$ s

$$\mu(x, y)=\begin{cases} 1, & \text{if} \\ -\sum_{x \leq z < y} \mu(x, z), & \text{if} \\ 0, & \text{if} \end{cases}$$

The essential properties of μ can
if L is a finite geometric lattice a
357].

The following facts from basic
An Abelian group is *free* if it i
groups. A *basis* of a free Ab
generators. The *rank* of a free A
bases. If G is free Abelian and
rank $K \leq$ rank G.

2. Order homology

Let L be a lattice with univ
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$x_0 < x_1 < \cdots < x_k$ in $L-\{0, 1\}$. B
the unique (-1)-simplex of $\Delta(L)$.
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$\tilde{H}_i(L) = \tilde{H}_i(\Delta(L), \mathbf{Z})$. This definiti
section 5.

The order homology of finite
Folkman [8, theorem 4.1].

THEOREM 2.1. *Let L be a g*

$$\tilde{H}_i(L) \cong \begin{cases} 0, & \text{if } i \neq r-2 \\ \mathbf{Z}^{|\mu(0,1)|}, & \text{if } i = r-2 \\ \mathbf{Z}^{|L|}, & \text{if } i = r-2 \end{cases}$$

variables which for all $x, y \in L$ satisfies

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y \\ -\sum\limits_{x \le z < y} \mu(x, z), & \text{if } x < y \\ 0, & \text{if } x \not\le y. \end{cases}$$

The essential properties of μ can be learned from Rota's paper [10]. For instance, if L is a finite geometric lattice and $x \in L_i$ then $|\mu(0, x)| = (-1)^i \mu(0, x) > 0$ [10, p. 357].

The following facts from basic Abelian group theory will be taken for granted. An Abelian group is *free* if it is isomorphic to a direct sum of infinite cyclic groups. A *basis* of a free Abelian group is a linearly independent set of generators. The *rank* of a free Abelian group is the common cardinality of all its bases. If G is free Abelian and K is a subgroup of G then K is also free and rank $K \le$ rank G.

2. Order homology

Let L be a lattice with universal bounds 0 and 1. We define a simplicial complex $\Delta(L)$ on the vertex set $L - \{0, 1\}$ by taking as k-simplices all chains $x_0 < x_1 < \cdots < x_k$ in $L - \{0, 1\}$. By convention, the empty set is considered to be the unique (-1)-simplex of $\Delta(L)$. By the *order homology* of L, written $\tilde{H}_*(L)$, we will understand the reduced simplicial homology of $\Delta(L)$ with integer coefficients: $\tilde{H}_i(L) = \tilde{H}_i(\Delta(L), \mathbf{Z})$. This definition is somewhat elaborated at the beginning of section 5.

The order homology of finite geometric lattices was first determined by J. Folkman [8, theorem 4.1].

THEOREM 2.1. *Let L be a geometric lattice of rank r. Then*

$$\tilde{H}_i(L) \cong \begin{cases} 0, & \text{if } i \ne r-2 \\ \mathbf{Z}^{|\mu(0,1)|}, & \text{if } i = r-2 \text{ and } L \text{ is finite,} \\ \mathbf{Z}^{|L|}, & \text{if } i = r-2 \text{ and } L \text{ is infinite.} \end{cases}$$

Proof. The first two cases are proved in Folkman's paper [8]. We will outline alternative arguments for them, and then prove the third case. Since $\Delta(L)$ is $(r-2)$-dimensional we get right away that $\tilde{H}_i(L)=0$ for $i>r-2$ and that $\tilde{H}_{r-2}(L)$ is free, being the subgroup of a free group.

Suppose first that L is finite. Then the order complex $\Delta(L)$ is *shellable* (see [4] for definition and proof), so a standard application of the Mayer-Vietoris exact sequence yields that $\tilde{H}_i(L)=0$ for $i<r-2$. It then follows that rank $\tilde{H}_{r-2}(L)=|\mu(0,1)|$, since $\mu(0,1)$ by P. Hall's theorem [10, p. 346] is the Euler characteristic of order homology.

Suppose now that L is infinite. Let us first prove the vanishing of lower-dimensional homology. Let ρ be an i-dimensional cycle, $0\le i\le r-3$; that is, $\rho=\sum_{k=1}^{n} t_k(y_0^k, y_1^k, \ldots, y_i^k)$, where for each k $t_k\in\mathbf{Z}$ and $0<y_0^k<y_1^k<\cdots<y_i^k<1$ in L, and $d\rho=0$. It is possible to select a finite set of points $P\subseteq L_1$ such that the join-sublattice of L that is join-generated in L by P, call it L', includes all elements y_j^k, $1\le k\le n$, $0\le j\le i$, and also the top element 1. L' is a finite geometric lattice [3, Lemma 3, p. 84] of rank r and ρ is an i-cycle in L', so since $\tilde{H}_i(L')=0$ there must be an algebraic $(i+1)$-chain τ in L' such that $d\tau=\rho$. But then τ is an $(i+1)$-chain also in L and $d\tau=\rho$ shows that ρ is an i-boundary. Hence, $\tilde{H}_i(L)=0$ for $i<r-2$.

It remains to prove that rank $\tilde{H}_{r-2}(L)=|L|$ when L is infinite. The following argument is based on Proposition 2.3. The result is implied also by Theorem 4.2.

$\tilde{H}_{r-2}(L)$ is a subgroup of the $(r-2)$-dimensional chain-group $C_{r-2}(L)$ which is freely generated by the maximal chains of $L-\{0,1\}$. Hence,

$$|L|=\text{rank } C_{r-2}(L)\ge \text{rank } \tilde{H}_{r-2}(L).$$

If $r=2$ then rank $\tilde{H}_0(L)=|L|$ since for any given $p\in L_1$, the set $\{p-q \mid q\in L_1-\{p\}\}$ is a basis of $\tilde{H}_0(L)$.

We proceed by induction on r. Assume that the result is true for lattices of rank $\le(r-1)$, $r\ge 3$. Pick a hyperplane x in L, and let $\mathbf{x}=\{p\in L_1 \mid p\not\le x\}$. Now use the induction assumption and Proposition 2.3 to settle the following two cases. If $|\mathbf{x}|=|L|$ then

$$\text{rank } \tilde{H}_{r-2}(L)\ge|\mathbf{x}|\cdot \text{rank } \tilde{H}_{r-3}([0,x])\ge|L|\cdot 1=|L|.$$

If $|\mathbf{x}|<|L|$ then, by Proposition 1, $|[0,x]|=|L_1-\mathbf{x}|=|L_1|=|L|$ and therefore

$$\text{rank } \tilde{H}_{r-2}(L)\ge|\mathbf{x}|\cdot \text{rank } \tilde{H}_{r-3}([0,x])\ge 1\cdot|L|=|L|.$$

In the remainder of this section we will present a method for manipulating cycles in order homology of geometric lattices and exemplify its usefulness.

Let $x\in L_{r-1}$ be a hyperpla
$(y_1, y_2, \ldots, y_{r-2})$ be an $(r-3)$-sim
$\cdots<y_{r-2}<x<1$ is a maximal ch

$$\sigma_p=(p, p\vee y_1, \ldots, p\vee y_{r-2})$$
$$+\sum_{\ell=1}^{r-2}(-1)^{\ell}(y_1, \ldots, y_\ell, p\vee$$
$$+(-1)^{r-1}(y_1, \ldots, y_{r-2}, x).$$

Thus σ_p is a linear combination o
linear combination of $(r-3)$-simpli
to show that passing from ρ to ρ_p
their dimension by one. Observe
dimensions we are considering so
cycles.

LEMMA 2.2. *Assume that L*
$p\in L_1$ *and* $p\not\le x$. *Then*

$$\rho\in\tilde{H}_{r-3}([0,x])\Rightarrow\rho_p\in\tilde{H}_{r-2}(L).$$

Proof. That

$$\rho=\sum_{i=1}^{k}a_i\sigma^i=\sum_{i=1}^{k}a_i(y_1^i, y_2^i, \ldots, y_r^i$$

belongs to $\tilde{H}_{r-3}([0,x])$ means that

$$d\rho=\sum_{i=1}^{k}a_i\,d\sigma^i=\sum_{i=1}^{k}a_i\sum_{j=1}^{r-2}(-1)^{j+1}($$
$$=\sum_{j=1}^{r-2}(-1)^{j+1}\sum_{i=1}^{k}a_i(y_1^i, \ldots, \hat{}_j, \ldots$$

(as is usual, "^" means deletion), wh

$(\alpha)\quad \sum_{i=1}^{k}a_i(y_1^i, \ldots, \hat{}_j, \ldots, y_{r-2}^i)=0$

Let $x \in L_{r-1}$ be a hyperplane of a geometric lattice L and let $\sigma = (y_1, y_2, \ldots, y_{r-2})$ be an $(r-3)$-simplex of the complex $\Delta([0, x])$, i.e., $0 < y_1 < y_2 < \cdots < y_{r-2} < x < 1$ is a maximal chain of L. If $p \in L_1$ and $p \not\leq x$ let

$$\sigma_p = (p, p \vee y_1, \ldots, p \vee y_{r-2}) + \sum_{\ell=1}^{r-2} (-1)^{\ell} (y_1, \ldots, y_\ell, p \vee y_\ell, \ldots, p \vee y_{r-2}) + (-1)^{r-1} (y_1, \ldots, y_{r-2}, x).$$

Thus σ_p is a linear combination of $(r-2)$-simplices of $\Delta(L)$. If $\rho = \sum_{i=1}^{k} a_i \sigma^i$ is a linear combination of $(r-3)$-simplices σ^i of $\Delta([0, x])$ let $\rho_p = \sum_{i=1}^{k} a_i \sigma_p^i$. We want to show that passing from ρ to ρ_p is a way of "blowing up" cycles by increasing their dimension by one. Observe that there are no non-zero boundaries in the dimensions we are considering so that our homology groups consist of "real" cycles.

LEMMA 2.2. *Assume that L is a geometric lattice of rank $r \geq 3$, $x \in L_{r-1}$, $p \in L_1$ and $p \not\leq x$. Then*

$$\rho \in \tilde{H}_{r-3}([0, x]) \Rightarrow \rho_p \in \tilde{H}_{r-2}(L).$$

Proof. That

$$\rho = \sum_{i=1}^{k} a_i \sigma^i = \sum_{i=1}^{k} a_i (y_1^i, y_2^i, \ldots, y_{r-2}^i)$$

belongs to $\tilde{H}_{r-3}([0, x])$ means that ρ has zero boundary: $d\rho = 0$. Thus,

$$d\rho = \sum_{i=1}^{k} a_i \, d\sigma^i = \sum_{i=1}^{k} a_i \sum_{j=1}^{r-2} (-1)^{j+1} (y_1^i, \ldots, \hat{\ }_j, \ldots, y_{r-2}^i) = \sum_{j=1}^{r-2} (-1)^{j+1} \sum_{i=1}^{k} a_i (y_1^i, \ldots, \hat{\ }_j, \ldots, y_{r-2}^i) = 0$$

(as is usual, "^" means deletion), which implies that

$$(\alpha) \quad \sum_{i=1}^{k} a_i (y_1^i, \ldots, \hat{\ }_j, \ldots, y_{r-2}^i) = 0 \quad \text{for all possible } j.$$

We want to show that $d\rho_p = 0$.

$$d\rho_p = d\Big(\sum_{i=1}^{k} a_i\sigma_p^i\Big) = \sum_{i=1}^{k} a_i\, d[(p, p\vee y_1^i, \dots, p\vee y_{r-2}^i)$$
$$+ \sum_{\ell=1}^{r-2} (-1)^{\ell}(y_1^i, \dots, y_\ell^i, p\vee y_\ell^i, \dots, p\vee y_{r-2}^i)$$
$$+ (-1)^{r-1}(y_1^i, \dots, y_{r-2}^i, x)]$$
$$= \sum_{j=1}^{r-1} (-1)^{j+1} \sum_{i=1}^{k} a_i(p, p\vee y_1^i, \dots, \underset{j}{\wedge}, \dots, p\vee y_{r-2}^i)$$
$$+ \sum_{\ell=1}^{r-2}\sum_{j=1}^{r-1} (-1)^{\ell+j+1} \sum_{i=1}^{k} a_i(y_1^i, \dots, \underset{j}{\wedge}, \dots, y_\ell^i, p\vee y_\ell^i, \dots, p\vee y_{r-2}^i)$$
$$+ \sum_{j=1}^{r-1} (-1)^{j+r} \sum_{i=1}^{k} a_i(y_1^i, \dots, \underset{j}{\wedge}, \dots, y_{r-2}^i, x).$$

By (α) we get that

$$(\beta)\quad \sum_{i=1}^{k} a_i(y_1^i, \dots, \underset{j}{\wedge}, \dots, y_{r-2}^i, x) = 0 \quad \text{for} \quad j \neq r-1.$$

Since the map $\varphi : [0, x] \to [p, 1]$ defined by $\varphi(y) = p \vee y$ is order-preserving and injective, (α) also entails that

$$(\gamma)\quad \sum_{i=1}^{k} a_i(p, p\vee y_1^i, \dots, \underset{j}{\wedge}, \dots, p\vee y_{r-2}^i) = 0 \quad \text{for} \quad j \neq 1,$$

and

$$(\delta)\quad \sum_{i=1}^{k} a_i(y_1^i, \dots, \underset{j}{\wedge}, \dots, y_\ell^i, p\vee y_\ell^i, \dots, p\vee y_{r-2}^i) = 0 \quad \text{for} \quad j \neq \ell \quad \text{and} \quad j \neq \ell+1.$$

Substituting (β), (γ) and (δ) into our earlier expression we get

$$d\rho_p = \sum_{i=1}^{k} a_i(p\vee y_1^i, p\vee y_2^i, \dots, p\vee y_{r-2}^i)$$
$$- \sum_{\ell=1}^{r-2}\sum_{i=1}^{k} a_i(y_1^i, \dots, y_{\ell-1}^i, p\vee y_\ell^i, \dots, p\vee y_{r-2}^i)$$

$$+ \sum_{\ell=1}^{r-2}\sum_{i=1}^{k} a_i(y_1^i, \dots, y_\ell^i, p\vee$$
$$- \sum_{i=1}^{k} a_i(y_1^i, y_2^i, \dots, y_{r-2}^i)$$
$$= \sum_{i=1}^{k} a_i[(p\vee y_1^i, \dots, p\vee y_{r-2}^i)$$
$$- \sum_{\ell=1}^{r-2} (y_1^i, \dots, y_{\ell-1}^i, p\vee y_\ell^i,$$
$$+ \sum_{\ell=2}^{r-1} (y_1^i, \dots, y_{\ell-1}^i, p\vee y_\ell^i,$$

It will be convenient in the seque
not on a given hyperplane x: for x

PROPOSITION 2.3. *Let L be a*
of L. Then

$$\operatorname{rank} \tilde{H}_{r-2}(L) \geq |\mathbf{x}| \cdot \operatorname{rank} \tilde{H}_{r-3}([0, .$$

Proof. Suppose that $r = 2$. Then
the set $\{p - x \mid p \in \mathbf{x}\}$ is a basis of $\tilde{H}$(
Suppose that $r \geq 3$. We know tha
Let $\{\rho^i \mid i \in I\}$ be a basis of $\tilde{H}_{r-3}([0.$
generated by $\{\rho_p^i \mid (i, p) \in I \times \mathbf{x}\}$. If we
dent, and hence a basis of G, it foll

$$\operatorname{rank} \tilde{H}_{r-2}(L) \geq \operatorname{rank} G = |I \times \mathbf{x}| = |.$$

which is what we want to prove.
Let $\{\rho^1, \rho^2, \dots, \rho^u\}$ be a finite line
and $\{p_1, p_2, \dots, p_v\}$ a finite subset of

$$\sum_{t=1}^{v}\sum_{s=1}^{u} b_{st}\rho_{p_t}^s = 0, \qquad b_{st} \in \mathbf{Z}.$$

To see that all b_{st} must be equal
$b_{11} \neq 0$. Then the linear independence
$\tau = \sum_{i=1}^{n} a_i\sigma^i$ be the expansion of τ as
of $\Delta([0, x])$. That $\tau \neq 0$ means that a

$$+\sum_{\ell=1}^{r-2}\sum_{i=1}^{k} a_i(y_1^i,\ldots,y_\ell^i, p\vee y_{\ell+1}^i,\ldots,p\vee y_{r-2}^i)$$

$$-\sum_{i=1}^{k} a_i(y_1^i, y_2^i,\ldots,y_{r-2}^i)$$

$$=\sum_{i=1}^{k} a_i[(p\vee y_1^i,\ldots,p\vee y_{r-2}^i)$$

$$-\sum_{\ell=1}^{r-2}(y_1^i,\ldots,y_{\ell-1}^i, p\vee y_\ell^i,\ldots,p\vee y_{r-2}^i)$$

$$+\sum_{\ell=2}^{r-1}(y_1^i,\ldots,y_{\ell-1}^i, p\vee y_\ell^i,\ldots,p\vee y_{r-2}^i)-(y_1^i,\ldots,y_{r-2}^i)]=0. \quad \square$$

It will be convenient in the sequel to have a special symbol for the set of points not on a given hyperplane x: for $x\in L_{r-1}$, let $\mathbf{x}=\{p\in L_1 \mid p\not\leq x\}$.

PROPOSITION 2.3. *Let L be a geometric lattice of rank r and x a hyperplane of L. Then*

$$\text{rank } \tilde{H}_{r-2}(L)\geq|\mathbf{x}|\cdot\text{rank } \tilde{H}_{r-3}([0,x]).$$

Proof. Suppose that $r=2$. Then $\Delta([0,x])=\{\varnothing\}$, so rank $\tilde{H}_{-1}([0,x])=1$, and the set $\{p-x \mid p\in\mathbf{x}\}$ is a basis of $\tilde{H}_0(L)$, so rank $\tilde{H}_0(L)=|\mathbf{x}|$.

Suppose that $r\geq 3$. We know that the two homology groups involved are free. Let $\{\rho^i \mid i\in I\}$ be a basis of $\tilde{H}_{r-3}([0,x])$ and let G be the subgroup of $\tilde{H}_{r-2}(L)$ generated by $\{\rho_p^i \mid (i,p)\in I\times\mathbf{x}\}$. If we can show that the ρ_p^i are linearly independent, and hence a basis of G, it follows that

$$\text{rank } \tilde{H}_{r-2}(L)\geq\text{rank } G=|I\times\mathbf{x}|=|I|\cdot|\mathbf{x}|,$$

which is what we want to prove.

Let $\{\rho^1,\rho^2,\ldots,\rho^u\}$ be a finite linearly independent set of cycles in $\tilde{H}_{r-3}([0,x])$ and $\{p_1,p_2,\ldots,p_v\}$ a finite subset of $\mathbf{x}$. Assume that

$$\sum_{t=1}^{v}\sum_{s=1}^{u} b_{st}\rho_{p_t}^s=0, \qquad b_{st}\in\mathbf{Z}.$$

To see that all b_{st} must be equal to zero; let us suppose that one is not, say $b_{11}\neq 0$. Then the linear independence of the ρ^s implies that $\tau=\sum_{s=1}^{u} b_{s1}\rho^s\neq 0$. Let $\tau=\sum_{i=1}^{n} a_i\sigma^i$ be the expansion of τ as a linear combination of $(r-3)$-simplices σ^i of $\Delta([0,x])$. That $\tau\neq 0$ means that $a_i\neq 0$ for some i, let us say that $a_1\neq 0$. The linear combination $(\sigma^1)_{p_1}$ contains a unique term $+\sigma^*$, where σ^* is an $(r-2)$-simplex of $\Delta(L)$ and $p_1 \in \sigma^*$. Since the map $\varphi: y \mapsto p_1 \vee y$ sends $[0, x]$ injectively into $[p_1, 1]$ the cycle $(\tau)_{p_1}$ must contain the term $+a_1\sigma^*$. But $(\tau)_{p_1} = \sum_{s=1}^{u} b_{s1}\rho_{p_1}^{s}$, and if $v \geq 2$ then no simplex occurring in the expansion of $\sum_{t=2}^{v}\sum_{s=1}^{u} b_{st}\rho_{p_t}^{s}$ can contain p_1. Hence, $\sum_{t=1}^{v}\sum_{s=1}^{u} b_{st}\rho_{p_t}^{s}$ would contain the term $+a_1\sigma^*$, $a_1 \neq 0$, which contradicts our assumption that it equals zero.

In the finite case Proposition 2.3 is equivalent to $|\mu(0, 1)| \geq |\mathbf{x}| \cdot |\mu(0, x)|$. This inequality was previously obtained by C. Greene [9, corollary 1].

3. Geometric bases and broken circuits

In this section we will define and construct for any geometric lattice L certain families of subsets of L_1 which we call *neat base-families* in L. Their significance for order homology will be demonstrated in section 4.

DEFINITION:

(3.1) If rank $L = 1$, then $\{\{1\}\}$ is a neat base-family.

(3.2) Assume that neat base-families are defined and exist in all geometric lattices of rank $\leq (r-1)$ and that rank $L = r$. Pick an arbitrary point $p \in L_1$ and for every hyperplane $h \in L_{r-1}$ such that $h \not\geq p$ let $\mathbf{B}_h$ be a neat base-family in $[0, h]$. Then

$$\mathbf{B} = \left\{ A \cup \{p\} \mid A \in \bigcup_{h \not\geq p} \mathbf{B}_h \right\}$$

is a neat base-family in L.

The point p used in the last step of the defining construction will be called the *distinguished point* of $\mathbf{B}$. E.g. in a rank 2 geometric lattice the collection of all 2-subsets of points containing a certain point p is a neat base-family with distinguished point p, and every neat base-family is of this form.

Let L be a geometric lattice of rank r. A set $\{b_1, b_2, \ldots, b_n\} \subseteq L_1$ is said to be a *base* of L if $n = r$ and $\bigvee_{i=1}^{n} b_i = 1$. The following facts are easy to prove by induction on rank L.

PROPOSITION 3.3. *Every member of a neat base-family in L with distinguished point p is a base of L containing p.*

PROPOSITION 3.4. *Let $p \in L_1$ and $\{A_i\}_{i \in I}$ be a collection of bases of L such that $p \in \bigcap_{i \in I} A_i$ and assume that $A_i - \{p\}$ and $A_j - \{p\}$ span different hyperplanes when $i \neq j$. Then there exists a neat base-family* $\mathbf{B}$ *in L such that $A_i \in \mathbf{B}$ for all $i \in I$ and p is the distinguished point of* $\mathbf{B}$.

PROPOSITION 3.5. *Let* $\mathbf{B}$ *be a neat base-family in L. Then* $\bigcup_{A \in \mathbf{B}} A = L_1$.

PROPOSITION 3.6. *A neat base-family* $\mathbf{B}$ *in a geometric lattice L has* $|\mu(0, 1)|$ *members if L is finite and $|L|$ members if L is infinite.*

Proof. First consider the finite case. The claim is true when $\operatorname{rank} L = 1$. For higher ranks let p and $\mathbf{B}_h$ have the same meaning as in (3.2). By induction then

$$|\mathbf{B}| = \sum_{h \not\geq p} |\mathbf{B}_h| = \sum_{h \not\geq p} |\mu(0, h)| = |\mu(0, 1)|.$$

The first equality, i.e. that $\mathbf{B}_{h_i} \cap \mathbf{B}_{h_j} = \emptyset$ when $h_i \neq h_j$, follows from the fact that members of $\mathbf{B}_{h_i}$ span h_i (Proposition 3.3). The second equality is the induction assumption, while the third is a consequence of Weisner's theorem [10, p. 351].

In the infinite case we have by Propositions 1 and 3.5 that $|\bigcup_{A \in \mathbf{B}} A| = |L|$, and by 3.3 that $|A| = \operatorname{rank} L < \infty$ for all $A \in \mathbf{B}$. Hence, $|\mathbf{B}| = |L|$.

Neat base-families can be constructed non-inductively by ordering the points of a geometric lattice. The idea underlying the following construction was first used by H. Whitney in studying chromatic polynomials of graphs and later generalized to finite geometric lattices by G.-C. Rota [10, p. 358].

Let L be a geometric lattice of rank r. Every set $A \subseteq L_1$ of points satisfies $\operatorname{rank}(\vee A) \leq |A|$. If equality holds A is said to be *independent*, otherwise *dependent*. Thus, A is a base of L if and only if A is maximal independent. The minimal dependent subsets of L_1 are called *circuits*. Now, let Ω be a well-ordering of the set L_1. If C is a circuit and p is the least element of C, then $C - \{p\}$ will be called a *broken circuit*.

LEMMA 3.7. *Let $x \in L$ and $D \subseteq L_1 \cap [0, x]$. Then D is a broken circuit in $[0, x]$ if and only if it is a broken circuit in L.*

Proof. The verification is straight-forward using the fact that $C \subseteq L_1 \cap [0, x]$ is a circuit in $[0, x]$ if and only if C is a circuit in L. Of course, the points of $[0, x]$ are well-ordered by the restriction of Ω.

PROPOSITION 3.8. *Let L be a geometric lattice of* rank r *and Ω a well-ordering of the points L_1. Then the collection* **BC** *of all r-element subsets of L_1 which contain no broken circuit is a neat base-family in L.*

Proof. The members of **BC** are bases of L since they may not contain any circuit. Let p be the least element of L_1. If $p \notin A$ for some $A \in \mathbf{BC}$ then since $\{p\} \cup A$ is dependent it contains a circuit C. In case $p \notin C$, then $D \subseteq C \subseteq A$ for the broken circuit D produced by C. In case $p \in C$, then since p is least in L_1 the set $C - \{p\}$ is a broken circuit and $C - \{p\} \subseteq A$. Hence, since $A \in \mathbf{BC}$ must not contain any broken circuit, we conclude that $p \in A$ for all $A \in \mathbf{BC}$. For each hyperplane $h \in L_{r-1}$ such that $p \not\leq h$ let $\mathbf{B}_h = \{A - \{p\} \mid A \in \mathbf{BC},\ \vee(A - \{p\}) = h\}$. Since the theorem is trivially true when rank $L = 1$, it will inductively be sufficient to show that $\mathbf{B}_h$ coincides with the family $\mathbf{C}_h$ of all $(r-1)$-element subsets of $L_1 \cap [0, h]$ which contain no broken circuit in $[0, h]$ under the restriction of Ω.

Suppose that $S \subseteq L_1 \cap [0, h]$. If $S \in \mathbf{B}_h$, then $S \cup \{p\}$ contains no broken circuit in L. By Lemma 3.7 therefore S cannot contain any broken circuit in $[0, h]$, that is, $S \in \mathbf{C}_h$. Suppose conversely that $S \in \mathbf{C}_h$. If $S \cup \{p\}$ contains a broken circuit D in L then $D \subseteq S$ since no element precedes p. But by Lemma 3.7 this contradicts $S \in \mathbf{C}_h$. Therefore, $S \cup \{p\} \in \mathbf{B}$ and $S \in \mathbf{B}_h$.

An immediate consequence is the following extension of the Whitney-Rota theorem [10, Corollary 1, p. 359].

COROLLARY 3.9. *The number of i-element subsets of L_1 which contain no broken circuit equals $\sum_{x \in L_i} |\mu(0, x)|$ if L is finite and equals $|L|$ if L is infinite, $i = 1, 2, \ldots, r = \text{rank } L$.*

Proof. Let $\mathbf{BC}_i = \{i$-element subsets of L_1 which contain no broken circuit in $L\}$ and for each $x \in L_i$ let $\mathbf{B}_x = \{i$-element subsets of $L_1 \cap [0, x]$ which contain no broken circuit in $[0, x]\}$. Then $\mathbf{BC}_i = \bigcup_{x \in L_i} \mathbf{B}_x$ and $\mathbf{B}_x \cap \mathbf{B}_y = \emptyset$ if $x \neq y$, as can be seen from Lemma 3.7 and the fact that $\vee A = x$ if $A \in \mathbf{B}_x$. Thus $|\mathbf{BC}_i| = \sum_{x \in L_i} |\mathbf{B}_x|$. Since by the foregoing theorem the collections $\mathbf{B}_x$ are neat base-families in their respective intervals the result follows from Propositions 1 and 3.6.

Not all neat base-families in a geometric lattice can be characterized as avoiding the broken circuits induced by a well-ordering of the points. Consider for instance the lattice of flats of the following combinatorial geometry (Figure 1):

Let **B** be the neat base-fam
from choosing $\{xa, xy\}$, $\{yb, yz$
nontrivial hyperplanes which lie
the members of **B** did not conta
under Ω, which is impossible.

4. Bases for homology

Our purpose in introducing
important algebraic role in the
lattice. However, before this can
$\tilde{H}_{r-2}(L)$ as a class of distinguish

Let $A = \{b_1, b_2, \ldots, b_r\}$, $r \geq 2$
each permutation $\pi \in \mathfrak{S}_r$ of th
$L - \{0, 1\}$, i.e., an $(r-2)$-simplex

$$\sigma_\pi = (b_{\pi(1)}, b_{\pi(1)} \vee b_{\pi(2)}, \ldots, b$$

We claim that

$$\rho_A = \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi \sigma_\pi$$

is an $(r-2)$-cycle ($(-1)^\pi$ equals
odd permutation). A direct com

$$d\rho_A = \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi \sum_{i=1}^{r-1} (-1)^{i+1}(b$$

$$= \sum_{i=1}^{r-1} (-1)^{i+1} \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi (b$$

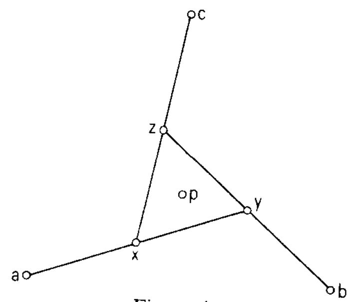

Figure 1.

Let $\mathbf{B}$ be the neat base-family with distinguished point p obtained by (3.2) from choosing $\{xa, xy\}$, $\{yb, yz\}$ and $\{zc, zx\}$ as the neat base-families in the nontrivial hyperplanes which lie outside p. If for some linear order Ω of the points the members of $\mathbf{B}$ did not contain any broken circuit, then $x < y$, $y < z$ and $z < x$ under Ω, which is impossible.

4. Bases for homology

Our purpose in introducing neat base-families is to show that they play an important algebraic role in the non-vanishing homology group of a geometric lattice. However, before this can be done we must explain how bases of L enter $\tilde{H}_{r-2}(L)$ as a class of distinguished cycles.

Let $A = \{b_1, b_2, \ldots, b_r\}$, $r \geq 2$, be an ordered base of a geometric lattice L. To each permutation $\pi \in \mathfrak{S}_r$ of the indices we associate a maximal chain σ_π of $L - \{0, 1\}$, i.e., an $(r-2)$-simplex of the complex $\Delta(L)$, by

$$\sigma_\pi = (b_{\pi(1)}, b_{\pi(1)} \vee b_{\pi(2)}, \ldots, b_{\pi(1)} \vee b_{\pi(2)} \vee \cdots \vee b_{\pi(r-1)}).$$

We claim that

$$\rho_A = \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi \sigma_\pi$$

is an $(r-2)$-cycle ($(-1)^\pi$ equals 1 or -1 depending on whether π is an even or odd permutation). A direct computation verifies this.

$$d\rho_A = \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi \sum_{j=1}^{r-1} (-1)^{j+1} (b_{\pi(1)}, \ldots, \hat{\ }_j, \ldots, b_{\pi(1)} \vee \cdots \vee b_{\pi(r-1)})$$
$$= \sum_{j=1}^{r-1} (-1)^{j+1} \sum_{\pi \in \mathfrak{S}_r} (-1)^\pi (b_{\pi(1)}, \ldots, b_{\pi(1)} \vee \cdots \vee b_{\pi(j-1)},$$
$$b_{\pi(1)} \vee \cdots \vee b_{\pi(j+1)}, \ldots, b_{\pi(1)} \vee \cdots \vee b_{\pi(r-1)}).$$

For any fixed j a given term in the sum to the right occurs exactly twice and then with $\pi(j)$ and $\pi(j+1)$ transposed and all other $\pi(i)$ equal, so that all terms cancel out in pairs. Hence, $d\rho_A = 0$.

We call the cycles ρ_A obtained from a base in this way *elementary cycles*. Another ordering of the base A can at most cause a change of sign of ρ_A. Since this is irrelevant for our purposes we may henceforth think of the relationship between unordered bases of L and elementary cycles of $\tilde{H}_{r-2}(L)$ as a one-one correspondence.

The following lemma points out that if an elementary cycle is "blown up" by the technique explained in section 2 (Lemma 2.2) we get a new elementary cycle.

LEMMA 4.1. *Let L be a geometric lattice of rank $r \geq 3$, $x \in L_{r-1}$, $p \in L_1$ and $p \not\leq x$. If C is a base of $[0, x]$ then*

$$(\rho_C)_p = \rho_{C \cup p}.$$

Proof. Naturally, the equality is only correct up to a + or − sign depending on the orderings chosen. Let $C = \{c_1, c_2, \ldots, c_{r-1}\}$ and give $C \cup \{p\}$, which is a base of L, the ordering $p < c_1 < c_2 < \cdots < c_{r-1}$. Then

$$\begin{aligned}(\rho_C)_p &= \sum_{\pi \in \mathfrak{S}_{r-1}} (-1)^{\pi} (\sigma_\pi(C))_p \\ &= \sum_{\pi \in \mathfrak{S}_{r-1}} (-1)^{\pi} \Big[(p, p \vee c_{\pi(1)}, \ldots, p \vee c_{\pi(1)} \vee \cdots \vee c_{\pi(r-2)}) \\ &\quad + \sum_{\ell=1}^{r-2} (-1)^{\ell} (c_{\pi(1)}, \ldots, c_{\pi(1)} \vee \cdots \vee c_{\pi(\ell)}, p \vee c_{\pi(1)} \vee \cdots \vee c_{\pi(\ell)}, \ldots, \\ &\qquad\qquad p \vee c_{\pi(1)} \vee \cdots \vee c_{\pi(r-2)}) \\ &\quad + (-1)^{r-1} (c_{\pi(1)}, \ldots, c_{\pi(1)} \vee \cdots \vee c_{\pi(r-1)}) \Big] \\ &= \sum_{\nu \in \mathfrak{S}_r} (-1)^{\nu} \sigma_\nu(C \cup p) = \rho_{C \cup p}.\end{aligned}$$

THEOREM 4.2. *Let L be a geometric lattice of rank $r \geq 2$ and let $\mathbf{B}$ be a neat base-family in L. Then the elementary cycles $\{\rho_A \mid A \in \mathbf{B}\}$ form a basis of the free group $\tilde{H}_{r-2}(L)$.*

Proof. Suppose that $A \in \mathbf{B}$. If p_1 is the distinguished point of $\mathbf{B}$ let $y_{r-1} = \vee(A - \{p_1\})$. If p_2 is the distinguished point of the neat base-family $\mathbf{B}_{y_{r-1}}$ in $[0, y_{r-1}]$, which exists by Definition (3.2), let $y_{r-2} = \vee(A - \{p_1, p_2\})$. Continuing like this we can inductively select a sequence of points $p_1, p_2, \ldots, p_r$ such that

$A = \{p_1, p_2, \ldots, p_r\}$ and a sequence of flats $y_{r-1}, y_{r-2}, \ldots, y_1 = p_r$, such that $\sigma_A^* = (y_1, y_2, \ldots, y_{r-1})$ is a maximal chain in $L - \{0, 1\}$ or, equivalently, an $(r-2)$-simplex. The following observation is crucial:

(4.3) $\pm\sigma_A^*$ is a term in $\rho_{A'}$, $A' \in \mathbf{B}$, if and only if $A = A'$.

Since $y_k = \bigvee_{i=r-k+1}^{r} p_i$, clearly $\pm\sigma_A^*$ appears in ρ_A. Suppose conversely that $\pm\sigma_A^*$ appears in $\rho_{A'}$ for some $A' \in \mathbf{B}$. A' must contain the distinguished point p_1. Our hypothesis demands that $y_{r-1} = \vee C$ for some subset C of A'. Since y_{r-1} has rank $(r-1)$ and p_1 cannot belong to C, we deduce that $C = A' - \{p_1\}$. Since p_2 is the distinguished point of $\mathbf{B}_{y_{r-1}}$, to which $A' - \{p_1\}$ belongs, we find that $p_2 \in A' - \{p_1\}$. Now, again $y_{r-2} = \vee D$ for some subset D of A' to which p_1 and p_2 cannot belong, so $D = A' - \{p_1, p_2\}$. Repeating this argument we eventually find that A' must contain $p_1, p_2, \ldots,$ and p_r, that is, $A' = A$.

It is an immediate consequence of (4.3) that $\{\rho_A \mid A \in \mathbf{B}\}$ is linearly independent. We will prove the generating property in two steps dealing with the finite and infinite cases respectively.

Suppose that L is finite. Let G be the subgroup of $\tilde{H}_{r-2}(L)$ generated by the linearly independent set $\{\rho_A \mid A \in \mathbf{B}\}$. Then G is a subgroup of maximal rank since $|\mathbf{B}| = |\mu(0, 1)|$ (Proposition 3.6) and rank $\tilde{H}_{r-2}(L) = |\mu(0, 1)|$ (Theorem 2.1). Consequently, every element of the quotient group $\tilde{H}_{r-2}(L)/G$ is of finite order. If $\rho \in \tilde{H}_{r-2}(L)$ there must then be integers $k \neq 0$ and t_A such that $k\rho = \sum_{A \in \mathbf{B}} t_A \rho_A$. All coefficients of terms in the expansion of the left member of this equation are divisible by k. This must then be true also for the right member. In particular, since by (4.3) the σ_A^*-term has coefficient $\pm t_A$, it follows that all t_A are divisible by k. Hence,

$$\rho = \sum_{A \in \mathbf{B}} \frac{t_A}{k} \rho_A, \qquad \frac{t_A}{k} \in \mathbf{Z},$$

so that $G = \tilde{H}_{r-2}(L)$.

Suppose next that L is infinite. We will then proceed as follows. Let $\mathbf{B}^p$ be the collection of all bases of L which contain p, the distinguished point of $\mathbf{B}$. By Proposition 3.3 $\mathbf{B} \subseteq \mathbf{B}^p$. We will prove first that $\{\rho_A \mid A \in \mathbf{B}^p\}$ generates $\tilde{H}_{r-2}(L)$ and then for every $A \in \mathbf{B}^p$ that ρ_A is a linear combination of some ρ_{C_i} with $C_i \in \mathbf{B}$.

(i) Let $\rho = \sum_{i=1}^{k} t_i (y_1^i, y_2^i, \ldots, y_{r-1}^i)$, $t_i \in \mathbf{Z} - \{0\}$, $y_j^i \in L_j$, be an $(r-2)$-cycle of $\tilde{H}_{r-2}(L)$. Let L' be the join-sublattice of L that is join-generated in L by $\{p, y_1^1, y_1^2, \ldots, y_1^k\}$. L' is also a geometric lattice (see e.g. [3, Lemma 3, p. 84]).

We claim that $y_j^i \in L'$ for all $1 \leq i \leq k$ and $1 \leq j \leq r-1$. This is easily seen as follows using induction on j. Assume that $y_{s-1}^i \in L'$ for $1 \leq i \leq k$. Since ρ is a cycle the non-zero term $\pm t_i(y_1^i, \ldots, \widehat{\ \ }_{s-1}, y_s^i, \ldots, y_{r-1}^i)$ of $d\rho$ must be cancelled by some other non-zero terms of the same form. This means that for some $\ell \neq i$ $y_j^\ell = y_j^i$ for $j \neq s-1$ and $y_{s-1}^\ell \neq y_{s-1}^i$. Hence, $y_s^i = y_{s-1}^1 \vee y_{s-1}^{\mathscr{L}}$ and, consequently, $y_s^i \in L'$. This argument also shows that rank $L' =$ rank L, hence any base of L' is also a base of L.

We have shown that ρ is a cycle also of $\tilde{H}_{r-2}(L')$. Since L' is finite it has already been shown that $\rho = \sum_{i=1}^n u_i \rho_{A_i}$, $u_i \in \mathbf{Z}$, for some bases A_i of L' belonging to a neat base-family in L' with distinguished point p. But this means that ρ is a linear combination of elementary cycles corresponding to bases A_i of L which all contain p. Hence, $\{\rho_A \mid A \in \mathbf{B}^p\}$ generates $\tilde{H}_{r-2}(L)$.

(ii) We will use induction over rank L to show for every infinite geometric lattice L and neat base-family $\mathbf{B}$ with distinguished point p in L that the span of $\{\rho_C \mid C \in \mathbf{B}\}$ in $\tilde{H}_{r-2}(L)$ includes $\{\rho_A \mid A \in \mathbf{B}^p\}$.

If rank $L = 2$ then $\mathbf{B} = \mathbf{B}^p$, so the statement is true.

Assume that the statement is true for rank$\leq (r-1)$ and that rank $L = r$. Pick a base $A \in \mathbf{B}^p$. Let h be the hyperplane spanned by $A - \{p\}$, and let $\mathbf{B}_h = \{D \subseteq L_1 \mid D \cup \{p\} \in \mathbf{B}, \vee D = h\}$. By (3.2) $\mathbf{B}_h$ is a neat base-family in $[0, h]$. $A - \{p\}$ is a base of $[0, h]$, so by what we have already proved together with the induction assumption $\rho_{A-p} = \sum_{i=1}^n r_i \rho_{D_i}$ for some $D_i \in \mathbf{B}_h$, $r_i \in \mathbf{Z}$. Therefore, using Lemma 4.1 we find that

$$\rho_A = (\rho_{A-p})_p = \left(\sum_{i=1}^n r_i \rho_{D_i} \right)_p = \sum_{i=1}^n r_i (\rho_{D_i})_p = \sum_{i=1}^n r_i \rho_{D_i \cup p}.$$

Since $D_i \in \mathbf{B}_h$ implies that $(D_i \cup \{p\}) \in \mathbf{B}$, we have shown that ρ_A is in the span of $\{\rho_C \mid C \in \mathbf{B}\}$.

Let us call a collection $\{A_i\}_{i \in I}$ of bases of a geometric lattice L such that $\{\rho_{A_i}\}_{i \in I}$ is a basis of $\tilde{H}_{r-2}(L)$ a *fundamental* base-family. Thus, Theorem 4.2 says that a family of bases is fundamental if it is neat. The converse is, however, not true. An easily available counterexample is that of a rank 2 lattice on n points $p_1, p_2, \ldots, p_n$: the base-family $\{\{p_1, p_2\}, \{p_2, p_3\}, \ldots, \{p_{n-1}, p_n\}\}$ is fundamental but not neat if $n \geq 4$. For another counterexample, let L be the rank 3 lattice on points $p_1, p_2, \ldots, p_5$ having circuits $\{p_1, p_2, p_3\}$, $\{p_3, p_4, p_5\}$ and $\{p_1, p_2, p_4, p_5\}$. L has 8 bases and $|\mu(0, 1)| = 4$. Out of the 70 collections of 4 bases of L 45 are fundamental base-families but only 13 are neat.

5. Whitney homology

Let P be a poset (partially ordered set) with least and greatest elements 0 and 1. Let $C_i(P)$ be the Abelian group which is freely generated by all i-chains

$a_0 < a_1 < \cdots < a_i$ of elements of $P - \{0, 1\}$. We consider the empty set to be the unique (-1)-chain so that $C_{-1}(P) \cong \mathbf{Z}$. Of course, $C_i(P) = 0$ whenever there is no i-chain in $P - \{0, 1\}$. Define a differential $d_i : C_i(P) \to C_{i-1}(P)$ as usual by

$$d_i(a_0, a_1, \ldots, a_i) = \sum_{j=0}^{i} (-1)^j (a_0, \ldots, \hat{a}_j, \ldots, a_i)$$

when $i \geq 0$ and $C_i(P) \neq 0$, and let $d_i = 0$ otherwise. The homology of the algebraic complex

$$(C(P), d) = \cdots \xrightarrow{d_{i+1}} C_i(P) \xrightarrow{d_i} C_{i-1}(P) \xrightarrow{d_{i-1}} \cdots$$

is the reduced *order homology* of P. We write

$$\tilde{H}_i(P) = H_i((C(P), d)) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}.$$

For a poset P with least element 0 let $D_i(P)$ be the Abelian group freely generated by all i-chains $a_0 < a_1 < \cdots < a_i$ of elements of $P - \{0\}$, this time however letting $D_{-1}(P) = 0$. For all i such that $i > 0$ and $D_i(P) \neq 0$ define a differential $d_i^W : D_i(P) \to D_{i-1}(P)$ by

$$d_i^W(a_0, a_1, \ldots, a_i) = \sum_{j=0}^{i-1} (-1)^j (a_0, \ldots, \hat{a}_j, \ldots, a_i).$$

Let $d_i^W = 0$ for all other i. The homology of the algebraic complex $(D(P), d^W)$ is the *Whitney homology* of P, studied by K. Baclawski in [1]. We write $H_i^W(P) = \operatorname{Ker} d_i^W / \operatorname{Im} d_{i+1}^W$.

In this section we construct an isomorphism which relates the Whitney homology groups of P to the order homology groups of lower intervals of P. The Whitney homology of geometric lattices is then easily computed.

THEOREM 5.1. *For every poset P with 0 and every $i \in \mathbf{Z}$*

$$H_i^W(P) \cong \bigoplus_{x \in P - \{0\}} \tilde{H}_{i-1}([0, x]).$$

Proof. For $x \in P - \{0\}$ let $D_i(x)$ be the subgroup of $D_i(P)$ generated by all i-chains $a_0 < a_1 < \cdots < a_i$ with $a_i = x$. Clearly, $D_i(P) = \bigoplus_{x>0} D_i(x)$ for all $i \in \mathbf{Z}$. Since the differential d_i^W is so defined that $d_i^W(D_i(x)) \subseteq D_{i-1}(x)$ it is also clear that $d_i^W = \bigoplus_{x>0} d_i^W(x)$, where $d_i^W(x)$ denotes the restriction of d_i^W to $D_i(x)$. Let a homomorphism $f_i : C_{i-1}([0, x]) \to D_i(x)$ be induced by the mapping $f_i : (a_0, a_1, \ldots, a_{i-1}) \to (a_0, a_1, \ldots, a_{i-1}, x)$ whenever $C_{i-1}([0, x]) \neq 0$, and let $f_i = 0$ otherwise. Clearly, f_i is an isomorphism for all i and the following diagram commutes

$$\begin{array}{ccccccc} \cdots \to C_i([0,x]) & \xrightarrow{d_i(x)} & C_{i-1}([0,x]) & \xrightarrow{d_{i-1}(x)} & C_{i-2}([0,x]) \to \cdots \\ \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} \\ \cdots \to D_{i+1}(x) & \xrightarrow{d^W_{i+1}(x)} & D_i(x) & \xrightarrow{d^W_i(x)} & D_{i-1}(x) \to \cdots \end{array}$$

where $d_i(x)$ is the differential of the complex $(C([0, x]), d)$ defined in the beginning of this section. Hence,

$$\begin{aligned} \bigoplus_{x>0} \tilde{H}_{i-1}([0, x]) &= \bigoplus_{x>0} (\operatorname{Ker} d_{i-1}(x)/\operatorname{Im} d_i(x)) \\ &\cong \bigoplus_{x>0} (\operatorname{Ker} d^W_i(x)/\operatorname{Im} d^W_{i+1}(x)) \\ &\cong \Big(\bigoplus_{x>0} \operatorname{Ker} d^W_i(x)\Big)\Big/\Big(\bigoplus_{x>0} \operatorname{Im} d^W_{i+1}(x)\Big) \\ &= \operatorname{Ker}\Big(\bigoplus_{x>0} d^W_i(x)\Big)\Big/\operatorname{Im}\Big(\bigoplus_{x>0} d^W_{i+1}(x)\Big) \\ &= \operatorname{Ker} d^W_i/\operatorname{Im} d^W_{i+1} = H^W_i(P). \end{aligned}$$

The finite case of the following theorem is due to K. Baclawski [1, theorem 3.5]. The infinite case was conjectured by him (personal communication).

THEOREM 5.2. *Let L be a geometric lattice of rank r. Then $H^W_i(L)$ is free for all $i \in \mathbf{Z}$ and*

$$\operatorname{rank} H^W_i(L) = \begin{cases} \sum_{x \in L_{i+1}} |\mu(0, x)|, & \text{if } 0 \leq i \leq r-1 \text{ and } L \text{ is finite,} \\ |L|, & \text{if } 0 \leq i \leq r-1 \text{ and } L \text{ is infinite,} \\ 0, & \text{otherwise} \end{cases}$$

Proof. Suppose that $0 \leq i \leq r-1$. Since $\tilde{H}_{i-1}([0, x]) \neq 0$ if and only if $x \in L_{i+1}$,

Theorem 5.1 gives

$$H^W_i(L) \cong \bigoplus_{x \in L_{i+1}} \tilde{H}_{i-1}([0, x$$

Since the summands on th $\sum_{x \in L_{i+1}} \operatorname{rank} \tilde{H}_{i-1}([0, x])$. Thu tion 1.

As was previously done fo now go on and describe an geometric structure. This w isomorphism in Theorem 5.1

Let P be a poset with 0.

(5.3) $\varphi_i : \bigoplus_{x \in P-\{0\}} \tilde{H}_{i-1}([0, x]) -$

as follows. An element of $\oplus_x$ cycle in $C_{i-1}([0, x])$ and $[\rho_x]$ non-zero.

If $\rho_x = \sum_{k=1}^{n} t_k(a_1^k, a_2^k, \ldots,$ 0 let $\bar{\rho}_x = 0$. Now define

$$\varphi_i\Big(\bigoplus_{x>0} [\rho_x]\Big) = \Big[\sum_{x>0} \bar{\rho}_x\Big].$$

It is easy to check that $\sum_{x>0} \bar{\rho}$ not depend on the choice of homomorphism. In fact, φ_i straightforward verification or ism constructed to prove The

Now, let L be a geometric be a nonvoid independent set

$$\bar{\rho}_A = \sum_{\pi \in \mathfrak{S}_{i+1}} (-1)^{\pi}(p_{\pi(1)}, p_{\pi($$

Just as in section 4 one verifie the cycles $\bar{\rho}_A$ of this form ele $x = \vee A$ and ρ_A is the elemen

Theorem 5.1 gives

$$H_i^W(L) \cong \bigoplus_{x \in L_{i+1}} \tilde{H}_{i-1}([0, x]).$$

Since the summands on the right are free so is $H_i^W(L)$, and rank $H_i^W(L) = \sum_{x \in L_{i+1}}$ rank $\tilde{H}_{i-1}([0, x])$. Thus the result follows from Theorem 2.1 and Proposition 1.

As was previously done for the order homology of a geometric lattice, we will now go on and describe an explicit basis for Whitney homology in terms of geometric structure. This will first require a more careful analysis of the isomorphism in Theorem 5.1.

Let P be a poset with 0. Define a map

$$(5.3) \quad \varphi_i : \bigoplus_{x \in P - \{0\}} \tilde{H}_{i-1}([0, x]) \to H_i^W(P)$$

as follows. An element of $\oplus_{x>0} \tilde{H}_{i-1}([0, x])$ is of the form $\oplus_{x>0} [\rho_x]$, where ρ_x is a cycle in $C_{i-1}([0, x])$ and $[\rho_x]$ is its homology class and only finitely many ρ_x are non-zero.

If $\rho_x = \sum_{k=1}^{n} t_k(a_1^k, a_2^k, \ldots, a_i^k) \neq 0$ let $\bar{\rho}_x = \sum_{k=1}^{n} t_k(a_1^k, a_2^k, \ldots, a_i^k, x)$ and if $\rho_x = 0$ let $\bar{\rho}_x = 0$. Now define

$$\varphi_i\left(\bigoplus_{x>0} [\rho_x]\right) = \left[\sum_{x>0} \bar{\rho}_x\right].$$

It is easy to check that $\sum_{x>0} \bar{\rho}_x$ is a cycle in $D_i(P)$ and that its homology class does not depend on the choice of representatives ρ_x. So φ_i is a well-defined group-homomorphism. In fact, φ_i is an isomorphism, as can be seen either by straightforward verification or by observing that φ_i coincides with the isomorphism constructed to prove Theorem 5.1.

Now, let L be a geometric lattice of rank r and let $A = \{p_1, p_2, \ldots, p_{i+1}\} \subseteq L_1$ be a nonvoid independent set. Define

$$\bar{\rho}_A = \sum_{\pi \in \mathfrak{S}_{i+1}} (-1)^{\pi} (p_{\pi(1)}, p_{\pi(1)} \vee p_{\pi(2)}, \ldots, p_{\pi(1)} \vee p_{\pi(2)} \vee \cdots \vee p_{\pi(i+1)}).$$

Just as in section 4 one verifies that $d_i^W \bar{\rho}_A = 0$, so $\bar{\rho}_A$ is a cycle in $D_i(L)$. We call the cycles $\bar{\rho}_A$ of this form *elementary cycles* in Whitney homology. Notice that if $x = \vee A$ and ρ_A is the elementary cycle in $\tilde{H}_{i-1}([0, x])$ in the sense of section 4, then the two meanings of $\bar{\rho}_A$, as defined in this and the preceding paragraph, coincide.

For $i = 1, 2, \ldots, r$ let $\mathbf{BC}_i$ denote the family of i-element subsets of L_1 which contain no broken circuit under some well-ordering of L_1.

THEOREM 5.4. *Let L be a geometric lattice of rank r and let $0 \le i \le r-1$. Then the homology classes of elementary cycles $[\bar{\rho}_A]$, $A \in \mathbf{BC}_{i+1}$, form a basis of the free group $H_i^W(L)$.*

Proof. By Theorem 4.2 each group $\tilde{H}_{i-1}([0, x])$, $x \in L_{i+1}$, has a basis consisting of elementary cycles ρ_A, where $A \in \mathbf{BC}_{i+1}$ (cf. also the proof of Corollary 3.9). Under the isomorphism (5.3) $\varphi_i : \oplus_{x \in L_{i+1}} \tilde{H}_{i-1}([0, x]) \to H_i^W(L)$ the basis element ρ_A is sent to $[\bar{\rho}_A]$ as explained above.

6. Combinatorial geometry of bases

The representation of bases A of a finite geometric lattice L as elementary cycles ρ_A in the free Abelian group $\tilde{H}_{r-2}(L)$ (cf. section 4) induces the structure of a *combinatorial geometry* (cf. [6] or [11]), let us call it $\mathscr{B}(L)$, on the set of bases of L by linear independence in $\tilde{H}_{r-2}(L)$. Theorem 4.2 shows that rank $\mathscr{B}(L) = |\mu(0, 1)|$. Clearly, a collection of bases in L is a base in the geometry $\mathscr{B}(L)$ if the corresponding elementary cycles form a basis of the group $\tilde{H}_{r-2}(L)$. We do not know whether also the converse is true, that is, whether $\mathscr{B}(L)$ must be a unimodular geometry (this question was raised by R. Stanley). In any case, every standard matrix representation of $\mathscr{B}(L)$ with respect to a neat base-family yields a matrix with entries 0 or ± 1, as can be seen from property (4.3). In this section we will prove a partition property of $\mathscr{B}(L)$ which is weaker than unimodularity (cf. Baclawski and White [2, Theorem 5]) but which will be useful in the next section.

A combinatorial geometry G on a set S is said to be 2-*partitionable* if for every $x \in S$ there is a partition $S - \{x\} = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$, such that $x \notin \bar{S}_1$ and $x \notin \bar{S}_2$ ($\bar{S}_i$ denotes the span of S_i in G).

PROPOSITION 6. *The geometry $\mathscr{B}(L)$ is* 2-*partitionable.*

Proof. Suppose that $A = \{a_1, a_2, \ldots, a_r\} \subseteq L_1$ is a base of L, and let $c_i = a_1 \vee a_2 \vee \cdots \vee a_i$ and $d_i = a_r \vee a_{r-1} \vee \cdots \vee a_{r-i+1}$ for $i = 1, 2, \ldots, r$. For every cycle $\rho = \sum t_i \sigma_i$, $t_i \in \mathbf{Z}$, in $\tilde{H}_{r-2}(L)$ denote the *support* of ρ, i.e., the set $\{\sigma_i \mid t_i \neq 0\}$, by supp ρ. Thus, the chains $\sigma = (c_1, c_2, \ldots, c_{r-1})$ and $\tau = (d_1, d_2, \ldots, d_{r-1})$ lie in the support of the elementary cycle ρ_A. In fact, σ and τ fully determine ρ_A (and hence A) in the sense of the following *claim*: If $\sigma, \tau \in \operatorname{supp} \rho_B$ for a base B of L then

$B = A$. Taking this claim momentarily for granted we can partition the set S_A of all bases of L except A into $S_1 = \{B \in S_A \mid \sigma \in \operatorname{supp} \rho_B\}$ and $S_2 = S_A - S_1$. Suppose that $n\rho_A = \sum_{i=1}^{e} t_i \rho_{B_i}$, $n, t_i \in \mathbf{Z}$, $n \neq 0$, for some bases B_i of L, $B_i \neq A$. Then it is necessary that $\sigma \in \operatorname{supp} \rho_{B_j}$ and $\tau \in \operatorname{supp} \rho_{B_k}$ for some bases B_j and B_k, $1 \leq j, k \leq e$. This implies that $B_j \in S_1$ and, because of the claim, that $B_k \in S_2$. Thus ρ_A is not in the linear span of $\{\rho_B \mid B \in S_1\}$ nor in the linear span of $\{\rho_B \mid B \in S_2\}$, i.e., $A \notin \bar{S}_1$ and $A \notin \bar{S}_2$.

To prove the claim we observe that if $E_1, E_2 \subseteq B$ and $x_1 = \vee E_1$ and $x_2 = \vee E_2$ then $x_1 \wedge x_2 = \vee(E_1 \cap E_2)$. The assumption that $\sigma, \tau \in \operatorname{supp} \rho_B$ implies that $c_i = \vee C_i$ and $d_i = \vee D_i$ for suitable subsets $C_i, D_i \subseteq B$, $i = 1, 2, \ldots, r$. Thus $\vee(C_i \cap D_{r+1-i}) = c_i \wedge d_{r+1-i} = (a_1 \vee a_2 \vee \cdots \vee a_i) \wedge (a_i \vee a_{i+1} \vee \cdots \vee a_r) = a_i$, so $\{a_i\} = C_i \cap D_{r+1-i} \subseteq B$ for $i = 1, 2, \ldots, r$, i.e., $A = B$.

The geometry structure we are considering can be extended from bases to all independent sets by representing these by elementary cycles in Whitney homology. For a geometric lattice L or rank r and $k = 1, 2, \ldots, r$ we define the combinatorial geometry $\mathcal{IN}_k(L)$ on the set of k-element independent sets $A \subseteq L_1$ by linear independence among the elements $[\bar{\rho}_A]$ in the free Abelian group $H_{k-1}^{W}(L)$ (cf. section 5). The analysis of Whitney homology in section 5 shows that $\mathcal{IN}_k(L) \cong \bigoplus_{x \in L_k} \mathcal{B}([0, x])$ (where the summation sign denotes matroid direct sum, cf. [11]) so the added generality yields nothing really new.

7. Independence numbers and Whitney numbers

Let L be a geometric lattice of rank r. For any integer k satisfying $1 \leq k \leq r$ consider the following three cardinal numbers:

$a_k = \sum_{x \in L_k} \operatorname{rank} \tilde{H}_{k-2}([0, x])$,

$b_k = \operatorname{rank} H_{k-1}^{W}(L)$,

$c_k =$ the number of k-element subsets of L_1 which contain no broken circuit under a given well-ordering of L_1.

It is known from the work of K. Baclawski [1], J. Folkman [8] and G.-C. Rota [10] that the three numbers are equal when L is finite. We have shown in section 5 that they are equal in general. Define the *Whitney numbers of the first kind* w_k for $k = 1, 2, \ldots, r$ as the common value $w_k = a_k = b_k = c_k$. By the *Whitney numbers of the second kind* W_k we understand the numbers $W_k = |L_k|$, $k = 1, 2, \ldots, r$. Finally, the *independence number* I_k, $k = 1, 2, \ldots, r$, is the number of k-element independent subsets of L_1. Clearly, $W_k \le w_k \le I_k$. While Whitney and independence numbers of finite geometric lattices pose several intriguing problems, we have seen that for infinite geometric lattices L and $1 \le k \le r$: $W_k = w_k = I_k = |L|$ (except $W_r = 1$).

In the remainder of this section we restrict attention to the finite case, for which we wish to derive refinements of the relation $w_k \le I_k$. Recall that a geometric lattice L is said to be *irreducible* if it is not the product of nontrivial factors. Also, L is said to be *r-uniform* if all r-element subsets of L_1 are bases of L.

THEOREM 7.1. *Let L be a finite geometric lattice of rank r having n points and b(L) bases. Then*

$$\text{(i)}\quad \frac{n}{r}\,|\mu(0,1)| \le b(L) \le \tfrac{1}{2}(|\mu(0,1)|+1)\,|\mu(0,1)|,$$

and if L is irreducible

$$\text{(ii)}\quad \frac{n}{r}(n-r)(r-1) < b(L).$$

Furthermore, equality occurs for the lower bound in (*i*) *if and only if L is r-uniform.*

COROLLARY 7.2. *For* $k = 1, 2, \ldots, r$:

$$\frac{m_k}{k}\, w_k \le I_k \le \frac{M_k+1}{2}\, w_k,$$

where $m_k = \min_{x \in L_k} |\{p \in L_1 \mid p \le x\}|$ *and* $M_k = \max_{x \in L_k} |\mu(0,x)|$.

Proof. For $p \in L_1$ let $b_p(L)$ denote the number of bases of L which contain p. Since any neat base-family in L with distinguished point p has $|\mu(0,1)|$ members which are bases of L containing p (Propositions 3.3 and 3.6), we get $|\mu(0,1)| \le b_p(L)$. Summing both sides over all points of L we obtain $n \cdot |\mu(0,1)| \le \sum_{p \in L_1} b_p(L) = r \cdot b(L)$, which is the first half of (i). This inequality shows that $|\mu(0,1)| \le b(L)$ with equality only if $n = r$. Hence, $|\mu(0,1)| = b(L)$ if and only if L is Boolean. For every $p \in L$, let $\mathbf{p} = \{h \in L_{r-1} \mid p \not\le h\}$. Using Weisner's theorem [10, p. 351] we get

$$|\mu(0,1)| = \sum_{h \in \mathbf{p}} |\mu(0,h)| \le \sum_{h \in \mathbf{p}} b([0,h]) = b_p(L).$$

Thus, $|\mu(0,1)| = b_p(L)$ if and only
Consequently, $n \cdot |\mu(0,1)| = r \cdot b(L)$
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To obtain the corollary we obse

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COROLLARY 7.3. *If L is irred*

$$\frac{n}{k}(n-k)(k-1) < I_k.$$

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19 (1979), 193–202.

Thus, $|\mu(0, 1)| = b_p(L)$ if and only if the intervals $[0, h]$ are Boolean for all $h \in \mathbf{p}$. Consequently, $n \cdot |\mu(0, 1)| = r \cdot b(L)$ if and only if the intervals $[0, h]$ are Boolean for all $h \in L_{r-1}$, which is equivalent with saying that L is r-uniform.

For the second half of (i) we utilize the geometry structure $\mathcal{B}(L)$ on the set of bases of L which was defined in the previous section. It has been shown by M. Feinberg [7, corollary 4.2] (cf. also Baclawski and White [2]) that a finite 2-partitionable vector geometry of rank ρ on ν points satisfies $\nu \leq \binom{\rho+1}{2}$. The geometry $\mathcal{B}(L)$ is of rank $|\mu(0, 1)|$ on $b(L)$ points and was shown in section 6 to be 2-partitionable.

Finally, T. Brylawski has shown [5] that if L is irreducible and not isomorphic to a certain geometric lattice H, then $|\mu(0, 1)| \geq (n-r)(r-1)$. Combining with (i) and observing that Brylawski's inequality is strict for r-uniform lattices we obtain (ii). The exceptional lattice H is the lattice of flats of the parallel connection of three 3-point lines, and it is easy to check that (ii) is valid also for H.

To obtain the corollary we observe that for every $x \in L_k$ the theorem gives

$$\frac{1}{k}\,|\{p \in L_1 \mid p \leq x\}| \cdot |\mu(0, x)| \leq I_k(x) \leq \tfrac{1}{2}(|\mu(0, x)| + 1)\,|\mu(0, x)|,$$

where $I_k(x)$ denotes the number of k-element independent sets $A \subseteq L_1$ for which $\vee A = x$. Since $w_k = \sum_{x \in L_k} |\mu(0, x)|$ and $I_k = \sum_{x \in L_k} I_k(x)$ the inequalities are obtained by summation over L_k.

G. Dinolt and U. Murty have shown for every irreducible finite geometric lattice L satisfying $n \geq 2r$ that $b(L) \geq \max(2(n-r)(r-1) - r + 3,\ 3n - 7)$, see pp. 298–300 of [11]. The inequality 7.1 (ii) improves this bound. Since the truncation of an irreducible geometric lattice is again irreducible we can derive from 7.1 (ii) also the following bounds.

COROLLARY 7.3. *If L is irreducible then for* $k = 1, 2, \ldots, r$:

$$\frac{n}{k}(n-k)(k-1) < I_k.$$

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University of Stockholm
Stockholm
Sweden

Addendum

(i) The question stated in section 6 has been answered in the negative by B. Lindström (Europ. J. Combinatorics, *2* (1981), 61–63). He shows that the geometry $\mathcal{B}(L)$ is not unimodular if L is the r-uniform geometric lattice on n points and $2<r<n-2$.

(ii) The homology of geometric lattices receives considerable attention in an interesting recent paper by P. Orlik and L. Solomon (Invent. Math. *56* (1980), 167–189). Their treatment of Whitney homology is closely related to ours.

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