Topological Methods

(A chapter for "Handbook of Combinatorics", ed. R. Graham, M. Grötschel and L. Lovász)

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CONTENT:

1. Introduction

Part I. EXAMPLES

2. The Evasiveness Conjecture
3. Fixed Points in Posets
4. Kneser's Conjecture
5. Discrete Applications of Borsuk's Theorem
6. Matroids and Greedoids
7. Oriented Matroids
8. Tutte's Homotopy Theorem

Part II. TOOLS

9. Combinatorial Topology
   Simplicial complexes and posets
   Homotopy and homology
10. Special Complexes
   Collapsible and shellable complexes
   Cohen–Macaulay complexes
   Induced subcomplexes
11. Combinatorial Homotopy Theorems
12. Cell Complexes
   Polyhedral complexes
   Regular cell complexes
13. Borsuk's Theorem

References

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1. Introduction

In this chapter we discuss some of the ways in which topology has been used in combinatorics. The emphasis is on methods for solving genuine combinatorial problems, and the selection of material reflects this aim.

The chapter is divided into two parts. In Part I a number of examples are presented which illustrate different uses of topology. In Part II we have gathered a number of combinatorial tools which have proven useful for manipulating the topological structure found in combinatorial situations. Also, a brief review of relevant parts of combinatorial topology is given. Part II is intended mainly for reference purposes.

Among the examples in Part I one can discern at least 4 ways in which topology enters the combinatorial sphere. Of course, it is in the nature of such comments that no rigid demarcation lines could or should be drawn. Also, other connections between topology and combinatorics may well follow different paths.

(1) Associate a simpl-complex then use topology

In the first three examples (Sections 2–4) topology enters this way. First a relevant simplicial complex is identified in the combinatorial context. Then it is shown that this complex has sufficiently favorable properties to allow application of some theorem of algebraic topology, which implies the combinatorial conclusion.

A different approach is seen in Section 5 and in Bárány's proof in Section 4. There the combinatorial configuration is represented in concrete fashion in $\mathbb{R}^d$ or on the d-sphere, and a topological result for Euclidean space (Borsuk's Theorem) has the desired effect on the configuration.

(2) Go for a configuration

The case of oriented matroids (Section 7) is unique. For these combinatorial objects there is a *topological representation theorem*, saying that oriented matroids are the same thing as arrangements of pseudospheres in a sphere. Of course, in this situation the topological perspective is always at hand as an alternative way of looking at these objects. Some non-trivial properties of oriented matroids find particularly simple proofs this way.

Homotopy results in combinatorics sometimes arise as follows. Say we want to define some property $\mathcal{P}$ at all vertices of a connected graph $G = (V, E)$. We start by defining $\mathcal{P}$ at some root node r, and then give a rule for how to define $\mathcal{P}$ at v's neighbors having already defined it at $v \epsilon V$. The problem of consistency arises: Can different paths from r to v lead to different definitions of $\mathcal{P}$ at v? One strategy for dealing with this is to define "elementary homotopies", meaning certain pairs of paths which can be exchanged without affecting the result (usually such pairs form small circuits such as triangles and squares). Then we need a "homotopy theorem" saying that any path from r to v can be deformed into any other such path using elementary homotopies. Tutte's and Maurer's homotopy theorems (Sections 6 and 8) are of this kind. From a topological point of view, the "elementary homotopies" mean that certain 2-cells are attached to the graph, and the homotopy theorem then says that the resulting 2-complex is simply connected.

Being topologically k-connected has a direct combinatorial meaning for $k = 0$ (connected), and as we have seen also for $k = 1$ (simply connected). The way that higher connectivity influences combinatorics is more subtle, see the examples in Sections 4 and 6.

Notation and terminology is explained in Part II. We treat simplicial complexes and posets almost interchangeably. The order complex of a poset and the poset of faces of a complex — these two constructions take posets to complexes and vice versa, and no ambiguity can arise from the topological point of view.

The author would like to thank L. Lovász for several helpful suggestions which have influenced the presentation in this chapter.

Part I. EXAMPLES

2. The Evasiveness Conjecture

By a *graph property* we shall understand a property which is isomorphism invariant: if $G_1 \cong G_2$ then G_1 has the property if and only if G_2 does. The following discussion will usually concern graphs having some fixed vertex set V. These graphs can simply be identified with the various subsets of $\binom{V}{2}$. Also, it is convenient to identify a graph property with the subset of the power set $\mathbf{2}^{\binom{V}{2}}$ which consists of all graphs having the property.

Let each graph G be represented by its $n \times n$ adjacency matrix M_G (the superdiagonal part of course suffices), $n =$ card V. Suppose that we are to algorithmically decide whether G has a certain given property $\mathcal{P}$ from this adjacency matrix. The allowed algorithms may inspect only one entry of M_G at a time. The problem is: How many entries in M_G must the best $\mathcal{P}$-testing algorithm inspect in the worst case G to decide whether $G \in \mathcal{P}$ (i.e., G has property $\mathcal{P}$)? This number is called the *argument complexity* of $\mathcal{P}$, written $c(\mathcal{P})$. Clearly, $0 \leq c(\mathcal{P}) \leq \binom{n}{2}$. The properties of complexity 0 are called *trivial*. They are the properties which all graphs have or which no graphs have. The properties of maximal complexity $\binom{n}{2}$ are called *evasive*. For them no algorithm can be guaranteed to perform with less than complete information about the graph.

Let us call a graph property $\mathcal{P} \subseteq \mathbf{2}^{\binom{V}{2}}$ *monotone* if it is preserved under deletion of edges. For instance, being circuit-free or planar are monotone properties. Since testing for the negation of a property $\mathcal{P}$ has the same argument complexity as testing for $\mathcal{P}$, it is reasonable to call monotone also those properties which are preserved under addition of edges, such as being connected or having chromatic number $\geq k$.

Work by several authors (for a survey, see Bollobás (1978), Chpt. 8) has led to the following conjecture.

(2.1) **Evasiveness Conjecture.** Every nontrivial monotone graph property is evasive.

The best partial result known is the following.

(2.2) **Theorem.** [Kahn, Saks and Sturtevant (1984)]
The evasiveness conjecture is true for properties of graphs on a prime-power number of vertices.

Let us sketch the proof of Kahn, Saks and Sturtevant to show the way in which topology is used.

Suppose that card $V = p^k$, p prime, and that $\mathcal{P} \neq \emptyset$ is a monotone nonevasive graph property. $\mathcal{P}$ is a family of subsets of $\binom{V}{2}$ closed under the formation of subsets — i.e., a simplicial complex. The conclusion we want to draw is that $\mathcal{P}$ is trivial, which, since $\mathcal{P} \neq \emptyset$, must mean that $\binom{V}{2} \epsilon \mathcal{P}$ — i.e., topologically $\mathcal{P}$ is the full simplex.

These two facts are crucial:

(2.3) The geometric realization $|\mathcal{P}|$ is contractible.

(2.4) There exists a group Γ of simplicial automorphisms of $\mathcal{P}$ which acts transitively on $\binom{V}{2}$ and which has a normal p-subgroup Γ_1, such that Γ/Γ_1 is cyclic.

For (2.3) one argues that the monotone property $\mathcal{P}$ is not evasive in the algorithmic sense defined above if and only if as a simplicial complex $\mathcal{P}$ is nonevasive in the recursive sense of (10.1). By (10.1) all nonevasive complexes are contractible.

The group Γ needed in (2.4) is constructed as follows. Identify V with the finite field $GF(p^k)$. Let $\Gamma = \{x \longmapsto ax+b \mid a,b \in GF(p^k), a \neq 0\}$ and $\Gamma_1 = \{x \longmapsto x+b \mid b \in GF(p^k)\}$. The assumption that $\mathcal{P}$ is an isomorphism-invariant property of graphs on V means that if γ is any permutation of V — in particular, if $\gamma \in \Gamma$ — then $A \in \mathcal{P}$ if and only if $\gamma(A) \in \mathcal{P}$. Hence, Γ is a group of simplicial automorphisms of $\mathcal{P}$. One checks that Γ is doubly transitive on $V = GF(p^k)$ and that the subgroup Γ_1 has the required properties.

By a theorem of R. Oliver, any action of a finite group Γ, having a subgroup Γ_1 with the stated properties, on a finite $\mathbf{Z}_p$-acyclic simplicial complex must have stationary points. Since our complex $\mathcal{P}$ is $\mathbf{Z}_p$-acyclic (being contractible), this means that there exists some point $x \in |\mathcal{P}|$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$. The point x is carried by the relative interior of a unique face $G \in \mathcal{P}$, and the fact that x is stationary implies that $\gamma(G) = G$ for all $\gamma \in \Gamma$. But since Γ is transitive on $\binom{V}{2}$ this is impossible unless $G = \binom{V}{2}$. Hence, $\binom{V}{2} \in \mathcal{P}$, and we are done.

For non-prime-power cardinalities the evasiveness conjecture remains open.

3. Fixed Points in Posets

A poset P is said to have the *fixed point property* if every order-preserving self-map $f : P \to P$ has a fixed point $x = f(x)$. It was shown by A. C. Davis and A. Tarski that a lattice has the fixed point property if and only if it is complete (meaning that meets and joins exist for subsets of arbitrary cardinality). It has long been an open problem to find some characterization of the finite posets which have the fixed point property. See Rival (1985) for references to work in this area. In the absence of such a characterization work has been directed toward finding nontrivial classes of finite posets which have the fixed point property. For this the Lefschetz fixed point theorem has proved to be useful.

Let L be a finite lattice and $z \in L$. Then y is said to be a *complement* of z, written $y \perp z$, if $y \wedge z = \hat{0}$ and $y \vee z = \hat{1}$. Let $Co(z) = \{y \in L \mid y \perp z\}$. The lattice L is called *complemented* if $Co(z) \neq \emptyset$ for all $z \in L$.

A finite lattice L has the fixed point property, this is very easy to see. It is more interesting to look at the *proper part* $\bar{L} = L - \{\hat{0}, \hat{1}\}$ of the lattice, which may or may not have the fixed point property.

(3.1) **Theorem.** [Baclawski and Björner (1979, 1981)]
Let L be a finite lattice and $z \in \bar{L}$. Then the poset $\bar{L} - Co(z)$ has the fixed point property. In particular, if L is noncomplemented then $\bar{L}$ has the fixed point property.

By (11.11) the order complex $\Delta(\bar{L} - Co(z))$ is contractible, and therefore by Lefschetz' theorem it has the *topological* fixed point property. From this the result easily follows.

Of course, the preceding argument is applicable to *any* $\mathbf{Q}$-acyclic finite poset (see (10.1) for some other combinatorially defined classes of such). Also, with this method one can prove more about the combinatorial structure of the *fixed point sets* $P^f = \{x \in P \mid x = f(x)\}$ than merely that they are nonempty.

Let Δ be an oriented simplicial complex and $f : \Delta \to \Delta$ a simplicial map. Say that a face $\tau \in \Delta \cup \{\emptyset\}$ is *fixed* if $f(\tau) = \tau$ as a set, and let $\varphi_i^+(f)$ be the number of fixed i-faces whose orientation is preserved and $\varphi_i^-(f)$ the number whose orientation is reversed. The

Hopf trace formula shows that if Δ is $\mathbf{Q}$-acyclic then $0 = \Sigma_{i\geq -1}(-1)^i(\varphi_i^+(f) - \varphi_i^-(f))$. Now, if $f : P \to P$ is an order-preserving poset map then a chain τ in P is fixed if and only if it is *point-wise* fixed, and orientation is always preserved. Hence, the right-hand side specializes to the *Möbius number* $\mu(P^f)$, or reduced Euler characteristic, i.e., the number of even length chains in P^f minus the number of odd length ones (where the empty chain counts as odd).

(3.2) **Theorem.** [Baclawski and Björner (1979)]
Let $f : P \to P$ be an order-preserving mapping of a finite $\mathbf{Q}$-acylic poset (e.g., a dismantlable poset or the proper part of a noncomplemented lattice). Then $\mu(P^f) = 0$.

The result shows that for instance two incomparable points cannot alone form a fixed point set in an acyclic poset. For other finite posets with the fixed point property such fixed point sets are, however, possible.

4. Kneser's Conjecture

Consider the collection of all n-element subsets of a $(2n + k)$-element set, $n \geq 1, k \geq 0$. It is easy to partition this collection into $k + 2$ classes so that every pair of n-sets within the same class has nonempty intersection. Can the same be done with only $k + 1$ classes? M. Kneser conjectured in 1955 that the answer is negative, and this was later confirmed by L. Lovász.

(4.1) **Theorem.** [Lovász (1978)]
If the n-subsets of a $(2n + k)$-element set are partitioned into $k + 1$ classes, then some class will contain a pair of disjoint n-sets.

Lovász's proof relies on Borsuk's Theorem (13.1). Soon after Lovász's breakthrough a simpler way of deducing Kneser's Conjecture from Borsuk's Theorem was discovered by Bárány (1978). However, Lovász's proof is applicable also to other situations and hence of greater general interest.

Let us first sketch Bárány's proof. By a theorem of D. Gale, for $n, k \geq 1$ there exist $2n + k$ points on the sphere $\mathbf{S}^k$ such that any open hemisphere contains at least n of them. Partition the n-subsets of these points into classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k$. For $0 \leq i \leq k$, let $\mathcal{O}_i$ be the set of all points $x \in \mathbf{S}^k$ such that the open hemisphere around x contains an n-subset from the class $\mathcal{C}_i$. Then $(\mathcal{O}_i)_{0\leq i\leq k}$ gives a covering of $\mathbf{S}^k$ by open sets. Part (i) of Borsuk's Theorem (13.1) implies that one of the sets, say $\mathcal{O}_k$, contains antipodal points. But the open hemispheres around these points are disjoint and both contain n-subsets from the class $\mathcal{C}_k$. Hence, $\mathcal{C}_k$ contains a pair of disjoint n-sets.

For Lovász's proof it is best to think of the problem in graph-theoretic terms. Define a graph $KG_{n,k}$ as follows: The vertices are the n-subsets of some fixed $(2n + k)$-element set X and the edges are formed by the pairs of disjoint n-sets. Then Theorem (4.1) can be reformulated: *The Kneser graph $KG_{n,k}$ is not $(k + 1)$-colorable.*

For any graph $G = (V, E)$ let $\mathcal{N}(G)$ denote the simplicial complex, called the *neighborhood complex*, whose vertex set is V and whose simplices are those sets of vertices which have a common neighbor. The topology of this complex has surprising combinatorial consequences.

(4.2) **Theorem.** [Lovász (1978)]
For any finite graph G, if $\mathcal{N}(G)$ is $(k-1)$-connected, then G is not $(k+1)$-colorable.

To prove Theorem (4.1) it will then suffice to show that $\mathcal{N}(KG_{n,k})$ is $(k-1)$-connected.

This is easily done as follows. Let $P = \{A \subseteq X | n \leq \text{ card } A \leq n + k\}$. Ordered by containment P is a subposet of the Boolean lattice $B(X)$ of all subsets of X. $B(X)$ is shellable (10.10) (iv), hence by (10.13) so is also P. As a k-dimensional shellable complex P is $(k-1)$-connected (10.2). Let C be the crosscut of n-element sets. By (11.4) P and the crosscut complex $\Gamma(P,C)$ are homotopy equivalent. It follows that $\Gamma(P,C)$, which is the same thing as $\mathcal{N}(KG_{n,k})$, is also $(k-1)$-connected.

The known proofs for Theorem (4.2) are more involved. A very elegant functorial argument was given by Walker (1983), which we will sketch here in briefest possible fashion. The same general argument was also found by Lovász (unpublished lecture notes) as a variation of his original proof. For more details see Walker (1983) and also the expository treatment in Björner (1985).

Take the following facts on faith: For each graph G there exists a subcomplex $\tilde{\mathcal{N}}(G)$ of the barycentric subdivision of $\mathcal{N}(G)$ and a simplicial mapping $\nu : \tilde{\mathcal{N}}(G) \to \tilde{\mathcal{N}}(G)$ with the following properties:

(1) $\tilde{\mathcal{N}}(G)$ and $\mathcal{N}(G)$ are of the same homotopy type,

(2) the pair $(\tilde{\mathcal{N}}(G), \nu)$ is an antipodality space,

(3) $(\tilde{\mathcal{N}}(K_m), \nu) \cong (\mathrm{S}^{m-2}, \alpha)$, i.e., complete graphs go to spheres as antipodality spaces,

(4) every graph map $g : G_1 \to G_2$ (mapping of the nodes which takes edges to edges) induces an equivariant map $\tilde{g} : \tilde{\mathcal{N}}(G_1) \to \tilde{\mathcal{N}}(G_2)$. (Walker shows that $\tilde{\mathcal{N}}$ (.) actually is a *functor* from the category of finite graphs and graph maps to the category of antipodality spaces and homotopy classes of equivariant maps.)

To prove Theorem (4.2), suppose that a graph G is $(k+1)$-colorable. This is clearly equivalent to the existence of a graph map $G \to K_{k+1}$ to the complete graph on $k+1$ nodes. Hence, we deduce the existence of an equivariant map $\tilde{\mathcal{N}}(G) \to \tilde{\mathcal{N}}(K_{k+1}) \cong \mathrm{S}^{k-1}$. So by part ($v$) of Borsuk's Theorem (13.1), we conclude that $\tilde{\mathcal{N}}(G)$, and hence $\mathcal{N}(G)$, is not $(k-1)$-connected.

A different application of Theorem (4.2) is given in Lovász (1983).

The following generalized "Kneser" conjecture was made by P. Erdös in 1973 and recently proved.

(4.3) **Theorem.** [Alon, Frankl and Lovász (1986)]
Let $n,t \geq 1$ and $k \geq 0$. If the n-subsets of a $(tn + (t-1)k)$-element set are partitioned into $k+1$ classes, then some class will contain t pairwise disjoint n-sets.

The proof is analogous to Lovász's proof of Theorem (4.1). For general t-uniform hypergraphs H a suitable *neighborhood* complex $\mathcal{C}(H)$ is defined. It is shown that if t is a prime and $\mathcal{C}(H)$ is $(k(t-1)-1)$-connected then H is not $(k+1)$-colorable. To prove this for odd primes t the Bárány – Shlosman – Szücs Theorem (13.3) is used rather than Borsuk's Theorem. See Alon, Frankl and Lovász (1986) for the details.

5. Discrete Applications of Borsuk's Theorem

One of the most famous consequences of Borsuk's Theorem is undoubtedly the Ham Sandwich Theorem (13.2). This result, or some version of the "ham sandwich" reasoning which leads to it, can be used in certain combinatorial situations to prove that composite configurations can be split in a balanced way. Two examples of this, due to N. Alon and coauthors, will be given in this section. For other applications of Borsuk's Theorem to combinatorics, see Bárány and Lovász (1982), Yao and Yao (1985), and Section 4.

Suppose that $2n$ points are given in general position in the plane $\mathbb{R}^2$, half colored red and the other half blue. It is a well known elementary problem to show that the red points can be connected to the blue points by n nonintersecting straight line segments. A quick argument goes like this. Of the $n!$ ways to match the blue and red points using straight line segments, choose one which minimizes the sum of the lengths. If two of its lines intersect, they could be replaced by the sides of the quadrilateral that they span, and a new matching of even shorter length would result. No such elementary proof is known for the following generalization to higher dimensions.

(5.1) **Theorem.** [Akiyama and Alon (1985)]
Let A be a set of $d \cdot n$ points in general position (no more that d points on any hyperplane) in $\mathbb{R}^d$. Let $A = A_1 \cup A_2 \cup \ldots \cup A_d$ be a partition of A into d pairwise disjoint sets of size n. Then there exist n pairwise disjoint $(d-1)$-dimensional simplices, such that each simplex intersects each set A_i in one of its vertices, $1 \leq i \leq d$.

The idea of Akiyama and Alon is to surround each point $p \epsilon A$ by a small ball of radius ε, where ε is small enough that no hyperplane intersects more than d such balls. Give each ball a uniform mass distribution of measure $1/n$. Then each color class $A_i, 1 \leq i \leq d$, is naturally associated with its n balls, forming a measurable set of measure 1. By the Ham Sandwich Theorem (13.2) there exists a hyperplane H which simultaneously bisects each color class. If n is odd, then H must intersect at least one ball from each A_i. General position immediately implies that H must intersect precisely one ball from each A_i, and in fact bisect this ball. By induction on n, the points on each side of H can now be assembled into disjoint simplices, and finally the points in H form one more such simplex. The argument if n is even is similar, but in that case H might have to be slightly moved to divide the points correctly for the induction step.

The other example has a more "applied" background. Suppose that k thieves steal a necklace with $k \cdot n$ jewels. There are t kinds of jewels on it, with $k \cdot a_i$ jewels of type $i, 1 \leq i \leq t$. The thieves want to divide the necklace fairly between them, wasting as little as possible of the precious metal in the links between jewels. They need to know in how many places they must cut the necklace? If the jewels of each kind appear contiguously on the opened necklace, then at least $t(k-1)$ cuts must be made. This number of cuts in fact always suffices. (Of course, what the thieves must *really* need is a fast algorithm for where to place these cuts.)

(5.2) **Theorem.** [Alon and West (1986), Alon (1987)]
Every open necklace with $k \cdot a_i$ beads of color $i, 1 \leq i \leq t$, can be cut in at most $t(k-1)$ places so that the resulting segments can be arranged into k piles with exactly a_i beads of color i in each pile, $1 \leq i \leq t$.

The idea for the proof is to turn the situation into a continuous problem by placing the open necklace (scaled to length 1) on the unit interval, and then to use a "ham sandwich" type argument there. For $k = 2$ this was done in Alon and West (1986) using Borsuk's Theorem. The extension to general k was achieved in Alon (1987) using the Bárány – Shlosman – Szücs Theorem (13.3).

6. Matroids and Greedoids

This and the next two sections are devoted to topological aspects of matroids and of two of their relatives — oriented matroids and greedoids. Basic topological facts about matroid complexes and geometric lattices are mentioned in (10.10).

The following result was proven by E. Györy and L. Lovász in response to a conjecture by A. Frank and S. Maurer.

(6.1) **Theorem.** [Lovász (1977), Györy (1978)]
Let $G = (V, E)$ be a k-connected graph, $\{v_1, v_2, \ldots, v_k\}$ a set of k vertices, and $n_1, n_2, \ldots, n_k$ positive integers with $n_1 + n_2 + \ldots + n_k = |V|$. Then there exists a partition $\{V_1, V_2, \ldots, V_k\}$ of V such that $v_i \in V_i, |V_i| = n_i$ and V_i spans a connected subgraph of $G, i = 1, 2, \ldots, k$.

The proof of Lovász uses topological methods, that of Györy does not. At the end of this section Lovász proof will be outlined for the case $k = 3$ in order to illustrate its use of topological reasoning. It relies on the connectivity of a certain polyhedral complex associated with certain forests in G. Similar complexes can be defined over the bases of a matroid, and more generally over the bases of a greedoid. Since the greedoid formulation contains the others as special cases, we shall use it in this section to develop the general result.

A set system $(E, \mathcal{F}), \mathcal{F} \subseteq 2^E$, is called a *greedoid* if the following axioms are satisfied:

$(G1)$ $\emptyset \in \mathcal{F}$,

$(G2)$ for all nonempty $A \in \mathcal{F}$ there exists an $x \in A$ such that $A - x \in \mathcal{F}$,

$(G3)$ if $A, B \in \mathcal{F}$ and $|A| > |B|$, then there exists an $x \in A - B$ such that $B \cup x \in \mathcal{F}$.

If also the extra condition $(G4)$ is satisfied, then $(E, \mathcal{F})$ is called an *interval greedoid*:

$(G4)$ if $A \subset B \subset C, A, B, C \in \mathcal{F}$ and $A \cup x, C \cup x \in \mathcal{F}$ for some $x \in E - C$, then also $B \cup x \in \mathcal{F}$.

The sets in $\mathcal{F}$ are called *feasible* and the maximal feasible sets *bases*. All bases have the same cardinality r, which is the *rank* of the greedoid.

The only examples which will be of concern here are *matroids* (feasible sets = independent sets) and *branching greedoids* of rooted graphs (feasible sets = edge sets which form a tree containing the root node). Both are interval greedoids. For other examples and further information about greedoids, see e.g. Korte and Lovász (1983) or Björner, Korte and Lovász (1985).

The feasible sets of a greedoid do not form a simplicial complex other than in the matroid case. However, as a poset, $\bar{\mathcal{F}} = \mathcal{F} - \{\emptyset\}$ ordered by inclusion gives a useful topology. A greedoid $(E, \mathcal{F})$ is called *k-connected* if for each $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ with $A \subseteq B$ and $|B - A| = \min(k, r - |A|)$ and such that $C \in \mathcal{F}$ for every $A \subseteq C \subseteq B$. Matroids are r-connected, and the branching greedoid of a k-connected rooted graph is k-connected.

(6.2) **Theorem.** [Björner, Korte and Lovász (1985)]
Let $(E, \mathcal{F})$ be a k-connected interval greedoid ($k \geq 2$). Then the poset of feasible sets $(\bar{\mathcal{F}}, \subseteq)$ is $(k-2)$-connected.

This result follows from (10.10) *(iii)* via (11.4), since for the crosscut C of minimal elements in $\bar{\mathcal{F}}$ the crosscut complex $\Gamma(\bar{\mathcal{F}}, C)$ is a matroid complex of rank $\geq k$.

Let $\mathcal{B}$ be the collection of all bases in a greedoid $(E, \mathcal{F})$ of rank r. Two bases B_1 and B_2 are *adjacent* if $B_1 \cap B_2 \in \mathcal{F}$ and $|B_1 \cap B_2| = r - 1$. Attaching edges between all adjacent pairs we get a graph with vertex set $\mathcal{B}$, the *basis graph*.

The shortest circuits in the basis graph can be explicitly described. There are two kinds of triangles and one kind of square (quadrilateral):

(6.3) Three bases $A \cup x, A \cup y, A \cup z$, where $A \in \mathcal{F}, |A| = r - 1$, span a triangle of the first kind.

(6.4) Three bases $A \cup x \cup y, A \cup x \cup z, A \cup y \cup z$, where $A \in \mathcal{F}, |A| = r - 2$, span a triangle of the second kind.

Amar aprender!!!

(6.5) Four bases $A \cup x \cup u, A \cup x \cup v, A \cup y \cup u, A \cup y \cup v$, where $A \in \mathcal{F}, |A| = r - 2$, span a square.

For branching greedoids triangles of the second kind cannot occur.

Now, attach a 2-cell (a "membrane") into each triangle and square. This gives a 2-dimensional regular cell complex $\mathcal{K}$, which we call the *basis complex*.

It is a straightforward combinatorial exercise to check that the basis complex of any 2-connected greedoid of rank ≤ 2 is 1-connected (i.e., connected and simply connected). For rank 2 (the only non-trivial case) this follows directly from the exchange axiom $(G3)$. In higher ranks the following is true.

(6.6) **Theorem.** [Björner, Korte and Lovász (1985)]
The basis complex $\mathcal{K}$ of any 3-connected interval greedoid is 1-connected.

In order to illustrate some of the tools given in Part II, we give a short proof of this. Let P be the poset of closed cells of $\mathcal{K}$ ordered by inclusion, and let Q be the top three levels of $(\mathcal{F}, \subseteq)$, i.e., the feasible sets of rank $r-2, r-1$ and r. Let $f : P \to Q$ be the order-reversing map which sends each cell τ to the intersection of the bases which span τ. By (6.2) and (10.12) the poset Q is 1-connected, so by (11.1) we only have to check that the fibers $f^{-1}(Q_{\geq A})$ are 1-connected for all $A \in Q$. But if $r(A) = r - i$, $i = 0, 1, 2$, then $f^{-1}(Q_{\geq A})$ is the basis complex of the rank i greedoid obtained by contracting A, and we have already checked that basis complexes of rank ≤ 2 greedoids are 1-connected.

Let $P = B_1 B_2 \dots B_d$ and $Q = B_d B_{d+1} \dots B_g$ be paths in the basis graph of a matroid, and let $PQ = B_1 B_2 \dots B_d B_{d+1} \dots B_g$ be their concatenation. Say that paths PQ and PRQ differ by an *elementary homotopy* if R is of the form $BCB, BCDB$ or $BCDEB$ with $B = B_d$.

(6.7) **Theorem.** [Maurer (1973)]
Let P and P' be any two paths with the same endpoints in the basis graph of a matroid. Then P can be transformed into P' by a sequence of elementary homotopies.

Maurer's "Homotopy Theorem" (6.7) is clearly a combinatorial reformulation of (6.6) in the matroid case. An application to oriented matroids will be given in the next section.

Time has come to return attention to Theorem (6.1). The following outline of the proof for the $k = 3$ case is quoted from Lovász (1979) (with some adjustments in square brackets to better suit the present discussion):

"So let G be a 3-connected graph, $v_1, v_2, v_3 \in V(G)$ and $n_1 + n_2 + n_3 = |V(G)|$. Take a new point a and connect it to v_1, v_2, and v_3. Consider the topological space $\mathcal{K}$ constructed for this new graph G'. [In our language, $\mathcal{K}$ is the basis complex of the branching greedoid determined by the rooted graph (G', a). This greedoid, whose bases are the spanning trees of G', is 3-connected.] For each spanning tree T of G', let $f_i(T)$ denote the number of points in T accessible from a along the edge $(a, v_i)(i = 1, 2)$. Then the mapping

$$f : T \mapsto (f_1(T), f_2(T))$$

maps the vertices of $\mathcal{K}$ onto lattice points of the plane. Let us subdivide each quadrilateral 2-cell in $\mathcal{K}$ by a diagonal into two triangles; in this way we obtain a triangulation $\bar{\mathcal{K}}$ of $\mathcal{K}$. Extend f affinely to each such triangle so as to obtain a continuous mapping of $\mathcal{K}$ into the plane. Obviously, the image of $\mathcal{K}$ is contained in the triangle $\Delta = \{x \geq 0, y \geq 0, x + y \leq n\}$. We are going to show that the mapping is onto Δ.

"Let us pick three spanning trees, T_1, T_2, T_3 first such that $f(T_1) = (n,0), f(T_2) = (0,n), f(T_3) = (0,0)$. Obviously, such trees exist. Next, by applying [the fact that the basis graph of a 2-connected greedoid is connected] to the graph $G' - (a, v_3)$, we select a polygon P_{12} in $\mathcal{K}$ connecting T_1 to T_2 and having $f_3(x) = 0$ at all points. Thus $f(P_{12})$ connects $(n,0)$ to $(0,n)$ along the side of the triangle Δ with these endpoints. Let P_{23} and P_{31} be defined analogously.

"By [Theorem (6.6)], $P_{12}+P_{23}+P_{31}$ can be contracted in $\mathcal{K}$ to a single point. Therefore, $f(P_{12}) + f(P_{23}) + f(P_{31})$ can be contracted in $f(\mathcal{K})$ to a single point. But "obviously" (or, rather, by applying the well-known fact that the boundary of a triangle cannot be contracted to a single point in the triangle with one interior point taken out), $f(\mathcal{K})$ must cover the whole triangle Δ. So in particular the point (n_1, n_2) belongs to the image of $\mathcal{K}$, and therefore it belongs to the image of a triangle of $\bar{\mathcal{K}}$. But it is easy to see that this implies that (n_1, n_2) is the image of one of the vertices of $\bar{\mathcal{K}}$; i.e., there exists a spanning tree T with

$$f_1(T) = n_1, \quad f_2(T) = n_2.$$

The three components of $T - a$ now yield the desired partition of $V(G)$."

Theorem (6.6) is a special case of a more general result saying that for any k-connected interval greedoid a certain higher-dimensional basis complex is $(k-2)$-connected. This more general result implies Theorem (6.1) for arbitrary k by extension of the ideas we have just seen in the $k = 3$ case. See Lovász (1977) and Björner, Korte and Lovász (1985) for complete details.

7. Oriented Matroids

Let E be a finite set with a fixed-point free involution $x \mapsto x^*$ (i.e., $x^* \neq x = x^{**}$ for all $x \in E$). Write $A^* = \{x^* | x \in A\}$, for subsets $A \subseteq E$. An *oriented matroid* $\mathcal{O} = (E, *, \mathcal{C})$ is such a set together with a family $\mathcal{C}$ of nonempty subsets such that

$(OM1)$ $\mathcal{C}$ is a clutter (i.e., $C_1 \neq C_2$ implies $C_1 \not\subseteq C_2$ for all $C_1, C_2 \in \mathcal{C}$),
$(OM2)$ If $C \in \mathcal{C}$ then $C^* \in \mathcal{C}$ and $C \cap C^* = \emptyset$,
$(OM3)$ If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2^*$ and $x \in C_1 \cap C_2^*$, then there exists $D \in \mathcal{C}$ such that $D \subseteq C_1 \cup C_2 - \{x, x^*\}$.

The sets in $\mathcal{C}$ are called *circuits* of the oriented matroid $\mathcal{O}$. For elements $x \in E$ let $\bar{x} = \{x, x^*\}$, and let $\bar{A} = \{\bar{x} | x \in A\}, A \subseteq E$, and $\bar{\mathcal{C}} = \{\bar{C} | C \in \mathcal{C}\}$. The system $\bar{\mathcal{C}}$ satisfies the usual matroid circuit exchange axioms, so $\bar{\mathcal{O}} = (\bar{E}, \bar{\mathcal{C}})$ is a matroid, called the *underlying matroid* of $\mathcal{O}$. Not all matroids arise from oriented matroids in this way; those that do are called *orientable*. A subset $B \subseteq E$ is called a *basis* of $\mathcal{O}$ if $\bar{B}$ is a basis of $\bar{\mathcal{O}}$. The *rank* of $\mathcal{O}$ equals the rank of $\bar{\mathcal{O}}$. See Bland and LasVergnas (1978) and Folkman and Lawrence (1978) for more information about oriented matroids.

Two topics from the theory of oriented matroids will be discussed in this section. Most important is the topological representation theorem of Folkman and Lawrence (1978), which states that every oriented matroid can be realized by an arrangement of pseudospheres. As an application we show how such realizations lead to quick proofs of some combinatorial properties of rank 3 oriented matroids. However, we begin by sketching (following LasVergnas (1978)) how Maurer's Homotopy Theorem (6.7) can be used to deduce the existence of a determinantal sign function.

The fundamental models for oriented matroids are sets of vectors in $\mathbb{R}^d$ and the relation of positive linear dependence (or more generally, positive linear dependence of vectors over *any* ordered field). Suppose that E is a finite subset of $\mathbb{R}^d - \{0\}$ such that $E = -E$, and for $x \in E$ let $x^* = -x$. A subset $A \subseteq E$ is *positive - linearly dependent* if $\Sigma_{x \in A} \lambda_x x = 0$ for some real coefficients $\lambda_x \geq 0$, not all equal to zero. Let $\mathcal{C}$ be the family of all inclusionwise minimal positive - linearly dependent subsets of E, except those of the form $\{x, x^*\}, x \in E$. Equivalently, $\mathcal{C}$ consists of all subsets of E which form the vertex set of a simplex of dimension ≥ 2 containing the origin in its relative interior. Oriented matroids $(E, *, \mathcal{C})$ which arise in this way are called *linear* over $\mathbb{R}$. Not all oriented matroids are isomorphic to linear ones.

Just like ordinary matroids, oriented matroids can be characterized in several ways. We shall discuss a characteristic property of the set of bases $\mathcal{B}$ of an oriented matroid, namely that a determinant can be defined up to sign (but not magnitude).

Let us review some essential features of the function $\delta : \tilde{\mathcal{B}} \to \{+1, -1\}$, taking *ordered* bases of a *linear* oriented matroid $(E, *, \mathcal{C}), E \subseteq \mathbb{R}^d$, to the sign of their determinants. A function η can be defined for certain pairs of ordered bases β and β' in $\mathbb{R}^d$ as follows:

(7.1) Suppose β and β' are permutations of the same basis B. Let $\eta(\beta, \beta') = +1$ if they are of the same parity and $= -1$ otherwise.

(7.2) Suppose $\beta = x_1 x_2 \ldots x_{r-1} y$ and $\beta' = x_1 x_2 \ldots x_{r-1} z$ with $y \neq z$. Let $\eta(\beta, \beta') = +1$ if y and z are on the same side of the hyperplane spanned by $\{x_1, \ldots, x_{r-1}\}$, and $= -1$ otherwise.

Now, once we choose an ordered basis β_0 and put $\det(\beta_0) := +1$, the function $\det(\beta)$ and its sign $\delta(\beta)$ is determined for all ordered bases β by the usual rules of linear algebra. But the function $\delta(\beta)$ is also *combinatorially* determined, because any pair of ordered bases can be connected by a chain of steps of type (7.1) or (7.2) and we have: *If β and β' are ordered bases as in (7.1) or (7.2) then $\delta(\beta) = \eta(\beta, \beta') \cdot \delta(\beta')$.*

The preceding discussion points the way how to generalize the determinantal sign function to all oriented matroids. First, to cast (7.2) in a form which is more compatible with the axiom system $(OM1) - (OM3)$, we replace it by the following reformulation:

(7.2)' Suppose $\beta = x_1 x_2 \ldots x_{r-1} y$ and $\beta' = x_1 x_2 \ldots x_{r-1} z$ with $y \neq z$, and if $y \neq z^*$ let $\{C, C^*\}$ be the unique pair of circuits such that in the underlying matroid $\{\bar{y}, \bar{z}\} \subseteq \bar{C} \subseteq \{\bar{x}_1, \ldots, \bar{x}_r, \bar{y}, \bar{z}\}$. Put $\eta(\beta, \beta') = +1$ if one of y and z lies in C and the other in C^*, and put $\eta(\beta, \beta') = -1$ otherwise.

(7.3) **Theorem.** [LasVergnas (1978)]
Let $\tilde{\mathcal{B}}$ be the set of ordered bases of an oriented matroid, and let $\beta_0 \in \tilde{\mathcal{B}}$. There exists a unique function $\delta : \tilde{\mathcal{B}} \to \{+1, -1\}$ such that $\delta(\beta_0) = +1$ and if $\beta, \beta' \in \tilde{\mathcal{B}}$ are related as in (7.1) or (7.2)' then $\delta(\beta) = \eta(\beta, \beta') \cdot \delta(\beta')$.

The proof runs as follows. Define a graph on the vertex set $\tilde{\mathcal{B}}$ by connecting pairs $\{\beta, \beta'\}$ which are related as in (7.1) or (7.2)' by an edge. The graph is clearly connected, and there is a projection $\pi : \tilde{\mathcal{B}} \to \mathcal{B}$ to the basis graph $\mathcal{B}$ of the underlying matroid. Now, put $\delta(\beta_0) := +1$, and for $\beta \in \tilde{\mathcal{B}}$ define

$$\delta(\beta) := \Pi_{i=1}^{n} \eta(\beta_{i-1}, \beta_i)$$

for some choice of path $\beta_0, \beta_1, \ldots, \beta_n = \beta$ in $\tilde{\mathcal{B}}$. The proof is complete once we show that this definition is independent of the choice of path from β_0 to β. If P_1 and P_2 are two such paths then by Theorem (6.7) their projections $\pi(P_1)$ and $\pi(P_2)$ in the basis graph differ by a sequence of elementary homotopies. Thus the checking is reduced to verifying

$$\Pi_{i=1}^{k} \eta(\alpha_{i-1}, \alpha_i) = 1$$

for closed paths $\alpha_0, \alpha_1, \ldots, \alpha_k = \alpha_0$ in $\tilde{\mathcal{B}}$ whose projection in $\mathcal{B}$ is an edge BCB, triangle $BCDB$ or square $BCDEB$. However, the basis configurations which give triangles or squares in the basis graph are explicitly characterized in (6.3) – (6.5), and this way the checking is brought down to a manageable size. See LasVergnas (1978) for the details.

To pave the way for the representation theorem for oriented matroids it is best to again look at the linear case for motivation. The representation theorem in fact says that any intuition gained from the linear case is going to be essentially correct (modulo some topological deformation which cannot be too bad) for general oriented matroids.

Let E be a finite subset of $\mathbb{R}^d - \{0\}$ such that $E = -E$, and let $\mathcal{O} = (E, *, \mathcal{C})$ be the linear oriented matroid as previously discussed. For each $e \in \bar{E} = \{\bar{x} = \{x, x^*\} | x \in E\}$, let H_e be the hyperplane orthogonal to the line spanned by e. The *arrangement of hyperplanes* $\mathcal{H} = \{H_e | e \in \bar{E}\}$ contains all information about $\mathcal{O}$, since one can go from H_e back to a pair of opposite normal vectors, and the definition of the sets which form circuits in $\mathcal{O}$ (i.e., the sets in $\mathcal{C}$) is independent of the length of vectors. By intersecting with the sphere $\mathbb{S}^{d-1} = \{y \in \mathbb{R}^d | \parallel y \parallel = 1\}$ we can alternatively look at the *arrangement of spheres* $\mathcal{S} = \{H_e \cap \mathbb{S}^{d-1} | e \in \bar{E}\}$, which is merely a collection of equatorial $(d-2)$-spheres inside the $(d-1)$-sphere. Clearly: *linear oriented matroids*, *arrangements of hyperplanes* and *arrangements of spheres* are the same thing.

When thinking about a linear oriented matroid $(E, *, \mathcal{C})$ as an arrangement of spheres it is useful to visualize elements $x \in E$ as closed hemispheres $\bar{H}_x = \{y \in \mathbb{S}^{d-1} | (y, x) \geq 0\}$. Then a subset $A \subseteq E$ belongs to $\mathcal{C}$ if and only if $A \cap A^* = \emptyset$ and A is minimal such that $\cup_{x \in A} \bar{H}_x = \mathbb{S}^{d-1}$.

We shall need the following terminology. Denote by $S^{j,0} = \{y \in \mathbb{S}^j | y_{j+1} = 0\}$ the standard equatorial subsphere of the standard sphere $\mathbb{S}^j = \{y \in \mathbb{R}^{j+1} | \parallel y \parallel = 1\}$, and let $S^{j,+} = \{y \in \mathbb{S}^j | y_{j+1} \geq 0\}$ and $S^{j,-} = -S^{j,+}$ denote the two closed hemispheres. A *sphere* S is a topological space for which there is a homeomorphism $f : \mathbb{S}^j \to S$ with the standard j-sphere, for some $j \geq 0$. A *pseudosphere* S' *in* S is any image $S' = f(S^{j,0})$ of the equatorial subsphere under such a homeomorphism. The pseudosphere S' is the intersection of its two *sides* $(S')^+ = f(S^{j,+})$ and $(S')^- = f(S^{j,-})$, which are homeomorphic to balls.

The crucial definition is this: An *arrangement of pseudospheres* $(\bar{E}, \mathcal{S})$ is a finite collection $\mathcal{S} = \{S_e | e \in \bar{E}\}$ of pseudospheres S_e in $\mathbb{S}^{d-1}$ such that

$(AP1)$ Every nonempty intersection $S_A = \cap_{e \in A} S_e$, $A \subseteq \bar{E}$, is a sphere.

$(AP2)$ For every nonempty intersection S_A and all $e \in \bar{E}$, either $S_A \subseteq S_e$ or $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

This definition is due to Folkman and Lawrence (1978). They actually required more, but the additional assumptions in their definition were proved to be redundant by Mandel (1982).

In analogy with the linear case (arrangement of spheres), an arrangement of pseudospheres $(\bar{E}, \mathcal{S})$ gives rise to an oriented matroid $(E, *, \mathcal{C})$ as follows: put $E = \{S_e^+ | e \in \bar{E}\} \cup \{S_e^- | e \in \bar{E}\}$, let $(S_e^+)^* = S_e^-$ and vice versa, and define $\mathcal{C}$ to be the collection of the minimal subsets $A \subseteq E$ such that $\cup A = \mathbb{S}^{d-1}$ and $A \cap A^* = \emptyset$. This way of constructing oriented matroids is surprisingly general:

(7.4) **Representation Theorem.** [Folkman and Lawrence (1978)]
Every oriented matroid of rank d arises from some arrangement of pseudospheres in S^{d-1} in the described fashion.

The proof of this result is fairly involved. First a poset is constructed from the oriented matroid, and then it is shown, for instance by using (12.6), that this poset is the poset of faces of some regular cell complex C. This complex C provides the $(d-1)$-sphere and various subcomplexes the $(d-2)$-subspheres forming the arrangement. The sphere C is constructible (Edmonds and Mandel (1978), Mandel (1982)), and even shellable (Lawrence (1984)), which implies that the whole construction of C and the relevant subcomplexes can be carried out in piecewise linear topology. In particular, this means that no topological pathologies need to be dealt with in representations of oriented matroids. Complete proofs of (7.4) can be found in Folkman and Lawrence (1978), Mandel (1982), and Björner, LasVergnas, Sturmfels, White and Ziegler (1988).

The representation theorem shows that all oriented matroids of rank 3 come from arrangements of "pseudocircles" on the 2-sphere. This representation can be used for quick proofs of some combinatorial properties as in the following application, which is due to Edmonds and Mandel (1978) and .... The first part is a generalization of the Sylvester-Gallai theorem (cf. Chapter ...), since a finite configuration of points in $\mathbb{R}^2$ is the same thing as a *linear* orientable matroid of rank 3.

(7.5) **Theorem.** [Edmonds and Mandel (1978),...]
Let M be an orientable matroid of rank 3. Then:
(*i*) M has a 2-point line,
(*ii*) if the points of M are 2-colored there exists a monochromatic line.

Here is how this result follows from (7.4). Represent the points of M as pseudocircles on the 2-sphere. Then lines are maximal collections of pseudocircles with nonempty intersection (which is necessarily a 0-sphere, i.e., two points). The arrangement of pseudocircles gives a graph G whose vertices are the points of intersection and edges the segments of pseudocircles between such points. Since this graph lies embedded in S^2 it is planar, and since $rk(M) = 3$ it is simple. We need the following lemma.

(7.6) **Lemma.** For any planarly embedded simple graph:
(*i*) some vertex has degree at most five,
(*ii*) if the edges are 2-colored then there exists a vertex around which the edges of each color class are adjacent.

Part (*i*) is a well-known consequence of Euler's formula (cf. Chapter ...). Part (*ii*) is also a consequence of Euler's formula, but perhaps not as well known. It was used by Cauchy in the proof of his Rigidity Theorem for 3-dimensional convex polytopes.

Now look at the graph G determined by the arrangement of pseudocircles. If all lines in M have at least 3 points, then every vertex in G will have degree at least 6, in violation of (*i*). If the pseudocircles are 2-colored and through every intersection point there is at least one pseudocircle of each color, then the induced coloring of the edges of G will violate (*ii*).

8. Tutte's Homotopy Theorem

A matroid is called *regular* if it can be coordinatized over every field. In Tutte (1958) a characterization is given of regular matroids in terms of forbidden minors. The proof relies in an essential way on a "Homotopy Theorem", expressing the 1-connectivity of certain 2-dimensional complexes. Tutte's Homotopy Theorem was also used by R. Reid (ca. 1971, unpublished) and Bixby (1979) to prove the forbidden minor characterization for representability over $GF(3)$. Both results (i.e., the characterizations of regular and

$GF(3)$ matroids) where also proven by Seymour (1979) using a different method which avoids use of the Homotopy Theorem.

Tutte's Homotopy Theorem seems to be the oldest topological result of its kind in combinatorics. Unfortunately it is quite technical both to state in full and to prove. Complete details can be found in Tutte (1958) and Tutte (1965).

In this section we shall state the Homotopy Theorem in sufficient detail to explain the way that it was used by Tutte to derive the characterization of regular matroids. Familiarity with basic matroid theory will be assumed. See Chapter ... for this material.

Let L be a finite geometric lattice of rank r, and write L^i for the set of flats of rank i; so L^{r-1} is the set of copoints, L^{r-2} the colines and L^{r-3} the coplanes. Flats $X \in L$ will be thought of as subsets of the point set L^1 via $\bar{X} = \{p \in L^1 | p \leq X\}$.

Given any point $a \in L^1$ we define a graph $TG(L,a)$ on the vertex set $L^{r-1}_{\not\geq a} = \{X \in L^{r-1} | a \not\leq X\}$ as follows: two copoints X and Y "off a" (i.e., in the set $L^{r-1}_{\not\geq a}$) span an edge if $X \wedge Y$ is a coline and $\bar{X} \cup \bar{Y} \neq L^1 - a$. On this graph we construct a 2-dimensional regular cell complex $TC(L,a)$ by attaching 2-cells into the triangles and squares of the following kinds:

(8.1) Triangles $XYZX$ for which $rk(X \wedge Y \wedge Z) \geq r-3$.

(8.2) Squares $XYZTX$ for which $rk(P) = r-3$, where $P = X \wedge Y \wedge Z \wedge T$, and *either* the coline $P \vee a$ is covered by exactly two copoints *or else* the interval $[P, \hat{1}]$ is isomorphic to the lattice of flats of the Fano matroid F_7 minus one of its points.

If L has no minor isomorphic to F_7^*, the dual of the Fano matroid, then (8.1) and (8.2) describe all the 2-cells of the *Tutte complex* $TC(L,a)$. This means that for the uses in representation theory, and hence also for the purposes of this exposition, the definition (8.1)–(8.2) of $TC(L,a)$ is sufficient. In general it is necessary to attach 2-cells also into certain squares $XYZTX$ for which $rk(X \wedge Y \wedge Z \wedge T) = r-4$. The definition of these squares (of the "corank 4 kind") is fairly complicated, so we refrain from describing them here.

(8.3) **Homotopy Theorem.** [Tutte (1958)]
The complex $TC(L,a)$ is 1-connected.

This statement of the Homotopy Theorem differs in form but not in content from the statement in Tutte (1958). Tutte has remarked about his theorem (in Tutte (1979), p. 446) that "the proof ... is long, but it is purely graph-theoretical and geometrical in nature. I am rather surprised that it seems to have acquired a reputation for extreme difficulty." No significant simplification of the original proof seems to be known, other than in special cases. One such case is if $\bar{X} \cup \bar{Y} \neq L^1 - a$ for all pairs X, Y of copoints "off a" such that $X \wedge Y$ is a coline. Then the top three levels of $L - [a, \hat{1}]$ form a poset which is 1-connected by (10.10) (*iv*) and (10.13), and the 1-connectivity can be transferred to $TC(L,a)$ by an application of the Fiber Theorem (11.1), similar to the proof of Theorem (6.6). A simpler and more conceptual proof of Tutte's theorem in full strength would be of definite interest.

The combinatorial meaning of the Homotopy Theorem is that any two copoints X and Y "off a" can be connected "off a" by a path in the Tutte graph $TG(L,a)$, and that any two such paths differ by a sequence of *elementary homotopies* of type $XYX, XYZX$ as in (8.1), or $XYZTX$ as in (8.2) or of the corank 4 kind. (Compare the discussion preceding Theorem (6.7).)

Let us now have a look at how this is used in representation theory.

(8.4) **Theorem.** [Tutte (1958)]
A matroid is regular if and only if it contains no minor isomorphic to the 4-point line U_2^4, the Fano plane F_7 or its dual F_7^*.

The difficult direction is to prove sufficiency of the exclusion of these three minors. The following coordinatizability criterion is used: A matroid $M = (E, \mathcal{I})$ is regular if and only if for every copoint X of the lattice of flats there is a function $f_X : E \to \{0, +1, -1\}$ such that $f_X^{-1}(0) = \bar{X}$ and for every triple X_1, X_2, X_3 of copoints whose intersection is a coline there exists $\gamma_1, \gamma_2, \gamma_3 \in \{+1, -1\}$ such that $\gamma_1 f_{X_1} + \gamma_2 f_{X_2} + \gamma_3 f_{X_3} = 0$.

The proof that any matroid with no minor of type U_2^4, F_7 or F_7^* is regular proceeds by induction on the size of the ground set. Suppose that $M = (E, \mathcal{I})$ is such a matroid and let L be its lattice of flats. Pick any point $a \in E = L^1$ and let $M' = M - a$ be the restriction of M to $E - a$ and L' its lattice of flats. By induction M' is regular and hence for all copoints $Y \in L'$ there exist functions $f'_Y : E - a \to \{0, +1, -1\}$ with the required properties. We may assume that a is not an isthmus, so rank L = rank L' (the isthmus case is easily handled by simpler means). The crux now is to construct good functions f_X for all $X \in L^{r-1}$ from the given functions $f'_Y, Y \in (L')^{r-1}$. We will describe here only how this is done when $X \not\geq a$, since this is where the Homotopy Theorem is used.

If $X \in L^{r-1}_{\not\geq a}$ then $X \in (L')^{r-1}$, so a function f'_X is already defined on all of $E - a$. Put $f_X := f'_X$ on $E - a$, and define $f_X(a)$ as follows.

First, define a function $t(X, Y)$ from edges $\{X, Y\}$ of the Tutte graph $TG(L, a)$ to $\{+1, -1\}$ by putting $t(X, Y) = f_X(p) \cdot f_Y(p)$ for any $p \in E - (\bar{X} \cup \bar{Y} \cup a)$. By the definition of $TG(L, a)$ the set $E - (\bar{X} \cup \bar{Y} \cup a)$ is nonempty, and one shows that the existence of two elements p and q in this set with $f_X(p) \cdot f_Y(p) \neq f_X(q) \cdot f_Y(q)$ implies the presence of a U_2^4 minor.

Next, pick some $X_0 \in L^{r-1}_{\not\geq a}$ and put $f_{X_0}(a) := 1$. For every other $X \in L^{r-1}_{\not\geq a}$, choose some path $X_0, X_1, \ldots, X_n = X$ in the graph $TG(L, a)$ and put

$$f_X(a) := \Pi_{i=1}^{n} t(X_{i-1}, X_i). \tag{8.5}$$

By Theorem (8.3) such paths exist and any two of them differ by a sequence of elementary homotopies. Hence, to verify that the definition (8.5) is independent of the choice of path from X_0 to X it suffices to check that

$$\Pi_{i=1}^{k} t(X_{i-1}, X_i) = 1$$

for all closed paths $X_0, X_1, \ldots, X_k = X_0, k = 2, 3, 4$, corresponding to the elementary homotopies. This brings the verification down to a few manageable cases, and it is here that the assumption about no F_7 or F_7^* minors is used.

Having sketched how the Homotopy Theorem is used to make definition (8.5) possible, our immediate aim with this section is accomplished and we leave the rest of the proof of Theorem (8.4) aside.

Part II. TOOLS

The rest of this chapter is devoted to a review of some definitions and results from combinatorial topology that have proven to be particularly useful in combinatorics. The material in Sections 9 (simplicial complexes), 12 (cell complexes) and 13 (Borsuk's theorem) is of a very general nature and detailed treatments can be found in many topology books. Specific references will therefore be given only sporadically. Most topics in Sections 10 and 11, on the other hand, are of a more special nature, and more substantial references (and even some proofs) will be given.

Many of the results mentioned have been discussed in a large number of papers and books. When relevant, our policy has been to reference the original source (when known to us) and some more recent papers that contribute simple proofs, extensions or up-to-date discussion (a subjective choice). We apologize for any inaccuracy or omission that may unintentionally have occured.

9. Combinatorial Topology

This section will review basic facts concerning simplicial complexes. Good general references are Munkres (1984a) and Spanier (1966). Basic notions such as *(topological) space*, *continuous map* and *homeomorphism* will be considered known.

Simplicial complexes and posets

(9.1) An *(abstract) simplicial complex* $\Delta = (V, \Delta)$ is a set V (the *vertex set*) together with a family Δ of nonempty finite subsets of V (called *simplices* or *faces*) such that $\emptyset \neq \sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$. Usually, $V = \cup\Delta$ (shorthand for $V = \cup_{\sigma\in\Delta}\sigma$) so V can be suppressed from the notation. The *dimension* of a face σ is $\dim \sigma = \text{card}\ \sigma - 1$, the *dimension* of Δ is $\dim \Delta = \max_{\sigma\in\Delta} \dim \sigma$. A d-dimensional complex is *pure* if every face is contained in a d-*face* (i.e., d-dimensional face).

Let $\Delta^k = \{k\text{-faces of } \Delta\}$ and $\Delta^{\leq k} = \cup_{j\leq k}\Delta^j$, for $k \geq 0$. The elements of $\Delta^0 = V$ and Δ^1 are called *vertices* and *edges*, respectively. If Δ is pure d-dimensional the elements of Δ^d are called *facets* (or *chambers*). $\Delta^{\leq k}$ is the k-*skeleton* of Δ. It is a *subcomplex* of Δ.

A (*geometric*) *simplicial complex* is a polyhedral complex in $\mathbb{R}^d$ (in the sense of (12.1)) whose cells are geometric simplices (the convex hull of affinely independent point-sets). If Γ is a geometric simplicial complex then the family of extreme-point-sets of cells in Γ form an abstract simplicial complex $\Delta(\Gamma)$ which is finite. Conversely, if Δ is a d-dimensional finite abstract simplicial complex then there exist geometric simplicial complexes Γ in $\mathbb{R}^{2d+1}$ such that $\Delta(\Gamma) \cong \Delta$. The underlying space $\cup\Gamma$ of any such Γ, unique up to linear homeomorphism, is called the *geometric realization* (or *space*) of Δ, denoted by $|\Delta|$. Thus, abstract and geometric simplicial complexes are equivalent notions in the finite case (to be precise, when finite-dimensional, denumerable and locally finite). The geometric realization $|\Delta|$ of arbitrary infinite abstract simplicial complexes Δ can be constructed as in Spanier (1966).

A *simplicial map* $f : \Delta_1 \to \Delta_2$ is a mapping $f : \Delta_1^0 \to \Delta_2^0$ such that $f(\sigma) \in \Delta_2$ for all $\sigma \in \Delta_1$. By affine extension across simplices it induces a continuous map $|f| : |\Delta_1| \to |\Delta_2|$.

(9.2) Let $P = (P, \leq)$ be a *poset* (partially ordered set). A totally ordered subset $x_0 < x_1 < \ldots < x_k$ is called a *chain* of *length* k. The supremum of this number over all chains in P is the *rank* (or *length*) of P. If all maximal chains have the same finite length then P is *pure*. P is a *lattice* if every pair of elements $x, y \in P$ has a least upper bound (*join*) $x \vee y$ and a greatest lower bound (*meet*) $x \wedge y$.

For $x \in P$, let $P_{\geq x}, P_{>x}, P_{\leq x}, P_{<x}$ be defined by $P_{\geq x} = \{y \in P : y \geq x\}$, etc.. For $x \leq y$ define the *open interval* $(x, y) = P_{>x} \cap P_{<y}$ and the *closed interval* $[x, y] = P_{\geq x} \cap P_{\leq y}$. A *bottom element* $\hat{0}$ and a *top element* $\hat{1}$ in P are elements satisfying $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$. If both $\hat{0}$ and $\hat{1}$ exist, P is *bounded*. Then $\bar{P} = P - \{\hat{0}, \hat{1}\}$ denotes the *proper part* of P. For arbitrary poset $P, \hat{P} = P \cup \{\hat{0}, \hat{1}\}$ denotes P extended by new top and bottom elements (so, card $(\hat{P} \backslash P) = 2$).

Let P be a pure poset of rank r. For $x \in P$, let $r(x) =$ rank $(P_{\leq x})$. The *rank function* $r : P \to \{0, 1, \ldots, r\}$ is bijective on each maximal chain. It decomposes P into *rank levels* $P^i = \{x \in P : r(x) = i\}, 0 \leq i \leq r$.

(9.3) The *face poset* $P(\Delta) = (\Delta, \subseteq)$ of a simplicial complex Δ is the set of faces ordered by inclusion. The *face lattice* of Δ is $\hat{P}(\Delta) = P(\Delta) \cup \{\hat{0}, \hat{1}\}$. It is a lattice. $P(\Delta)$ is pure iff Δ is pure, and rank $P(\Delta) = \dim \Delta$.

The *order complex* $\Delta(P)$ of a poset P is the simplicial complex on vertex set P whose k-faces are the k-chains $x_0 < x_1 < \ldots < x_k$ in P. A poset map $f : P_1 \to P_2$ which is *order-preserving* ($x \leq y$ implies $f(x) \leq f(y)$) or *order-reversing* ($x \leq y$ implies $f(x) \geq f(y)$) is simplicial $f : \Delta(P_1) \to \Delta(P_2)$, and therefore induces a continuous map $|f| : |\Delta(P_1)| \to |\Delta(P_2)|$.

For a simplicial complex Δ, let $sd\Delta = \Delta(P(\Delta))$, called the *(first) barycentric subdivision* (due to its geometric version). A basic fact is that Δ and $sd\Delta$ are homeomorphic. Therefore, passage between simplicial complexes and posets via the mappings $P(\cdot)$ and $\Delta(\cdot)$ does not affect the topology, and from a topological point of view simplicial complexes and posets can be considered essentially equivalent notions.

The geometric realization $|P| = |\Delta(P)|$ associates a topological space with every poset P. In this chapter, whenever we make topological statements about a poset P we have the space $|P|$ in mind. [There exists at least one other way of associating an interesting topology with a poset P; namely, let the *order-ideals* (sets A satisfying $x \leq y \in A$ implies $x \in A$) in P be the open sets of a topology on P. In this *ideal topology* the continuous maps are precisely the order-preserving maps and "homotopy" has a combinatorial meaning. See Stong (1966). The ideal topology is also relevant for sheaf cohomology over posets (Baclawski (1975), Yuzvinsky (1987)) and has surprising connections with the order complex topology (McCord (1966)).]

(9.4) Let T be a topological space, $\approx$ an equivalence relation on T, and $\pi : T \to T/\approx$ the projection map. The quotient $T/\approx$ is made into a topological space by letting $A \subseteq T/\approx$ be open iff $\pi^{-1}(A)$ is open in T. If $S_i, i \in I$, are pairwise disjoint subsets of T, then $T/(S_i)_{i \in I}$ denotes the *quotient space* obtained by identifying the points within each set $S_i, i \in I$. E.g., cone $(T) = T \times [0,1]/(T \times \{1\})$ is the *cone over* T, and susp $(T) = T \times [0,1]/(T \times \{0\}, T \times \{1\})$ is the *suspension* of T. The d-ball modulo its boundary is homeomorphic to the d-sphere: $\mathbf{B}^d/\mathbf{S}^{d-1} \cong \mathbf{S}^d$.

If $(T_i, x_i)_{i \in I}, x_i \in T_i$, is a family of pointed pairwise disjoint spaces, then the *wedge* of this family is $\cup_{i \in I} T_i / (\cup_{i \in I}\{x_i\})$. The *join* of two spaces T_1 and T_2 is the space $T_1 * T_2 = T_1 \times T_2 \times [0,1]/(\{(t,x,0) | x \in T_2\}, \{(y,s,1) | y \in T_1\})_{t \in T_1, s \in T_2}$.

The *join* of two simplicial complexes Δ_1 and Δ_2 (with $\Delta_1^0 \cap \Delta_2^0 = \emptyset$) is the complex $\Delta_1 * \Delta_2 = \Delta_1 \cup \Delta_2 \cup \{\sigma \cup \tau | \sigma \in \Delta_1 \text{ and } \tau \in \Delta_2\}$. Further, the *cone over* Δ and *suspension* of Δ are the complexes cone $(\Delta) = \Delta * \Gamma_1$, susp $(\Delta) = \Delta * \Gamma_2$, where Γ_i is the 0-dimensional complex with i vertices, $i = 1, 2$. There is a homeomorphism

(9.5) $$|\Delta_1 * \Delta_2| \cong |\Delta_1| * |\Delta_2|.$$

(In case Δ_1 and Δ_2 are not locally finite the topology of the right-hand side may need to be modified to the associated compactly generated topology, see Walker (1988).) In particular, $|\text{ cone }(\Delta)| \cong \text{cone }(|\Delta|)$ and $|\text{ susp }(\Delta)| \cong \text{susp }(|\Delta|)$.

The *direct product* $P \times Q$ of two posets is the cartesian product set ordered by $(x,y) \le (x',y')$ if $x \le x'$ in P and $y \le y'$ in Q. The *join* (or *ordinal sum*) $P * Q$ of two posets is their disjoint union ordered by making each element of P earlier than each element of Q and otherwise keeping the given orderings whithin P and Q. Clearly, $\Delta(P * Q) = \Delta(P) * \Delta(Q)$.

There are the following homeomorphisms (Quillen (1978), Walker (1988)):

(9.6) $$|P \times Q| \cong |P| \times |Q|,$$

(9.7) $$|(P \times Q)_{>(x,y)}| \cong |P_{>x}| * |Q_{>y}|.$$

(9.8) $$|((x,y),(x',y'))| \cong \text{ susp }(|(x,x')| * |(y,y')|), \text{ if } x < x' \text{ in } P \text{ and } y < y' \text{ in } Q.$$

(Again, special care has to be taken with the topology of the right-hand sides if the participating order complexes are not locally finite.)

(9.9) Let Δ be a simplicial complex and $\sigma \in \Delta \cup \{\emptyset\}$. Then define the subcomplexes: *deletion* $dl_\Delta(\sigma) = \{\tau \in \Delta | \tau \cap \sigma = \emptyset\}$, *star* $st_\Delta(\sigma) = \{\tau \in \Delta | \tau \cup \sigma \in \Delta\}$ and *link* $lk_\Delta(\sigma) = \{\tau \in \Delta | \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$. Clearly, $dl(\sigma) \cap st(\sigma) = lk(\sigma)$ and $\sigma * lk(\sigma) = st(\sigma)$. If $\sigma \in \Delta^0$ then also $dl(\sigma) \cup st(\sigma) = \Delta$; and $dl(\emptyset) = st(\emptyset) = lk(\emptyset) = \Delta$.

Homotopy and homology

(9.10) Two mappings $f_0, f_1 : T_1 \to T_2$ of topological spaces are *homotopic* (written $f_0 \sim f_1$) if there exists a mapping (called a *homotopy*) $F : T_1 \times [0,1] \to T_2$ such that $F(t,0) = f_0(t)$ and $F(t,1) = f_1(t)$ for all $t \in T_1$. (All mappings between topological spaces are assumed to be continuous.) The spaces T_1 and T_2 are of the *same homotopy type* (or *homotopy equivalent*) if there exist mappings $f_1 : T_1 \to T_2$ and $f_2 : T_2 \to T_1$ such that $f_2 \circ f_1 \sim id_{T_1}$ and $f_1 \circ f_2 \sim id_{T_2}$. Denote this by $T_1 \simeq T_2$. A space which is homotopy equivalent to a point is called *contractible*.

Let $\mathbf{S}^{d-1} = \{x \in \mathbb{R}^d | \|x\| = 1\}$ and $\mathbf{B}^d = \{x \in \mathbb{R}^d | \|x\| \le 1\}$ denote the standard $(d-1)$-*sphere* and d-*ball*, respectively. Note that $\mathbf{S}^{-1} = \emptyset, \mathbf{S}^0 = \{ \text{ two points } \}$ and $\mathbf{B}^0 = \{ \text{ point } \}$. A space T is k-*connected* if for $0 \le i \le k$ each mapping $f : \mathbf{S}^i \to T$ can be extended to a mapping $\hat{f} : \mathbf{B}^{i+1} \to T$ such that $\hat{f}(x) = f(x)$ for all $x \in \mathbf{S}^i$. In particular, 0-connected means arcwise connected. The property of being k-connected is a *homotopy invariant* (i.e., is transferred to other spaces of the same homotopy type). $\mathbf{S}^d$ is $(d-1)$-connected but not d-connected, $\mathbf{B}^d$ is contractible. It is convenient to define the following degenerate cases: (-1)-*connected* means "nonempty", and every space (whether empty or not) is k-*connected* for $k \le -2$.

A simplicial complex Δ is contractible iff Δ is k-connected for all $k \ge 0$ (or, for all $0 \le k \le \dim \Delta$). The corresponding statement for general spaces is false in the nontrivial direction. Furthermore, a simplicial complex is k-connected iff its $(k+1)$-skeleton is k-connected.

Let $\pi_i(T) = \pi_i(T,x)$ denote the set of homotopy classes of maps $f : \mathbf{S}^i \to T$ such that $f((1,0,\ldots,0)) = x$, from the i-sphere to a pointed topological space $(T,x), x \in T$, $i \geq 0$. For $i \geq 1$ there exists a composition that makes $\pi_i(T)$ into a group, the *i-th homotopy group* of T (at the point x). For $i \geq 2$, the group $\pi_i(T)$ is Abelian. $\pi_1(T)$ is the *fundamental group*, and T is *simply connected* if $\pi_1(T) = 0$. The space T is k-connected iff $\pi_i(T,x) = 0$ for all $0 \leq i \leq k$ and $x \in T$. So, 1-connected means simply connected and arcwise connected.

(9.11) For the definitions of *simplicial homology groups* $H_i(\Delta, G)$ and *reduced simplicial homology groups* $\tilde{H}_i(\Delta, G)$ of a complex Δ with coefficients in an Abelian group G, we refer to Munkres (1984a) or Spanier (1966).

Let $\tilde{H}_i(\Delta) = \tilde{H}_i(\Delta, Z)$. The degenerate case

$$\tilde{H}_i(\emptyset) = \begin{cases} Z & , \quad i = -1, \\ 0 & , \quad i \neq -1, \end{cases}$$

should be noted. For $\Delta \neq \emptyset$, $\tilde{H}_i(\Delta) = 0$ for all $i < 0$ and all $i > \dim \Delta$, and $\tilde{H}_0(\Delta) = Z^{c-1}$, where c is the number of connected components of Δ. $H_i(\Delta) = \tilde{H}_i(\Delta)$ for all $i \neq -1, 0, H_{-1}(\Delta) = 0$ and $H_0(\Delta) = \tilde{H}_0(\Delta) \oplus Z$.

For a finite simplicial complex Δ let $\beta_i =$ rank $H_i(\Delta) = \dim_Q H_i(\Delta, Q), i \geq 0$. The *Betti numbers* β_i satisfy the *Euler-Poincaré formula*

$$\Sigma_{i \geq 0}(-1)^i \text{ card } (\Delta^i) = \Sigma_{i \geq 0}(-1)^i \beta_i. \tag{9.12}$$

Either side of (9.12) can be taken as the definition of the *Euler characteristic* $\chi(\Delta)$. The *reduced Euler characteristic* is $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$. Formula (9.12) is valid with $\beta_i = \dim_{\mathbf{k}} H_i(\Delta, \mathbf{k})$ for an arbitrary field $\mathbf{k}$, although the individual integers β_i may depend on $\mathbf{k}$. Additional relations exist between the face-count numbers $f_i =$ card (Δ^i) and the Betti numbers β_i (Björner and Kalai (1988)). Much is known about the *f-vectors* $f(\Delta) = (f_0, f_1, \ldots)$ for various special classes of complexes Δ, see Chapter XX.

Two complexes of the same homotopy type have isomorphic homology groups in all dimensions. A complex Δ is *k-hom-connected* (over G) if $\tilde{H}_i(\Delta, G) = 0$ for all $i \leq k$. So, (-1)-*hom-connected* means nonempty and 0-*hom-connected* means nonempty and connected. Further, Δ is *G-acyclic* if $\tilde{H}_i(\Delta, G) = 0$ for all $i \in Z$.

We now list some relations between homotopy properties and homology of a complex Δ, which are frequently useful. They are consequences of the theorems of Hurewicz and Whitehead (Spanier (1966)).

(9.13) Δ is k-connected iff Δ is k-hom-connected (over Z) and simply connected, $k \geq 1$.

(9.14) Δ is contractible iff Δ is Z-acyclic and simply connected.

(9.15) If Δ is simply connected, $\tilde{H}_i(\Delta) = 0$ for $i \neq d > 1$, and $\tilde{H}_d(\Delta) \cong Z^k$, then Δ is homotopy equivalent to a wedge of k d-spheres.

(9.16) Assume $\dim \Delta = d \geq 0$. Then Δ is $(d-1)$-connected iff Δ is homotopy equivalent to a wedge of d-spheres.

The analogues of (9.14)–(9.16) may fail for non-triangulable spaces.

The following is a consequence of Alexander duality on the $(n-2)$-sphere. For combinatorial applications of Poincaré duality and the related hard Lefschetz theorem, see Stanley (1980).

(9.17) **Combinatorial Alexander Duality.** Let $\Delta \subseteq 2^V, V \notin \Delta$, be a simplicial complex, card $V = n$, and let $\Delta^* = \{S \in 2^V \backslash \{\emptyset, V\} | V \backslash S \notin \Delta\}$. (Note that $\Delta^{**} = \Delta$.) Let $\mathbf{k}$ be a field. Then $\tilde{H}_i(\Delta, \mathbf{k}) \cong \tilde{H}_{n-3-i}(\Delta^*, \mathbf{k})$, for all $i \in Z$.

We end this section with some very useful elementary lemmas

Suppose Δ is a simplicial complex and T a space. Let $C : \Delta \to 2^T$ be order-preserving (i.e., $C(\sigma) \subseteq C(\tau) \subseteq T$, for all $\sigma \subseteq \tau$ in Δ). A mapping $f : |\Delta| \to T$ is *carried* by C if $f(|\sigma|) \subseteq C(\sigma)$ for all $\sigma \in \Delta$. Let $k \in Z_+ \cup \{\infty\}$.

(9.18) **Carrier Lemma.** Assume that $C(\sigma)$ is $\min(k, \dim(\sigma))$-connected for all $\sigma \in \Delta$. Then:
(i) if $f, g : |\Delta^{\leq k}| \to T$ are both carried by C, then $f \sim g$,
(ii) there exists a mapping $|\Delta^{\leq k+1}| \to T$ carried by C.

In particular, if $C(\sigma)$ is always contractible then $|\Delta|$ can replace the skeleta in (i) and (ii) ($k = \infty$ case). Carrier lemmas of various kinds are common in topology. For proofs of this version, see Lundell and Weingram (1969) or Walker (1981b).

(9.19) **Contractible Subcomplex Lemma.** If Δ_0 is a contractible subcomplex of a simplicial complex Δ, then the projection map $|\Delta| \to |\Delta|/|\Delta_0|$ is a homotopy equivalence.

This is a consequence of the homotopy extension property for simplicial pairs (see Brown (1968) and Björner and Walker (1983)).

(9.20) **Gluing Lemmas.** Examples of simple gluing lemmas for simplicial complexes Δ_1 and Δ_2 are:
(i) if Δ_1 and $\Delta_1 \cap \Delta_2$ are contractible, then $\Delta_1 \cup \Delta_2 \simeq \Delta_2$,
(ii) if Δ_1 and Δ_2 are k-connected and $\Delta_1 \cap \Delta_2$ $(k-1)$-connected, then $\Delta_1 \cup \Delta_2$ is k-connected,
(iii) if $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are k-connected, then so are also Δ_1 and Δ_2.

Such results are often special cases of the theorems in Section 11, especially (11.2). Otherwise they can be deduced from the Mayer-Vietoris long exact sequence and the Seifert-van Kampen theorem.

A general principle for gluing homotopies appears in Brown (1968), p. 240, and Mather (1966). It gives a convenient proof for part (i) of the following lemma. For the remaining parts use (9.19) and (9.20).

(9.21) **Lemma.** Let $\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$ be a simplicial complex, and assume $\Delta_i \cap \Delta_j \subseteq \Delta_0$ for all $1 \leq i < j \leq n$.
(i) If Δ_i is contractible, $1 \leq i \leq n$, then

$$\Delta \simeq \Delta_0 \cup \cup_{i=1}^{n} \text{ cone } (\Delta_0 \cap \Delta_i).$$

(I.e., raise a cone independently over each subcomplex $\Delta_0 \cap \Delta_i$.)
(ii) If Δ_i is contractible, $0 \leq i \leq n$, then

$$\Delta \simeq \text{ wedge }_{1 \leq i \leq n} \text{ susp } (\Delta_0 \cap \Delta_i).$$

(iii) If Δ_i is k-connected and $\Delta_0 \cap \Delta_i$ $(k-1)$-connected, $0 \leq i \leq n$, then Δ is k-connected.

(*iv*) If Δ and $\Delta_0 \cap \Delta_i$ are k-connected, $1 \le i \le n$, then all Δ_i are k-connected, $0 \le i \le n$.

10. Special Complexes

Some classes of complexes which are frequently encountered in combinatorics will be reviewed.

Collapsible and shellable complexes

(10.1) Let Δ be a simplicial complex, and suppose that $\sigma \in \Delta$ is a proper face of exactly one simplex $\tau \in \Delta$. Then the complex $\Delta' = \Delta\backslash\{\sigma,\tau\}$ is obtained from Δ by an *elementary collapse* (and Δ is obtained from Δ' by an *elementary anticollapse*). Note that $\Delta' \simeq \Delta$. If Δ can be reduced to a single point by a sequence of elementary collapse steps, then Δ is *collapsible.*

The class of *nonevasive* complexes is recursively defined as follows: (i) a single vertex is nonevasive, (ii) if for some $x \in \Delta^0$ both $lk_\Delta(x)$ and $dl_\Delta(x)$ are nonevasive, then so is Δ.

The following logical implications are strict (i.e., converses are false):

cone $\Longrightarrow$ nonevasive $\Longrightarrow$ collapsible $\Longrightarrow$ contractible $\Longrightarrow$ Z-acyclic.

(Furthermore, for an arbitrary field $\mathbf{k}$: Z-acyclic $\Longrightarrow$ k-acyclic $\Longrightarrow$ Q-acyclic $\Longrightarrow$ $\tilde{\chi} = 0$.)

Nonevasive complexes were defined by Kahn, Saks and Sturtevant (1984) to model the notion of argument complexity discussed in Section 2. Collapsibility has long been studied in combinatorial topology. Noteworthy is the fact that two simply connected finite complexes Δ and Δ' are homotopy equivalent iff a sequence of elementary collapses and elementary anticollapses can transform Δ into Δ' (see Cohen (1973)). In particular, the contractible complexes are precisely the complexes that collapse/anticollapse to a point.

An element x in a poset P is *irreducible* if $P_{>x}$ has a least element or $P_{<x}$ a greatest element. A finite poset is *dismantlable* if successive removal of irreducibles leads to a single-element poset. A dismantlable poset is nonevasive. [By Stong (1966), "dismantlable" is equivalent to "contractible" in the *ideal topology* mentioned in (9.3).] A *directed* poset (for all $x, y \in P$ there exists $z \in P$ such that $x, y \le z$) is contractible.

(10.2) Let Δ be a pure d-dimensional simplicial complex, and suppose that the k-face σ is a subset of exactly one d-face τ. Then the complex $\Delta' = \Delta\backslash\{\gamma|\sigma \subseteq \gamma \subseteq \tau\}$ is obtained from Δ by a (k,d)-*collapse.* If $\sigma \neq \tau$, then $\Delta' \simeq \Delta$. If Δ can be reduced to a single d-simplex by a sequence of (k,d)-collapses, $0 \le k \le d$, then Δ is *shellable.*

A pure simplicial complex Δ is *vertex-decomposable* if either (i) $\Delta = \emptyset$, (ii) Δ consists of a single vertex, or (iii) for some $x \in \Delta^0$ both $lk_\Delta(x)$ and $dl_\Delta(x)$ are vertex-decomposable. E.g., every simplex and simplex-boundary is vertex-decomposable. The class of *constructible* complexes is defined by: (i) every simplex and $\emptyset$ is constructible, (ii) if Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are constructible and $\dim \Delta_1 = \dim \Delta_2 = 1 + \dim \Delta_1 \cap \Delta_2$, then $\Delta_1 \cup \Delta_2$ is constructible.

The following logical implications among these properties of a pure d-dimensional complex are strict:

vertex-decomposable $\Longrightarrow$ shellable $\Longrightarrow$ constructible $\Longrightarrow$ $(d-1)$-connected.

The first implication as well as the definition of vertex-decomposable complexes is due to Provan and Billera (1980). The concept of shellability has an interesting history going back to the 19th century, see Grünbaum (1967). Constructible complexes were defined by M. Hochster, see Stanley (1977).

Shellability is usually regarded as a way of *putting together* (rather than collapsing — taking apart) a complex. Therefore the following alternative definition is more common: A finite pure d-dimensional complex Δ is *shellable* if its d-faces can be ordered $\sigma_1, \sigma_2, \ldots, \sigma_t$ so that $(\delta\sigma_1 \cup \ldots \cup \delta\sigma_{k-1}) \cap \delta\sigma_k$ is a pure $(d-1)$-dimensional complex for $2 \leq k \leq t$, where $\delta\sigma_j = 2^{\sigma_j} \backslash \{\emptyset\}$. In words, the requirement is that the k-th facet σ_k intersects the union of the preceding ones along a part of its boundary which is a union of maximal proper faces of σ_k. Such an ordering of the facets is called a *shelling*.

If $\sigma \in \Delta$ and Δ is a shellable (or constructible) complex, then so is $lk_\Delta(\sigma)$. Shellability is also preserved by some other constructions on complexes and posets such as (10.13). Several basic properties of simplicial shellability (also for infinite complexes) are reviewed in Björner (1984b). Shellability of cell complexes is discussed in Danaraj and Klee (1974) and Björner (1984a). To establish shellability of (order complexes of) posets, a special method exists called *lexicographic shellability*. See Björner, Garsia and Stanley (1982) or Björner and Wachs (1983) for details.

(10.3) Simplicial PL spheres and PL balls are defined in (12.2), (PL = piecewise linear). The property of being PL is a combinatorial property — whether a geometric simplicial complex Δ is PL depends only on the abstract simplicial complex Δ. For PL topology see Hudson (1969).

For showing that specific complexes are homeomorphic to spheres or balls, the following result is frequently useful.

(10.4) **Theorem.** Let Δ be a constructible d-dimensional simplicial complex.
(*i*) If every $(d-1)$-face is contained in exactly two d-faces, then Δ is a PL sphere.
(*ii*) If every $(d-1)$-face is contained in one or two d-faces, both cases occurring, then Δ is a PL ball.

(10.4) follows from some basic PL topology (see Hudson (1969), p. 39), and should probably be considered a folk theorem. For shellable Δ it appears in Bing (1965) and Danaraj and Klee (1974).

If Δ is a triangulation of the d-sphere and $\sigma \in \Delta^k$, then $lk_\Delta(\sigma)$ has the same homology as the $(d-1-k)$-sphere. If $\sigma \in \Delta^0$, then there is even homotopy equivalence between $lk_\Delta(\sigma)$ and S^{d-1}. However, if Δ is a PL d-sphere and $\sigma \in \Delta^k$, then $lk_\Delta(\sigma)$ is itself a PL $(d-1-k)$-sphere.

Cohen-Macaulay complexes

(10.5) Let $\mathbf{k}$ be a field or the ring of integers Z. A finite-dimensional simplicial complex Δ is *Cohen-Macaulay over* $\mathbf{k}$ (written $CM/\mathbf{k}$ or CM if $\mathbf{k}$ is understood or irrelevant) if $lk_\Delta(\sigma)$ is $(\dim lk_\Delta(\sigma) - 1)$-hom-connected over $\mathbf{k}$ for all $\sigma \in \Delta \cup \{\emptyset\}$. Further, Δ is *homotopy-Cohen-Macaulay* if $lk_\Delta(\sigma)$ is $(\dim lk_\Delta(\sigma) - 1)$-connected for all $\sigma \in \Delta \cup \{\emptyset\}$.

The following implications are strict:

$$\text{constructible} \Longrightarrow \text{homotopy-}CM \Longrightarrow CM/Z \Longrightarrow CM/\mathbf{k} \Longrightarrow CM/Q,$$

for an arbitrary field $\mathbf{k}$ of characteristic $\neq 0$. The first implication follows from the fact that constructibility implies $(d-1)$-connectivity and is inherited by links, the second implication follows from (9.13), and the rest via the Universal Coefficient Theorem. In particular, shellable complexes are CM.

An important aspect of *finite CM* complexes Δ is that they have an equivalent ring-theoretic definition. Suppose that $\Delta^0 = \{x_1, x_2, \dots, x_n\}$, and consider the ideal I in the polynomial ring $\mathbf{k}[x_1, x_2, \dots, x_n]$ generated by monomials $x_{i_1} x_{i_2} \dots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \notin \Delta, 1 \le i_1 < i_2 < \dots < i_k \le n, k \ge 1$. Let $\mathbf{k}[\Delta] = \mathbf{k}[x_1, \dots, x_n]/I$, called the *Stanley-Reisner ring* (or *face ring*) of Δ. Then Δ is $CM/\mathbf{k}$ iff the ring $\mathbf{k}[\Delta]$ is Cohen-Macaulay in the sense of commutative algebra (Reisner (1976)). An exposition of the ring-theoretic aspects of simplicial complexes, and their combinatorial use, can be found in Stanley (1983). There other ring-theoretically motivated classes of complexes, such as *Gorenstein complexes* and *Buchsbaum complexes*, are also discussed. Other approaches to the ring-theoretic aspects of complexes and to Reisner's theorem can be found in Baclawski and Garsia (1981) and Yuzvinsky (1987).

Cohen-Macaulay complexes and posets were introduced around 1974–75 in the work of Baclawski (1976) (1980), Hochster (1977), Reisner (1976) and Stanley (1975) (1977). The notion of homotopy-CM first appeared in Quillen (1978). Björner, Garsia and Stanley (1982) give an elementary introduction to CM posets. A notable combinatorial application of Cohen-Macaulayness is Stanley's proof of tight upper bounds for the number of faces that can occur in each dimension for triangulations with n vertices of the d-sphere (Stanley (1975), (1983)). An application to lower bounds is given in Stanley (1987).

(10.6) Define a pure d-dimensional complex Δ to be *strongly connected* (or *dually connected*) if each pair of facets $\sigma, \tau \in \Delta^d$ can be connected by a sequence of facets $\sigma = \sigma_0, \sigma_1, \dots, \sigma_n = \tau$, so that $\dim(\sigma_{i-1} \cap \sigma_i) = d - 1$ for $1 \le i \le n$. *Every CM complex is pure and strongly connected.* This follows from the following lemma, which is proved by induction on $\dim \Delta$.

(10.7) **Lemma.** Let Δ be a finite-dimensional simplicial complex, and assume that $lk_\Delta(\sigma)$ is connected for all $\sigma \in \Delta \cup \{\emptyset\}$ such that $\dim(lk_\Delta(\sigma)) \ge 1$. Then Δ is pure and strongly connected.

The property of being CM is *topologically invariant*: whether Δ is $CM/\mathbf{k}$ or not depends only on the topology of $|\Delta|$. This is implied by the following reformulation of CMness, due to Munkres (1984b).

(10.8) **Theorem.** A finite-dimensional complex Δ is $CM/\mathbf{k}$ iff its space $T = |\Delta|$ satisfies: $\tilde{H}_i(T; \mathbf{k}) = H_i(T, T \backslash p; \mathbf{k}) = 0$ for all $p \in T$ and $i < \dim \Delta$.

In this formulation $\tilde{H}_i$ denotes reduced singular homology and H_i relative singular homology with coefficients in $\mathbf{k}$. A consequence of (10.8) is that if M is a triangulable manifold (with or without boundary) and $\tilde{H}_i(M) = 0$ for $i < \dim M$, then every triangulation of M is CM. For instance: (1) every triangulation of the d-sphere, d-ball or $\mathbb{R}^d$ is CM/Z, but not necessarily homotopy-CM (beware: homotopy-CM is *not* topologically invariant), (2) a triangulation of real projective d-space is $CM/\mathbf{k}$ iff char $\mathbf{k} \ne 2$.

(10.9) The definition of Cohen-Macaulay posets (posets P such that $\Delta(P)$ is CM) deserves a small comment. Let P be a poset of finite rank and $\sigma : x_0 < x_1 < \dots < x_k$ a chain in P. Then $lk_{\Delta(P)}(\sigma) = P_{<x_0} * (x_0, x_1) * \dots * (x_{k-1}, x_k) * P_{>x_k}$. For arbitrary complexes Δ_1 and Δ_2, if Δ_1 is k_1-(hom)-connected and Δ_2 k_2-(hom)-connected then $\Delta_1 * \Delta_2$ is $(k_1 + k_2 + 2)$-(hom)-connected. Therefore, P is CM (resp. homotopy-CM) iff every open interval (x, y) in $\hat{P}$ is (rank $(x, y) - 1$)-hom-connected (resp. (rank $(x, y) - 1$)-connected).

(10.10) An abundance of shellable and CM simplicial complexes appear in combinatorics. Only a few examples can be mentioned here.

(*i*) The boundary complex of a simplicial convex polytope is shellable (Bruggesser and Mani (1971), Danaraj and Klee (1974)). Every simplicial PL sphere is the boundary of a shellable ball (Pachner (1986)). There exist non-shellable triangulations of the 3-ball (Rudin (1958)) and of the 5-sphere (see below). Shellability of spheres and balls is surveyed in Danaraj and Klee (1978).

(*ii*) The following implications are valid for any simplicial sphere: constructible $\Longrightarrow$ $PL \Longrightarrow$ homotopy-CM. The 5-sphere admits triangulations that are non-homotopy-CM (Edwards (1975)), and also PL triangulations that are non-constructible (Mandel (1982)).

(*iii*) The complex of independent sets in a matroid is constructible (Stanley (1977)) and vertex-decomposable (Provan and Billera (1980)). More generally, the complex generated by the basis-complements is a greedoid is vertex-decomposable (Björner, Korte and Lovász (1985)).

(*iv*) Every semimodular (in particular, every geometric or modular) lattice of finite rank is CM (Folkman (1966)) and shellable (Björner (1980)). For any element x in a geometric lattice L, the poset $L\backslash[x,\hat{1}]$ is shellable (Wachs and Walker (1986)).

(*v*) Tits buildings are CM (Solomon-Tits, see Solomon (1969)) and shellable (Björner (1984b)). The topology of more general group geometries has been studied by M. Ronan, S. Smith, J. Tits and others with a view to uses in group theory. See Smith (1985) for information and further references.

(*vi*) The poset of elementary Abelian p-subgroups of a finite group was shown by Quillen (1978) to be homotopy-CM in some cases. The full subgroup lattice of a finite group G is shellable (or CM) iff G is supersolvable (Björner (1980)). Various posets of subgroups have been studied from a topological point of view by K. Brown, J. Thévenaz and others. See Thévenaz (1987) for a guide to this literature.

Induced subcomplexes

Connectivity, Cohen-Macaulayness, etc., is under certain circumstances inherited by suitable subcomplexes. For a simplicial complex Δ and $A \subseteq \Delta^0$, let $\Delta_A = \{\sigma \in \Delta \mid \sigma \subseteq A\}$ (the *induced subcomplex* on A).

(10.11) **Lemma.** Let Δ be a finite-dimensional complex, and $A \subseteq V = \Delta^0$. Assume that $lk_\Delta(\sigma)$ is k-connected for all $\sigma \in \Delta_{V\backslash A}$. Then Δ_A is k-connected iff Δ is k-connected.

(10.12) **Lemma.** Let P be a poset of finite rank and A a subset. Assume that $P_{>x}$ is k-connected for all $x \in P\backslash A$. Then A is k-connected iff P is k-connected.

Proof. These lemmas are equivalent. We start with (10.12). Let $f : A \to P$ be the embedding map. For $x \in P$,

$$f^{-1}(P_{\geq x}) = \begin{cases} A_{\geq x} & , \text{ if } x \in A, \\ P_{>x} \cap A & , \text{ if } x \notin A. \end{cases}$$

Now, $A_{\geq x}$ is contractible (being a cone), and $P_{>x} \cap A$ is k-connected by induction on rank (P). The result therefore follows by (11.1) (*ii*).

To prove (10.11), let $P = P(\Delta)$ and $Q = \{\tau \in \Delta \mid \tau \cap A \neq \emptyset\} \subseteq P$. Since $P_{>\sigma} \cong P(lk_\Delta(\sigma))$ is k-connected for all $\sigma \in P \backslash Q$, (10.12) applies. On the other hand, by (11.8) the map $f(\tau) = \tau \cap A$ on Q induces homotopy equivalence between Q and $f(Q) = P(\Delta_A)$. □

The homology versions of (10.11) and (10.12), obtained by using k-hom-connectivity throughout, can be proven by a parallel method. Also, if the hypothesis "k-connected" were replaced by "contractible" in (10.11) and (10.12), then the conclusion would be that Δ_A and Δ (resp. A and P) are homotopy equivalent.

(10.13) **Type-selection Theorem.** Let Δ be a pure simplicial complex, $A \subseteq \Delta^0$ and $m \in \mathbb{N}$. Suppose that card $(A \cap \sigma) = m$ for every facet $\sigma \in \Delta$. If Δ is $CM/\mathbf{k}$, homotopy-CM or shellable, then the same property is inherited by Δ_A.

For CMness this result was proven in varying degrees of generality by Baclawski (1980), Munkres (1984b), Stanley (1979) and Walker (1981a). It follows at once from (10.11). For shellability, proofs appear in Björner (1980) (1984b).

11. Combinatorial Homotopy Theorems

In this section we collect some tools for manipulating homotopies and the homotopy type of complexes, which have proven to be useful in combinatorics.

Some of these results concern *simplicial* maps $f : \Delta \to P$ from a simplicial complex Δ to a poset P. Such a map sends vertices of Δ to elements of P in such a way that each $\sigma \in \Delta$ is mapped to a chain in P. In particular, an order-preserving or order-reversing mapping of posets $Q \to P$ is of this type.

(11.1) **Fiber Theorem.** [Quillen (1978), Walker (1981b)]
Let $f : \Delta \to P$ be a simplicial map from a simplicial complex Δ to a poset P.
(i) Suppose all fibers $f^{-1}(P_{\geq x}), x \in P$, are contractible. Then f induces homotopy equivalence between Δ and P.
(ii) Suppose all fibers $f^{-1}(P_{\geq x}), x \in P$, are k-connected. Then Δ is k-connected if and only if P is k-connected.

Proof. Suppose that all fibers are contractible. Then the mapping $C(\sigma) = f^{-1}(P_{\geq \min \sigma})$, $\sigma \in \Delta(P)$, is a contractible carrier from $\Delta(P)$ to $|\Delta|$. By Lemma (9.18) (ii) there exists a continuous map $g : \Delta(P) \to \Delta$ carried by C, i.e., $g(|\sigma|) \subseteq |f^{-1}(P_{\geq \min \sigma})|$, for every chain $\sigma \in \Delta(P)$. One sees that g is a homotopy inverse to f as follows, using (9.18) (i): $C'(\sigma) = |P_{\geq \min \sigma}|, \sigma \in \Delta(P)$, is contractible and carries $f \circ g$ and id_P, and $C''(\pi) = |f^{-1}(P_{\geq \min f(\pi)})|, \pi \in \Delta$, is contractible and carries $g \circ f$ and id_Δ. Hence, $f \circ g \sim id_P$ and $g \circ f \sim id_\Delta$.

The second part is proved analogously by passing to $(k+1)$-skeleta and using k-connected carriers in Lemma (9.18). □

The *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $\mathcal{N} = \mathcal{N}(A_i)$ defined on the vertex set I so that a finite subset $\sigma \subseteq I$ is in $\mathcal{N}$ precisely when $\cap_{i \in \sigma} A_i \neq \emptyset$.

(11.2) **Nerve Theorem.** [Borsuk (1948), Björner, Korte and Lovász (1985)]
Let Δ be a simplicial complex (or, a regular cell complex) and $(\Delta_i)_{i \in I}$ a family of subcomplexes such that $\Delta = \cup_{i \in I} \Delta_i$.
(i) Suppose every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_t}$ is contractible. Then Δ and the nerve $\mathcal{N}(\Delta_i)$ are homotopy equivalent.
(ii) Suppose every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_t}$ is $(k-t+1)$-connected. Then Δ is k-connected if and only if $\mathcal{N}(\Delta_i)$ is k-connected.

Proof. For convenience, assume that the covering of Δ by the Δ_i's is *locally finite*, meaning that each vertex of Δ belongs to only finitely many subcomplexes Δ_i. (The case of more general coverings requires a slightly different argument.)

Let $Q = P(\Delta)$ and $P = P(\mathcal{N}(\Delta_i))$ be the face posets. Define a mapping $f : Q \to P$ by $\pi \longmapsto \{i \in I | \pi \in \Delta_i\}$. Clearly f is order-reversing, so $f : \Delta(Q) \to P$ is simplicial. The fiber at $\sigma \in P$ is $f^{-1}(P_{\geq \sigma}) = \cap_{i \in \sigma} \Delta_i$. Part (i) now follows from Theorem (11.1). Also, if all nonempty finite intersections are k-connected, part (ii) follows the same way. In the stated generality, part (ii) can be proven by the method used for Theorem 4.10 in Björner, Korte and Lovász (1985). □

The Nerve Theorem has several versions for coverings of a topological space by subspaces. The earliest of these seem to be due to Leray (1945) and Weil (1952). Discussions of results of this kind can be found in Wu (1962) and McCord (1967). We state here a version which seems suitable for possible use in combinatorics. An application to oriented matroids appears in Edelman (1984).

(11.3) **Nerve Theorem.** [Weil (1952), Wu (1962), McCord (1967)]
Let X be a triangulable space and $(A_i)_{i \in I}$ a locally finite family of open subsets (or a *finite* family of closed subsets) such that $X = \cup_{i \in I} A_i$. If every nonempty intersection $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_t}$ is contractible, then X and the nerve $\mathcal{N}(A_i)$ are homotopy equivalent.

By *locally finite* is meant that each point of X lies in at most finitely many sets A_i. We warn that Theorem (11.3) is false for locally finite coverings by *closed* sets and also for too general spaces X. For a counterexample in the first case, take X to be the unit circle and $A_i = \{e^{2\pi i t} | \frac{1}{i+1} \leq t \leq \frac{1}{i}\}, i = 1, 2, \ldots$. In the second case one can e.g. let X be the wedge of two topologist's combs A_1 and A_2 (as in Spanier (1966),Ex. 5, p. 56).

The conclusions in part (ii) of Theorems (11.1) and (11.2) can be strengthened: In (11.1), if all fibers are k-connected, then f induces isomorphisms of homotopy groups $\pi_i(\Delta) \cong \pi_i(P)$, for all $i \leq k$. Consequently, if in (11.2) all nonempty finite intersections $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_t}$ are k-connected, then $\pi_i(\Delta) \cong \pi_i(\mathcal{N}(\Delta_i))$, for all $i \leq k$. A similar k-connectivity version of Theorem (11.3) appears in Wu (1962).

Let P be a poset. A subset $C \subseteq P$ is called a *crosscut* if (1) C is an antichain, (2) for every finite chain σ in P there exists some element in C which is comparable to each element in σ, (3) if $A \subseteq C$ is *bounded* (i.e., has an upper bound or a lower bound in P) then the join $\vee A$ or the meet $\wedge A$ exists in P. For instance, the atoms of a lattice L of finite length form a crosscut in L and in $\bar{L}$.

A crosscut C in P determines the simplicial complex $\Gamma(P, C)$ consisting of the bounded subsets of C.

(11.4) **Crosscut Theorem.** [Rota (1964), Folkman (1966), Lakser (1971), Björner (1981)]
The crosscut complex $\Gamma(P, C)$ and P are homotopy equivalent.

Proof. For $x \in C$, let $\Delta_x = \Delta(P_{\leq x} \cup P_{\geq x})$. Then $(\Delta_x)_{x \in C}$ is a covering of $\Delta(P)$, by condition (2), and every nonempty intersection is a cone, by condition (3), and hence contractible. Since $\Gamma(P, C) = \mathcal{N}(\Delta_x)$, Theorem (11.2) implies the result. □

(11.5) **Bipartite Relation Theorem.**[Dowker (1952), Mather (1966)]
Suppose $G = (V_0, V_1, E), E \subseteq V_0 \times V_1$, is a bipartite graph, and let $\Delta_i, i = 0, 1$, be the simplicial complex whose faces are all finite subsets $\sigma \subseteq V_i$ which have a common neighbor in V_{1-i}. Then Δ_0 and Δ_1 are homotopy equivalent.

Proof. First delete any isolated vertices from G. This does not affect Δ_0 and Δ_1. Now, for every $x \in V_1$ let Δ_x consist of all finite subsets of $\{y \in V_0 | (y,x) \in E\}$. Then $(\Delta_x)_{x \in V_1}$ is a covering of Δ_0 with contractible nonempty intersections. The nerve of this covering is Δ_1, so Theorem (11.2) applies. □

Theorems (11.2) (i), (11.4) and (11.5) are equivalent in the sense that either one implies the other two. The following is a variation of the Fiber Theorem (11.1).

(11.6) **Ideal Relation Theorem.** [Quillen (1978)]
Let P and Q be posets and suppose that $R \subseteq P \times Q$ is a relation such that $(x,y) \leq (x',y') \in R$ implies that $(x,y) \in R$. (I.e., R is an order ideal in the product poset.) Suppose furthermore that $R_x = \{y \in Q | (x,y) \in R\}$ and $R_y = \{x \in P | (x,y) \in R\}$ are contractible for all $x \in P$ and $y \in Q$. Then P and Q are homotopy equivalent.

Proof. By symmetry if suffices to show that P and R are homotopy equivalent. By Theorem (11.1) it suffices for this to show that the fiber $\pi^{-1}(P_{\geq x})$ is contractible for all $x \in P$, where $\pi : R \to P$ is the projection map $\pi(x,y) = x$. Let $F_x = \pi^{-1}(P_{\geq x}) = \{(z,y) \in R | z \geq x\}$, and let $\rho : F_x \to R_x$ be the projection $\rho(z,y) = y$. Now, $\rho^{-1}((R_x)_{\geq y}) = \{(z,w) \in F_x | w \geq y\} = \{(z,w) \in R | (z,w) \geq (x,y)\}$ is a cone and hence contractible, for all $y \in R_x$. So by the Fiber Theorem F_x is homotopic to R_x, which by assumption is contractible. (Remark: There is an obvious k-connectivity version also of this result.) □

(11.7) **Order Homotopy Theorem.**[Quillen (1978)]
Let $f, g : \Delta \to P$ be simplicial maps from a simplicial complex Δ to a poset P. If $f(x) \leq g(x)$ for every vertex x of Δ, then f and g are homotopic.

Proof. For each face $\sigma \in \Delta$, let $C(\sigma) = f(\sigma) \cup g(\sigma)$. The minimal element in the chain $f(\sigma)$ is below every other element in $C(\sigma)$. So the order complex of $C(\sigma)$ is a cone, and hence contractible. Since $C(\,.\,)$ carries both f and g, these maps are homotopic by Lemma (9.18). □

(11.8) **Corollary.** Let $f : P \to P$ be an order-preserving map such that $f(x) \geq x$ for all $x \in P$. Then f induces homotopy equivalence between P and $f(P)$.

If also $f^2(x) = f(x)$ for all $x \in P$ (f is a *closure operator* on P) then $f(P)$ is a strong deformation retract of P. The hypotheses of (11.7) and (11.8) can be weakened to that $f(x)$ and $g(x)$ [resp., $f(x)$ and x] are comparable for all x.

Call a poset P *join-contractible* (*via* p), if for some element $p \in P$ the join (least upper bound) $p \vee x$ exists for all $x \in P$. Define *meet-contractible* in dual fashion.

(11.9) **Corollary.** [Quillen (1978)]
If P is join-contractible then P is contractible.

Proof. Since $x \leq p \vee x \geq p$, for all $x \in P$, Theorem (11.7) shows that $id \sim p \vee id \sim p$, i.e., the identity map on P is homotopic to the constant map p. □

(11.10) **Lemma.**[Björner and Walker (1983)]Let $f, g : \Delta \to P$ be simplicial maps. Suppose for each pair of vertices (x,y) such that $\{x,y\} \in \Delta$ that either $\{f(x), g(y)\}$ has an upper bound u in P such that the meet $u \wedge z$ exists for all $z \geq f(x)$, or $\{f(x), g(y)\}$ has a lower bound ℓ in P such that the join $\ell \vee z$ exists for all $z \leq g(y)$. Then f and g are homotopic.

Proof. For $\sigma \in \Delta$, let $C(\sigma) = \{z \in P | z \geq \min f(\sigma)\} \cup \{z \in P | z \leq \max g(\sigma)\}$. If $x, y \in \sigma$ are such that $f(x) = \min f(\sigma)$ and $g(y) = \max g(\sigma)$, and $u, \ell \in P$ are as in the statement of the lemma, one sees that either $C(\sigma)$ is meet-contractible via u or else $C(\sigma)$ is join-contractible via ℓ. So, C is contractible and carries f and g. By Lemma (9.18) the conclusion follows. ▫

The set of complements $Co(z)$ of an element z in a bounded lattice L is defined in Section 3. Recall that $\bar{L} = L - \{\hat{0}, \hat{1}\}$.

(11.11) **Homotopy Complementation Theorem.** [Björner (1981), Walker (1981b), Björner and Walker (1983)]
Let L be a bounded lattice and $z \in \bar{L}$.
(*i*) The poset $\bar{L} - Co(z)$ is contractible. In particular, if L is noncomplemented then $\bar{L}$ is contractible.
(*ii*) If $Co(z)$ is an antichain, then

$$\bar{L} \simeq \underset{y \in Co(z)}{\text{wedge}} \text{ susp } (\bar{L}_{<y} * \bar{L}_{>y}).$$

Proof. For part (*i*) it suffices to observe that the constant map $f(x) = z$ and the identity map $g(y) = y$ on the poset $P = \bar{L} - Co(z)$ satisfy the hypothesis of Lemma (11.10). Part (*ii*) then follows by (9.21) (*ii*). ▫

Suppose that L is a bounded lattice whose proper part is not contractible. Then by part (*i*) every element x has a complement in L. This conclusion can be strengthened in at least two ways: (1) [Lovász and Schrijver (unpublished)]Every chain $x_0 < x_1 < \ldots < x_k$ in $\bar{L}$ has a complementing chain $y_0 \geq y_1 \geq \ldots \geq y_k$ (i.e., $x_i \perp y_i$ for $0 \leq i \leq k$); (2) [Björner (1981)]Every element $x \in \bar{L}$ has a complement which is a join of atoms (assuming atoms exist).

12. Cell Complexes

Most classes of cell complexes differ from the simplicial case in that a purely combinatorial description of these objects *as such* cannot be given. However, the two classes defined here, polyhedral complexes and regular CW complexes, are sufficiently close to the simplicial case to allow a similar combinatorial approach in many cases. For simplicity only *finite* complexes will be considered.

Good general references for polyhedral complexes are Grünbaum (1967) and Hudson (1969), and for cell complexes Lundell and Weingram (1969). Cell complexes are discussed in many books on algebraic topology such as Munkres (1984a) and Spanier (1966).

Polyhedral complexes

(12.1) A *convex polytope* π is a bounded subset of $\mathbb{R}^d$ which is the solution set of a finite number of linear equalities and inequalities. Any nonempty subset obtained by changing some of the inequalities to equalities is a *face* of π. Equivalently, $\pi \subseteq \mathbb{R}^d$ is a convex polytope iff π is the convex hull of a finite set of points in $\mathbb{R}^d$. See Chapter XX (Polytopal complexes and their relatives) for more information about convex polytopes.

A *polyhedral complex* (or *convex cell complex*) Γ is a finite collection of convex polytopes in $\mathbb{R}^d$ such that (*i*) if $\pi \in \Gamma$ and σ is a face of π then $\sigma \in \Gamma$, and (*ii*) if $\pi, \tau \in \Gamma$ and $\pi \cap \tau \neq \emptyset$ then $\pi \cap \tau$ is a face of both π and τ. The members of Γ are called *cells*. The *underlying space* of Γ is $|\Gamma| = \cup\Gamma$, with the topology induced as a subset of $\mathbb{R}^d$. If every cell in Γ is a *simplex* (the convex hull of an affinely independent set of points) then Γ is called a (*geometric*) *simplicial complex*. The *dimension* of a cell equals the linear dimension of its affine span, and $\dim \Gamma = \max_{\pi \in \Gamma} \dim \pi$. Further terminology, such as *vertices, edges, facets, pure, k-skeleton, face poset, face lattice*, etc., is defined just as in the simplicial case, see (9.1) and (9.3).

(12.2) A polyhedral complex Γ_1 is a *subdivision* of another such complex Γ_2 if $|\Gamma_1| = |\Gamma_2|$ and every cell of Γ_1 is a subset of some cell of Γ_2. The abstract simplicial complex $\Delta(P(\Gamma))$, i.e., the order complex of Γ's face poset, has geometric realizations (by choosing as new vertices an interior point in each cell) that subdivide Γ. Every polyhedral complex can also be simplicially subdivided without introducing new vertices.

Let Σ^d denote the complex consisting of a geometric d-simplex and all its faces, and let $\delta\Sigma^d$ denote its boundary. These complexes provide the simplest triangulations of the d-ball and the $(d-1)$-sphere, respectively. A polyhedral complex Γ is called a *PL d-ball* (or *PL $(d-1)$-sphere*) if it admits a subdivision whose face poset is isomorphic to the face poset of some subdivision of Σ^d (resp. $\delta\Sigma^d$). This is equivalent to saying that there exists a homeomorphism $|\Gamma| \to |\Sigma^d|$ (resp. $|\Gamma| \to |\delta\Sigma^d|$) which is induced by a simplicial map (a *piece-wise linear*, or *PL, map*). The boundary complex of a convex d-polytope is a PL $(d-1)$-sphere. PL balls and spheres enjoy several good properties that are not shared by general polyhedral decompositions of the d-ball and $(d-1)$-sphere and that make them favorable to work with. See also (10.3) and (10.4).

Regular cell complexes

(12.3) By "cell complex" we will here understand what in topology is usually called a "finite CW complex".

Let X be a Hausdorff space. A subset σ is called an *open d-cell* if there exists a mapping $f : \mathbf{B}^d \to X$ whose restriction to the interior of the d-ball is a homeomorphism $f : \text{Int}(\mathbf{B}^d) \to \sigma$. The dimension $\dim\sigma = d$ is well-defined by this. The closure $\bar{\sigma}$ is the corresponding *closed cell*. It is true that $f(B^d) = \bar{\sigma}$, but $\bar{\sigma}$ is not necessarily homeomorphic to $\mathbf{B}^d$. We write $\dot{\sigma} = \bar{\sigma} \backslash \sigma$.

A *cell complex* C is a finite collection of pairwise disjoint sets together with a Hausdorff topology on their union $|C| = \cup C$ such that:

(*i*) each $\sigma \in C$ is an open cell in $|C|$, and
(*ii*) $\dot{\sigma} \subseteq C^{<\dim\sigma}$ (the union of all cells in C of dimension less than $\dim\sigma$), for all $\sigma \in C$.

Then C is also called a *cell decomposition* of the space $|C|$. Furthermore, C is *regular* if each closed cell $\bar{\sigma}$ in C is homeomorphic to a ball.

The cell decomposition of the d-sphere into one 0-cell and one d-cell (a point and its complement in $\mathbf{S}^d$) is not regular. Every polyhedral complex is a regular cell complex (the interiors of the convex polytopes are the open cells). Regular cell complexes are more general than polyhedral complexes in several ways. For instance, it is allowed that the intersection of two closed cells can have nontrivial topological structure.

(12.4) From now on only regular cell complexes will be considered. Two particular properties make a regular complex C favorable from a combinatorial point of view (see Lundell and Weingram (1969) for proofs):

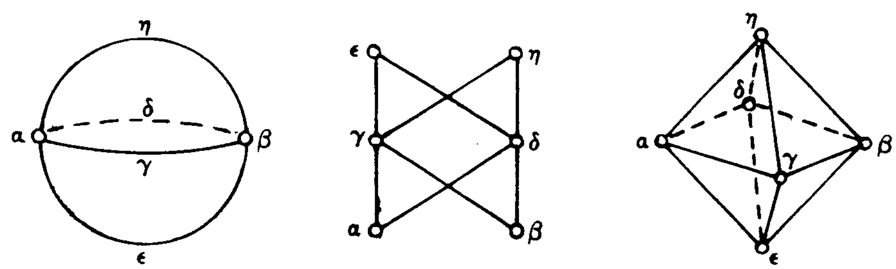
(*i*) The boundary $\dot{\sigma}$ of each cell $\sigma \in C$ is a union of lower-dimensional cells (a subcomplex). Hence, the situation resembles that of polyhedral complexes: each closed d-cell $\bar{\sigma}$ is homeomorphic to $\mathbf{B}^d$, and its boundary $\dot{\sigma}$ (homeomorphic to $\mathbf{S}^{d-1}$) has a regular cell decomposition provided by the cells that intersect $\dot{\sigma}$. Define the *face poset* $P(C)$ as the set of all closed cells ordered by containment. The poset rank in $P(C)$ of any $\sigma \in C$ equals $\dim\sigma$.
(*ii*) $|C| \cong |\Delta(P(C))|$, i.e., the order complex of $P(C)$ is homeomorphic to $|C|$. Geometrically, this means that regular cell complexes admit "barycentric subdivisions". From a

combinatorial point of view it means that regular cell complexes can be interpreted as a class of posets without any loss of topological information.

Because of (*i*), regular cell complexes can be characterized in the following way: A family of balls (homeomorphs of $B^d, d \geq 0$) in a Hausdorff space X is the set of closed cells of a regular cell complex iff the interiors of the balls partition X and the boundary of each ball is a union of other balls. This is what Mandel (1982) calls a "ball complex".

An important consequence of (*ii*) is that a d-dimensional regular cell complex C can always be "realized" in $\mathbb{R}^{2d+1}$ by a simplicial complex, so that every closed cell in C is a triangulated ball (a cone over a simplicial sphere). We will call C a *PL cell complex* (suppressing the word "regular") if every such ball is PL.

The figure shows a regular cell decomposition C of the 2-sphere, its face poset $P(C)$, and its simplicial representation $\Delta(P(C))$, where each original 2-cell is triangulated into 4 triangles, etc.



(12.5) Given a finite poset P, does there exist a regular cell complex (or even a polyhedral complex) C such that $P = P(C)$; and if so, what is its topology and how can C be constructed from P? This question is discussed in Björner (1984a) and Mandel (1982). One answer is that P is isomorphic to the face poset of some regular cell complex iff $\Delta(P_{<x})$ is homeomorphic to a sphere for all $x \in P$. However, since it has been proven that simplicial spheres cannot be recognized algorithmically this is not a fully satisfactory answer. The question of how to recognize the face posets of polyhedral complexes is one version of the Steinitz problem (see Chapter XX, Section 4).

For the cellular interpretation of posets the following result, derivable from (10.4), has proven useful in practice. See Björner (1984a) and Mandel (1982) for further details and other similar results. Let us call a poset P *thin* if every closed interval of rank 2 has 4 elements (two "in the middle"). Also, $P \cup \{\hat{0}\}$ will denote P with a new minimum element $\hat{0}$ adjoined, and $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ as usual.

(12.6) **Theorem.** Let P be a pure finite poset of rank d. Assume that $\Delta(P)$ is constructible.
(*i*) If $P \cup \{\hat{0}\}$ is thin, then $P \cong P(C)$ for some PL cell complex C homotopy equivalent to a wedge of d-spheres.
(*ii*) If $\hat{P}$ is thin, then $P \cong P(C)$ for some PL cell decomposition of the d-sphere.

13. Borsuk's Theorem

Let p be a prime. By a $\mathbf{Z}_p$*-space* we understand a pair (T, ν) where T is a topological space and $\nu : T \to T$ is a fixed-point free continuous mapping of order p (i.e., $\nu^p = id$). A mapping $f : T_1 \to T_2$ of $\mathbf{Z}_p$-spaces $(T_i, \nu_i), i = 1,2$, is *equivariant* if $\nu_2 \circ f = f \circ \nu_1$. A $\mathbf{Z}_2$-space is often called an *antipodality space*. The standard example is $(\mathbf{S}^d, \alpha)$, the d-sphere with its antipodal map $\alpha(x) = -x$.

Borsuk's Theorem has several applications in combinatorics (see Sections 4 and 5). We state five equivalent versions. Proofs and ramifications appear in many topology books, see e.g. Dugundji and Granas (1982). Steinlein (1985) gives an extensive survey of generalizations, applications and references.

(13.1) **Borsuk's Theorem.** [Borsuk (1933)]
(*i*) If $\mathbf{S}^d$ is covered by $d+1$ subsets, all closed or all open, then one of these must contain a pair of antipodal points. (Borsuk-Liusternik-Schnirelman)
(*ii*) For every continuous mapping $f : \mathbf{S}^d \to \mathbb{R}^d$ there exists a point x such that $f(x) = f(-x)$. (Borsuk-Ulam)
(*iii*) For every odd ($f(-y) = -f(y)$ for all y) continuous mapping $f : \mathbf{S}^d \to \mathbb{R}^d$ there exists x for which $f(x) = 0$. (Borsuk-Ulam)
(*iv*) There exists no equivariant map $\mathbf{S}^n \to \mathbf{S}^d$, if $n > d$.
(*v*) For any d-connected antipodality space T, there exists no equivariant map $T \to \mathbf{S}^d$.

(13.2) **Corollary: "Ham Sandwich Theorem".**
Given d bounded and Lebesgue measurable sets in $\mathbb{R}^d$ there exists some affine hyperplane that simultaneously bisects them all.

We end by stating a useful generalization of the Borsuk-Ulam Theorem to $\mathbf{Z}_p$-spaces for $p > 2$. First a few definitions, see Bárány, Shlosman and Szücs (1981) for complete details. Let p be a prime and $n \geq 1$. Take p disjoint copies of the $n(p-1)$-dimensional ball and identify their boundaries. Call this space $\mathbf{X}_{n,p}$. There exists a mapping $\nu : \mathbf{S}^{n(p-1)-1} \to \mathbf{S}^{n(p-1)-1}$ of the identified boundary which makes it into a $\mathbf{Z}_p$-space. Extend this mapping to $\mathbf{X}_{n,p}$ as follows. If (y, r, q) denotes the point of $\mathbf{X}_{n,p}$ from the q:th ball with radius r and $\mathbf{S}^{n(p-1)-1}$-coordinate y, then put $\nu(y, r, q) = (\nu y, r, q+1)$, where $q+1$ is reduced modulo p. This mapping ν makes $\mathbf{X}_{n,p}$ into a $\mathbf{Z}_p$-space. (Note that $(\mathbf{X}_{n,2}, \nu) \cong (\mathbf{S}^n, \alpha)$.)

(13.3) **Theorem.** [Bárány, Shlosman and Szücs (1981)]
For every continuous mapping $f : \mathbf{X}_{n,p} \to \mathbb{R}^n$ there exists a point x such that $f(x) = f(\nu x) = \ldots = f(\nu^{p-1} x)$.

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