COMBINATORIAL REMARKS ON PARTITIONS OF A MULTIPARTITE NUMBER

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1. Introduction. A partition of an integer m is uniquely determined by listing the multiplicity of each summand occurring in it. Thus the representation $15 = 1 + 1 + 1 + 2 + 5 + 5$ is equivalent to the function v given by $v(0) = 0, v(1) = 3, v(2) = 1, v(3) = 0, v(4) = 0, v(5) = 2, v(6) = \cdots = v(15) = 0$. Evidently partitions of m into nonnegative parts correspond to functions defined on $\{0, 1, \cdots, m\}$ and partitions of m into nonzero (i.e. positive) parts to functions defined on $\{1, 2, \cdots, m\}$. This definition has an obvious generalization to partitions of multipartite numbers where it simplifies combinatorial matters.

2. Partitions considered as functions and as lattice points of convex solids. Lower case Latin letters denote real numbers or functions, Latin capital letters sets. $A \times B$ is the Cartesian product of A and B. $A \simeq B$ means that A and B have the same number of elements. $\varnothing$ is the empty set. $F_n = \{\beta = (b_1, \cdots, b_n) : b_j$ is a nonnegative integer for each j, $1 \leq j \leq n\}$ is the set of n-partite numbers. By definition $|\beta| = b_1 + \cdots + b_n$ is the weight of β and $G_n = \{\mu = (m_1, \cdots, m_n) \varepsilon F_n : 0 < |\mu|\}$. $\theta = (0, \cdots 0)$ is the zero element of F_n. Let $e_{jj} = 1$ for each j, $e_{ji} = 0$ if $j \neq i$. Then $\epsilon(j) = (e_{j1}, \cdots, e_{jn})$ for each j, $1 \leq j \leq n$. $\beta \leq \mu$ means $b_j \leq m_j$ for each j, $1 \leq j \leq n$. $\beta < \mu$ means $\beta \leq \mu$ and $\beta \neq \mu$.

Lower case Greek letters always represent elements of F_n and when one occurs as an index in a sum, product or union it runs through all values in F_n satisfying stated restrictions, if any. If $T \subset F_n$ then $B(T) = \{v : T \to F_1\}$ is the set of nonnegative integer valued functions with domain T. If $T = \{\xi : \theta < \xi \leq \mu\}$ then $B(T)$ is written $B(\mu)$. If $T = \{\xi : \xi \leq \mu\}$ then $B(T)$ is written $B_\theta(\mu)$.

Definition. Let $r \varepsilon F_1$, Let $T = T(\mu) = \{\xi \varepsilon G_n : \xi \leq \mu\}$. If S is a finite subset of G_n, let $S^* = S \cup \{\epsilon(1), \epsilon(2), \cdots, \epsilon(n)\}$.

$$P_S(\mu) = \{v \varepsilon B(S^*) : \sum_{\xi \varepsilon S^*} \xi v(\xi) = \mu\};$$

$$P(\mu) = P_T(\mu);$$

$$U_r(\mu) = \{v \varepsilon B(\mu) : \sum_{\theta < \xi \leq \mu} v(\xi) = r\};$$

$$P_r(\mu) = P(\mu) \cap U_r(\mu);$$

$$Q_r(\mu) = \{v \varepsilon B_\theta(\mu) : \sum_{\xi \leq \mu} \xi v(\xi) = \mu, \quad \sum_{\xi \leq \mu} v(\xi) = r\}.$$

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$P_S(\mu)$ is thus the set of partitions of μ into elements taken from S^* or, in other words, into elements which either belong to S or are of weight 1. This odd-looking definition is motivated by the fact that not all non-zero multipartite numbers μ can be partitioned into elements of S in general although all can be partitioned by choosing a collection of summands (with multiplicities) from S and letting elements of weight 1 "take up the slack" in a unique way. Another reason for this approach is given after Theorem 3 below. $P(\mu)$ is the set of partitions of μ into nonzero parts. $P_r(\mu)$ the set of partitions of μ into precisely r nonzero parts, $Q_r(\mu)$ the set of partitions of μ into precisely r parts, some of which may be zero. From the ordered pair definition of a function it follows easily that $P_\varnothing(\mu)$, $P(\theta)$, $P_0(\theta)$, $Q_0(\theta)$, $Q_1(\theta)$, $Q_2(\theta)$, $\cdots$ are singleton sets. Furthermore if $\mu \neq \theta$ then $Q_0(\mu) = P_0(\mu) = P_1(\theta) = P_2(\theta) = \cdots = \varnothing$. These facts are assumed without explicit mention throughout §3.

It is easy to see that

$$P(\mu) = \bigcup_{i=0}^{|\mu|} P_i(\mu), \qquad Q_r(\mu) \backsimeq \bigcup_{i=0}^{r} P_i(\mu)$$

and

$$P_S(\mu) \backsimeq \{v \in B(S) : \sum_{\xi \in S} \xi v(\xi) \leq \mu\}.$$

If

$$S = \{\sigma(1) = (s_{11}, \cdots, s_{1n}), \cdots, \sigma(t) = (s_{t1}, \cdots, s_{tn})\},$$

then the last relation has an obvious interpretation putting $P_S(\mu)$ into one-to-one correspondence with the lattice points of a convex solid. To be specific

$$P_S(\mu) \backsimeq \left\{(x_1, x_2, \cdots, x_t) \in F_t : \text{for each } j, \quad 1 \leq j \leq n, \quad \sum_{l=1}^{t} s_{lj} x_l \leq m_j\right\}.$$

Look at the particular case of $P(\mu) = P_T(\mu)$, where $T(\mu) = \{\beta \in G_n : \beta \leq \mu\}$. There are $(\prod_{1 \leq i \leq n} (m_i + 1)) - n - 1$ elements of $T(\mu)$ and they could be arranged in a linear order. But, after all, a vector is only a function defined on a finite set and no geometrical properties of a solid are changed by a renumbering of coordinate axes. Consequently, it is sufficient from a geometrical point of view to say that

$$P(\mu) = \{\{(\beta, v_\beta) : \beta \in T(\mu)\} : \text{for each } j, \quad 1 \leq j \leq n, \quad \sum_{\beta \in T(\mu)} b_j v_\beta \leq m_j\}.$$

No similar geometrical representations of $P_r(\mu)$ and $Q_r(\mu)$ are apparent. However if $n = 1$, whence $\mu = m$, the solid is a simplex and

THEOREM 1. *If* $2 \leq r, m$ *and* $S = \{s_2, \cdots, s_r\}$, *then*

$$P_S(m) \backsimeq \{(x_2, \cdots, x_r) \in F_{r-1} : s_2 x_2 + \cdots + s_r x_r \leq m\}$$

$$P(m) \backsimeq \{(x_2, \cdots, x_m) \in F_{m-1} : 2x_2 + \cdots + m x_m \leq m\}$$

$$U_r(m) \backsimeq \{(x_2, \cdots, x_m) \varepsilon F_{m-1} : x_2 + \cdots + x_m \leq r\}$$
$$P_r(m) \backsimeq \{(x_2, \cdots, x_r) \varepsilon F_{r-1} : 2x_2 + \cdots + rx_r \leq m - r\}$$
$$Q_r(m) \backsimeq \{(x_2, \cdots, x_r) \varepsilon F_{r-1} : 2x_2 + \cdots + rx_r \leq m\}.$$

Proof. The case of $P_S(m)$ has aleady been discussed. $P(m)$ is the special case $S = \{2, \cdots, m\}$ of this. If $S = \{2, \cdots r\}$, it is well known that $P_r(m + r) \simeq Q_r(m) \simeq P_S(m)$. The relation for $U_r(m)$ follows directly from its definition and the fact that

$$\{(x_1, \cdots, x_m) \varepsilon F_m : x_1 + \cdots + x_m = r\} \backsimeq \{(x_2, \cdots, x_m) \varepsilon F_{m-1} : x_2 + \cdots + x_m \leq r\}.$$

Let $p_S(\mu)$ be the number of elements in $P_S(\mu)$. Similarly $p(\mu), u_r(\mu), p_r(\mu), q_r(\mu)$.

3. **Recurrence formulas.** Theorem 1 and the discussion preceding it not only tell what certain sets of partitions "look like" but also give a representation of partitions which seems suitable to computer use, of which more in a forthcoming paper. However, for enumerating partitions of fairly small multipartite numbers recurrence formulas provide a more useful tool. Furthermore, it is possible to use inductive arguments based on these formulas to get information about the asymptotic behavior of the functions mentioned above, although this is not necessarily the best approach.

If $k \varepsilon F_1$, $1 \leq k \leq n$ is fixed, let $E(j, k, r, \mu) = \{v \varepsilon Q_r(\mu) : \sum v(\gamma)\ j$, the sum being over γ such that $\epsilon(k) \leq \gamma\}$. To return to the language in which partitions are usually discussed, $E(j, k, r, \mu)$ is the set of partitions of the multipartite number μ into r summands (summands equal to zero allowed) of which precisely j have strictly positive k-th positions. Evidently there are no more than m_k such summands since otherwise they could not add to μ. And every partition $v \varepsilon Q_r(\mu)$ must belong to $E(j, k, r, \mu)$ for some j, $1 \leq j \leq r$. Therefore

$$Q_r(\mu) = \bigcup_{j \leq \min\{r, m_k\}} E(j, k, r, \mu).$$

But if $j \leq \min\{r, m_k\}$, then

$$E(j, k, r, \mu) \backsimeq \bigcup_{\beta \leq \mu - m_k \epsilon(k)} Q_{r-j}(\beta) \times Q_j(\mu - \beta - j\epsilon(k)).$$

Let

$$A(r, \mu) = \{(i, \beta) : i \leq \min\{r, m_k\},\quad \beta \leq \mu - m_k\epsilon(k)\}.$$

Then

$$Q_r(\mu) \backsimeq \bigcup_{A(r,\mu)} Q_{r-i}(\beta) \times Q_i(\mu - \beta - i\epsilon(k)).$$

Similarly

$$P_r(\mu) \backsimeq \bigcup_{A(r,\mu)} P_{r-i}(\beta) \times Q_i(\mu - \beta - i\epsilon(k)).$$

Consequently

LEMMA 1.

$$q_r(\mu) = \sum_{A(r,\mu)} q_{r-i}(\beta)q_i(\mu - \beta - i\epsilon(k))$$

$$p_r(\mu) = \sum_{A(r,\mu)} \sum_{j=0}^{i} p_{r-i}(\beta)p_j(\mu - \beta - i\epsilon(k)).$$

Now $q_0(\theta) = 1$ and $q_0(\mu) = 0$ if $\mu \neq \theta$. A routine calculation shows that

THEOREM 2.

$$q_r(\mu + r!\, s\epsilon(k)) = q_r(\mu) + \sum q_y(\beta)q_{r-y}(\mu - \beta + (r!\, t + rx + y)\epsilon(k)),$$

where the summation is over $\{(t, x, y, \beta) : 0 \le t \le s - 1, 0 \le x \le (r-1)! - 1, 1 \le y \le r - 1, \beta \le \mu - m_k\epsilon(k)\}$.

If $\xi = (x_1, x_2, \cdots, x_n)$, then let $\mu^\xi = \prod_{1 \le i \le n} m_i^{x_i}$ and let $\mu \equiv \xi \bmod \beta$ if $m_j \equiv x_j \bmod b_j$ for each j, $1 \le j \le n$. Wright [4] has shown that if r, β are fixed and μ is a variable obeying the restriction $\mu \equiv \beta \bmod (r!, r!, \cdots, r!)$, then $q_r(\mu)$ is a polynomial in μ in the sense that there is some fixed γ such that

$$q_r(\mu) = \sum_{\xi \le \gamma} c_{r,\beta,\xi}\mu^\xi,$$

where the $c_{r,\beta,\xi}$ depend only on r, β, ξ.

An alternative proof of this follows from Theorem 2. For $q_1(\mu) = 1$ identically. From the induction hypothesis (that for each i, $1 \le i \le r - 1$, the function $q_i(\mu)$ is a polynomial in μ when the m_j are suitably restricted to residue classes mod $i!$) it follows in a straightforward fashion that the right-hand side of the equation in Theorem 2 is a polynomial in s for fixed μ. Thus so is the left. But it is an elementary matter to verify that if $q_r(\mu)$ is a polynomial in each m_k separately, then it is a polynomial in μ.

Since $p_r(\mu) = q_r(\mu) - q_{r-1}(\mu)$, it is not difficult to see that a similar statement holds regarding $p_r(\mu)$. [1] contains a detailed proof of this based on an analog of Theorem 2.

Let

$$H(\mu, \beta) = \{j \in F_1 : j\beta \le \mu\}$$

and

$$J(\mu, \beta) = \{j \in F_1 : \theta < j\beta \le \mu\}.$$

THEOREM 3. *If β is not an element of the finite set T and if $T \cup \{\beta\} = S \subset G_n$, then*

$$p_S(\mu) = \sum_{j \in H(\mu,\beta)} p_T(\mu - j\beta).$$

Proof. If $j \in H(\mu, \beta)$, then there are precisely $p_T(\mu - j\beta)$ partitions $v \in P_S(\mu)$ such that $v(\beta) = j$.

Let $S = \{\sigma(1) = (s_{11}, \cdots, s_{1n}), \cdots, \sigma(k) = (s_{k1}, \cdots, s_{kn})\}$ be a finite subset of G_n and let $\lambda = \lambda(S) = (l_1, \cdots, l_n)$ be defined by setting $l_i = \prod s_{ii}$, the product being over i, $1 \leq i \leq k$, such that $s_{ii} > 0$. It can be proved by means of a finite induction based on Theorem 3 that S uniquely determines a finite set $Y = \{K_1, \cdots, K_q\}$ of half cones with vertices at θ such that $\bigcup_{r=1}^{q} K_r = F_n$ and such that if the variable μ is confined to the cone K_w for some fixed w and is further restricted by the requirement that $\mu \equiv \beta \bmod \lambda$, where β is arbitrary but fixed, then $p_S(\mu)$ is a polynomial of degree k in μ. In other words, to fixed S, β, w there corresponds a representation

$$p_S(\mu) = \sum_{\xi \in X} c_{S,w,\beta,\xi}\mu^{\xi}$$

where $|\xi| \leq k$ for each ξ in the (therefore finite) set X, and where $c_{S,w,\beta,\xi}$ depends only on S, w, β, ξ. For example, if $S = \{(1, 3)\}$, then the number $p_S((m_1, m_2))$ of partitions of $\mu = (m_1, m_2)$ into parts taken from the set $\{(0, 1), (1, 0), (1, 3)\}$ is given by the rule

$$p_S((m_1, m_2)) = 1 + m_1, \qquad m_2 \geq 3m_1;$$

$$p_S((m_1, m_2)) = 1 + \left[\frac{m_2}{3}\right], \qquad m_2 \leq 3m_1.$$

This gives an example of what can be learned from a recurrence formula like Theorem 2 and also gives another argument for regarding the definition of $p_S(\mu)$ as a natural one, for k is the number of elements of S, not of S^*. This result is known for $n = 1$ [2] in which case, of course, there is no hint of the existence of the half cones K_w.

LEMMA 2. *If $\theta < \beta$, then*

$$\sum_{v \in P(\mu)} v(\beta) = \sum_{j \in J(\mu,\beta)} p(\mu - j\beta).$$

Proof. The number of $v \in P(\mu)$ such that $1 \leq j \leq v(\beta)$ is just $p(\mu - j\beta)$.

By definition $\delta \mid \gamma$ if there is some $b \in F_1$ such that $\gamma = b\delta$. $\sigma(\gamma) = \sum_{\delta \mid \gamma} \delta$. If $\xi \neq \theta$, then d_ξ is the positive greatest common divisor of the components of ξ. It is easy to see that

LEMMA 3.

$$\{(\delta, \gamma) : \theta < \gamma \leq \mu, \delta \mid \gamma\} = \{(\beta, j\beta) : \theta < \beta \leq \mu, j \in J(\mu, \beta)\}.$$

LEMMA 4.

$$\mu p(\mu) = \sum_{\theta < \beta \leq \mu} \sigma(\beta) p(\mu - \beta).$$

Proof. It follows from Lemmas 2 and 3 that

$$\begin{aligned}\mu p(\mu) &= \sum_{\theta<\beta\le\mu}\ \sum_{v\in P(\mu)} \beta v(\beta)\\ &= \sum_{\theta<\beta\le\mu}\ \sum_{j\in J(\mu,\beta)} \beta p(\mu - j\beta)\\ &= \sum_{\theta<\gamma\le\mu}\ \sum_{\delta\mid\gamma} \delta p(\mu-\gamma)\\ &= \sum_{\theta<\gamma\le\mu} \sigma(\gamma)p(\mu-\gamma).\end{aligned}$$

Lemma 4 is a generalization of an identity in [3]. It can also be proved by the use of generating functions.

THEOREM 4. *If* $\epsilon(k) \le \mu$, *then*

$$p(\mu) = \frac{1}{m_k}\sum\sum b_k p(\mu - j\beta).$$

The sum above is over

$$\left\{(j, \beta) : 1 \le j \le m_k\ ,\ \beta \le \frac{1}{j}\,\mu\right\}.$$

Proof. If $\xi \ne \theta$, then

$$\sigma(\xi) = \sum_{d\mid d_\xi} \frac{1}{d}\,\xi.$$

Consequently Lemma 4 gives

$$\begin{aligned}m_k p(\mu) &= \sum\sum x_k p(\mu-\xi)/d\\ &= \sum\sum\sum x_k p(\mu - d\beta - x_k\epsilon(k))/d,\end{aligned}$$

where the double sum above is over $\{(\xi, d) : \epsilon(k) \le \xi \le \mu,\ d \mid d_\xi\}$ and the triple sum is over

$$\left\{(x_k\ , d, \beta)\colon 1 \le x_k \le m_k\ ,\ d \mid x_k\ ,\ \beta \le \frac{1}{d}(\mu - m_k\epsilon(k))\right\}.$$

The theorem now follows from the one-dimensional case of Lemma 3.

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