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Orientability of Matroids

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In this paper we define oriented matroids and develop their fundamental properties, which lead to generalizations of known results concerning directed graphs, convex polytopes, and linear programming. Duals and minors of oriented matroids are defined. It is shown that every coordinatization (representation) of a matroid over an ordered field induces an orientation of the matroid. Examples of matroids that are orientable but not coordinatizable and of matroids that are not orientable are presented. We show that a binary matroid is orientable if and only if it is unimodular (regular), and that every unimodular matroid has an orientation that is induced by a coordinatization and is unique in a certain straightforward sense.

1. INTRODUCTION

Let F be a field, let E be a finite set, and denote by F^E the vector space o mappings from E to F. The *support* of $\alpha \in F^E$ is defined to be the set $\underline{S}(\alpha) = \{e \in E: \alpha(e) \neq 0\}$.

Let $\mathscr{R}$ be a vector subspace of F^E. A nonzero vector $\alpha \in \mathscr{R}$ is an *elementar* vector of $\mathscr{R}$ if $\underline{S}(\alpha)$ is minimal (with respect to inclusion) among the support of all nonzero vectors in $\mathscr{R}$. The set $\mathscr{C}$ of supports of elementary vectors of has the following properties:

(C1) $C \in \mathscr{C}$ implies $C \neq \varnothing$, and

$C_1, C_2 \in \mathscr{C}$ and $C_1 \subseteq C_2$ imply $C_1 = C_2$;

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(C2) for all $C_1, C_2 \in \mathscr{C}$ and $x \in C_1 \cap C_2$, $y \in C_1 \backslash C_2$, there exists $\in \mathscr{C}$ such that $y \in C_3 \subseteq (C_1 \cup C_2) \backslash x$.

Whitney [15] used the properties (C1) and (C2) to abstract linear depen-ence, calling a set E together with *a* set $\mathscr{C}$ of subsets of E satisfying (C1) d (C2) a *matroid*. (The term *combinatorial pregeometry* is also used to scribe such systems.) Not all matroids arise as above from vector spaces, t matroids retain much of the fundamental structure of vector spaces. For ample, the notions of rank, bases, flats, hyperplanes, and orthogonals eralize in the context of matroids. However, matroids do not capture tain sign properties of vector spaces over ordered fields. For example, let $= (V, E)$ be a 2-connected graph, let A be the $(0, \pm 1)$-vertex-edge inci-ence matrix of an orientation of G and let $\mathscr{R}$ be the null space of A in $\mathbb{R}^E$. hen the orientation of G is lost in passing from the elementary vectors of $\mathscr{R}$ their supports, but is deducible (up to reversing all edges) from the *signed pports* $(S^+(\alpha), S^-(\alpha))$ of the elementary $\alpha \in \mathscr{R}$, which distinguish the subsets $(\alpha) = \{e \in E: \alpha(e) > 0\}$ and $S^-(\alpha) = \{e \in E: \alpha(e) < 0\}$ of $\underline{S}(\alpha)$.

In this paper we introduce and develop a theory of oriented matroids at generalizes the structure of signed supports of elementary vectors of a ctor space over an ordered field. Oriented matroids thus provide a richer straction than matroids of vector spaces over ordered fields. In particular, e can generalize in the context of oriented matroids notions usually sociated with oriented graphs, linear programming and convex polyhedra. Camion [4], Fulkerson [8], and Rockafellar [13] previously investigated e combinatorial nature of a number of interesting theorems concerning ctor spaces over ordered fields. Several of the theorems and proofs in , 8, 13] translate directly into the context of oriented matrois. In fact, ockafellar in [13] suggested that one should be able to axiomatize a system "signed" or "oriented" matroids that would abstract the combinatorial ructure of signed supports of elementary vectors in ordered vector spaces. inty's work on digraphoids [12], which gave the first notion of matroid ientations and partially motivated Camion, Fulkerson, and Rockafellar, as clearly too restrictive for this purpose. The broader notion of orienta-lity presented here achieves the abstraction that Rockafellar foresaw.

In the next section we present five axiomatizations of oriented matroids d prove their equivalence. The subject of oriented matroid duality is turally developed within the establishment of that equivalence. The maining four sections concern: (3) examples and interpretations; (4) inors of oriented matroids; (5) systems whose minimal elements form an iented matroid; and (6) binary oriented matroids (Minty's digraphoids 12]). It is assumed that the reader has some familiarity with matroid theory. hitney's original paper on the subject [15], the paper by Tutte [14], and the ook by Crapo and Rota [7] are appropriate references.

2. Oriented Matroids

We define a *signed set* X to be a set $\underline{X}$, called the set *underlying* X, and [illegible] mapping $sg_X(x) : \underline{X} \to \{-1, 1\}$, called the *signature of* X. Let X be a sign[illegible] set. The sets $X^+ = \{x \in \underline{X} : sg_X(x) = 1\}$ and $X^- = \{x \in \underline{X} : sg_X(x) =$ [illegible] describe X in a convenient way. The *opposite of* X, denoted $-X$, is the sig[illegible] set having $(-X)^+ = X^-$ and $(-X)^- = X^+$; we write $Y = \pm X$ if eit[illegible] $Y = X$ or $Y = -X$. If $\underline{X}$ is a subset of some set E, then X will be called [illegible] *signed subset of* E, and if $\underline{X} = \varnothing$, then we write $X = \varnothing$.

Theorem 2.1. *Let E be a finite set and let $\mathcal{O}$ be a set of signed subsets of* [illegible] *such that*

(0) *for all $X \in \mathcal{O}$, $X \neq \varnothing$ and $-X \in \mathcal{O}$; and for all $X_1, X_2 \in \mathcal{O}$, $\underline{X}_2 \subseteq$* [illegible] *implies $X_1 = \pm X_2$.*

Then the following two properties are equivalent:

(I) *for all $X_1, X_2 \in \mathcal{O}$ such that $X_1 \neq -X_2$, and all $x \in (X_1^+ \cap X_2^-)$* [illegible] *$(X_1^- \cap X_2^+)$, there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x$ and X_3^-* [illegible] *$(X_1^- \cup X_2^-)\backslash x$;*

(II) *for all $X_1, X_2 \in \mathcal{O}$, $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+$* [illegible] *$X_2^-) \cup (X_1^- \backslash X_2^+)$, there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash$* [illegible] *$X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x$, and $y \in \underline{X}_3$.*

Theorem 2.1 will be proved in the second part of this section.

We define an *oriented matroid* to be a structure $(E, \mathcal{O})$, as above, that satis[illegible] fies (0) and (I).

For $\mathcal{O}$ a set of signed sets, let $\underline{\mathcal{O}} = \{\underline{X} : X \in \mathcal{O}\}$. If $M = (E, \mathcal{O})$ is an oriente[illegible] matroid, then $\underline{M} = (E, \underline{\mathcal{O}})$ is a matroid, since (0) and (I) clearly imply [illegible] Lehman's circuit axioms for $\underline{M}$ [11]. Note that (0) and (II) imply Whitney's [illegible] circuit axioms [15]. If one relaxes (II) by requiring that $y \in \underline{X}_1 \backslash \underline{X}_2$, rather than [illegible] $y \in (X_1^+ \backslash X_2^-) \cup (X_1^- \backslash X_2^+) \supseteq \underline{X}_1 \backslash \underline{X}_2$, then the resulting property $(\mathrm{I}\frac{1}{\mathrm{II}})$ is [illegible] obviously stronger than (I) but weaker than (II), and is, by Theorem 2.1, [illegible] equivalent to both, under condition (0). In the form $(\mathrm{I}\frac{1}{\mathrm{II}})$, the elimination [illegible] property for oriented matroids most closely resembles Whitney's circuit [illegible] elimination axiom. In Section 5 we will see that when the condition

$$\underline{X}_2 \subseteq \underline{X}_1 \text{ implies } X_1 = \pm X_2$$

is dropped from (0), then (I) and (II) are no longer equivalent, while $(\mathrm{I}\frac{1}{\mathrm{II}})$ [illegible] and (II) are.

Let M be a matroid on E with circuits $\mathscr{C}$ and let $\mathcal{O}$ be a set of signed subsets [illegible] of E. If $(E, \mathcal{O})$ is an oriented matroid and $\underline{\mathcal{O}} = \mathscr{C}$, then $\mathcal{O}$ is called an *orientation* [illegible] of M and each $X \in \mathcal{O}$ is called a *(signed) circuit* of $(E, \mathcal{O})$. If there exists an [illegible] orientation of M, then M is called *orientable*.

[illegible] he key condition of the *signed elimination properties* (I) and (II) that [illegible] tes to orientation is that

$$X_3^+ \subseteq X_1^+ \cup X_2^+ \quad \text{and} \quad X_3^- \subseteq X_1^- \cup X_2^-. \tag{2.1}$$

[illegible] he signed elimination properties, the underlying matroid structure and [illegible] structure pertaining specifically to orientation are intimately tied. By [illegible] oking matroid duality (orthogonality), one can define oriented matroids [illegible] uch a way that properties pertaining solely to orientation are divorced in [illegible] atural way from those properties that stem only from the underlying [illegible] troid structure.

[illegible] et $M = (E, \mathscr{C})$ be a matroid. A set $\mathcal{O}$ of signed sets satisfying $\underline{\mathcal{O}} = \mathscr{C}$ and [illegible] $= -\mathcal{O} = \{-X : X \in \mathcal{O}\}$ will be called a *circuit signature of M*. Accordingly, [illegible] *ocircuit signature of M* is a circuit signature of $M^\perp$, the *dual* (or *orthogonal*) [illegible] *M*.

[illegible] Theorem 2.2. *Let M be a matroid on a finite set E, let $\mathcal{O}$ be a circuit signa-* [illegible] *re of M and let $\mathcal{O}'$ be a cocircuit signature of M.*

(a) *Then the following three properties are equivalent:*

(III) *for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ such that $|\underline{X} \cap \underline{Y}| = 2$ or 3,*

$$(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing \quad \textit{and} \quad (X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing;$$

(IV) *for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ such that $\underline{X} \cap \underline{Y} \neq \varnothing$,*

$$(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing \quad \textit{and} \quad (X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing;$$

(V) *for all $e \in E$ and all partitions of E into subsets R, G, B, W with* [illegible] *$R \cup G$, exactly one of the following holds:*

(i) *there exists $X \in \mathcal{O}$ such that*

$$e \in \underline{X} \subseteq R \cup G \cup B \quad \textit{and} \quad X^- \cap R = X^+ \cap G = \varnothing$$

(ii) *there exists $Y \in \mathcal{O}'$ such that*

$$e \in \underline{Y} \subseteq R \cup G \cup W \quad \textit{and} \quad Y^- \cup R = Y^+ \cap G = \varnothing.$$

(b) *Furthermore, $\mathcal{O}$ is an orientation of M if and only if there exists a* [illegible] *ocircuit signature $\mathcal{O}^\perp$ of M such that for $\mathcal{O}' = \mathcal{O}^\perp$ the properties (III), (IV),* [illegible] *nd (V) are satisfied. In fact, if $\mathcal{O}$ is an orientation of M, then $\mathcal{O}^\perp$ is unique,* [illegible] *nd by symmetry $\mathcal{O}^\perp$ is an orientation of $M^\perp$.*

It is evident from Theorem 2.2 that a matroid is orientable if and only if its dual is orientable. Given an orientation $\mathcal{O}$ of M, the orientation $\mathcal{O}^{\perp}$ of M^* described in part (b) of the theorem will be called the *dual* (*orthogonal*) of $\mathcal{O}$. Similarly $(E, \mathcal{O}^{\perp})$ is called the *dual* (*orthogonal*) of $(E, \mathcal{O})$. Note that the uniqueness result in Theorem 2.2b implies that $(\mathcal{O}^{\perp})^{\perp} = \mathcal{O}$, thus we speak of dual pairs of orientations and dual pairs of oriented matroids.

The properties (III), (IV), and (V) of Theorem 2.2 are related to conditions that Minty gave for digraphoids [12]. (That relationship is discussed in Section 6.) We will see in the next section that (III) and (IV) abstract the notion of orthogonality. Accordingly, signed sets X and Y having either $\underline{X} \cap \underline{Y} = \varnothing$, or $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing$ will be called *orthogonal*, and (IV) will be called the *orthogonality property of dual pairs of oriented matroids*.

In the remainder of this section we will, after briefly introducing some useful operations on matroid signatures, prove Theorems 2.1 and 2.2. The reader may wish to read Section 3, which provides examples and interpretations of oriented matroids, before reading these proofs.

Given X, a signed subset of E, and $A \subseteq E$, the signed set Z having $Z^+ = (X^+ \backslash A) \cup (X^- \cap A)$ and $Z^- = (X^- \backslash A) \cup (X^+ \cap A)$ is said to be obtained from X by *reversing signs* on A and is denoted by $Z = {}_AX$. Thus $-X = {}_EX$. For $\mathcal{O}$ a circuit signature of a matroid M on E and $A \subseteq E$, the circuit signature ${}_A\mathcal{O}$ of M obtained from $\mathcal{O}$ by reversing signs on A is defined by ${}_A\mathcal{O} = \{{}_AX : X \in \mathcal{O}\}$.

Note that properties (I) and (II) of Theorem 2.1 are invariant under this operation. Similarly, properties (III), (IV), and (V) of Theorem 2.2 obviously hold for $\mathcal{O}, \mathcal{O}'$ if and only if they hold for ${}_A\mathcal{O}, {}_A\mathcal{O}'$ for all $A \subseteq E$.

Let $\mathcal{O}$ be a circuit signature of a matroid M on E and let $e \in E$. The set $\mathcal{O}\backslash e$ obtained by *deleting* e in $\mathcal{O}$ is defined by $\mathcal{O}\backslash e = \{X \in \mathcal{O} : e \notin \underline{X}\}$. Note that $\mathcal{O}\backslash e$ is a circuit signature of the matroid minor of M obtained by deleting e. In order to define the corresponding *contraction* operation we adopt the following notation. If X is a signed set, then $X\backslash e$ denotes the signed set Z having $Z^+ = X^+\backslash e$ and $Z^- = X^-\backslash e$. For $\mathcal{O}$ a set of signed sets, define $\mathrm{Min}(\mathcal{O})$ to be the set of *minimal* members of $\mathcal{O}$, i.e., $\mathrm{Min}(\mathcal{O}) = \{X \in \mathcal{O} : X' \in \mathcal{O}$ and $\underline{X}' \subseteq \underline{X}$ imply $\underline{X}' = \underline{X}\}$. The set $\mathcal{O}/e$ obtained by *contracting* e *in* $\mathcal{O}$ is defined to be $\mathrm{Min}\{X\backslash e : X \in \mathcal{O}, X\backslash e \neq \varnothing\}$. Of course, $\mathcal{O}/e$ is a circuit signature of M/e, the matroid minor of M obtained by contracting e. The single element deletion and contraction operations described above will be very useful in the following proofs. The general subject of oriented matroid minors will be addressed directly in Section 4.

Proof of Theorem 2.1

It is clear that (II) implies (I). We will use the contraction and deletion operations to inductively prove that (I) implies (II).

Let $\mathcal{O}$ be a circuit signature of a matroid M on E and suppose that $\mathcal{O}$ satisfies (I). It is obvious that $\mathcal{O}|e$ also satisfies (I). In order to prove that $\mathcal{O}/e$ satisfies (I) we give two preliminary results.

LEMMA 2.1.1. *Let $X_1 \in \mathcal{O}$ with $X_1^- = \varnothing$ and let $X_2 \in \mathcal{O}$ have $\underline{X}_2\backslash\underline{X}_1 = \{e\}$ with $e \in X_2^+$ and $X_2^- \neq \varnothing$. Then there is a signed circuit $X \in \mathcal{O}$ having $X^- = \varnothing$ and $(\underline{X}_1\backslash X_2^-) + e \subseteq \underline{X}$.*

Proof. Let $x \in X_2^-$. By (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x = X_2^-\backslash x$. It follows that $e \in \underline{X}_3$, otherwise $\underline{X}_3 \subsetneqq \underline{X}_1$, and since $e \in X_2^+\backslash\underline{X}_1$, we have $e \in X_3^+$. Also

$$\underline{X}_1\backslash\underline{X}_2 \subseteq \underline{X}_3, \tag{2.2}$$

otherwise by eliminating e from $\underline{X}_2$ and $\underline{X}_3$ we get a circuit of $\mathcal{O}$ properly contained in $\underline{X}_1$.

Now suppose that $y \in X_2^+\backslash\underline{X}_3$. If we use (I) to eliminate e from X_2 and $-X_3$, then we get $X_1' \in \mathcal{O}$ having $\underline{X}_1' \subseteq \underline{X}_2 \cup \underline{X}_3\backslash e \subseteq \underline{X}_1$, thus by (0) it must be that $X_1' = \pm X_1$. Note that $x, y \in X_1^+$, since $x, y \in \underline{X}_2$ and $X_1^- = \varnothing$. But $x \in X_2^-\backslash\underline{X}_3$, and $y \in X_2^+\backslash\underline{X}_3$, so by (I) x and y do not agree in sign in X_1', a contradiction. Therefore $X_2^+ \subseteq \underline{X}_3$, so by (2.2) we have $(\underline{X}_1\backslash X_2^-) + e \subseteq X_3^+$. If $X_3^- = \varnothing$, then the conclusion of the lemma is satisfied by $X = X_3$. Otherwise, we can repeat the argument above with X_3 in place of X_2. Thus we obtain $X_4 \in \mathcal{O}$ having $X_4^+ \supseteq (\underline{X}_1\backslash X_3^-) + e \supseteq (\underline{X}_1\backslash X_2^-) + e$ and $X_4^- \subsetneqq X_3^- \subsetneqq X_2^-$. The procedure can be repeated at most $|X_2^-|$ times until it terminates with a circuit $X_k \in \mathcal{O}$ satisfying $(\underline{X}_1\backslash X_2^-) + e \subseteq \underline{X}_k$ and $X_k^- = \varnothing$.

LEMMA 2.1.2. *Let $X \in \mathcal{O}$ and $e \in E$. For all $x \in \underline{X}\backslash e$ there is a circuit $\hat{X} \in \mathcal{O}/e$ such that $x \in \underline{\hat{X}} \subseteq \underline{X}\backslash e$ and $\hat{X}^+ \subseteq X^+$, $\hat{X}^- \subseteq X^-$.*

Proof. By reversing signs on X^- we see that it suffices to establish the lemma in the case $X^- = \varnothing$. If $X\backslash e \in \mathcal{O}/e$, then obviously $\hat{X} = X\backslash e$ satisfies the conditions of the lemma. Suppose that $X\backslash e \notin \mathcal{O}/e$, so there exists $Z \in \mathcal{O}$ having $\varnothing \neq \underline{Z}\backslash e \subsetneqq \underline{X}$ and $e \in Z^+$. If $Z^- = \varnothing$, then set $Z_1 = Z$. If $Z^- \neq \varnothing$, then by Lemma 2.1.1 with $X_1 = X$ and $X_2 = Z$ there exists $Z_1 \in \mathcal{O}$ such that $Z_1^- = \varnothing$ and $e \in \underline{Z}_1 \subseteq \underline{X} + e$. By reversing e in $\mathcal{O}$, applying Lemma 2.1.1 with $X_1 = X$ and $X_2 = -Z_1$, then reversing e back again, we see that there is a $Z_2 \in \mathcal{O}$ such that $e \in \underline{Z}_2 \subseteq \underline{X} + e$, $Z_2^-\backslash e = \varnothing$ and $\underline{X}\backslash\underline{Z}_1 \subseteq \underline{Z}_2$. Now $Z_i\backslash e \in \mathcal{O}/e$ and $Z_i^-\backslash e = \varnothing$ for $i = 1, 2$, and $\underline{Z}_1 \cup \underline{Z}_2 = \underline{X}$, so $x \in \underline{Z}_1\backslash e$ or $x \in \underline{Z}_2\backslash e$.

LEMMA 2.1.3. *For any $e \in E$ both $\mathcal{O}/e$ and $\mathcal{O}\backslash e$ satisfy* (I).

Proof. It is clear that $\mathcal{O}\backslash e$ satisfies (I), since $\mathcal{O}$ satisfies (I). Let $\hat{X}_1, \hat{X}_2 \in \mathcal{O}/e$

with $\hat{X}_1 \neq -\hat{X}_2$ and $x \in \hat{X}_1^+ \cap \hat{X}_2^-$. There must exist $X_1, X_2 \in \mathcal{C}$ such that $\hat{X}_1 = X_1\backslash e$, $\hat{X}_2 = X_2\backslash e$, hence $X_1 \neq -X_2$ and $x \in X_1^+ \cap X_2^-$. By (I) we get $X_3 \in \mathcal{C}$ having $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x$. Lemma 2.1.2 implies that there is an $\hat{X}_3 \in \mathcal{C}/e$ satisfying $\hat{X}_3^+ \subseteq (\hat{X}_1^+ \cup \hat{X}_2^+)\backslash x$, $\hat{X}_3^- \subseteq (\hat{X}_1^- \cup \hat{X}_2^-)\backslash x$.

Now we can establish that $\mathcal{C}$ satisfies (II). This is trivial when $|E| = 1$; suppose that it holds whenever $|E| \leqslant p$. Let $|E| = p + 1 \geqslant 2$. Note that the inductive hypothesis and Lemma 2.1.3 imply that $\mathcal{C}/e$ and $\mathcal{C}\backslash e$ satisfy (II) for all $e \in E$. Let $X_1, X_2 \in \mathcal{C}$ have the smallest possible value of $|\underline{X}_2\backslash\underline{X}_1|$ subject to the existence of elements $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+\backslash X_2^-) \cup (X_1^-\backslash X_2^+)$ such that there is no $X_3 \in \mathcal{C}$ satisfying

$$X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x, \qquad X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x, \qquad \text{and} \qquad y \in \underline{X}_3. \tag{2.3}$$

Since properties (I) and (II) are invariant under reversal of signs on a subset of E, there is no loss of generality in assuming that $X_1^- = \varnothing$, $X_2^-\backslash\underline{X}_1 = \varnothing$. Note that if $|\underline{X}_2\backslash\underline{X}_1| = 1$, then Lemma 2.1.1 implies that (2.3) can be satisfied by some $X_3 \in \mathcal{C}$, hence $|\underline{X}_2\backslash\underline{X}_1| \geqslant 2$.

Suppose that $e \in X_2^-\backslash x$, implying that $e \in X_1^+$. Then $\hat{X}_1 = X_1\backslash e$ and $\hat{X}_2 = X_2\backslash e$ are in $\mathcal{C}/e$, which satisfies (II). Thus there is some $\hat{X}_3 \in \mathcal{C}/e$ such that $\hat{X}_3^+ \subseteq (\hat{X}_1^+ \cup \hat{X}_2^+)\backslash x$, $\hat{X}_3^- \subseteq (\hat{X}_1^- \cup \hat{X}_2^-)\backslash x$, and $y \in \hat{\underline{X}}_3$. Now the circuit $X_3 \in \mathcal{C}$ having $\hat{X}_3 = X_3\backslash e$ satisfies (2.3), since the sign of e is not constrained by (2.3). So we may assume that $X_2^- = \{x\}$, and (2.3) reduces to

$$X_3^- = \varnothing, \qquad \underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash x, \qquad \text{and} \qquad y \in \underline{X}_3. \tag{2.4}$$

Let $e \in \underline{X}_2\backslash\underline{X}_1$. By Lemma 2.1.2 there is an $\hat{X}_1 \in \mathcal{C}/e$ such that $\hat{X}_1^- = \varnothing$ and $y \in \hat{\underline{X}}_1 \subseteq \underline{X}_1$. If $x \notin \hat{\underline{X}}_1$, set $\hat{Z} = \hat{X}_1$. If $x \in \hat{\underline{X}}_1$, since $\hat{X}_2 = X_2\backslash e \in \mathcal{C}/e$, which satisfies (II), there is a nonnegative $\hat{Z} \in \mathcal{C}/e$ such that $y \in \hat{\underline{Z}} \subseteq (\hat{\underline{X}}_1 \cup \hat{\underline{X}}_2)\backslash x$. Let $Z \in \mathcal{C}$ have $\hat{Z} = Z\backslash e$. If $Z^- = \varnothing$, then (2.4) is satisfied by $X_3 = Z$, so $Z^- = \{e\}$. Since $e \in X_2^+$, we can apply (I) to establish the existence of $X_2' \in \mathcal{C}$ such that $X_2'^+ \subseteq (X_2^+ \cup Z^+)\backslash e$ and $X_2'^- \subseteq (X_2^- \cup Z^-)\backslash e = \{x\}$.

Now there are two cases to consider:

(i) Suppose that $x \notin \underline{X}_2'$, which implies $X_2'^- = \varnothing$. If $y \in \underline{X}_2'$, then $X_3 = X_2'$ satisfies (2.4), so assume that $y \notin \underline{X}_2'$. Now $x \in \underline{X}_1\backslash\underline{X}_2'$, so there is an element $e' \in \underline{X}_2'\backslash\underline{X}_1$, and $e' \neq e$ since $e \notin \underline{X}_2'$. By repeating the arguments above for $\mathcal{C}/e'$, rather than $\mathcal{C}/e$, we either construct an $X_3 \in \mathcal{C}$ satisfying (2.4), or a $Z' \in \mathcal{C}$ such that $y \in \underline{Z}' \subseteq (\underline{X}_1 \cup \underline{X}_2')\backslash x$ and $Z'^- = \{e'\}$. However, $Z', X_2' \in \mathcal{C}\backslash x$, which satisfies (II) by the inductive hypothesis. Note that $\{e'\} = (Z'^+ \cap X_2'^-) \cup (Z'^- \cap X_2'^+)$, $\underline{Z}' \cup \underline{X}_2' \subseteq \underline{X}_1 \cup \underline{X}_2$, and $y \in \underline{Z}'\backslash\underline{X}_2'$. So applying (II) with $X_1 = Z'$ and $X_2 = X_2'$ gives an X_3 that satisfies (2.4).

(ii) If $x \in \underline{X}_2'$, then $X_2'^- = \{x\}$, so certainly $X_2' \neq -X_1$. But since $\underline{X}_2' \subseteq (\underline{X}_2 \cup \underline{Z})\backslash e \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash e$, we have $\underline{X}_2'\backslash\underline{X}_1 \subseteq (\underline{X}_2\backslash\underline{X}_1)\backslash e$. Therefore, by the choice of X_1 and X_2, the elimination property (II) must hold for X_1, X_2'. In particular since $x \in X_1^+ \cap X_2'^-$ and $y \in X_1^+\backslash X_2'^-$, there exists X_3 satisfying $X_3^+ \subseteq (X_1^+ \cup X_2'^+)\backslash x$, $X_3^- \subseteq (X_1^- \cup X_2'^-)\backslash x$, and $y \in \underline{X}_3$. Since $X_2'^- = \{x\}$, $X_1^- = \varnothing$, we see that $X_3^- = \varnothing$. Moreover, $\underline{X}_2' \subseteq \underline{X}_2 \cup \underline{Z} \subseteq \underline{X}_2 \cup \underline{X}_1$ so $\underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash x$, and (2.4) is satisfied by X_3. Thus Theorem 2.1 is established.

Proof of Theorem 2.2

This proof is broken into several parts. First we establish the equivalence of (IV) and (V) in part (a) of the theorem. We refer to a partition R, G, B, W of E with $R \cup G \neq \varnothing$, as in (V), as a 4-*painting* of E into red, green, blue, and white elements.

Proof of (IV) $\Rightarrow$ (V). Assume given a 4-painting R, G, B, W of E and a distinguished element $e \in R \cup G$.

Suppose that $X \in \mathcal{C}$ satisfies alternative (i) of (V) and $Y \in \mathcal{C}'$ satisfies alternative (ii) of (V). Then $e \in (X^+ \cap Y^-) \cup (X^- \cap Y^-)$ and $(X^- \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, a contradiction. Hence alternatives (V.i) and (V.ii) cannot both hold. We will now show by induction on the cardinality of $R \cup G$ that at least one of (V.i) and (V.ii) must hold.

Suppose that $|R \cup G| = 1$, i.e., $R \cup G = \{e\}$, and (V.i) fails. Then e is not in the closure of B, so there is a hyperplane H of M such that $B \subseteq H$ and $e \notin H$. Thus for some $Y \in \mathcal{C}'$ we have $e \in \underline{Y} = E\backslash H \subseteq W + e$ and either Y or $-Y$ satisfies (V.ii).

Now assume that the result holds for all 4-paintings having no more than p red and green elements, where $p \geqslant 1$, and that it fails for the 4-painting R, G, B, W with $e \in R \cup G$ the distinguished element and $|R \cup G| = p + 1 \geqslant 2$.

Select $e' \in R \cup G$, $e' \neq e$, and let R', G', B', W' and R'', G'', B'', W'', respectively, be the 4-paintings obtained from R, G, B, W by repainting e' first blue and then white. Since $|R' \cup G'| = p$, either (V.i) or (V.ii) is satisfied with respect to R', G', B', W' and $e \in R' \cup G'$. But a $Y \in \mathcal{C}'$ satisfying (V.ii) for this painting would also satisfy (V.ii) for the original painting, a contradiction. Hence (V.i) is satisfied by some $X \in \mathcal{C}$ having $e \in \underline{X} \subseteq R' \cup G' \cup B'$, $X^- \cap R' = X^+ \cap G' = \varnothing$. Furthermore, since (V.i) fails for the original painting, $e' \in (X^- \cap R) \cup (X^+ \cap G)$. Similarly, since $|R'' \cup G''| = p$, we know that there is a $Y \in \mathcal{C}'$ such that $e \in \underline{Y} \subseteq R'' \cup G'' \cup W''$, $Y^- \cap R'' = Y^+ \cap G'' = \varnothing$, and $e' \in (Y^- \cap R) \cup (Y^+ \cap G)$. But then $\{e, e'\} \subseteq (X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, a contradiction of the orthogonality condition (IV).

Proof of (V) $\Rightarrow$ (IV). Suppose that $\mathcal{C}, \mathcal{C}'$ satisfies (V) and that $X \in \mathcal{C}$, $Y \in \mathcal{C}'$ with X and Y not orthogonal. Replacing Y by $-Y$, if necessary, we can assume that $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$. Let $R = X^+ \cup Y^+$, $G = E \backslash R \supseteq X^- \cup Y^-$, $B = W = \varnothing$, and distinguish any $e \in (X^+ \cap Y^+) \cup (X^- \cap Y^-)$. Then X satisfies alternative (V.i) and Y satisfies (V.ii), a contradiction.

It is clear that property (IV) implies (III). Before completing the proof of part (a) of Theorem 2.2 by showing that (III) implies (IV), it will be useful to prove the following lemma, which establishes one of the implications in part (b) of the theorem.

LEMMA 2.2.1. *If $\mathcal{O}$, $\mathcal{O}'$ is a pair of circuit and cocircuit signatures of a matroid and $\mathcal{O}$, $\mathcal{O}'$ satisfies* (V), *then each of $\mathcal{O}$ and $\mathcal{O}'$ satisfies* (I) *and is, therefore, a matroid orientation.*

Proof. By symmetry, it is enough to show that $\mathcal{O}$ satisfies (I). Let $X_1, X_2 \in \mathcal{O}$, with $X_1 \neq -X_2$ and $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$. Consider the following 4-painting of E:

$$R = (X_1^+ \backslash X_2^-) \cup (X_2^+ \backslash X_1^-), \qquad G = (X_1^- \backslash X_2^+) \cup (X_2^- \backslash X_1^+),$$
$$B = [(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)] \backslash x, \qquad W = [E \backslash (\underline{X}_1 \cup \underline{X}_2)] + x,$$

and distinguish any $e \in \underline{X}_1 \backslash \underline{X}_2 \subseteq R \cup G$. Suppose that $Y \in \mathcal{O}'$ satisfies alternative (V.ii) with respect to this 4-painting. Then $e \in (X_1^+ \cap Y^+) \cup (X_1^- \cap Y^-)$ and $(X_1^+ \cap Y^-) \cup (X_1^- \cap Y^+) \subseteq \{x\}$. Since the equivalence of (IV) and (V) has been established and the pair $\mathcal{O}$, $\mathcal{O}'$ satisfies (V), it must also satisfy (IV), implying that

$$x \in (X_1^+ \cap Y^-) \cup (X_1^- \cap Y^+). \tag{2.5}$$

But $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ so by (2.5) $x \in (X_2^- \cap Y^-) \cup (X_2^+ \cap Y^+)$, yet $(X_2^+ \cap Y^-) \cup (X_2^- \cap Y^+) = \varnothing$, a contradiction. So alternative (V.ii) fails and (V.i) must hold. This implies the existence of some $X \in \mathcal{O}$ having $\underline{X} \subseteq R \cup G \cup B = (\underline{X}_1 \cup \underline{X}_2) \backslash x$ and $X^+ \subseteq R \cup B \subseteq X_1^+ \cup X_2^+$, $X^- \subseteq G \cup B \subseteq X_1^- \cup X_2^-$. Therefore the signed elimination property (I) is satisfied by $\mathcal{O}$. Since $\mathcal{O}$ is a matroid signature, it also satisfies (0) of Theorem 2.1, hence $\mathcal{O}$ is a matroid orientation.

Note that this proof, with no further work, indicates directly that $\mathcal{O}$ and $\mathcal{O}'$ satisfy the stronger elimination property (II).

We will now use Lemma 2.2.1 to complete the proof of part (a) of Theorem 2.2 by showing that (III) implies (IV).

Proof of (III) $\Rightarrow$ (IV). Let $\mathcal{O}$, $\mathcal{O}'$ satisfy (III). Note that for any choice of $e \in E$ it follows that $\mathcal{O}/e$, $\mathcal{O}'\backslash e$ and $\mathcal{O}\backslash e$, $\mathcal{O}'/e$ also satisfy (III). For $|E|$ sufficiently small the result must hold. Suppose that it fails for the pair $\mathcal{O}$, $\mathcal{O}'$, but holds for all pairs of matroid signatures on fewer than $|E|$ elements. Let $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ with X and Y not orthogonal. We assume without loss of generality that $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, since if that is not the case for X and Y, it is the case for X and $-Y$. Furthermore, we can reverse signs in $\mathcal{O}$ and $\mathcal{O}'$ on $X^- \cup Y^-$, since (III) and (IV) are invariant under such reversals, so we may assume that $X^- = Y^- = \varnothing$.

Suppose that $e \in \underline{X} \backslash \underline{Y}$. Then $X \backslash e \in \mathcal{O}/e$ and $Y \in \mathcal{O}'\backslash e$. But $X \backslash e$ and Y are not orthogonal, yet the pair $\mathcal{O}/e$, $\mathcal{O}'\backslash e$ satisfies (III) and, by the inductive hypothesis, (IV). Thus $\underline{X} \subseteq \underline{Y}$. Similarly $\underline{Y} \subseteq \underline{X}$, so $\underline{X} = \underline{Y}$.

Suppose $u, v \in E \backslash \underline{X}$, $u \neq v$. Since $Y \in \mathcal{O}'\backslash u$ and $\mathcal{O}/u$, $\mathcal{O}'\backslash u$ satisfies (III), and hence (IV), it follows that $X \backslash u \notin \mathcal{O}/u$. Thus there is some $U \in \mathcal{O}$ such that

$$\{u\} \subsetneq \underline{U} \subsetneq \underline{X} + u. \tag{2.6}$$

Select U so that $|U^- \cap \underline{Y}|$ is minimized subject to (2.6). Note that $U^- \cap \underline{Y} \neq \varnothing$ since $U \backslash u \in \mathcal{O}/u$ must be orthogonal to $Y \in \mathcal{O}'\backslash u$. Let $w \in U^- \cap \underline{Y}$ and observe that $U, X \in \mathcal{O}\backslash v$. Now $\mathcal{O}\backslash v$, $\mathcal{O}'/v$ satisfies (III) and, by the inductive hypothesis, it satisfies (IV). By Lemma 2.2.1 $\mathcal{O}\backslash v$ satisfies (I) and, therefore, (II). Hence there exists $\hat{V} \in \mathcal{O}\backslash v$ such that $\hat{V}^+ \subseteq (X^+ \cup U^+) \backslash w$ and $\hat{V}^- \subseteq (X^- \cup U^-) \backslash w$. Let $V \in \mathcal{O}$ such that $\hat{V} = V \backslash v$ and observe that $\underline{V} \subseteq (\underline{X} \cup \underline{U}) \backslash w \subseteq \underline{X} + u \backslash w$ so $u \in \underline{V}$. But then $\{u\} \subsetneq \underline{V} \subsetneq \underline{X} + u$ and $V^- \cap \underline{Y} = V^- \cap \underline{X} \subsetneq U^- \cap \underline{X}$, contradicting the choice of U. Hence there exist no distinct u and v in $E \backslash \underline{X}$, implying that $|E| \leqslant |\underline{X}| + 1 = |\underline{Y}| + 1$. If r is the Whitney rank of the matroid $(E, \underline{\mathcal{O}})$ then $|\underline{X}| \leqslant r + 1$ and $|\underline{Y}| \leqslant |E| - r + 1$ since $\underline{X} \in \underline{\mathcal{O}}$ and $\underline{Y} \in \underline{\mathcal{O}}^\perp$. Therefore $|E| \leqslant |\underline{Y}| + 1 \leqslant |E| - r + 2$ so $r \leqslant 2$ and $|E| \leqslant |\underline{X}| + 1 \leqslant r + 2 \leqslant 4$. So, $|\underline{X} \cap \underline{Y}| \leqslant 3$ and orthogonality of X and Y follows from (III).

The following example indicates that if (III) is relaxed to require orthogonality only for those $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ having $|\underline{X} \cap \underline{Y}| = 2$, then (IV) is no longer implied. Let M be the four-point line, the self-dual matroid on a four-element set E having as its circuits the four triples in E. Let $\mathcal{O}$ and $\mathcal{O}'$ both be given by the rows and the opposites of the rows in the following 4×4 array.

e_1	e_2	e_3	e_4
+	−	+	0
+	−	0	+
+	0	−	−
0	+	[illegible]	+

Note that $\mathcal{O}$, $\mathcal{O}'$ does not satisfy (IV) (no oriented matroid can be self-dual), but orthogonality is satisfied for all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ having $|\underline{X} \cap \underline{Y}| = 2$.

To complete the proof of Theorem 2.2 it suffices to establish

LEMMA 2.2.2. *If $M = (E, \mathcal{O})$ is an oriented matroid, then there is a unique cocircuit signature $\mathcal{O}^{\perp}$ of $\underline{M}$ such that $\mathcal{O}, \mathcal{O}^{\perp}$ satisfies the orthogonality condition* (IV).

To prove Lemma 2.2.2. we first recall a familiar property of matroids.

LEMMA 2.2.3. *Let $M = (E, \mathscr{C})$ be a matroid. For any $C \in \mathscr{C}$ and $e, e' \in C$, $e \neq e'$, there exists $D \in \mathscr{C}^{\perp}$ such that $C \cap D = \{e, e'\}$.*

Proof. The set $C \backslash e$ is independent in M, so $(E \backslash C) + e$ contains a dual base $B^{\perp}$. Therefore there is a cocircuit $D \subseteq B^{\perp} + e' \subseteq (E \backslash C) \cup \{e, e'\}$ and $e' \in D$. Now $e' \in C \cap D \subseteq \{e, e'\}$, so $e \in C \cap D$, otherwise $| C \cap D | = 1$.

From Lemma 2.2.3 we see that for any circuit signature $\mathcal{O}$ of a matroid M there exists a cocircuit signature $\mathcal{O}'$ of M satisfying the condition

> for every $Y \in \mathcal{O}'$ there exists an element $e \in \underline{Y}$ such that for all $y \in \underline{Y}$, $y \neq e$, there is some $X \in \mathcal{O}$ having $\underline{X} \cap \underline{Y} = \{e, y\}$ and X orthogonal to Y. (2.7)

Proof of Lemma 2.2.2. Let $\mathcal{O}'$ be any cocircuit signature of $\underline{M}$ satisfying (2.7). Suppose that $X \in \mathcal{O}$, $Y \in \mathcal{O}'$, X and Y are not orthogonal, and $|\underline{X} \cap \underline{Y}|$ is as small as possible, subject to the conditions above. Since X and Y are not orthogonal, $\underline{X} \cap \underline{Y} \neq \varnothing$ and thus $|\underline{X} \cap \underline{Y}| \geqslant 2$, because $\underline{X}$ is a circuit and $\underline{Y}$ is a cocircuit of the matroid $\underline{M}$. Let $x, y \in \underline{X} \cap \underline{Y}$, $x \neq y$. By reversing signs in $\mathcal{O}$ and $\mathcal{O}'$ and replacing X or Y by its opposite, if necessary, we can assume that $X^- = Y^- = \varnothing$. Thus, $x, y \in X^+ \cap Y^+$.

We will first show that there is a signed circuit $Z \in \mathcal{O}$ such that

$$x \in Z^+, \qquad y \in Z^-, \qquad \text{and } \underline{Z} \cap \underline{Y} = \{x, y\}. \tag{2.8}$$

Now $Y \in \mathcal{O}'$ and the pair $\mathcal{O}, \mathcal{O}'$ satisfies (2.7). Hence there is an $e \in \underline{Y}$ such that for each $z \in \underline{Y}$, $z \neq e$, there exists $X_z \in \mathcal{O}$ having $\underline{X}_z \cap \underline{Y} = \{e, z\}$ and X_z orthogonal to Y. If $e = x$, then either $Z = X_y$ or $Z = -X_y$ satisfies (2.8); and if $e = y$ then $Z = X_x$ or $Z = -X_x$ satisfies (2.8). Suppose that $e \neq x, y$. Then, replacing X_x or X_y by its opposite, if necessary, we have

$$e \in X_x^- \cap X_y^-, \qquad x \in X_x^+ \backslash \underline{X}_y, \qquad y \in X_y^+ \backslash \underline{X}_x,$$

and

$$(\underline{X}_x \cup \underline{X}_y) \cap \underline{Y} \subseteq \{x, y, e\}.$$

By (II) there is a $Z \in \mathcal{O}$ with $Z^- \subseteq (X_x^- \cup X_y^-) \backslash e$, $Z^+ \subseteq (X_x^+ \cup X_y^+) \backslash e$, and $x \in Z^+$. So $x \in \underline{Z} \cap \underline{Y} \subseteq \{x, y\}$, thus $y \in Z^-$ and (2.8) is satisfied.

Now we have $x \in X^+ \cap Z^+$ and $y \in X^+ \cap Z^-$. By (II) there exists a signed circuit $X' \in \mathcal{O}$ with $x \in X'^+ \subseteq (X^+ \cup Z^+) \backslash y$ and $X'^- \subseteq (X^- \cup Z^-) \backslash y$. Now $X^- \cap \underline{Y} = \varnothing$ and $Z^- \cap \underline{Y} = \{y\}$ so $X'^- \cap \underline{Y} = \varnothing$, implying that $X' \in \mathcal{O}$ is not orthogonal to Y, since $x \in X'^+ \cap Y^+$ and $Y^- = \varnothing$. Moreover, $\underline{X}' \cap \underline{Y} \subsetneqq \underline{X} \cap \underline{Y}$, contradicting the choice of X. Therefore, all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ are orthogonal, so (IV) is satisfied by $\mathcal{O}$, $\mathcal{O}'$, i.e., $\mathcal{O}'$ is *dual to* $\mathcal{O}$. Moreover, by Lemma 2.2.3 there can be at most one cocircuit signature $\mathcal{C}'$ of $\underline{M}$ that has X and Y orthogonal for all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ such that $|\underline{X} \cap \underline{Y}| = 2$. Hence $\mathcal{O}'$ is the unique cocircuit signature of $\underline{M}$ that is dual to $\mathcal{C}$. This completes the proof of Theorem 2.2.

3. EXAMPLES

EXAMPLE 3.1. Oriented matroids coordinatizable over an ordered field. Let F be an ordered field let E be a finite set, and let $\mathscr{R}$ be a vector subspace of F^E. Consider the set $\mathcal{O}$ of signed supports of elementary vectors of $\mathscr{R}$ and the set $\mathcal{C}'$ of signed supports of elementary vectors of $\mathscr{R}^{\perp}$, the orthogonal complement of $\mathscr{R}$. Clearly $\mathcal{O}$ is a circuit signature and $\mathcal{C}'$ is a cocircuit signature of the matroid $(E, \underline{\mathcal{O}})$. If $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$, then there are elementary vectors $\alpha \in \mathscr{R}$ and $\beta \in \mathscr{R}^{\perp}$ such that $X^+ = S^+(\alpha)$, $X^- = S^-(\alpha)$ and $Y^+ = S^+(\beta)$, $Y^- = S^-(\beta)$. It follows that X and Y are orthogonal as signed sets, since α and β are orthogonal vectors in F^E. Thus the orthogonality property of Theorem 2.2 is satisfied by $\mathcal{O}, \mathcal{O}'$, $(E, \mathcal{O})$ is an oriented matroid and $\mathcal{C}^{\perp} = \mathcal{C}'$; we denote by $S(\mathscr{R})$ the oriented matroid $(E, \mathcal{C})$.

An oriented matroid $M = (E, \mathcal{O})$ that arises in this way is said to be *coordinatizable* (or *representable*) over F. If, for a given ordering of E, A is an $m \times n$ matrix over F with $\mathscr{R}$ as its null space, then A is called a coordinatization of M. (More properly, we might call A a *Whitney coordinatization on of* M and a *Tutte coordinatization of* $M^{\perp}$.) In this case E can be considered to be the family $\{e_1 = a_1, \ldots, e_n = a_n\}$ of points in F^m, where $a_1, \ldots, a_n$ are the columns of A, and we say that M is *the oriented matroid on E determined by linear dependence in F^m*.

Example 3.1 yields

PROPOSITION 3.2. *All matroids coordinatizable over an ordered field are orientable.*

For additional generality in Example 3.1, and Proposition 3.2, we could let F be noncommutative, i.e., an ordered division ring, and let $\mathscr{R}$ be a left (right) vector subspace of the left (right) vector space F^E. In fact the reader familiar with [4] will recognize that Example 3.1 generalizes when F is an ordered unitary ring and $\mathscr{R}$ is a *unimodular module* (see [4]). Thus, for example, any integral chain group $\mathscr{R}$ describes an oriented matroid.

EXAMPLE 3.3. Graphic oriented matroids.

Let A be the $(0, \pm 1)$-vertex-edge incidence matrix of a directed graph $\Gamma = (V, E)$ and let $M = (E, \mathcal{O})$ be the oriented matroid coordinatized by A. Then $\mathcal{O}(\mathcal{O}^{\perp})$ is the collection of edge sets of elementary circuits (cocircuits) in Γ. If $X \in \mathcal{O}$ and the corresponding circuit is traversed so that some $e \in X^+$ is encountered as a forward edge or some $e \in X^-$ is encountered as a reverse edge, then the set of all forward (reverse) edges so encountered will be $X^+(X^-)$. For $Y \in \mathcal{O}^{\perp}$, removal of the edges of $\underline{Y}$ *cuts* a previously connected component of Γ into two connected components, with every edge of $\underline{Y}$ having one vertex in each. Then Y^+ is the subset of $\underline{Y}$ crossing the cut in one direction and Y^- consists of those edges of $\underline{Y}$ crossing the cut in the opposite direction.

Hence the following proposition, which follows immediately from Theorem 2.2 with $G = \varnothing$ in (V), is a generalization of Minty's painting lemma for directed graphs (see [12]).

PROPOSITION 3.4. *Let $M = (E, \mathcal{O})$ be an oriented matroid. Distinguish an element $e \in E$ and partition E into subsets $e \in R$, B, W. Then exactly one of the following alternatives holds:*

(i) *there is a signed circuit $X \in \mathcal{O}$ having $e \in \underline{X} \subseteq R \cup B$ and $X^- \cap R = \varnothing$; or*

(ii) *there is a signed cocircuit $Y \in \mathcal{O}$ having*

$$e \in \underline{Y} \subseteq R \cup W \quad \text{and} \quad Y^- \cap R = \varnothing.$$

Minty's extension of his painting lemma from directed graphs to digraphoids [12] is the special case of Proposition 3.4 for binary oriented matroids (as we shall see in Section 6). Camion [4], Fulkerson [8], and Rockafellar [13] extended the result further, to the case of oriented matroids coordinatizable over an ordered field.

EXAMPLE 3.5. Affine coordinatizations of oriented matroids.

Let A be an $m \times n$ matrix over an ordered field F and let $M = (E, \mathcal{O})$ be the oriented matroid coordinatized by A. Let $\hat{A}$ be the $(m+1) \times n$ matrix over F obtained from A by adding as a row the vector $(1,\ldots, 1) \in F^n$ and let $\hat{M} = (E, \hat{\mathcal{O}})$ be the oriented matroid coordinatized by $\hat{A}$. We say that A is an *affine* coordinatization of $\hat{M}$. Think of E as the family of points in F^m described by the columns of A. Then $\hat{\mathcal{O}}$ is the set of signed supports of elementary vectors of the subspace $\{\alpha \in F^E : \sum_{e \in E} \alpha(e)\, e = 0 \text{ and } \sum_{e \in E} \alpha(e) = 0\}$ of F^E; we say that $\hat{M}$ is *the oriented matroid on E determined by affine dependence over F.*

We call a matroid orientation $\mathcal{O}$ that arises as in Example 3.1 (or 3.5) a canonical orientation of $(E, \mathcal{O})$. In Example 3.3 we saw that a canonical

orientation that is induced by the $(0, \pm 1)$-vertex-edge incidence matrix of a directed graph has a simple graphical interpretation. We will now give a general geometric interpretation of canonical matroid orientations.

Let F be an ordered field, let m be a positive integer, let E be a finite family of points in F^m, and let $\mathcal{O}$ be the canonical matroid orientation determined by linear dependence over F in E. Recall that $\mathcal{O}$ is the set of signed supports of elementary vectors of the subspace $\mathcal{R} \subseteq F^E$ consisting of all $\alpha \in F^E$ having $\sum_{e \in E} \alpha(e)\, e$ equal to the zero vector in F^m. Let $X \in \mathcal{O}$. If $|\underline{X}| = 1$, then the subset $\{X, -X\} \subseteq \mathcal{O}$ can be trivially described. Suppose that $|\underline{X}| \geqslant 2$. Then for some elementary vector $\alpha \in \mathcal{R}$ we have $X = (S^+(\alpha), S^-(\alpha))$ and $\sum_{e \in \underline{X}} \alpha(e)\, e = 0$. Let $x, y \in \underline{X}$, $x \neq y$, so $\alpha(x) \neq 0$, $\alpha(y) \neq 0$. If $\alpha(x)$ and $\alpha(y)$ have the same sign, then $\alpha(x) + \alpha(y) \neq 0$ so we have

$$\begin{aligned}[]&[\alpha(x) + \alpha(y)]^{-1} [\alpha(x)\, x + \alpha(y)\, y] \\ &\quad = -[\alpha(x) + \alpha(y)]^{-1} \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e)\, e \Big]. \end{aligned} \tag{3.1}$$

In other words, if $\alpha(x)$ and $\alpha(y)$ have the same sign, then the vector subspace of F^m generated by $\underline{X} \setminus \{x, y\}$ intersects the line segment between x and y. (We adopt the convention that the subspace generated by the empty set consists of the zero vector.) The converse can also be easily verified.

Having characterized $\mathcal{O}$ as above, we can give a geometric characterization of $\mathcal{O}^{\perp}$. Recall that the cocircuits of a matroid are the complements of hyperplanes. Assume that the rank of E in F^m, i.e., the rank of $\underline{M}$, is $r \leqslant m$. Then the hyperplanes of $\underline{M}$ correspond to the $(r-1)$-dimensional subspaces of F^m generated by independent subsets of E; of course, if $r = m$ then these $(r-1)$-dimensional subspaces are hyperplanes in F^m. It follows from Lemma 2.2.3, orthogonality, and the characterization of $\mathcal{O}$ above that if $Y \in \mathcal{O}^{\perp}$ and $u, v \in \underline{Y}$, $u \neq v$, then u and v have the same sign in Y if and only if they are on the same side of the vector subspace of F^m generated by $E \setminus \underline{Y}$.

These geometric interpretations remain interesting when the notion of linear dependence is replaced by affine dependence. In the case when M is determined by affine dependence, we have $\sum_{e \in \underline{X}} \alpha(e) = 0$ in (3.1). Thus (3.1) can be rewritten as

$$\begin{aligned}[]&[\alpha(x) + \alpha(y)]^{-1} [\alpha(x)\, x + \alpha(y)\, y] \\ &\quad = \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e) \Big]^{-1} \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e)\, e \Big]. \end{aligned}$$

Therefore we have

PROPOSITION 3.6. *Let F be an ordered field and let E be a finite family of points in F^m. Suppose that $M = (E, \mathcal{O})$ is the oriented matroid on E determined by linear (affine) dependence over F. For $X \in \mathcal{O}$ and $x, y \in \underline{X}$, $x \neq y$, x and y*

agree in sign in X if and only if the linear (affine) subspace of F^m *generate* $\underline{X}\backslash\{x, y\}$ *intersects the line segment between x and y. Furthermore, for* Y *and* $u, v \in \underline{Y}$, $u \neq v$, *u and v have the same sign in Y if and only if u a are on the same side of the linear (affine) subspace of* F^m *generated by* $E\backslash\underline{Y}$

EXAMPLE 3.7. Minors of the Möbius geometry of Cheung and Crap

Let E be a finite subset of $\mathbb{R}^2$ and let $\mathscr{C}$ be the set of all subsets of E c sisting of either four points on a common line, four points on a comm circle, or five points with no four on a common line or circle. Then M $(E, \mathscr{C})$ is a matroid minor of the Möbius geometry introduced by Cheung a Crapo [5].

M is coordinatizable over the reals. Let $f: E \to \mathbb{R}^4$ be defined by $f(a, b)$ $(a^2 + b^2, a, b, 1)$ for $(a, b) \in E$. Then a subset $T \subseteq E$ is dependent in M if a only if $f(T) = \{f(e) : e \in T\}$ is linearly dependent in $\mathbb{R}^4$. Hence Proposition 3.2 M is orientable.

The canonical orientation $\mathscr{O}$ of M induced by f, which can be interpreted $\mathbb{R}^4$ as in Proposition 3.6, has an interesting interpretation in $\mathbb{R}^2$. Hyperplan of M are intersections of E with circles and lines. Each triple in E determin a hyperplane H that partitions $E\backslash H$ into two subsets (interior and exteri points of a circle H, or the sets of points on either side of a line H), whic form the positive and negative elements of the cocircuit $E\backslash H$. It follows fro orthogonality that if $X \in \mathscr{O}$ has $|X| = 4$, then the signs in X of the element of $\underline{X}$ alternate along the line or circle that they define. Suppose that $X \in \mathscr{O}$ ha $|\underline{X}| = 5$ and $x, y \in \underline{X}$, $x \neq y$. Then $\underline{X}\backslash\{x, y\}$ defines a line or circle H, and and y have the same sign in X if and only if the line segment joining the crosses H.

EXAMPLE 3.8. Alternating orientations.

A circuit signature $\mathscr{O}$ of a matroid $(E, \mathscr{O})$ is said to be alternating with respect to an order $H: e_1 < e_2 < \cdots < e_n$ of the elements of E if for every $X \in \mathscr{O}$ with, say, $\underline{X} = \{e_{i_1}, ..., e_{i_s}\}$, $i_1 < i_2 < \cdots < i_s$, $sg_X(e_{i_{j+1}}) = -sg_X(e_{i_j})$ $j = 1, ..., s-1$.

PROPOSITION 3.9. *Let the elements of E be denoted* $e_1, ..., e_n$ *and let H be the order* $e_1 < e_2 < \cdots < e_n$. *If* $M = (E, \mathscr{C})$ *is a matroid with the property*

$$\text{for all } C \in \mathscr{C},\ D \in \mathscr{C}^{\perp} \text{ with } |C \cap D| = 2 \text{ or } 3, \text{ there exist } e', e'' \in C \cap D,\ e' < e'' \text{ such that } e' < e < e'' \text{ implies } e \notin C \cap D \text{ and } |\{e \in E : e' < e < e'' \text{ and } e \notin C \cup D\}| \text{ is even,} \tag{3.2}$$

then the alternating circuit signature $\mathscr{O}$ *of* $(E, \mathscr{C})$ *with respect to H is an orientation of* $(E, \mathscr{C})$.

of. Let $\mathscr{O}'$ be the cocircuit signature of M having for each $Y \in \mathscr{O}'$ with $e_{j_1}, ..., e_{j_t}\}$, $j_1 < j_2 < \cdots < j_t$, $sg_Y(e_{j_m}) = (-1)^{(m+j_m)-(l+j_l)} sg_Y(e_{j_l})$, for $\leqslant l, m \leqslant t$. We will show that $\mathscr{O}, \mathscr{O}'$ satisfies (III), hence $\mathscr{O}$ is an orienta- nd $\mathscr{O}^{\perp} = \mathscr{O}'$. Let Y be as above and suppose that $X \in \mathscr{O}$ with $\underline{X} =$ $, e_{i_s}\} \in \mathscr{C}$, $i_1 < i_2 < \cdots < i_s$, and $|\underline{X} \cap \underline{Y}| = 2$ or 3. Let e' and e'' y (3.2) with $e' = e_{i_p} = e_{j_l}$ and $e'' = e_{i_q} = e_{j_m}$. Then

$$sg_X(e'') = (-1)^{q-p} sg_X(e')$$

$$sg_Y(e'') = (-1)^{(m+j_m)-(l+j_l)} sg_Y(e').$$

ffices to show that $d = (q - p) + (m - l) + (j_m - j_l)$ is odd, since this lies that e' and e'' have opposite signs in one of X and Y and the same in the other. Let $S = \{e \in E : e' < e < e''\}$, so that $j_m - j_l = 1 + |S|$. w $S \cap \underline{X} \cap \underline{Y} = \varnothing$ and $S \cap [E\backslash(\underline{X} \cup \underline{Y})] = c$ is even by (3.2). Therefore $j_l = 1 + c + |S \cap \underline{X}| + |S \cap \underline{Y}| = 1 + c + (q - p - 1) + (m -$ l), so $d = 2(q - p) + 2(m - l) - 1 + c$.

he reader will note that $\mathscr{O}^{\perp}$ is obtainable from the alternating cocircuit ature of M with respect to H by reversing signs on either of the sets : h is odd$\}$ or $\{e_h : h$ is even$\}$.

EXAMPLE 3.8.1. Free matroids.

Let E be an n-element set, and suppose that $1 \leqslant r \leqslant n - 1$. The *free* *atroid of rank r* on E, denoted $\mathscr{F}_n{}^r$, has as its bases all r-element subsets E.

COROLLARY 3.9.1. *The free matroid* $\mathscr{F}_n{}^r$ *has, for each order H of its ements, an alternating orientation with respect to H.*

Proof. Let C and D be a circuit and cocircuit, respectively, of $\mathscr{F}_n{}^r$, so $C| = r + 1$ and $|D| = n - r + 1$. If $|C \cap D| = 2$, then $C \cup D = E$ d (3.2) is obviously satisfied for the pair C, D. Suppose that $C \cap D =$ $, y, z\}$, with $x < y < z$. Then $|E\backslash(C \cup D)| = 1$, say $\{e\} = E\backslash(C \cup D)$. If $> y$, then $e' = x$ and $e'' = y$ satisfy (3.2), otherwise $e' = y$ and $e'' = z$, tisfy (3.2).

Corollary 3.9.1 can be established directly by verifying that for any n real mbers $t_1 < t_2 < \cdots t_n$, the matrix

$$\begin{bmatrix} t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & & \vdots \\ t_1^{r-1} & t_2^{r-1} & & t_n^{r-1} \end{bmatrix}$$

s an affine coordinatization of $\mathscr{F}_n{}^r$ that induces an alternating orientation.

An example of a nonfree matroid that satisfies the hypothesis of Propos[illegible] tion 3.9 is the matroid M determined by affine dependence in $\mathbb{R}^2$ on the si[illegible] points in Fig. 1.

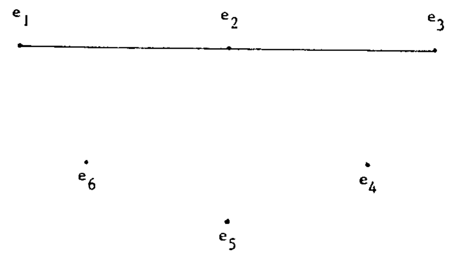

FIG. 1. A nonfree matroid with an alternating orientation.

EXAMPLE 3.10. The noncoordinatizable Vámos matroid is orientable. Let $M_1 = (E, \mathcal{O}_1)$ be the oriented matroid determined by affine dependence over the reals on the set $E = \{e_1, \ldots, e_8\} \subseteq \mathbb{R}^3$ given in Fig. 2.

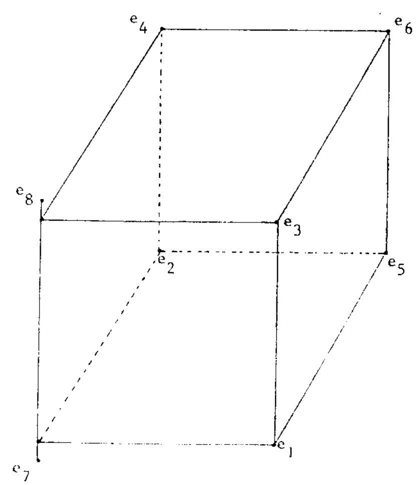

FIG. 2. A pre-Vámos oriented matroid M_1.

Note that six of the eight points in E are vertices of the unit cube, while e_7 and e_8 are translations of the remaining two vertices of the cube some small distance $\epsilon > 0$ along the line determined by that pair of vertices. Thus $\{e_3, e_4, e_6, e_8\}$ and $\{e_1, e_2, e_5, e_7\}$ are independent sets in $\underline{M}_1$. The circuit $C^* = \{e_1, e_2, e_3, e_4\}$ and the cocircuit $D^* = \{e_5, e_6, e_7, e_8\}$ of $\underline{M}_1$ play a special role in what follows.

Let $(E, \mathscr{C}_2)$ be the matroid having $\mathscr{C}_2 = (\underline{\mathcal{O}}_1 \cup \{C_5, C_6, C_7, C_8\}) \backslash \{C^*\}$,

[illegible]ere $C_i = C^* + e_i$, $i = 5, \ldots, 8$. Note that $\mathscr{C}_2^\perp = (\underline{\mathcal{O}}_1^\perp \cup \{D_1, D_2, D_3, D_4\}) \backslash \{D^*\}$, where $D_j = D^* + e_j$, $j = 1, \ldots, 4$. Vámos (see [9]) showed that $(E, \mathscr{C}_2)$ is not coordinatizable over any field (or division ring). We will [illegible]ploit the close resemblance between $(E, \mathscr{C}_2)$ and the coordinatizable matroid M_1 to describe how an orientation of $(E, \mathscr{C}_2)$ can be constructed. Let $\mathcal{O}_2$ and $\mathcal{O}_2'$ be a circuit signature and a cocircuit signature, respectively, of $(E, \mathscr{C}_2)$ [illegible]aving $X \in \mathcal{O}_2$ for all $X \in \mathcal{O}_1$ with $\underline{X} \in \mathscr{C}_2$ and $Y \in \mathcal{O}_2'$ for all $Y \in \mathcal{O}_1^\perp$ with $\underline{Y} \in \mathscr{C}_2^\perp$. The remaining circuits in $\mathscr{C}_2$ yet to be signed into $\mathcal{O}_2$ are C_5, C_6, C_7, C_8 and the remaining cocircuits are D_1, D_2, D_3, D_4. Let $X^* \in \mathcal{O}_1$ and $Y^* \in \mathcal{O}_1^\perp$ with $\underline{X}^* = C^*$ and $\underline{Y}^* = D^*$. For each C_i, $i = 5, \ldots, 8$, include in $\mathcal{O}_2$ the signed sets X_i and $-X_i$ described by

$$\begin{aligned} sg_{X_i}(e) &= sg_{X^*}(e) && \text{if } e \neq e_i, \\ &= sg_{Y^*}(e) && \text{if } e = e_i, \end{aligned}$$

[illegible]nd for each Y_j, $j = 1, \ldots, 4$ include in $\mathcal{O}_2'$ the signed sets Y_j and $-Y_j$ having

$$\begin{aligned} sg_{Y_j}(e) &= sg_{Y^*}(e) && \text{if } e \neq e_j, \\ &= -sg_{X^*}(e) && \text{if } e = e_j. \end{aligned}$$

[illegible]ote that for $1 \leqslant j \leqslant 4$ and $5 \leqslant i \leqslant 8$, X_i and Y_j are orthogonal since e_i has the same sign in X_i and Y_j and e_j has opposite signs in X_i and Y_j. Moreover, for any $X \in \mathcal{O}_2$, $Y \in \mathcal{O}_2'$ such that either $\underline{X} \in \mathscr{C}_1$ or $\underline{Y} \in \mathscr{C}_1^\perp$, orthogonality of X and Y follows from the fact that M_1 is an oriented matroid. Therefore $\mathcal{O}_2$ is an orientation of the Vámos matroid $(E, \mathscr{C}_2)$ and $\mathcal{O}_2' = \mathcal{O}_2^\perp$.

The orientation $\mathcal{O}_2$ is given explicitly by the set of signed sets described in Table I and their opposites. Each entry in the table gives the index set of the elements in a circuit of the Vámos matroid. The signed set X represented by

TABLE I

The Orientation $\mathcal{O}_2$ of the Vámos Matroid

$1\bar{3}\bar{5}6$	$1\bar{2}\bar{3}45$	$123\bar{6}\bar{7}$	$\bar{1}\bar{2}567$	$\bar{1}\bar{3}467$	$\bar{2}\bar{3}457$	$\bar{2}\bar{3}578$
$1\bar{3}\bar{7}8$	$1\bar{2}\bar{3}46$	$1\bar{2}\bar{3}68$	$12\bar{5}6\bar{8}$	$1\bar{3}\bar{4}68$	$234\bar{5}\bar{8}$	$\bar{2}\bar{3}678$
$2\bar{4}\bar{5}6$	$1\bar{2}\bar{3}4\bar{7}$	$\bar{1}\bar{2}457$	$\bar{1}\bar{2}578$	$145\bar{6}\bar{7}$	$\bar{2}\bar{3}467$	$345\bar{6}\bar{7}$
$2\bar{4}\bar{7}8$	$1\bar{2}\bar{3}4\bar{8}$	$1\bar{2}4\bar{5}\bar{8}$	$12\bar{6}\bar{7}8$	$14\bar{5}6\bar{8}$	$2\bar{3}\bar{4}68$	$\bar{3}\bar{4}568$
$5\bar{6}\bar{7}8$	$1\bar{2}\bar{3}\bar{5}\bar{7}$	$124\bar{6}\bar{7}$	$1\bar{3}\bar{4}57$	$\bar{1}\bar{4}578$	$235\bar{6}\bar{7}$	$34\bar{5}7\bar{8}$
	$1\bar{2}\bar{3}58$	$1\bar{2}4\bar{6}\bar{8}$	$134\bar{5}\bar{8}$	$\bar{1}\bar{4}678$	$23\bar{5}6\bar{8}$	$\bar{3}\bar{4}678$

an entry has as X^- those elements whose indices are overlined, e.g., $1\bar{3}\bar{5}6$ represents an oriented circuit X having $X^+ = \{1, 6\}$ and $X^- = \{3, 5\}$. The special circutis $C_5, \ldots, C_8$ of the Vámos matroid correspond to the first four entries in the second column of Table I. Table II similarly describes $\mathcal{O}_2^{\perp}$.

TABLE II

$\mathcal{O}_2^{\perp}$, the Dual of $\mathcal{O}_2$

1356	$\bar{1}56\overline{78}$	$12\overline{36}7$	$125\bar{6}7$	$13\overline{46}7$	$2\bar{3}457$	$2\bar{3}57\bar{8}$
1378	$256\overline{78}$	$1\bar{2}368$	$1256\bar{8}$	13468	$\bar{2}34\bar{5}8$	$2\overline{36}7\bar{8}$
2456	$356\overline{78}$	12457	$1257\bar{8}$	$\bar{1}456\bar{7}$	$2\bar{3}4\bar{6}7$	$3456\bar{7}$
2478	$\bar{4}56\overline{78}$	$\bar{1}24\bar{5}8$	$12\bar{6}78$	$1\bar{4}56\bar{8}$	23468	$34\bar{5}68$
$1\bar{2}3\bar{4}$	12357	$12\overline{46}7$	$13\bar{4}57$	$1\bar{4}57\bar{8}$	$\bar{2}356\bar{7}$	$34\bar{5}78$
	$1\bar{2}3\bar{5}8$	$\bar{1}2468$	$\bar{1}34\bar{5}8$	$\bar{1}4\bar{6}78$	$2\bar{3}56\bar{8}$	$34\bar{6}78$

The cocircuits $D_1, \ldots, D_4$ of the Vámos matroid correspond to the first four entries in the second column.

In demonstrating that $\mathcal{O}_2$ is an orientation we have relied on the structure of $\underline{\mathcal{O}}_2$ and the fact that $\mathcal{O}_1$ is an orientation of M_1; we have not specifically invoked the sign properties of $\mathcal{O}_1$ that distinguish it from other orientations of M_1. Hence, any orientation of M_1 induces, as above, an orientation of M_2.

Other examples of noncoordinatizable orientable matroids are the non-Desargues matroid, [9, Example 2], and a modification of the non-Pappus matroid, [9, Example 3].

EXAMPLE 3.11. Some nonorientable matroids.

Let $r \geqslant 3$ be an integer and let E be a set of cardinality $2r$, $E = \{e_1, \ldots, e_r, e_1', \ldots, e_r'\}$. We denote by M_r the matroid on E with the following circuits: $\{e_i, e_i', e_j, e_j'\}$ for $1 \leqslant i < j \leqslant r$, $\{e_1, \ldots, e_{i-1}, e_i', e_{i+1}, \ldots, e_r\}$ for $1 \leqslant i \leqslant r$, $\{e_1', \ldots, e_r'\}$ and all $(r+1)$-subsets of E not containing any of the preceding $[r(r-1)/2] + r + 1$ sets.

LEMMA 3.11.1. *The involution τ on E defined by $\tau(e_i) = e_i'$ for $1 \leqslant i \leqslant r$ is an isomorphism from M_r to $(M_r)^{\perp}$.*

The proof is left to the reader.

Let $\mathbb{Q}$ denote the field of rational numbers.

LEMMA 3.11.2. *For all $e \in E$, $M_r \backslash e$ and M_r / e are coordinatizable over $\mathbb{Q}$.*

Proof. By the symmetry of M_r with respect to $e_1, \ldots, e_r$ and by Lemma 3.11.1 it suffices to prove Lemma 3.11.2 for $e = e_r'$.

A coordinatization of $M_r \backslash e_r'$. Let $\alpha_1, \ldots, \alpha_r$ be the canonical basis of $\mathbb{Q}^r$ and let $\alpha_i' = \alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r$ for $1 \leqslant i \leqslant r-1$. Then $M_r \backslash e_r'$ is isomorphic to the matroid on $\{\alpha_1, \ldots, \alpha_r, \alpha_1', \ldots, \alpha_{r-1}'\}$ determined by linear dependence in $\mathbb{Q}^r$.

A coordinatization of M_r / e_r'. Let $\alpha_1, \ldots, \alpha_{r-2}, \alpha_r$ be the canonical basis of $\mathbb{Q}^{r-1}$, $\alpha_{r-1} = \alpha_1 + \cdots + \alpha_{r-2}$, $\alpha_i' = \alpha_i + \alpha_r$ for $1 \leqslant i \leqslant r-2$, and $\alpha_{r-1}' = \alpha_{r-1} + (r-2)\alpha_r$. Then M_r / e_r' is isomorphic to the matroid on $\{\alpha_1, \ldots, \alpha_r, \alpha_1', \ldots, \alpha_{r-1}'\}$ determined by linear dependence in $\mathbb{Q}^{r-1}$.

PROPOSITION 3.12. *For $r \geqslant 4$, M_r is not orientable.*

Before proving Proposition 3.12 it will be useful to state the following simple consequence of the signed elimination property (II).

LEMMA 3.12.1. *Let $M = (E, \mathcal{O})$ be an oriented matroid. Suppose that $X_1, X_2 \in \mathcal{O}$, $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, and that there is a unique circuit of $\underline{M}$ contained in $(\underline{X}_1 \cup \underline{X}_2) \backslash x$. Then there are exactly two signed circuits $X_3 \in \mathcal{O}$ and $-X_3 \in \mathcal{O}$ contained in $(\underline{X}_1 \cup \underline{X}_2) \backslash x$, and $(X_1^+ \cup X_2^+) \backslash (X_1^- \cup X_2^-) \subseteq X_3^+$, $(X_1^- \cup X_2^-) \backslash (X_1^+ \cup X_2^+) \subseteq X_3^-$.*

Proof of Proposition 3.12. Suppose, contrary to the proposition, that $\mathcal{O}$ is an orientation of M_r, $r \geqslant 4$. Denote by X_j, for $2 \leqslant j \leqslant r$, and Z signed circuits having $\underline{X}_j = \{e_1, e_1', e_j, e_j'\}$ and $\underline{Z} = \{e_1', e_2, \ldots, e_r\}$. By reversing signs in $\mathcal{O}$ on a subset of E and appropriately choosing each X_j and Z from the pair of opposite signed circuits having the given underlying circuit, we can assume with no loss of generality that $X_2^+ = \{e_1, e_1'\}$, $X_2^- = \{e_2, e_2'\}$, $e_1 \in X_j^+$ and $\{e_j, e_j'\} \subseteq X_j^-$ for $3 \leqslant j \leqslant r$, and $e_1' \in Z^-$.

Note that $C_j = \{e_2, e_2', e_j, e_j'\}$ is the unique circuit of M_r contained in $(\underline{X}_2 \cup \underline{X}_j) \backslash e_1$ (because $r \geqslant 4$). Since $e_1, e_1' \notin C_j$, $e_1 \in X_2^+ \cap (-X_j)^-$, and $e_1' \in X_2^+$, it follows from Lemma 3.12.1 that $e_1' \in (-X_j)^- = X_j^+$. Thus $X_j^+ = \{e_1, e_1'\}$, $X_j^- = \{e_j, e_j'\}$ for $j = 2, \ldots, r$.

Similarly, for $2 \leqslant j \leqslant r$ $\{e_1, \ldots, e_{j-1}, e_j', e_{j+1}, \ldots, e_r\}$ is the unique circuit of M_r contained in $(\underline{Z} \cup \underline{X}_j) \backslash e_1'$. By Lemma 3.12.1 $e_j \in Z^+$, hence $Z^- = \{e_1'\}$ and $Z^+ = \{e_2, \ldots, e_r\}$.

Now, by Lemma 3.11.1 $\{e_1, \ldots, e_r\}$ is a cocircuit of M_r. Let $Y \in \mathcal{O}^{\perp}$ have $\underline{Y} = \{e_1, \ldots, e_r\}$ and $e_1 \in Y^+$. It follows from orthogonality of $X_j \in \mathcal{O}$ and $Y \in \mathcal{O}^{\perp}$ that $e_j \in Y^+$, $j = 2, \ldots, r$, so $Y_j^- = \varnothing$. But this contradicts orthogonality of $Z \in \mathcal{O}$ and $Y \in \mathcal{O}^{\perp}$.

Proposition 3.12, Lemma 3.11.2, and Proposition 3.2 indicate that for all $r \geqslant 4$ M_r is not orientable, but all proper minors of M_r are orientable.

Therefore the matroids that collectively characterize orientable matroids their exclusion as minors (in the spirit of [14]) are infinite in numb Examples of rank 3 nonorientable matroids with all proper minors orienta include the MacLane matroid (see [9]), as has been verified by Yves Kodra (CNRS, Paris) with the aid of a computer, and the Fano matroid.

The matroids M_r are related to well-known matroids introduced Lazarson (see [9]). Let $p \geqslant 2$ be a prime number, let $GF(p)$ be the Gal field $\mathbb{Z}/p\mathbb{Z}$, and let $E = \{e_1, \ldots, e_{p+1}, e_1', \ldots, e_{p+1}', f\} \subseteq (GF(p))^{p+1}$, wh $\{e_1, \ldots, e_{p+1}\}$ is the canonical basis of $(GF(p))^{p+1}$, $f = e_1 + \cdots + e_{p+1}$, a $e_i' = f - e_i$, $i = 1, \ldots, p+1$. Let L_p be the matroid on E determined linear dependence in $(GF(p))^{p+1}$. Lazarson showed that L_p is coordinatizab over a division ring F if and only if F has characteristic p. L_2 is the Fa matroid.

PROPOSITION 3.13. *For all prime numbers $p \geqslant 2$ M_{p+1} is isomorphic* $L_p \backslash f$.

The proof is left to the reader.

It follows from Propositions 3.12 and 3.13 that for all prime numbe $p \geqslant 3$ the matroid L_p is not orientable. L_2 is also nonorientable as alrea noted.

Ingleton observed in [9] that for all prime numbers $p \geqslant 3$ $L_p \backslash f$ (and hen M_{p+1}) is coordinatizable over a division ring F if and only if F has chara teristic p. It is not difficult to show that if the integer $p \geqslant 3$ is not prime, th M_{p+1} is not coordinatizable over any division ring.

4. MINORS OF ORIENTED MATROIDS

It is clear from Lemma 2.1.3 that minors of orientable matroids ar orientable. In this section we will discuss oriented matroid minors. First w recall some notation from Section 2: (1) if X is a signed subset of E an $A \subseteq E$, then $(X\backslash A)$ denotes the signed set having $(X\backslash A)^+ = X^+\backslash A$ and $(X\backslash A)$ $= X^-\backslash A$; (2) if $\mathcal{O}$ is a collection of signed subsets of E, then $\mathrm{Min}(\mathcal{O})$ denotes th set of $X \in \mathcal{O}$ such that $\underline{X}$ is a (set-wise) minimal element of $\underline{\mathcal{O}}$.

PROPOSITION 4.1. *Let $M = (E, \mathcal{O})$ be an oriented matroid and let A and be disjoint subsets of E. Then*

$$\hat{\mathcal{O}} = \mathrm{Min}\{X\backslash A : X \in \mathcal{O}, X\backslash A \neq \varnothing \text{ and } \underline{X} \cap B = \varnothing\}$$

and

$$\hat{\mathcal{O}}^{\perp} = \mathrm{Min}\{Y\backslash B : Y \in \mathcal{O}^{\perp}, Y\backslash B \neq \varnothing \text{ and } \underline{Y} \cap A = \varnothing\}$$

are matroid orientations and $\hat{\mathcal{O}}^{\perp} = (\hat{\mathcal{O}})^{\perp}$.

oof. Note that $\underline{\hat{M}} = (E, \underline{\hat{\mathcal{O}}})$ is the matroid minor of $\underline{M}$ obtained by racting A and deleting B and $\underline{\hat{M}}^{\perp} = (E, \underline{\hat{\mathcal{O}}}^{\perp})$. It is easy to see that the $\hat{\mathcal{O}}$, $\hat{\mathcal{O}}^{\perp}$ satisfies the orthogonality property (IV), since any pair $\hat{X} \in \hat{\mathcal{O}}$, $\hat{\mathcal{O}}^{\perp}$ corresponds to a pair $X \in \mathcal{O}$, $Y \in \mathcal{O}^{\perp}$ having $\hat{X} = X\backslash A$, $\hat{Y} = Y\backslash B$ and $\underline{Y} \subseteq \underline{\hat{X}} \cap \underline{\hat{Y}}$. Thus it suffices to show that $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^{\perp}$ are signatures of $\underline{\hat{M}}$. rly $\hat{\mathcal{O}} = -\hat{\mathcal{O}}$. We must show that $\hat{X}_1, \hat{X}_2 \in \hat{\mathcal{O}}$ and $\underline{\hat{X}}_1 = \underline{\hat{X}}_2$ imply $\hat{X}_1 =$. It then follows that $\hat{\mathcal{O}}$ is a circuit signature of $\underline{\hat{M}}$ and by symmetry $\hat{\mathcal{O}}^{\perp}$ cocircuit signature of $\underline{\hat{M}}$.

uppose that $\hat{X}_1, \hat{X}_2 \in \hat{\mathcal{O}}$ and $\underline{\hat{X}}_1 = \underline{\hat{X}}_2$. There exist $X_1, X_2 \in \mathcal{O}$ such that $X_i\backslash A$ and $\underline{X}_i \cap B = \varnothing$ for $i = 1, 2$. Suppose that $\hat{X}_1 \neq \pm\hat{X}_2$, so e is an element $e \in (\hat{X}_1^+ \cap \hat{X}_2^-) \cup (\hat{X}_1^- \cap \hat{X}_2^+)$ and an element $e' \in$ $\cap \hat{X}_2^+) \cup (\hat{X}_1^- \cap \hat{X}_2^-)$. Since $\mathcal{O}$ is an orientation and both e and e' have same sign in X_i and $\hat{X}_i$, $i = 1, 2$, by the signed elimination property (II) $\mathcal{O}$ there is some $X_3 \in \mathcal{O}$ having $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$, $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$, $e' \in \underline{X}_3\backslash A$. Therefore $\underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash e$, so $\underline{X}_3 \cap B = \varnothing$ and $e' \in (\underline{X}_3\backslash A) \subseteq$ $\cup \underline{X}_2)\backslash(A + e) = \underline{\hat{X}}_1\backslash e$, contradicting the minimality of $\hat{X}_1$ in $\hat{\mathcal{O}}$.

Given $M = (E, \mathcal{O})$, A and B as in Proposition 4.1 we say that $\hat{M}$ is the *ented matroid minor* of M obtained by *contracting* A and deleting B, and, course, $\hat{M}^{\perp}$ is the oriented matroid minor of $M^{\perp}$ obtained by contracting B d deleting A. If $A = \varnothing$, then $\hat{\mathcal{O}} = \{X \in \mathcal{O} : \underline{X} \cap B = \varnothing\}$, denoted $\mathcal{O}\backslash B$, and $B = \varnothing$, then $\hat{\mathcal{O}} = \mathrm{Min}\{X\backslash A : X \in \mathcal{O}$ and $\underline{X}\backslash A \neq \varnothing\}$, denoted $\hat{\mathcal{O}} = \mathcal{O}/A$. m the analogous properties of matroid minors we easily get

PROPOSITION 4.2. *Let M be an oriented matroid on a set E and let A and B disjoint subsets of E. Then*

(i) $(M\backslash A)\backslash B = M\backslash(A \cup B)$;

(ii) $(M/A)/B = M/(A \cup B)$;

(iii) $(M/A)\backslash B = (M\backslash B)/A$.

us Proposition 4.1 can be restated in the form

THEOREM 4.3. *If $M = (E, \mathcal{O})$ is an oriented matroid and $A \subseteq E$, then A and $M\backslash A$ are oriented matroids and $(M/A)^{\perp} = M^{\perp}\backslash A$, $(M\backslash A)^{\perp} = M^{\perp}/A$.*

Proposition 4.2 and Lemma 2.1.2 imply the following useful result.

PROPOSITION 4.4. *If $M = (E, \mathcal{O})$ is an oriented matroid, $A \subseteq E$, $X \in \mathcal{O}$, and $\in \underline{X}\backslash A$, then there exists $\hat{X} \in \mathcal{O}/A$ such that $x \in \underline{\hat{X}}$ and $\hat{X}^+ \subseteq X^+$, $\hat{X}^- \subseteq X^-$.*

5. CARRIERS AND SPANS OF ORIENTED MATROIDS

In this section we will discuss certain sets of signed sets whose minimal onempty elements are the signed circuits of an oriented matroid. First we examine the effect of relaxing the requirement in (0) of Theorem 2.1 [redacted] $X_1, X_2 \in \mathcal{O}$ and $\underline{X}_2 \subseteq \underline{X}_1$ imply $X_1 = \pm X_2$.

PROPOSITION 5.1. *Let $\mathcal{O}$ be any set of nonempty signed sets such* [redacted] *satisfies* (I) *and has $\mathcal{O} = -\mathcal{O}$. Then for each $X \in \mathcal{O}$ there exists $X' \in$* Min [redacted] *such that $X'^+ \subseteq X^+$ and $X'^- \subseteq X^-$.*

Proof. Let $X_1 \in \mathcal{O}$ have $X_1^+ \subseteq X^+$, $X_1^- \subseteq X^-$, and $|\underline{X}_1|$ as small as poss [redacted] If $X_1 \in \text{Min}(\mathcal{O})$, then $X' = X_1$ satisfies the conclusion of the proposit [redacted] Suppose that $X_1 \notin \text{Min}(\mathcal{O})$, so there exists $X_2 \in \mathcal{O}$ having $\underline{X}_2 \subsetneq \underline{X}_1$. Se [redacted] $X_2 \in \mathcal{O}$ such that $\underline{X}_2 \subsetneq \underline{X}_1$ and $|(X_2^+ \cap X_1^-) \cup (X_2^- \cap X_1^+)|$ is minimized [redacted] $e \in (X_2^+ \cap X_1^-) \cup (X_2^- \cap X_1^+)$, which is nonempty by the choice of X_1 [redacted] (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X$ [redacted] $X_2^-)\backslash e$. But then $\underline{X}_3 \subsetneq \underline{X}_1$ and $(X_3^+ \cap X_1^-) \cup (X_3^- \cap X_1^+) \subseteq [(X_2^+ \cap X_1$ [redacted] $(X_2^- \cap X_1^+)]\backslash e$, contradicting the choice of X_2.

THEOREM 5.2. *Let $\mathcal{O}$ be a set of nonempty signed sets such that $\mathcal{O}$ satis* [redacted] *the elimination property* (I) *and has $\mathcal{O} = -\mathcal{O}$. Then* Min($\mathcal{O}$) *is the set of sig* [redacted] *circuits of an oriented matroid.*

Proof. Clearly $X \in \text{Min}(\mathcal{O})$ implies $X \neq \varnothing$ and $-X \in \text{Min}(\mathcal{O})$.

Suppose that $X_1, X_2 \in \text{Min}(\mathcal{O})$ with $\underline{X}_2 \subseteq \underline{X}_1$ and $X_1 \neq \pm X_2$. Let $e \in (X$ [redacted] $X_2^-) \cup (X_1^- \cap X_2^+)$, which is nonempty since $X_1 \neq \pm X_2$ and $\varnothing \neq \underline{X}_2 \subseteq$ [redacted] By (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X$ [redacted] $X_2^-)\backslash e$, so $\underline{X}_3 \subseteq \underline{X}_1\backslash e$, contradicting $X_1 \in \text{Min}(\mathcal{O})$.

Now let $X_1, X_2 \in \text{Min}(\mathcal{O})$, $X_1 \neq \pm X_2$, and $e \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X$ [redacted] By (I) there is some $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X_1$ [redacted] $X_2^-)\backslash e$. By Proposition 5.1 there exists $X_3' \in \text{Min}(\mathcal{O})$ such that $X_3'^+ \subseteq X_3$ [redacted] $(X_1^+ \cup X_2^+)\backslash e$ and $X_3'^- \subseteq X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$. Hence Min($\mathcal{O}$) satisfies [redacted]

Signed sets X_1, X_2 having $(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+) = \varnothing$ will be call [redacted] *compatible*. The *union* $X_1 \cup X_2$ of compatible signed sets X_1 and X_2 is defi [redacted] to be the signed set having $(X_1 \cup X_2)^+ = X_1^+ \cup X_2^+$ and $(X_1 \cup X_2)^-$ [redacted] $X_1^- \cup X_2^-$. Given mutually compatible signed sets $X_1, \ldots, X_k$ in some se [redacted] signed sets $\mathscr{P}$, the union $X_1 \cup X_2 \cup \cdots \cup X_k$ is said to have a *confor* [redacted] *decompsition* in $\mathscr{P}$.

PROPOSITION 5.3. *If $\mathcal{O}$ is a set of signed sets such that $\mathcal{O}$ satisfies* ($\text{I}\frac{\text{I}}{\text{II}}$) [redacted] *has $\mathcal{O} = -\mathcal{O}$, then every $X \in \mathcal{O}$ has a conformal decomposition in* Min($\mathcal{O}$).

Proof. Suppose that $|\underline{X}|$ is minimal subject to $X \in \mathcal{O}$ having no conform [redacted] decomposition in Min($\mathcal{O}$). There is no loss of generality in assuming $X^- =$ [redacted] since all of property ($\text{I}\frac{\text{I}}{\text{II}}$), Min($\mathcal{O}$), and the subset of $\mathcal{O}$ having conform [redacted] decompositions in Min($\mathcal{O}$) are invariant under the reversal of signs on a [redacted] subset of E. By Proposition 5.1 there exists $X_1 \in \text{Min}(\mathcal{O})$ having $X_1^+ \subseteq X^+$ a [redacted]

[redacted] $X^- = \varnothing$. It suffices to show that for each $e \in \underline{X}\backslash\underline{X}_1$ there is some [redacted] having $X_e^- = \varnothing$ and $e \in X_e^+ \subsetneq X^+$. Then by the choice of X each X_e [redacted] conformal decomposition in Min($\mathcal{O}$) giving (with X_1) a conformal [redacted] position of X in Min($\mathcal{O}$).

[redacted] $e \in \underline{X}\backslash\underline{X}_1$ and let $X_2 \in \mathcal{O}$ have $e \in X_2^+ \subsetneq X^+ \cup X_1^- = X^+$, $X_2^- \subsetneq X^- \cup$ [redacted] X_1^+, and $|X_2^-|$ as small as possible (property ($\text{I}\frac{\text{I}}{\text{II}}$) ensures that we [redacted] d such an $X_2 \in \mathcal{O}$). Suppose $e' \in X_2^-$. Then by ($\text{I}\frac{\text{I}}{\text{II}}$) there exists $X_3 \in \mathcal{O}$ [redacted] that $e \in X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e' \subseteq X^+$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e' \subseteq X_2^-\backslash e'$, [redacted] dicting the choice of X_2. Thus $X_2^- = \varnothing$ and $e \in X_2^+ \subseteq X^+$.

[redacted] OREM 5.4. *If $\mathcal{O}$ is a set of nonempty signed subsets of E that has $\mathcal{O} =$* [redacted] *hen $\mathcal{O}$ satisfies* ($\text{I}\frac{\text{I}}{\text{II}}$) *if and only if $\mathcal{O}$ satisfies* (II).

[redacted] *of.* Clearly (II) implies ($\text{I}\frac{\text{I}}{\text{II}}$). Suppose that $\mathcal{O}$ satisfies ($\text{I}\frac{\text{I}}{\text{II}}$) and has [redacted] $\mathcal{O}$. Let $X_1, X_2 \in \mathcal{O}$ with $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+ \cap$ [redacted] $(X_1^- \cap X_2^-)$. We must show that there exists

$$[\text{redacted}] \in \mathcal{O} \text{ such that } y \in \underline{X}_3,\ X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x, \text{ and } X_3^- \subseteq (X_1^- \cup [\text{redacted}])\backslash x. \tag{5.1}$$

[redacted] oposition 5.3 X_1 and X_2 have conformal decompositions in Min($\mathcal{O}$); in [redacted] ular there must exist $X_1', X_2' \in \text{Min}(\mathcal{O})$ such that $X_i'^+ \subseteq X_i^+$, $X_i'^- \subseteq X_i^-$, [redacted] $\underline{X}_i'$, $i = 1, 2$. If $x \notin \underline{X}_i'$ for $i = 1$ or 2, then $X_3 = X_i'$ satisfies (5.1). If [redacted] $\cap \underline{X}_2'$, then it must be that $x \in (X_1'^+ \cap X_2'^-) \cup (X_1'^- \cap X_2'^+)$. Now, [redacted] $= -\mathcal{O}$ and $\mathcal{O}$ satisfies ($\text{I}\frac{\text{I}}{\text{II}}$), and hence (I), by Theorem 5.2 Min($\mathcal{O}$) is [redacted] ented matroid. So by property (II) for Min($\mathcal{O}$), there exists $X_3 \in \text{Min}(\mathcal{O})$ [redacted] that $y \in \underline{X}_3$, $X_3^+ \subseteq (X_1'^+ \cap X_2'^+)\backslash x$ and $X_3^- \subseteq (X_1'^- \cup X_2'^-)\backslash x$ and X_3 [redacted] es (5.1).

[redacted] reader will note that under the hypothesis of Theorem 5.4, the elimina- [redacted] roperty (I) is not equivalent to ($\text{I}\frac{\text{I}}{\text{II}}$) and (II). For example, let $E =$ [redacted] 3} and let $\mathcal{O}$ consist of the six signed subsets of E described by

$$123,\ \overline{123},\ 2,\ \bar{2},\ 3,\ \bar{3}.$$

[redacted] $\mathcal{O} = -\mathcal{O}$ and $\mathcal{O}$ satisfies (I), but not (II).

[redacted] et $\mathcal{O}$ of signed sets satisfying ($\text{I}\frac{\text{I}}{\text{II}}$) and $\mathcal{O} = -\mathcal{O}$ will be called a *carrier* [redacted] matroid orientation Min($\mathcal{O}$). Proposition 5.3 indicates that for a given [redacted] ed matroid $M = (E, \mathcal{O})$ every carrier of $\mathcal{O}$ is a subset of the set $\mathscr{K}(\mathcal{O})$, [redacted] *gned span of* $\mathcal{O}$, consisting of all signed subsets of E having conformal [redacted] positions in $\mathcal{O}$.

[redacted] is an ordered field and $\mathcal{O}$ is the set of signed supports of elementary [redacted] s in a vector subspace $\mathscr{R} \subseteq F^E$, then $\mathscr{K}(\mathcal{O})$ is the set of signed supports [redacted] ctors in $\mathscr{R}$. In fact, for any oriented matroid $M = (E, \mathcal{O})$, $\mathscr{K}(\mathcal{O})$ retains [redacted] of the familiar properties that hold for the coordinatizable case. For example : $\varnothing \in \mathscr{K}(\mathcal{O})$, since the empty signed set decomposes into an empty union of signed circuits; $\mathscr{K}(\mathcal{O}) = -\mathscr{K}(\mathcal{O})$; $\mathcal{O} \subseteq \mathscr{K}(\mathcal{O})$; $\mathscr{K}(\mathcal{O})$ satisfies the elimination properties (I) and (II) of Theorem 2.1; and the pair $\mathscr{K}(\mathcal{O}), \mathscr{K}(\mathcal{O}^\perp)$ satisfies the painting property (V) and the orthogonality properties (III) and (IV) of Theorem 2.2 (in fact $\mathscr{K}(\mathcal{O})$ is precisely the set of all signed subsets X of E having X orthogonal to all $Y \in \mathcal{O}^\perp$). These conditions on signed spans of orientations are among the properties that Rockafellar [13] recognized oriented matroids ought to have.

The effect of contractions and deletions on the signed span of an orientation $\mathcal{O}$ on E is particularly easy to describe. Obviously $\mathscr{K}(\mathcal{O}\backslash e) = \{X \in \mathscr{K}(\mathcal{O}): e \notin \underline{X}\}$ for all $e \in E$. Furthermore, Proposition 4.4 implies that $X \backslash e \in \mathscr{K}(\mathcal{O}/e)$ for every $X \in \mathcal{O}$ and $e \in E$. Thus we have

PROPOSITION 5.5. *Let $M = (E, \mathcal{O})$ be an oriented matroid, let A and B be disjoint subsets of E, and let $\hat{\mathcal{O}} = (\mathcal{O}/A)\backslash B$. Then $\mathscr{K}(\hat{\mathcal{O}}) = \{X \backslash A: X \in \mathscr{K}(\mathcal{O})$ and $\underline{X} \cap B = \varnothing\}$.*

6. BINARY ORIENTED MATROIDS

Let E be a finite set. Recall that a subspace $\mathscr{R}$ of $\mathbb{R}^E$ is *unimodular* (*regular*) if all elementary vectors of $\mathscr{R}$ are proportional to $(0, \pm 1)$-vectors. A matroid M on E is *binary* if M is coordinatizable over $GF(2)$ and M is *unimodular* (*regular*) if $M = \underline{S}(\mathscr{R})$ for some unimodular subspace $\mathscr{R} \subseteq R^E$.

A *digraphoid* as defined by Minty in [12] is a dual pair of matroids $M, M^\perp$ together with circuit signatures $\mathcal{O}$ and $\mathcal{O}^\perp$ of M and $M^\perp$, respectively, such that the following axiom is satisfied:

$$\text{for all } X \in \mathcal{O},\ Y \in \mathcal{O}^\perp,\quad |(X^+ \cap Y^+) \cup (X^- \cap Y^-)| = |(X^+ \cap Y^-) \cup (X^- \cap Y^+)|.$$

It is obvious from the orthogonality property (IV) that a digraphoid is a dual pair of oriented matroids.

Actually digraphoids constituted the first attempt at axiomatizing oriented matroids. However, the above axiom is too restrictive—Minty showed that digraphoids are precisely the dual pairs of oriented matroids $S(\mathscr{R}), S(\mathscr{R}^\perp)$ for unimodular subspaces $\mathscr{R}$ of vector spaces $\mathbb{R}^E$ (see [12, App. 1]). The main result of this section is that binary oriented matroids are precisely the oriented matroids $S(\mathscr{R})$ that arise from unimodular subspaces $\mathscr{R}$ of $\mathbb{R}^E$, and digraphoids are, therefore, equivalent to dual pairs of binary oriented matroids.

THEOREM (Tutte [14, Proposition 7.51]). *A matroid M is unimodular if and only if M is binary and has no minor isomorphic to the Fano matroid (L_2 of Example 3.11) or its dual.*

PROPOSITION 6.1. *A binary matroid is orientable if and only if it is a unimodular matroid.*

Proof. Since minors of orientable matroids are orientable (see Section 4), an orientable matroid can have no minor isomorphic to the Fano, matroid or its dual. Hence by Tutte's theorem above a binary orientable matroid is unimodular. The converse is clear.

PROPOSITION 6.2. *Let M and M' be binary oriented matroids on a set E having $\underline{M} = \underline{M}'$. Then there exists $A \subseteq E$ such that $M' = {}_AM$.*

In order to prove Proposition 6.2 we will first give some preliminary results.

Let M be a matroid on a set E. Whitney showed that the following two properties are equivalent [15, Theorem 19]:

(i) for all $x, y \in E$, $x \neq y$, there are circuits $C_0, C_1, \ldots, C_k$ of M such that $x \in C_0$, $y \in C_k$ and $C_i \cap C_{i+1} \neq \varnothing$, for $i = 0, \ldots, k-1$;

and

(ii) for all $x, y \in E$, $x \neq y$, there is a circuit C of M containing x and y.

A matroid M having these properties is said to be *connected* (or *irreducible*).

A pair of circuits C, C' of a matroid M with rank function ρ is called *modular* if $\rho(C) + \rho(C') = \rho(C \cup C') + \rho(C \cap C')$.

LEMMA 6.2.1 (Tutte [14, Proposition 4.34]). *Let M be a matroid on a set E and let e be an element of E such that M/e is connected. Suppose that C and C' are distinct circuits of M having $e \in C \cap C'$. Then there are circuits $C = C_0$, $C_1, \ldots, C_k = C'$ of M such that $\{e\} \subsetneq C_i \cap C_{i+1}$ and the pair C_i, C_{i+1} is modular for $i = 0, 1, \ldots, k-1$.*

LEMMA 6.2.2. *Let M be a connected matroid on E with no 2-element circuits. Then there is an element $e \in E$ such that M/e is connected.*

Proof. The proof is by induction on $|E|$. The lemma is clearly true for $|E| = 3$. Suppose that $|E| \geqslant 4$. Crapo [6] showed that for every $e \in E$ either M/e or $M\backslash e$ is connected. Let $e \in E$ and suppose that M/e is not connected. Then $M\backslash e$ is connected, and by the inductive hypothesis there exists

$e' \in E\backslash e$ such that $(M\backslash e)/e'$ is connected. Since M is connected there is some circuit C of M having $e, e' \in C$. By the hypothesis of the lemma, $|C| \geq 3$, so $C\backslash e'$ is a circuit of M/e', $e \in C\backslash e'$, and $(C\backslash e') \cap (E\backslash\{e, e'\}) \neq \varnothing$. Since $(M\backslash e)/e' = (M/e')\backslash e$ is connected, it follows that M/e' is connected.

LEMMA 6.2.3 (Tutte [14, Proposition 5.35]). *A matroid is binary if and only if for all modular pairs of circuits C, C' of M such that $C \cap C' \neq \varnothing$ and $C \neq C'$, there are exactly three circuits contained in $C \cup C'$, namely, C, C', and $C \Delta C'$, the symmetric difference of C and C'.*

From Lemma 6.2.3 and the signed elimination property (I) we get

LEMMA 6.2.4. *Let $M = (E, \mathcal{O})$ be a binary oriented matroid. If $X, Z \in \mathcal{O}$, $\underline{X}, \underline{Z}$ is a modular pair in $\underline{M}$ and $x, z \in \underline{X} \cap \underline{Z}$, then $sg_X(x) \cdot sg_X(z) = sg_Z(x) \cdot sg_Z(z)$.*

We will need one more lemma. We say that a signed set X is *carried* by its underlying set $\underline{X}$.

LEMMA 6.2.5. *Let M and M' be binary oriented matroids on a set E having $\underline{M} = \underline{M}'$. Suppose that X_1, X_2 are distinct signed circuits of M such that $\underline{X}_1, \underline{X}_2$ is a modular pair of circuits and $e \in \underline{X}_1 \cap \underline{X}_2$.*

(i) *If X_1 and X_2 are signed circuits of M', then the opposite pair of signed circuits of M carried by $\underline{X}_1 \Delta \underline{X}_2$ are signed circuits of M'.*

(ii) *If $|\underline{X}_1 \cap \underline{X}_2| \geq 2$, X_1 is a signed circuit of M' and $X_2\backslash e$ is a signed circuit of M'/e, then X_2 is a signed circuit of M'.*

Proof. (i) By the signed elimination property (I), the signatures of the signed circuits of M and M' carried by $\underline{X}_1 \Delta \underline{X}_2$ are completely determined, in the same way, by X_1 and X_2, and are thus equal.

(ii) Let X_2' be a signed circuit of M' such that $\underline{X}_2 = \underline{X}_2'$ and $X_2\backslash e = X_2'\backslash e$ and let $x \in (\underline{X}_1 \cap \underline{X}_2)\backslash e$. By Lemma 6.2.4 we have $sg_{X_1}(e)\, sg_{X_1}(x) = sg_{X_2}(e)\, sg_{X_2}(x)$ and $sg_{X_1}(e)\, sg_{X_1}(x) = sg_{X_2'}(e)\, sg_{X_2'}(x)$. On the other hand $sg_{X_2}(x) = sg_{X_2'}(x)$, hence $sg_{X_2}(e) = sg_{X_2'}(e)$, and therefore $X_2 = X_2'$.

Proof of Proposition 6.2

The proof is by induction on $|E|$. Without loss of generality we may suppose that $|E| \geq 2$ and $\underline{M} = \underline{M}'$ is connected.

We consider two cases

(1) Suppose first that $\underline{M} = \underline{M}'$ has a 2-element circuit $\{e, e'\}$. We have $\underline{M\backslash e} = \underline{M'\backslash e}$, hence by the inductive hypothesis there exists $A' \subseteq E\backslash e$ such that $M'\backslash e = {}_{\overline{A'}}(M\backslash e)$. Let X_0 and X_0' be signed circuits of M and M', respectively, carried by $\{e, e'\}$ and having $e' \in X_0^+ \cap X_0'^+$. We set $A = A'$ if $X_0 = X_0'$ and $A = A' \cup \{e\}$ otherwise.

We show that $M' = {}_{\bar{A}}M$. Let X' be a signed circuit of M'. Since $M'\backslash e = {}_{\bar{A}}(M\backslash e)$ and $X_0' = {}_{\bar{A}}X_0$ we need only consider the case where $e \in \underline{X}'$ and $\underline{X}' \neq \{e, e'\}$. Now $\underline{X}_1 = \underline{X}' \Delta \{e, e'\} = \underline{X}'\backslash e + e'$ is a circuit of $\underline{M}\backslash e = \underline{M}'\backslash e$ and $\underline{X}_1, \{e, e'\}$ is a modular pair of circuits. Hence by (i) of Lemma 6.2.5, X' is a signed circuit of ${}_{\bar{A}}M$.

(2) Suppose now that $\underline{M} = \underline{M}'$ has no 2-element circuit. By Lemma 6.2.2 there exists $e \in E$ such that $\underline{M}/e = \underline{M}'/e$ is connected. Since $\underline{M/e} = \underline{M'/e}$, by the inductive hypothesis there exists $A' \subseteq E\backslash e$ such $M'/e = {}_{\overline{A'}}(M/e)$. Let X_0 be a signed circuit of M such that $e \in \underline{X}_0$. $|\underline{X}_0| \geq 2$, hence $X_0\backslash e$ is a signed circuit of M/e. Now ${}_{\overline{A'}}(X_0\backslash e)$ is a signed circuit of M'/e. Let X_0' be the signed circuit of M' such that $\underline{X}_0' = \underline{X}_0$ and $X_0'\backslash e = {}_{\overline{A'}}(X_0\backslash e)$. We set $A = A'$ if $X_0' = {}_{\overline{A'}}X_0$, $A = A' \cup \{e\}$ otherwise.

We will now show that $M' = {}_{\bar{A}}M$. Let X' be a signed circuit of M'. Since $X_0' = {}_{\bar{A}}X_0$ and $M'/e = {}_{\bar{A}}(M/e)$ we have only to consider the case where $X' \neq \pm X_0'$ and X' is not a circuit of M'/e.

(2a) $e \in \underline{X}'$.

By Lemma 6.2.1 there are signed circuits $X_1, X_2, \ldots, X_k = X$ of M such that $\underline{X}_k = \underline{X}'$, $\{e\} \subsetneq \underline{X}_i \cap \underline{X}_{i+1}$, and $\underline{X}_i, \underline{X}_{i+1}$ is a modular pair of circuits, for $i = 0, 1, \ldots, k-1$. Now ${}_{\bar{A}}X_0 = X_0'$ is a signed circuit of ${}_{\bar{A}}M$ and M', ${}_{\bar{A}}X_1$ is a signed circuit of ${}_{\bar{A}}M$, and $({}_{\bar{A}}X_1)\backslash e$ is a signed circuit of M'/e, since $M'/e = {}_{\bar{A}}(M/e)$. Hence by (ii) of Lemma 6.2.5 ${}_{\bar{A}}X_1$ is a signed circuit of M'. By induction on k we show in this way that ${}_{\bar{A}}X_k = {}_{\bar{A}}X$ is a circuit of M'. Since $\underline{X} = \underline{X}'$, X' is a signed circuit of ${}_{\bar{A}}M$.

(2b) $e \notin \underline{X}'$.

There is a signed circuit X_1' of M' such that $e \in \underline{X}_1'$ and $\underline{X}_1'\backslash e \subsetneq \underline{X}'$. $\underline{X}_1', \underline{X}'$ is a modular pair of circuits. Since $\underline{M}'$ is binary there is a signed circuit X_2' of M' carried by $\underline{X}' \Delta \underline{X}_1'$ and we have $\underline{X}' = \underline{X}_1' \Delta \underline{X}_2'$. Now $e \in \underline{X}_1'$, $e \in \underline{X}_2'$, hence X_1' and X_2' are signed circuits of ${}_{\bar{A}}M$ by (2a). Therefore by (i) of Lemma 6.2.5 X' is a signed circuit of ${}_{\bar{A}}M$.

COROLLARY 6.2.6. *Let M be a binary oriented matroid on a set E. Then there is a unimodular subspace $\mathscr{R}$ of $\mathbb{R}^E$ such that $M = S(\mathscr{R})$.*

Proof. By Proposition 6.1 there is a unimodular subspace $\mathscr{R}$ of $\mathbb{R}^E$ such that $\underline{M} = \underline{S(\mathscr{R})}$. By Proposition 6.2 we have $M = {}_{\bar{A}}S(\mathscr{R})$ for some subset A of E. Hence $M = S(\epsilon \cdot \mathscr{R})$, where $\epsilon : E \to \{1, -1\}$ is defined by $\epsilon(x) = -1$ if $x \in A$ and $\epsilon(x) = 1$ if $x \in E\backslash A$.

From Proposition 6.2 and Corollary 6.2.6 we immediately get

COROLLARY 6.2.7 (Camion [3, Th. 4, Sect. 5.2], Brylawski and Lucas [2 [illegible] Prop. 4.2]). *Let $\mathscr{R}$ and $\mathscr{R}'$ be unimodular subspaces of $\mathbb{R}^E$ having $\underline{S}(\mathscr{R}) = \underline{S}(\mathscr{R}'$[illegible] Then there is a mapping $\epsilon : E \to \{1, -1\}$ such that $\mathscr{R}' = \epsilon\mathscr{R}$.*

Corollary 6.2.7 is also implied by the recent work of both Bixby and Seymo[illegible] on matroids coordinatizable over $GF(3)$.

COROLLARY 6.2.8. *Let G be an undirected graph. Then every orientatio[illegible] of the polygon–matroid (respectively, the bond-matroid) of G corresponds t[illegible] some orientation of the edges of G.*

Minty's digraphoid axiom is the strengthening to the binary case of th[illegible] orthogonality axiom (IV). It should be clear from Proposition 6.2 and it[illegible] corollaries that the corresponding strengthening of the circuit elimination axiom (I) is:

for all $X_1, X_2 \in \mathscr{O}$, $X_1 \neq -X_2$, having $(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+) \neq \varnothing$, there exists $X_3 \in \mathscr{O}$ such that $X_3^+ \subseteq (X_1^+ \backslash X_2^-) \cup (X_2^+ \backslash X_1^-)$ and $X_3^- \subseteq (X_1^- \backslash X_2^+) \cup (X_2^- \backslash X_1^+)$.

Let M be an orientable matroid on a set E. The operation of sign reversal on subsets of E clearly describes an equivalence relation on the set of orientations of M. Proposition 6.2 indicates that if M is binary, then all pairs of orientations of M are related by sign reversal, i.e., there is exactly one class under this relation.

Problem. Let M be an orientable matroid. How many classes of orientations of M are there?

PROPOSITION 6.3. *Let n and r be positive integers, $2 \leqslant r \leqslant n-2$. Then the free matroid $\mathscr{F}_n{}^r$ of rank r on n elements has at least $(n-1)!/2$ classes of orientations.*

Proof. Let $\mathscr{O}$ be an orientation of a matroid M on a set E. Let $G(\mathscr{O})$ be the set of 2-element subsets $\{x, y\} \subseteq E$, $x \neq y$, such that either $sg_X(x) = sg_X(y)$ for all $X \in \mathscr{O}$ such that $\{x, y\} \subseteq \underline{X}$ or $sg_X(x) = -sg_X(y)$ for all $X \in \mathscr{O}$ such that $\{x, y\} \subseteq \underline{X}$. Clearly $G(\mathscr{O}) = G({}_A\mathscr{O})$ for any $A \subseteq E$.

Let $M = \mathscr{F}_n{}^r$, $2 \leqslant r \leqslant n-2$, and let $\mathscr{O}$ be the alternating orientation of M with respect to some order $e_1 < e_2 < \cdots < e_n$ of E. It is easy to see that $G(\mathscr{O}) = \{\{e_i, e_{i+1}\} : i = 1, 2, \ldots, n, e_{n+1} = e_1\}$. Proposition 6.3 follows.

Note added in proof. Separate papers based on additional results from [1] ("A combinatorial abstraction of linear programming") and [10] ("Bases of oriented matroids" and "Convexity in oriented matroids") will appear in this journal.

[illegible]*ote added in proof.* Recently, previously unpublished work on oriented matroids [illegible]e late Jon Folkman has appeared in summary from in the Ph.D. Thesis of Jim Lawrence [illegible]versity of Washington, Seattle, Summer 1975). Although Folkman's approach to [illegible]ted matroids differs noticeably from ours, his axiomatization is based on an elimination [illegible]erty that is clearly equivalent to (II) of our Theorem 2.1. Thus it is clear that the [illegible]matizations represented by Theorems 2.1 and 2.2, each of which one or both of us [illegible]developed before learning of Folkman's work, are equivalent to Folkman's axiomatiza-[illegible] It appears from his unpublished notes that Folkman was aware of the possibility of [illegible]xiomatization of oriented matroids based on the orthogonality property (IV) of [illegible]orem 2.2, but, apparently, he never pursued it.

ACKNOWLEDGMENT

[illegible]his paper is a synthesis of work that was undertaken separately by the authors and [illegible] in each case, completed in early 1974, shortly before we learned from S. B. Maurer of [illegible]common interests. Our initial announcements of these results appeared in [1, 10].

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Oriented Matroids

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In this paper, the basic properties of oriented matroids are examined. A topological representation theorem for oriented matroids is proven, utilizing the notion of an "arrangement of pseudo-hemispheres." The duality theorem of linear programming is extended to oriented matroids.

I. INTRODUCTION

The study of matroids was begun by Whitney in [16], where one may see how a finite subset of a vector space yields, in a natural way, a matroid. The matroids embody a combinatorial presentation of familiar properties of the linear dependence relations in such subsets of vector spaces.

In a vector space over an *ordered* field one may wish to study the properties of *positive* dependence relations (Davis, [3]). The natural analogue of the matroid in this setting is the "oriented matroid." From a finite subset S of such a vector space comes an oriented matroid. From this, in turn, one may derive the incidence structure associated with the cones generated by subsets of S. Certain of these cones may be linear subspaces; indeed, underlying each oriented matroid will be an ordinary matroid structure (Theorem 2).

The definition for oriented matroids used here is taken from unpublished notes of Jon Folkman, who died before completing them. Results from his notes have been incorporated into this paper and enlarged upon in sections II and V.

In sections II, III, and VI we derive the basic properties of oriented matroids. These properties are all well-known for the oriented matroids arising from sets in vector spaces; for instance, for such oriented matroids Theorem 22 is the duality theorem for linear programming. (See Rockafellar [12].)

The oriented matroids have been studied also by Bland in his doctoral dissertation [1]. He worked from a different definition, but it is the case that

* Much of this is taken from the doctoral dissertation of the second author: it was completed at the University of Washington, where the author held a National Science Foundation Graduate Fellowship.

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the oriented matroids he studied are the same as those described he[illegible] showed that those satisfying his definition also satisfy Folkman's, and [illegible] we shall see that our oriented matroids have his properties.

Certain classes of oriented matroids had previously been studied. [illegible] [10] generalized the notion of a "directed graph" to that of a "digraph[illegible] Digraphoids are easily seen to correspond to (dual pairs of) oriented ma[illegible] whose underlying matroids are "regular."

Rockafellar [12] felt that a broader theory of orientation ought to [illegible] which did not require that the underlying matroid be regular. He [illegible] examples arising from real vector spaces of what one might mean b[illegible] "orientation" of a matroid, and presented in this context a numbe[illegible] interesting theorems, including the linear programming duality theo[illegible]

Bland's presentation closely followed the ideas of Minty and Rockafe[illegible] He showed that his oriented matroids have several of the properties give[illegible] "realizable" oriented matroids by Rockafellar's theorems. Here we h[illegible] succeeded in showing that Rockafellar's Theorem 7, the linear programm[illegible] duality theorem, is also valid in the more general framework of orien[illegible] matroids. This is Theorem 22, below. Thus, analogues of all the theorems [illegible] Rockafellar's paper are seen to hold.

Given a set of n points in a real vector space we can form the "G[illegible] transform" of the set to obtain a set of n points corresponding to these [illegible] another real vector space. (See Grünbaum [5].) These sets carry "du[illegible] oriented matroid structures. Bland's definition in [1] is in terms of such du[illegible] pairs, as is Minty's definition for digraphoids in [10]. We discuss this dualit[illegible] in Chapter III, where the "hull functions" for oriented matroids will [illegible] described and a class of hull functions forming a common generalization [illegible] both the oriented and ordinary matroid hull functions will be examine[illegible] These are the hull functions of the "gatroids."

Section IV contains, perhaps, our most interesting result. Here there is [illegible] description of a means of representing oriented matroids in terms of "arrang[illegible] ments of pseudohemispheres." In the two-dimensional case these objects a[illegible] the arrangements of pseudolines, a good discussion of which may be found [illegible] Grünbaum [6]. We show that any arrangement of pseudo-hemisphe[illegible] carries the structure of an oriented matroid (Theorem 16), and that from an[illegible] oriented matroid there comes such an arrangement (Theorem 20). Thus w[illegible] establish a correspondence between the oriented matroids and the arrange[illegible] ments of pseudo-hemispheres.

Halsey, in his doctoral dissertation [7], was interested in certain complexe[illegible] in the n-cube which could be obtained as the inverse image of the boundary [illegible] of the image of the cube under an orthogonal projection. In trying to describe [illegible] these complexes combinatorially, he discovered a class of complexes whose [illegible] topological duals may be identified with the "simple" arrangements of [illegible] pseudo-hemispheres. He found that there is a natural, bijective correspon-

[illegible]ce between the simple d-arrangements of $2n$ pseudo-hemispheres, where d [illegible] dimension of the complex, and the simple $(n-d)$-arrangements of [illegible]seudo-hemispheres. (His terminology was slightly different.) Our duality [illegible]riented matroids may be viewed as an extension of this correspondence, [illegible]ll as an extension of the similar correspondence given for arrangements [illegible]enuine hyperplanes by McMullen in [9].

[illegible] Section V we develop a different characterization of the simple oriented [illegible]roids. This characterization is from Folkman's notes.

[illegible] this presentation of the oriented matroids, basic properties of matroids [illegible] be found to be of use, particularly in view of the underlying matroid [illegible]cture associated with an oriented matroid. Good discussions of matroids [illegible] be found in Tutte [14], Klee [8], Crapo and Rota [2], and in the original [illegible]icle by Whitney [16]. Tutte [14] also describes the regular matroids.

II. Circuits of Oriented Matroids

An *involution* on a set E is a function $*: E \to E$ such that $(x^*)^* = x$, for [illegible]ch element x of E. If also $x^* \neq x$, for each element x of E, then the involu-[illegible]on is said to be *fixed-point free*. If $*$ is an involution on E and S is a subset [illegible] E, then the set $\{x^* \mid x \in S\}$ will be denoted by S^*.

A *clutter* of subsets of a set E is a collection $\mathscr{C}$ of subsets of E such that if [illegible] and B are in $\mathscr{C}$ with $A \subset B$, then $A = B$.

An *oriented matroid* is a triple $(E, \mathscr{C}, *)$, where E is a finite set, $\mathscr{C}$ is a [illegible]ollection of non-empty subsets of E, and $*$ is a fixed-point free involution [illegible]n E, such that:

(1) $\mathscr{C}$ is a clutter;

(2) If $S \in \mathscr{C}$ then $S^* \in \mathscr{C}$ and $S \cap S^* = \varnothing$;

(3) If $S, T \in \mathscr{C}$, $x \in S \cap T^*$, and $S \neq T^*$, then there is a set $C \in \mathscr{C}$ with [illegible] $C \subset (S \cup T) \sim \{x, x^*\}$. We call the sets in $\mathscr{C}$ the *circuits* of the oriented matroid $(E, \mathscr{C}, *)$.

Suppose that E is a finite set of non-zero vectors in a vector space over an ordered field, and suppose $E = -E$. Such a set is endowed with the structure of an oriented matroid in the following way. For x in E let $x^* = -x$. Let $\mathscr{C}$ be the collection of subets C of E, minimal (with respect to set inclusion) such that:

(a) $C \cap C^* = \varnothing$;

(b) There exist positive real numbers α_s $(s \in C)$ such that $\sum_{s \in C} \alpha_s s = 0$.

That is, a subset C of E is in $\mathscr{C}$ if and only if it is the vertex set of a simplex containing the origin in its relative interior but is not of the form $\{u, -u\}$.

THEOREM 1. *The triple $(E, \mathscr{C}, *)$ is an oriented matroid.*

□ The first two conditions of the definition are obviously satisfied. In order to verify the third, it is convenient to have the following notions. A *relation* on E is a real-valued function ρ on E such that $\sum_{x \in E} \rho_x x = 0$. The relation is *positive* if ρ_x is nonnegative for each x in E. Its *support* is the set $\{x \in E \mid \rho_x \neq 0\}$. A positive relation is *minimal* if its support is non-empty and properly contains the support of no other such positive relation. Obviously two minimal positive relations have the same support if and only if they differ by a scalar multiple; and any non-zero positive relation may be written as a sum of minimal positive relations.

Now, suppose S and T are in $\mathscr{C}$, x is in $S \cap T^*$, and $S \neq T^*$. Since S and T are in $\mathscr{C}$, S and T are the supports of minimal positive relations α and β. We may choose the relations α and β so that $\alpha_x = \beta_{x^*} = 1$ by taking appropriate scalar multiples. Let ρ be the function on E with $\rho_s = \alpha_s + \beta_s$ if s is neither x nor x^*, and $\rho_s = 0$ if s is x or x^*. Then ρ is a positive relation, since:

$$\sum_{s \in E} (\rho_s s) = \sum_{s \in S \sim \{x\}} (\alpha_s s) + \sum_{s \in T \sim \{x^*\}} (\beta_s s) = \sum_{s \in E} (\alpha_s s) + \sum_{s \in E} (\beta_s s) = 0.$$

S is not contained in T^*, so there is an element y of $S \sim T^*$. Then $\rho_y = \alpha_y > 0$ since y is in the support of α but not of β, so ρ is non-zero. We may write:

$$\rho = \rho_1 + \rho_2 + \cdots + \rho_k,$$

where the ρ_i's are minimal positive relations. At least one of the ρ_i's, say ρ_1, is positive on y. Then the support C of ρ_1 cannot be of the form $\{u, -u\}$, since $-y$ cannot be in it. Therefore C is in $\mathscr{C}$, and:

$$y \in C \subset (S \cup T) \sim \{x, x^*\}.$$

Hence $(E, \mathscr{C}, *)$ is an oriented matroid. □

Actually, the proof shows somewhat more than the definition requires; namely:

(3′) If S and T are in $\mathscr{C}$, $x \in S \cap T^*$, and $y \in S \sim T^*$, then there is a circuit C with:

$$y \in C \subset (S \cup T) \sim \{x, x^*\}.$$

This is true for all oriented matroids arising in the way described above. We shall see that it is, in fact, true for any oriented matroid. (This is Theorem 4.)

Oriented matroids that may arise as in Theorem 1 will be called *realizable* oriented matroids. We shall see that many properties of the realizable oriented matroids are shared by all oriented matroids.

It is well-known that any finite subset of a vector space is endowed with the structure of a matroid (Whitney [16]; Tutte [14]). If E is a finite set of non-zero vectors in a vector space over an ordered field, with $E = -E$, and $(E, \mathscr{C}, *)$ is the oriented matroid arising from this set, then it is possible to describe the matroid associated with this set in terms of the oriented matroid structure. We show now that the analogous construction in any oriented matroid yields a matroid.

Let $\mathscr{O} = (E, \mathscr{C}, *)$ be an oriented matroid. For elements x of E, let $\bar{x} = \{x, x^*\}$. For subsets S of E let $\bar{S}$ be the set $\{\bar{x} \mid x \in S\}$, and let $\bar{\mathscr{C}} = \{\bar{C} \mid C \in \mathscr{C}\}$. The pair $(\bar{E}, \bar{\mathscr{C}})$ will be called the *underlying matroid* of $\mathscr{O}$. We will prove that it is actually a matroid; i.e., that:

(1) $\bar{\mathscr{C}}$ is a clutter of non-empty subsets of $\bar{E}$;

and:

(2) If $\bar{S}$, $\bar{T} \in \bar{\mathscr{C}}$, $\bar{x} \in \bar{S} \cap \bar{T}$, and $\bar{S} \neq \bar{T}$, then there is a set $\bar{C} \in \bar{\mathscr{C}}$ with $\bar{C} \subset (\bar{S} \cup \bar{T}) \sim \{\bar{x}\}$.

First, we need a lemma.

LEMMA. *Suppose $(E, \mathscr{C}, *)$ is an oriented matroid. Suppose S and T are circuits with $S \subset T \cup T^*$. Then either $S = T$ or $S = T^*$.*

□ Suppose, on the contrary, that there is a circuit T for which we can find a circuit $S \subset T \cup T^*$ with $S \neq T$ and $S \neq T^*$. Let such a circuit S be chosen with $|S \cap T^*|$ as small as possible. $|S \cap T^*| \neq 0$ since otherwise we would have $S \subset T$. Let x be an element of $S \cap T^*$. Since $S \neq T^*$, there is a circuit C contained in $(S \cup T) \sim \{x, x^*\}$. This circuit C can be neither T nor T^* since it contains neither x nor x^*. But:

$$C \cap T^* \subset (S \cap T^*) \sim \{x\},$$

so that:

$$|C \cap T^*| < |S \cap T^*|.$$

This cannot be the case, since $|S \cap T^*|$ was to be minimal, so no such circuit T can exist, and the lemma is established. □

THEOREM 2. *If $(E, \mathscr{C}, *)$ is an oriented matroid then the underlying matroid $(\bar{E}, \bar{\mathscr{C}})$ is, indeed, a matroid.*

□ It is immediate from the lemma that $\bar{\mathscr{C}}$ is a clutter of subsets of $\bar{E}$. Now suppose $A, B \in \bar{\mathscr{C}}$, $x \in E$ with $\bar{x} \in A \cap B$, and $A \neq B$. Circuits S and T of the oriented matroid may be chosen so that $A = \bar{S}$, $B = \bar{T}$, $x \in S$, and

$x^* \in T$. Then $x \in S \cap T^*$ and, since $\bar{S} \neq \bar{T}$, $S \neq T^*$. Therefore there is a circuit C with:

$$C \subset (S \cup T) \sim \{x, x^*\}.$$

Then $\bar{C} \in \bar{\mathscr{C}}$ and $\bar{C} \subset (A \cup B) \sim \{\bar{x}\}$. □

Suppose, again, that $(E, \mathscr{C}, {}^*)$ is an oriented matroid arising from a subset E of a vector space over an ordered field. If $A \subset E$ and $x \in E$, then x is in the conical hull of A if and only if x is already in A or there is a circuit C with:

$$(*) \quad x^* \in C \subset A \cup \{x^*\}.$$

For $A \subset E$, let $h(A)$ be the union of the set A and $\{x \in E \mid$ there is a circuit C such that (*) holds$\}$. Then for realizable oriented matroids:

(1) $A \subset h(A)$, for each subset A of E;

(2) $h(A) \subset h(B)$ if $A \subset B$;

and

(3) $h(h(A)) = h(A)$.

Clearly the first two conditions also hold if h is defined similarly for an oriented matroid that is not realizable. The third also holds, but is not so immediate. That this is true is Theorem 5 below. First we need some other results.

LEMMA. *Suppose $(E, \mathscr{C}, {}^*)$ is an oriented matroid. Suppose S and T are circuits, $p \in T \subset \{p\} \cup S$, and $S \cap T \neq \varnothing$. Then there is a circuit C with $p^* \in C \subset \{p^*\} \cup S$ with $S \subset T \cup C$.*

□ Let z be an element of $S \cap T = T \sim \{p\}$. Then there is a circuit C_1 contained in $(S \cup T^*) \sim \{z, z^*\}$. $C_1 \cap T^*$ has fewer elements than T^*, since z^* is not in C_1.

Let C be a circuit chosen so that $C \cap T^*$ has as few elements as possible, with:

(a) $C \subset S \cup T^* \cup \{p\}$;

(b) $C \neq S$, $C \neq T$.

Since C_1 satisfies (a) and (b), C can have no more elements than C_1; therefore $C \neq T^*$.

Suppose there is an element x other than p^* contained in the set $C \cap T^*$. Then there is a circuit C_0 that is contained in the set $(C \cup T) \sim \{x, x^*\}$. Then C_0 is contained in the set $S \cup T^* \cup \{p\}$. x^* is in S and in T but not in C_0, so C_0 is neither S nor T. But $C_0 \cap T^*$ is contained in $(C \cap T^*) \sim \{x\}$, so $C_0 \cap T^*$ has fewer elements than $C \cap T^*$. This cannot be the case, so $C \cap T^* \subset \{p^*\}$; i.e., $C \subset S \cup \{p, p^*\}$.

Suppose $p \in C$. Then $p \in C \cap T$ and $C \neq T$, so there is circuit D with $D \subset (C \cup T^*) \sim \{p, p^*\}$. Then $D \subset S \cup S^*$, so $D = S$ or $D = S^*$.

$D \neq S$, since otherwise $S \subset C \cup T^*$; i.e., $S \subset C$. This cannot be the case since $C \neq S$.

Suppose $D = S^*$. Then $S^* \subset C \cup T^*$. Then $S^* \subset T^*$. This cannot be the case.

Therefore $p \notin C$, so $p^* \in C \subset S \cup \{p^*\}$.

Now, $\{p^*\} = C \cap T^*$ and $C \neq T^*$, so there is a circuit contained in $(C \cup T) \sim \{p, p^*\} \subset S$. Since $\mathscr{C}$ is a clutter, this circuit must equal S, and $S \subset T \cup C$. □

THEOREM 3. *Let $(E, \mathscr{C}, {}^*)$ be an oriented matroid, $p \in E$, and $\{p\} \notin \mathscr{C}$. Let $\mathscr{C}_0 = \{C \in \mathscr{C} \mid C \subset E \sim \{p, p^*\}\}$. Let $\mathscr{C}_1$ be the collection of all minimal sets D contained in $E \sim \{p, p^*\}$ for which there is a set $C \in \mathscr{C}$ with $D = C \sim \{p, p^*\}$. Then both $(E \sim \{p, p^*\}, \mathscr{C}_0, {}^*)$ and $(E \sim \{p, p^*\}, \mathscr{C}_1, {}^*)$ are oriented matroids.*

□ It is obvious that $(E \sim \{p, p^*\}, \mathscr{C}_0, {}^*)$ is an oriented matroid, so we verify only that $(E \sim \{p, p^*\}, \mathscr{C}_1, {}^*)$ is an oriented matroid.

The collection $\mathscr{C}_1$ is a clutter of non-empty sets since its elements are the minimal elements in another collection of non-empty sets.

If $S \in \mathscr{C}_1$ then $S = C \sim \{p, p^*\}$ for some circuit $C \in \mathscr{C}$, so $S^* = C^* \sim \{p, p^*\}$, and S^* is also a minimal set of this form. Therefore $S^* \in \mathscr{C}_1$. Clearly $S \cap S^* = \varnothing$.

Finally, suppose S and T are in $\mathscr{C}_1$, $x \in S \cap T^*$, and $S \neq T^*$. Let $S = U \sim \{p, p^*\}$ and $T = V \sim \{p, p^*\}$, where U and V are in $\mathscr{C}$. Then $x \in U \cap V^*$, and $U \neq V^*$. There is a circuit C in $\mathscr{C}$ contained in $(U \cup V) \sim \{x, x^*\}$. Then $C \sim \{p, p^*\}$ contains some element of $\mathscr{C}_1$, which is contained in $(S \cup T) \sim \{p, p^*\}$, as required. □

We will say that $(E \sim \{p, p^*\}, \mathscr{C}_0, {}^*)$ arises from $(E, \mathscr{C}, {}^*)$ be *deletion of* $\{p, p^*\}$, and that the oriented matroid $(E \sim \{p, p^*\}, \mathscr{C}_1, {}^*)$ arises by *contraction of* $\{p, p^*\}$. If $(E, \mathscr{C}, {}^*)$ arises as in Theorem 1, so that E is in a vector space over an ordered field, then the deletion of $\{p, p^*\}$ gives rise to the oriented matroid determined by the set $E \sim \{p, p^*\}$ in the vector space, while the contraction corresponds to the oriented matroid determined by the orthogonal projection of $E \sim \{p, p^*\}$ to the orthogonal complement of the line through the points p and p^*.

THEOREM 4. *If $(E, \mathscr{C}, {}^*)$ is an oriented matroid, S and T are circuits,*

$x \in S \cap T^*$, and $y \in S \sim T^*$, then there is a circuit C with $y \in C \subset (S \cup T) \sim \{x, x^*\}$.

□ Suppose this is false. Pick an oriented matroid $(E, \mathscr{C}, *)$, so that E has as few elements as possible, in which the theorem fails, for some choice of S, T, x, and y. Let U be a circuit contained in $(S \cup T) \sim \{x, x^*\}$. There is an element p of U with neither p nor p^* in T, since otherwise $U \subset T \cup T^*$. p is in $S \sim \{x, x^*\}$. $p \neq y^*$ since $y^* \notin S \cup T$. Since we have assumed the theorem does not hold here, y cannot be in U, so p is not y.

Let $(E_1, \mathscr{C}_1, *)$ be the contraction of $(E, \mathscr{C}, *)$ at $\{p, p^*\}$. T contains some element of $\mathscr{C}_1$, so there is an element V of $\mathscr{C}$ with $V \sim \{p, p^*\} \in \mathscr{C}_1$ and $V \sim \{p, p^*\} \subset T$. By the lemma, we may assume x^* is in V. The set $S \sim \{p\}$ must itself be in $\mathscr{C}_1$. Then x is in $S \sim \{p\}$ and in $(V \sim \{p, p^*\})^*$; y is in $S \sim \{p\}$ but not in $(V \sim \{p, p^*\})^*$.
Since E_1 has fewer elements than E, there must be an element $W \sim \{p, p^*\}$ of $\mathscr{C}_1$, where W is in $\mathscr{C}$, with $y \in W \sim \{p, p^*\}$, and with $W \sim \{p, p^*\}$ contained in the set:

$$(S \sim \{p\}) \cup (V \sim \{p, p^*\}) \sim \{x, x^*\}.$$

We cannot have $W \subset (S \cup T) \sim \{x, x^*\}$, so $p^* \in W$.

Let $(E_0, \mathscr{C}_0, *)$ be the oriented matroid derived from $(E, \mathscr{C}, *)$ by deletion of $\{x, x^*\}$. Then U and W are contained in E_0, and are therefore elements of $\mathscr{C}_0$. $p \in U \cap W^*$ and y is in $W \sim U^*$. Therefore there is a circuit R of $\mathscr{C}_0$ (and thus in $\mathscr{C}$) with $y \in R \subset (U \cup W) \sim \{p, p^*\}$. But $(U \cup W) \sim \{p, p^*\}$ is a subset of $(S \cup T) \sim \{x, x^*\}$. It follows that R is a circuit satisfying the requirements of our theorem, contrary to the assumption that there was no such circuit. □

Let $(E, \mathscr{C}, *)$ be an oriented matroid. For each subset A of E let $h(A) = A \cup \{x \in E \mid$ there is a circuit C with x^* in C and with C contained in $A \cup \{x^*\}\}$.

THEOREM 5. *$h(h(A)) = h(A)$, for each subset A of E.*

□ Clearly $h(A) \subset h(h(A))$.
Suppose y is not in $h(A)$. We must show that y is not in $h(h(A))$. Let M be a maximal set such that:

(a) $A \subset M \subset h(A)$;

and

(b) y is not in $h(M)$.

The set A satisfies (a) and (b), so there is such a maximal set.

Suppose $M \neq h(A)$. Then there is an element x in $h(A)$ but not in M. Then x is in $h(A)$, but not in A, so there is a circuit T with $x^* \in T \subset A \cup \{x^*\}$.
Note that x is in $h(A)$ and y is not in $h(A)$, so $y \neq x$.
By the choice of M, $y \in h(M \cup \{x\}) \sim M$. There is a circuit S with $y^* \in S \subset M \cup \{x, y^*\}$. $x \in S$, since otherwise we would have $y \in h(M)$.
Then $x \in S \cap T^*$ and $y^* \in S \sim T^*$. There is a circuit C with y^* in C and with C contained in $(S \cup T) \sim \{x, x^*\}$, but this is contained in $M \cup \{y^*\}$. This cannot be the case, since y is not in $h(M)$, so it must be the case that $M = h(A)$, and y is not in the set $h(M) = h(h(A))$. □

The function h will be called the *hull function* for the oriented matroid $(E, \mathscr{C}, *)$. A characterization of such hull functions will be given in the next section.

III. HULL FUNCTIONS

Let E be a finite set. Let $P(E)$ denote the collection of all subsets of E. A *bumper function* b for E is a function b: $P(E) \to P(E)$ which satisfies:

(1) $A \subset b(A)$, for each subset A of E;

and:

(2) If $A \subset B$ then $b(A) \subset b(B)$.

The pair (E, b) is called a *bumpered set*.
The bumper function is called a hull function if it also satisfies:

(3) $b(b(A)) = b(A)$, for each subset A of E.

The following theorem provides a characterization of hull functions that will prove useful.

THEOREM 6. *If (E, b) is a bumpered set then b is a hull function if and only if:*

(3′) *If $q \in b(A \cup \{p\})$ and $p \in b(A)$, then $q \in b(A)$.*

□ Suppose that (E, b) does not satisfy (3). Let M be a maximal subset of E with $b(b(M)) \neq b(M)$. Then $b(b(M))$ properly contains $b(M)$, and $b(M)$ properly contains M. Choose elements p and q, with p in $b(M)$ but not in M, and q in $b(b(M))$ but not in $b(M)$. Then q is in $b(b(M \cup \{p\})) = b(M \cup \{p\})$ and p is in $b(M)$, but q is not in $b(M)$. Therefore (3′) is not satisfied.
Suppose that (E, b) satisfies (3). Suppose $p \in b(A)$ and $q \in b(A \cup \{p\})$. Then $q \in b(b(A)) = b(A)$, and (3′) is satisfied. □

Let (E, b) be a bumpered set and let $*$ be an involution on E. Then the triple $(E, b, *)$ is called an *involuted bumpered set* if:

(4) For $p \in E$ with $p \neq p^*$, if $p \in b(A \cup \{p^*\})$ then $p \in b(A)$.

For instance, if $(E, \mathscr{C}, *)$ is an oriented matroid and h is its hull function then it is obvious that $(E, h, *)$ is an ivoluted bumpered set.

THEOREM 7. *Let $(E, b, *)$ be an involuted bumpered set. For any subset A of E, let $\hat{b}(A) = A \cup \{x \in E \mid x^*$ is not in the set $b(E \sim (A \cup \{x^*\}))\}$. Then $(E, \hat{b}, *)$ is an involuted bumpered set, and $\hat{\hat{b}} = b$.*

□ Condition (1) is clearly satisfied.
If $A \subset B$ then $\{x \in E \mid x^* \notin b(E \sim (A \cup \{x^*\}))\}$ is contained in

$$\{x \in E \mid x^* \notin b(E \sim (B \cup \{x^*\}))\},$$

so the second condition is satisfied and (E, b) is a bumpered set.
Suppose $p^* \neq p$ and $p \in \hat{b}(A \cup \{p^*\})$. If $p \in A$ then $p \in \hat{b}(A)$. Suppose $p \notin A$. Since $p \in \hat{b}(A \cup \{p^*\})$, p^* is not in $b(E \sim (A \cup \{p^*\}))$. But then $p \in \hat{b}(A)$. Condition (4) is satisfied, and $(E, \hat{b}, *)$ is an involuted bumpered set.
That $\hat{\hat{b}} = b$ is obvious when one notes that b is the bumper function such that, if $A \cup B \cup \{p, p^*\}$ is a disjoint union, $= E$, $p \in b(A)$ if and only if $p^* \notin \hat{b}(B)$. □

The involuted bumpered set $(E, \hat{b}, *)$ is called the *dual* of $(E, b, *)$. We shall soon see that if $(E, b, *)$ arises from an oriented matroid, then so does its dual; then both b and $\hat{b}$ will be hull functions.

If (E, b) is a bumpered set and $p \in E$ then the function b_0 such that $b_0(A) = b(A) \sim \{p\}$ for each set $A \subset E \sim \{p\}$ is also a bumper function. The pair $(E \sim \{p\}, b_0)$ is the *elementary minor of* (E, b) *obtained by deletion of* p.

If b_1 is the function such that $b_1(A) = b(A \cup \{p\}) \sim \{p\}$ for each subset A of $E \sim \{p\}$, then the bumpered set $(E \sim \{p\}, b_1)$ is the *elementary minor of* (E, b) *obtained by contraction of* p.

A *minor* of (E, b) is any pair (E_0, b_0), where there is a set $C \subset E \sim E_0$ with $b_0(A) = b(C \cup A) \sim C$ for each subset A of E_0. If $E = E_0 \cup C \cup D$ is a disjoint union, this minor is obtained from (E, b) by *deletion* of D and *contraction* of C. (E, b) is a minor of itself, and any other minor of (E, b) may be obtained as an elementary minor of some minor of (E, b).

If $(E, h, *)$ is an involuted bumpered set and if (E_0, b_0) is a minor with $E_0 = E_0^*$, then $(E_0, b_0, *)$ is also an involuted bumpered set. If $p \in E$ there are four possibilities for such involuted minors on $E \sim \{p, p^*\}$. We may delete p and p^*, delete p and contract p^*, delete p^* and contract p, or contract both. Each involuted minor so obtained, being an involuted bumpered set, will have a dual; this dual will be a minor of the dual of

$(E, b, *)$ which can be obtained from $(E, \hat{b}, *)$ by switching the operations of deletion and contraction used to obtain the minor of $(E, b, *)$. For instance, if $(E \sim \{p, p^*\}, b_0, *)$ is obtained from $(E, b, *)$ by deleting p and contracting p^*, then $(E \sim \{p, p^*\}, \hat{b}_0, *)$ can be obtained from $(E, \hat{b}, *)$ by contracting p and deleting p^*.

If $(E, b, *)$ is an involuted bumpered set then Theorem 6 insures that b is a hull function if and only if $(E, b, *)$ fails to have certain involuted minors on sets E_0 of the form $\{p, p^*, q, q^*\}$. Since minors of $(E, \hat{b}, *)$ are duals of minors of $(E, b, *)$, there is a similar condition on minors of $(E, b, *)$ that insures that $\hat{b}$ is a hull function. This is given by the following theorem.

THEOREM 8. *Let $(E, b, *)$ be an involuted bumpered set. Then $\hat{b}$ is a hull function if and only if:*

(5) *If $q \in b(A \cup \{p^*\}) \sim b(A)$, then $p \in b((A \cup \{q^*\}) \sim \{p\})$.*

□ Suppose $\hat{b}$ is a hull function. Suppose $A \subset E$ with:

(a) $q \in b(A \cup \{p^*\})$;

and

(b) $q \notin b(A)$.

We must show:

(c) $p \in b((A \cup \{q^*\}) \sim \{p\})$.

Suppose not. Let $B = E \sim (A \cup \{p^*, q^*\})$. Since (c) does not hold, $p^* \in \hat{b}(B \cap \{p\})$, and therefore $p^* \in \hat{b}(B)$. From (b) it follows that $q^* \in \hat{b}(B \cap \{p^*\})$. Therefore $q^* \in \hat{b}(B)$, since $\hat{b}$ is a hull function. But then (a) cannot hold.

Suppose b satisfies (5). We must show that $\hat{b}$ is a hull function. Suppose $p \in \hat{b}(A \cup \{q\})$ and $q \in \hat{b}(A)$. We must show that $p \in \hat{b}(A)$.

Suppose not. We may assume $p^* \in A$. Let $B = E \sim (A \cup \{p, q\})$. Then since $p \notin \hat{b}(A)$, $p^* \in b(B \cup \{q\})$. Since $p \in \hat{b}(A \cup \{q\})$ but not in $A \cup \{q\} \subset \hat{b}(A) \cup \{q\}$, $p^* \notin b(B)$. Since $q \in \hat{b}(A) \sim A$, $q^* \notin b((B \cup \{p\}) \sim \{q^*\})$. This cannot be the case, since (5) holds. □

An involuted bumpered set $(E, h, *)$ will be called a *gatroid* (*generalized matroid*) if both h and $\hat{h}$ are hull functions; i.e., if h is a hull function and $(E, h, *)$ satisfies condition (5). In case the involution on E fixes each element of E, these conditions reduce to the usual definition of a matroid in terms of its hull function, condition (5) reducing to the exchange axiom. Such gatroids will sometimes be called *ordinary*. Gatroids will sometimes be called *ordinary*. Gatroids in which no point is left fixed by the involution will be called *oriented*. Condition (5) will be called the *exchange axiom* for gatroids.

If $(E, \mathscr{C}, *)$ is an oriented matroid and h is its hull function, then $(E, h, *)$

is easily seen to satisfy the exchange axiom, so $(E, h, *)$ is an orient gatroid. In such a gatroid we have, for any subset A of E, $h(A^*) = h(A)$. Any gatroid for which this is true will be called *symmetric*. Soon we wi see that the symmetric, oriented gatroids are precisely the ones that com from oriented matroids. The map that takes the oriented matroid $(E, \mathscr{C},$ to the symmetric, oriented gatroid $(E, h, *)$, where h is the hull function o the oriented matroid, is a bijection.

First, we give a decomposition theorem for the symmetric gatroids. The following lemma will be required.

LEMMA. *Suppose* $(E, h, *)$ *is a gatroid*, $A \subset E$, $p \in h(A \cup \{q\})$, $p \neq p^*$, *and* $q = q^*$. *Then* $p \in h(A)$.

□ Suppose not. Then, by the exchange axiom, q is in the set $h((A \cup \{p^*\}) \sim \{q\})$. Then $h(A \cup \{p^*\})$ contains q, so $h(A \cup \{p^*\})$ contains $h(A \cup \{q\})$. Therefore $p \in h(A \cup \{p^*\})$, so $p \in h(A)$, contradicting the assumption. □

Suppose that $(E_1, b_1, *)$ and $(E_2, b_2, *)$ are involuted bumpered sets, with E_1 and E_2 disjoint. For subsets A of E, where $E = E_1 \cup E_2$, let:

$$b(A) = b_1(A \cap E_1) \cup b_2(A \cap E_2).$$

Then $(E, b, *)$ is an involuted bumpered set, called the *free join* of the two bumpered sets.

THEOREM 9. *Let* $(E, h, *)$ *be a symmetric gatroid. Then* $(E, h, *)$ *is the free join of a symmetric, ordinary gatroid and a symmetric, oriented gatroid.*

□ Let $E_1 = \{x \in E \mid x \neq x^*\}$. Let $E_2 = \{x \in E \mid x = x^*\}$. For $A \subset E_i$ ($i = 1$ or 2) let $h_i(A) = h(A) \cap E_i$. Then $(E, h_i, *)$ is the minor obtained from $(E, h, *)$ by deleting $E \sim E_i$. Therefore, $(E, h_i, *)$ is a symmetric gatroid. It is oriented if $i = 1$ and it is ordinary if $i = 2$. We will show that the gatroid $(E, h, *)$ is the free join of these two gatroids.

Obviously $E = E_1 \cup E_2$. It is also clear that we have the inclusion:

$$h(A) \supset h_1(A \cap E_1) \cup h_2(A \cap E_2),$$

for each subset A of E. It is necessary only to verify the reverse inclusion.

Suppose $p \in h(A)$, and $p \neq p^*$. It is immediate from the lemma that if M is a minimal subset of A with $p \in h(M)$ then M is contained in E_2. Therefore $p \in h(A \cap E_2)$, as required.

Now suppose $p \in h(A)$ with $p \neq p^*$. We must show that p is in $h(A \cap E_1)$. Let B be a minimal subset of A with p in $h(B \cup B^*)$. Suppose $q \in B$ with $q \neq q^*$. Then p is in $h(B \cup B^*)$ but not in $h((B \cup B^*) \sim \{q, q^*\})$. Either:

(a) $p \in h((B \cup B^*) \sim \{q\}) \sim h((B \cup B^*) \sim \{q, q^*\})$,

r:

(b) $p \in h(B \cup B^*) \sim h((B \cup B^*) \sim \{q\})$.

n either case it follows from the exchange axiom and symmetry that $\{q, q^*\} \subset h((B \cup B^* \cup \{p\}) \sim \{q, q^*\})$. By the lemma, $\{q, q^*\} \subset h((B \cup B^*) \sim \{q, q^*\})$. This cannot be the case, since then $p \in h((B \sim \{q\}) \cup (B \sim \{q\})^*)$. Therefore $B = B^* \subset A \cap E_1$, and $p \in h(A \cap E_1)$. □

Let $(E, h, *)$ be a symmetric oriented gatroid. Let $\mathscr{C}$ be the collection of minimal subsets C of E with the following properties:

(1) $C \neq \varnothing$ and $C \cap C^* = \varnothing$;

and

(2) $C^* \subset h(C)$.

$\mathscr{C}$ is the set of *circuits* of $(E, h, *)$. If h is the hull function for an oriented matroid it is obvious that $\mathscr{C}$ is precisely the set of circuits of this oriented matroid. We will show below that if $(E, h, *)$ and $\mathscr{C}$ are as we have it here then $(E, \mathscr{C}, *)$ will always be an oriented matroid. This is Theorem 11. Some other results will be useful.

LEMMA. *Let* $(E, h, *)$ *be a gatroid. Suppose* $p \neq p^*$, $q \neq q^*$, $p \in h(A \cup \{q\})$, *and* $p \in h(A \cup \{q^*\})$. *Then* $p \in h(A)$.

□ Suppose not. Then $p \in h(A \cup \{q^*\}) \sim h(A)$, so q is in $h(A \cup \{p^*\})$. But then $h(A \cup \{p^*\})$ contains $h(A \cup \{q\})$, and $p \in h(A \cup \{p^*\})$. But then $h(A \cup \{p^*\})$ contains $h(A \cup \{q\})$, and $p \in h(A \cup \{p^*\})$. This means $p \in h(A)$. □

LEMMA. *Let* $(E, h, *)$ *be a gatroid. Suppose* $p \neq p^*$, $q \neq q^*$, *and* $p \in h(A \cup \{q, q^*\})$. *Then* $p \in h(A \cup \{q\})$ *or* $p \in h(A \cup \{q^*\})$.

□ This lemma is dual to the preceding one. In any nontrivial case, we may choose a set B so that E is the union of the pair-wise disjoint sets A, B, and $\{p, p^*, q, q^*\}$. That p is in $h(A \cup \{q, q^*\})$ is, in the dual gatroid, simply that p is not in $h(B)$. Therefore, p fails to be in at least one of the sets $h(B \cup \{q\})$ and $h(B \cup \{q^*\})$, by the lemma. This is clearly equivalent to the desired conclusion. □

LEMMA. *Suppose* $p \in h(S) \sim S$. *Then there is a set* T *with* $p^* \in T \subset S \cup \{p^*\}$ *such that* $T^* \subset h(T)$ *and* $T \cap T^* \neq \varnothing$.

□ Let T_0 be a minimal subset of S with $p \in h(T_0)$. Let $T = T_0 \cup \{p^*\}$. If $q \in T_0$ then $p \in h(T_0) \sim h(T_0 \sim \{q\})$, so by the exchange axiom $q^* \in h(T)$. Therefore $T^* \subset h(T)$.

Clearly $p^* \notin T_0$. Suppose T_0 contains elements u and u^*. Then, by the preceding lemma, $p \in h(T_0 \sim \{u\})$ or $p \in h(T_0 \sim \{u^*\})$, contradicting the minimality of T_0. Therefore $T \cap T^* = \varnothing$. □

THEOREM 10. *Let $(E, h, *)$ be a symmetric, oriented gatroid. Suppose $p \in E$ and S is a minimal subset of E satisfying:*

(a) $p \in S \sim S^*$;

and

(b) $p^* \in h(S)$.

Then S is a circuit of the gatroid.

□ Suppose this is not correct. Pick a symmetric, oriented gatroid $(E, h, *)$, S, and p, with E having as few elements as possible, so that the conditions, but not the conclusion, of the theorem are satisfied. Note that $S \cap S^*$ must be empty.

If $u \in S \sim \{p\}$ then $p^* \in h(S) \sim h(S \sim \{u\})$, by the minimality of S, so $u^* \in h(S)$. Therefore $S^* \subset h(S)$. We must show that if T is a proper subset of S, with $T \neq \varnothing$, then T^* is not contained in $h(T)$. For this it will suffice to show that, given any element q of $S \sim \{p\}$, q^* is not in $h(S \sim \{p\})$, for p cannot be in such a set T.

For $A \subset E \sim \{q, q^*\}$, let:

$$h_0(A) = h(A \cup \{q, q^*\}) \sim \{q, q^*\}.$$

Then $(E \sim \{q, q^*\}, h_0, *)$ is the symmetric, oriented gatroid obtained by contracting q and q^*. $E \sim \{q, q^*\}$ has fewer elements than E, so the theorem holds here.

Let $R = S \sim \{q\}$. Then:

(a) $p \in R \sim R^*$;

and

(b) $p^* \in h_0(R)$.

Suppose U is contained in R properly. Then $p^* \notin h(U \cup \{q\})$, if $p \in U$, since then $U \cup \{q\}$ is contained in S properly, and must fail to satisfy one of the conditions. Also, p^* cannot be in $h(U \cup \{q^*\})$, since this is contained in $h(S \sim \{q\})$. Therefore, by a lemma, $p^* \notin h(U \cup \{q, q^*\})$, so $p^* \notin h_0(U)$. It follows that R is a minimal set satisfying (a) and (b), so it is a minimal non-empty set with $R^* \subset h_0(R)$.

Suppose $q^* \in h(S \sim \{p\})$. Let T be a minimal subset of $S \sim \{p\}$ with $q^* \in h(T)$. T is non-empty, since if q^* were in $h(\varnothing)$, then, by symmetry, q would be in $h(\varnothing)$, so p^* would be in $h(S \sim \{q\})$. If $u \in T$ then $u^* \in h((T \sim \{u\}) \cup \{q\})$. By the preceding lemma $T \cup \{q\}$ contains a non-empty subset V such that $V^* \subset h(V)$. If $W = V \sim \{q\}$, $W^* \subset h_0(W)$. This is contrary to the minimality of R, since $W \subset R \sim \{p\}$. This cannot be the case. Therefore $q^* \notin h(S \sim \{p\})$, and the theorem is established. □

This theorem has two important corollaries.

COROLLARY. *Let S be a subset of E, and suppose, for each element p of S, $p^* \in h(S \sim \{p^*\})$. Then S is the union of circuits.*

□ For $p \in S$ let T be a minimal subset of $S \sim \{p^*\}$ with $p^* \in h(T)$ and $p \in T \sim T^*$. By the theorem, T is a circuit. Then for each element p of S there is a circuit T with $p \in T \subset S$. □

COROLLARY. *Let A be a subset of E with $h(A) = A$ and $A \neq E$. Then A is the intersection of maximal such sets.*

□ This is dual to the preceding corollary. The set A satisfies the properties $h(A) = A$ and $A \neq E$ if and only if, in the dual symmetric, oriented gatroid, the set $S = E \sim A$ is nonempty and has the property that for each p in S, p^* is included in the set $h(S \sim \{p^*\})$. □

We are now in a position to establish that the triples $(E, \mathscr{C}, *)$ will indeed be oriented matroids, for any symmetric, oriented gatroid $(E, h, *)$.

THEOREM 11. *Let $(E, h, *)$ be a symmetric, oriented gatroid. Let $\mathscr{C}$ be the set of circuits of $(E, h, *)$. Then $(E, \mathscr{C}, *)$ is an oriented matroid.*

□ $\mathscr{C}$ is a clutter of non-empty subsets of E. Clearly, if $C \in \mathscr{C}$ then $C^* \in \mathscr{C}$ and $C \cap C^* = \varnothing$.

Suppose S and T are in $\mathscr{C}$, $x \in S \cap T^*$, and $S \neq T^*$. Let y be an element of $S \sim T^*$. Then $x \in h(T \sim \{x^*\})$ and $x^* \in h(S \sim \{x\})$, so:

$$h((S \cup T) \sim \{x, x^*\}) = h(S \cup T).$$

Therefore $y^* \in h((S \cup T) \sim \{x, x^*\})$. If C is a minimal subset of $(S \cup T) \sim \{x, x^*\}$ with $y \in C$ and $y^* \in h(C)$ then, by Theorem 10, C is a circuit. $y \in C \subset (S \cup T) \sim \{x, x^*\}$. □

The following theorem will complete our description of the correspondence between the oriented matroids and the symmetric, oriented gatroids.

THEOREM 12. *Let $(E, h, *)$ be a symmetric oriented gatroid with circuits $\mathscr{C}$. Let g be the hull function for the oriented matroid $(E, \mathscr{C}, *)$. Then $g = h$.*

□ If A is a subset of E we must show that $g(A) = h(A)$. Suppose that x is an element of $g(A)$. If it is in A then, of course, it is in $h(A)$. If it is not in A, then there must be a circuit C with $s^* \in C \subset A \cup \{x^*\}$. Then:

$$x \in C^* \subset h(C \sim \{x^*\}) \subset h(A).$$

Suppose $x \in h(A) \sim A$. Then there is a circuit S with $x^* \in S \subset A \cup \{x^*\}$, by Theorem 10. Therefore, $x \in g(A)$. □

It follows from the results established that the correspondences we have described are bijective. The mapping that gives, for an oriented matroid $(E, \mathscr{C}, *)$ with hull function h, the symmetric, oriented gatroid $(E, h, *)$ is the inverse of the mapping that gives, for a symmetric, oriented gatroid, the oriented matroid with the same circuits.

By the *dual* of an oriented matroid $(E, \mathscr{C}, *)$, with hull function h, we will mean the oriented matroid $(E, \hat{\mathscr{C}}, *)$, where $\hat{\mathscr{C}}$ is the set of circuits of $(E, \hat{h}, *)$.

The following is a generalization of Minty's "colored arc lemma." It was proven by Rockafellar in [12] for the realizable oriented matroids. Bland [1] has already proven that it is true for oriented matroids, but we will find it convenient to have it here in order to show that our oriented matroids are in fact the same as those of Bland.

THEOREM 13. *Let $(E, \mathscr{C}, *)$ be an oriented matroid with dual $(E, \hat{\mathscr{C}}, *)$. Suppose $p \in E$, and that A and B are disjoint subsets of $E \sim \{p, p^*\}$ with $A \cup B \cup \{p, p^*\} = E$. Then exactly one of the following holds:*

(a) *There is $C \in \mathscr{C}$ with $p \in C \subset A \cup \{p\}$;*

(b) *There is $D \in \hat{\mathscr{C}}$ with $p^* \in D \subset B \cup \{p^*\}$.*

□ Let h and $\hat{h}$ be the corresponding hull functions. The theorem follows at once from the observation that it is equivalent to the statement that $p^* \in h(A)$ if and only if p is not in $\hat{h}(B)$. □

THEOREM 14. *Let $(E, \mathscr{C}, *)$ be an oriented matroid, with dual $(E, \hat{\mathscr{C}}, *)$. Let $(\bar{E}, \bar{\mathscr{C}})$ and $(\bar{E}, \bar{\hat{\mathscr{C}}})$ be the underlying matroids. Suppose $\bar{p} \in \bar{E}$, and $\bar{A}$ and $\bar{B}$ are disjoint subsets of $\bar{E} \sim \{\bar{p}\}$ with $\bar{A} \cup \bar{B} \cup \{\bar{p}\} = \bar{E}$. Then exactly one holds:*

(a) *There is $\bar{C} \in \bar{\mathscr{C}}$ with $\bar{p} \in \bar{C} \subset \bar{A} \cup \{\bar{p}\}$;*

(b) *There is $\bar{D} \in \bar{\hat{\mathscr{C}}}$ with $\bar{p} \in \bar{D} \subset \bar{B} \cup \{\bar{p}\}$.*

□ This follows immediately from Theorem 13 by taking for A the set $\{x \in E \mid \bar{x} \in \bar{A}\}$, for B the set $\{x \in E \mid \bar{x} \in \bar{B}\}$, and p such that $\bar{p} = \{p, p^*\}$. □

It is an immediate consequence of this theorem that the matroids $(\bar{E}, \bar{\mathscr{C}})$ and $(\bar{E}, \bar{\hat{\mathscr{C}}})$ are dual.

Bland [1] has given an axiomatization for dual pairs of oriented matroids. Using our terminology this may be given in the following conditions, in terms of the 4-tuple $(E, \mathscr{C}, \hat{\mathscr{C}}, *)$:

(a) $\mathscr{C}$ and $\hat{\mathscr{C}}$ are collections of subsets of E such that if A and B are in $\mathscr{C}$(or, in $\hat{\mathscr{C}}$) and $A \subset B \cup B^*$, then $A = B$ or $A = B^*$;

(b) $(\bar{E}, \bar{\mathscr{C}})$ and $(\bar{E}, \bar{\hat{\mathscr{C}}})$ are dual matroids:

and

(c) If $C \in \mathscr{C}$ and $D \in \hat{\mathscr{C}}$ then if $C \cap D \neq \varnothing$, $C^* \cap D \neq \varnothing$.

He has shown that, given such a 4-tuple, $(E, \mathscr{C}, *)$ and $(E, \hat{\mathscr{C}}, *)$ are (dual) oriented matroids. We have seen already that if these are dual oriented matroids, then the 4-tuple $(E, \mathscr{C}, \hat{\mathscr{C}}, *)$ satisfies (a) and (b). The following theorem shows that it also satisfies (c), completing the demonstration that the two kinds of oriented matroids are the same.

THEOREM 15. *Let $(E, \mathscr{C}, *)$ and $(E, \hat{\mathscr{C}}, *)$ be dual oriented matroids. Suppose $C \in \mathscr{C}$ and $D \in \hat{\mathscr{C}}$. Then if $C \cap D \neq \varnothing$, $C^* \cap D \neq \varnothing$.*

□ Suppose $C^* \cap D = \varnothing$. Let h be the hull function for the oriented matroid $(E, \mathscr{C}, *)$. $C^* \subset E \sim D$, so:

$$C \subset h(C^*) \subset h(E \sim D) = E \sim D.$$

Therefore, $C \cap D = \varnothing$. □

IV. ARRANGEMENTS OF PSEUDO-HEMISPHERES

Let $(E, \mathscr{C}, *)$ be the oriented matroid arising from a subset E of the real vector space R^d, as in Theorem 1. It is convenient here to view such realizable oriented matroids in a different way.

Let S^{d-1} be the unit sphere centered at the origin in R^d. For $p \in E$ let $\sigma(p) = \{x \in S^{d-1} \mid x \cdot p \geqslant 0\}$, so that $\sigma(p)$ is a closed hemisphere in S^{d-1}. The circuits $C \in \mathscr{C}$ may now be described as the minimal non-empty subsets C of E with $C \cap C^* = \varnothing$ and $\bigcup_{p \in C} \sigma(p) = S^{d-1}$.

By an *arrangement of hemispheres* we mean a collection ξ of finitely many closed hemispheres in S^{d-1} such that if $s \in \xi$ then also $-s \in \xi$. Any such arrangement yields an oriented matroid $(\xi, \mathscr{C}, *)$ if we take $s^* = -s$ for $s \in \xi$ and if $\mathscr{C}$ is the collection of minimal subsets C of ξ with:

(1) $C \neq \varnothing$ and $C \cap C^* = \varnothing$

and

(2) $\bigcup_{s \in C} s = S^{d-1}$.

Let $(\xi, \mathscr{C}, *)$ be such an oriented matroid. Its hull function h is easy to describe. If $s \in \xi$ then $s \in h(A)$ if and only if s contains $\bigcap_{x \in A} x$.

By considering minors of such oriented matroids, one gets a large class of oriented gatroids which are not symmetric. Let ξ be an arrangement of hemispheres in S^{d-1} and let R be a subset of S^{d-1} which is the intersection of finitely many hemispheres of S^{d-1}. For $A \subset \xi$ let $h(A) = \{s \in \xi \mid s$ contains the set $R \cap (\bigcap_{t \in A} t)\}$. Then $(\xi, h, *)$ is such a gatroid.

Many interesting results concerning arrangements of hemispheres may be derived from the results of Shannon in [13], where arrangements of hyperplanes in projective space are studied.

It is possible to derive oriented matroids from objects topologically similar to arrangements of hemispheres. In fact, one can represent *any* oriented matroid as that arising from some such "arrangement of pseudo-hemispheres."

A *topological cell complex* (Whitehead [15]) is a triple (X, P, φ), where P is a finite partially ordered set with a least element, denoted by O, X is a Hausdorff topological space, and φ is a function from P to subsets of X, such that:

(1) $\varphi(O) = \varnothing$;

(2) If c and d are in P with $c \neq d$ then $\varphi(c) \cap \varphi(d) = \varnothing$;

(3) If c is in P then $\bigcup_{d \leqslant c} \varphi(d)$ is homeomorphic to a closed ball whose interior is $\varphi(c)$ and whose boundary is $\bigcup_{d < c} \varphi(d)$.

Note that if (X, P, φ) and (Y, Q, τ) are topological cell complexes with $P = Q$, then X and Y may be identified by a homeomorphism in such a way that the complexes are also identified.

A *closed subcomplex* of (X, P, φ) is a triple (Y, Q, τ), where Y is a closed subset of X, Q is a subset of P such that if $a \in Q$ and $b \in P$ with $b \leqslant a$, then $b \in Q$, τ is the restriction of φ to Q, and $Y = \bigcup_{a \in Q} \tau(a)$.

An *arrangement of pseudo-hemispheres* is a topological cell complex (X, P, φ), where X is a sphere provided with an Involutive homeomorphism * without fixed-points, together with a collection ξ of closed sub-complexes each homeomorphic to a ball of the same dimension as X, such that:

(1) If $s \in \xi$ then $s^* \in \xi$ and $s \cap s^*$ is the sphere bounding each;

(2) If $A \subset \xi$ with $A = A^*$ then $\bigcap_{s \in A} s$ is empty or a sphere; if also $t \in \xi$ then either $t \supset \bigcap_{s \in A} s$ or $t \cap (\bigcap_{s \in A} s)$ is a closed ball;

and

(3) If $p \in P$ then $\bigcup_{q \leqslant p} \varphi(q)$ is the intersection of elements of ξ

We will show that any such arrangement ξ determines an oriented matroid $(\xi, \mathscr{C}, *)$, where $\mathscr{C}$ is the collection of minimal subsets C, called *circuits*, of ξ such that:

(a) $C \neq \varnothing$, and $C \cap C^* = \varnothing$;

and

(b) $\bigcup_{s \in C} s = X$.

This will be Theorem 16. First, we need three lemmas.

LEMMA. *Let ξ be an arrangement of pseudo-hemispheres. Suppose $A \subset \xi$. Then the set $V = X \sim (\bigcup_{s \in A} s)$ is connected, or empty.*

□ Let $A = \{s_1, s_2, \ldots, s_n\}$. We proceed by induction on n, noting that for $n = 1$ the result holds, since then V is homeomorphic to R^d. Suppose $n > 1$, and that the result is valid for sets of smaller cardinality. In particular, $V' = X \sim \bigcup_{i=1}^{n-1} s_i$ is connected; and, since $\xi' = \{s_n \cap s_n^* \cap s \mid s \in \xi\}$ is an arrangement of pseudo-hemispheres in the sphere $s_n \cap s_n^*$, $W = (s \cap s^*) \sim (\bigcup_{i=1}^{n-1} s_i)$ is connected. Let d be the dimension of X. For each element t of V' choose an open neighborhood $U(t)$ with $t \in U(t) \subset V'$, $U(t)$ homeomorphic to R^d, and such that: (a) $U(t) \subset V' \sim s_n$ if $t \notin s_n$; (b) $U(t) \subset V' \sim s_n^*$ if $t \notin s_n^*$; and (c) $U(t) \cap s_n$ and $U(t) \cap s_n^*$ both homeomorphic to closed halfspaces in R^d if $t \in s_n \cap s_n^*$. Suppose p and q are in V. Then p and q are in V', and since V' is connected, there are elements t_i of V' $(0 \leqslant i \leqslant m)$ with $p \in U(t_0)$, $q \in U(t_m)$, and $u(t_{i-1}) \cap U(t_i) \neq \varnothing$ for $1 \leqslant i \leqslant m$. Since W is connected we may assume that no t_i is in $X \sim s_n^*$, and that if t_i and t_{i-1} are in $s_n \cap s_n^*$ then $U(t_i) \cap U(t_{i+1}) \cap s_n \cap s_n^* \neq \varnothing$. Then $(U(t_{i-1}) \sim s_n) \cap (U(t_i) \sim s_n)$ is nonempty, for $1 \leqslant i \leqslant m$. Since these subsets $U(t_i) \sim s_n$ of V are connected, p and q are in the same component of V, and V is connected. □

(The set V above can be shown to be homeomorphic to R^d; however, we don't use this here.)

Let ξ be an arrangement with circuits $\mathscr{C}$. Let the arrangement $\xi' = \{s \cap p \cap p^* \mid s \in \xi \sim \{p, p^*\}\}$ have circuits $\mathscr{C}'$.

LEMMA. *Suppose $C \in \mathscr{C}'$, $U \subset \xi$, and $C = \{s \cap p \cap p^* \mid s \in U\}$. Furthermore, suppose that if s and t are in U then $s \cap p \cap p^*$ and $t \cap p \cap p^*$ are the same sets if and only if $s = t$. Then U, $U \cup \{p\}$, or $U \cup \{p^*\}$ is in $\mathscr{C}$.*

□ Since $C \in \mathscr{C}'$, $(p \cap p^*) \sim (\bigcup_{s \in U} s) = \varnothing$, and U is a minimal set for which this holds. If $X \sim (\bigcup_{s \in U} s) = \varnothing$, then $U \in \mathscr{C}$. Otherwise $X \sim (\bigcup_{s \in U} s)$ is in one of the connected components of $X \sim (p \cap p^*)$, so that one of $U \cup \{p\}$ and $U \cup \{p^*\}$ is in $\mathscr{C}$. □

LEMMA. *Suppose $D \in \mathscr{C}$ and $p \in D$. Then if:*

$$C = \{s \cap p \cap p^* \mid s \in D \sim \{p\}\},$$

C is in $\mathscr{C}'$, unless $|C| = 2$ and $C = C^*$.

□ Let $U = D \sim \{p\}$. Since $D \in \mathscr{C}$, we have:

(a) $\varnothing \neq X \sim (\bigcup_{s \in U} s) \subset p$.

Since this is an open set, it is contained in the interior of p; i.e., it misses $p \cap p^*$. Then:

(b) $(p \cap p^*) \sim \bigcup_{s \in U}(s \cap p \cap p^*) = \varnothing$.

If V is a proper subset of U then it is not true that $X \sim (\bigcup_{s \in V} s) \subset p$, since $D \in \mathscr{C}$. This open, connected set meets both p and p^*, and so must meet $p \cap p^*$. Therefore U is a minimal set for which (b) holds, so the conclusion follows. □

THEOREM 16. *$(\xi, \mathscr{C}, *)$ is an oriented matroid.*

□ We verify only the third condition of the definition, the others being obvious. We preceed by induction on the dimension of X.

Suppose S and T are in $\mathscr{C}$, $x \in (S \cap T^*)$, and $S \neq T^*$.

First, suppose $S \cap T^*$ contains nothing other than x. Then, if $U = (S \cup T) \sim \{x, x^*\}$, we have:

$$X \sim \bigcup_{s \in U} s = \Big(X \sim \bigcup_{s \in S \sim \{x\}} (s)\Big) \cap \Big(X \sim \bigcup_{t \in T \sim \{x^*\}} (t)\Big)$$
$$\subset (X \sim \{x^*\}) \cap (X \sim \{x\}) = \varnothing.$$

Therefore, since $U \cap U^* = \varnothing$, U contains some element of $\mathscr{C}$, as required.

Now suppose there is in $S \cap T^*$ an element $p \neq x$. Let $(\xi', \mathscr{C}', *)$ be the oriented matroid corresponding to the arrangement $\xi' = \{s \cap p \cap p^* \mid s \in \xi \sim \{p, p^*\}\}$. Let:

$$S_0 = \{s \cap p \cap p^* \mid s \in S \sim \{p\}\}$$

and:

$$T_0 = \{t \cap p \cap p^* \mid t \in T \sim \{p^*\}\}.$$

Let $x_0 = x \cap p \cap p^*$. Either, say, $|S_0| = 2$, or S_0 and T_0 are in the collection $\mathscr{C}'$. In the first case, there is an element y of ξ such that $S = \{x, y, p\}$, so that $x_0{}^*$ is the set $y \cap p \cap p^*$. In this case, letting W be the set $(T \sim \{x^*, p^*\}) \cup \{y\}$, we see that:

$$(p \cap p^*) \sim \bigcup_{s \in W} s = (p \cap p^*) \sim \bigcup_{s \in T \sim \{p^*\}} (s) = \varnothing,$$

and W is a minimal set for which this is true. Then the set $W_0 = \{s \cap p \cap p^* \mid s \in W\}$ is in $\mathscr{C}'$. Then, by the lemma, W, $W \cup \{p\}$, or $W \cup \{p^*\}$ is in $\mathscr{C}$, and this is the required circuit. Suppose the other case holds.

If $S_0 = T_0^*$ then there are elements u of $S \sim \{x\}$ and v of $T \sim \{x^*\}$ with $u \neq v^*$ and $u \cap p \cap p^* = v^* \cap p \cap p^*$. Then either $\{u, v, p\} \in \mathscr{C}$ or $\{u, v, p^*\} \in \mathscr{C}$.

If $S_0 \neq T_0^*$ then there is C in $\mathscr{C}'$ with:

$$C \subset (S_0 \cup T_0) \sim \{x_0, x_0{}^*\}.$$

By the lemma there is $D \in \mathscr{C}$ with $D \subset (S \cup T) \sim \{x, x^*\}$. □

Suppose ξ is actually an arrangement of genuine hemispheres of S^d. ξ is *proper* if $\bigcap_{s \in \xi} s = \varnothing$. In this case ξ determines a subdivision of the sphere, a topological cell complex (and may also be viewed as an arrangement of pseudohemispheres). If h is the hull function of the corresponding oriented matroid, then the sets $A \subset \xi$ with $h(A) = A$ correspond to the regions of the sphere that may be represented as the intersection of the hemispheres that contain them. The maximal such proper subsets of ξ correspond to minimal such regions—single points which may be represented as the intersection of hemispheres in the arrangement. These points are called the *vertices* of the arrangement. (See Shannon [13].)

If A is such a maximal, proper subset of ξ with $h(A) = A$ we have seen that $\xi \sim A$ is a circuit of the dual oriented matroid. Thus vertices of the arrangement correspond to circuits of the dual oriented matroid. Similarly, cells of the complex determined by the arrangement correspond to sets C contained in ξ which may be represented as a union of circuits of the dual oriented matroid and such that $C \cap C^* = \varnothing$.

Let $\mathscr{O} = (E, \mathscr{C}, *)$ be an oriented matroid and let $\hat{\mathscr{C}}$ be the collection of circuits of the dual oriented matroid. We call elements of $\hat{\mathscr{C}}$ *points* of $\mathscr{O}$. If S is a union of points of and $S \cap S^* = \varnothing$, then S is called a *cell* of $\mathscr{O}$.

Let P be the partially ordered set of cells of $\mathscr{O}$ ordered by inclusion. (P includes the empty set.) We will show that there is a topological cell complex (X, P, φ), where X is a sphere. First, however, we need to develop some other results.

Let A be a subset of E. Let $r(A)$, the *rank* of A, be the maximum number of elements of a subset of R that is independent in the underlying matroid. The *rank* of the oriented matroid is $r(E)$; i.e., it is the rank of the underlying matroid.

THEOREM 17. *Let $\mathscr{O} = (E, \mathscr{C}, *)$ be an oriented matroid of rank 2. Let U, S, and T be points of $\mathscr{O}$ with $U \subset S \cup T$. Then $S \cap T$ is a subset of U.*

□ Suppose $p \in S \cap T$. If $S = T$ then $U \subset S$ so $U = S = T$. We may suppose $S \neq T$. Then there is a point C contained in $(S \cup T^*) \sim \{p, p^*\}$. p^* is not in $S \cup T$, so it is not in U. If p is not in U then $\bar{p}$ is not in $\bar{C} \cup \bar{U}$, contradicting the assumption that the underlying matroid has rank 2. Therefore, $p \in U$. □

For an oriented matroid $\mathcal{O}$, let $G(\mathcal{O})$ denote the graph whose vertices are the points of $\mathcal{O}$, with U and V adjacent provided U is neither V nor V^* and $U \cup V$ contains no points of the oriented matroid other than U and V. Let $G_p(\mathcal{O})$ denote the subgraph of $G(\mathcal{O})$ spanned by vertices which contain p.

THEOREM 18. *Let $\mathcal{O} = (E, \mathscr{C}, *)$ and let p be in E. Then if the rank of $\mathcal{O}$ is at least 2, $G(\mathcal{O})$ and $G_p(\mathcal{O})$ are connected graphs.*

□ We proceed by induction on the rank.

If the rank of $\mathcal{O}$ is 2 and U and V are points of $\mathcal{O}$, we must show there is a path in $G(\mathcal{O})$ from U to V. We may assume that $U \neq V$ and $U \neq V^*$.

Suppose that U is not adjacent to V. Then there is a point S with $S \subset U \cup V$, other than U and V. By Theorem 17, S contains $U \cap V$. Therefore, since S cannot contain V, $U \cup S$ is properly contained in $U \cup V$.

Pick S_1 so that $U \cup S_1$ has as few elements as possible, with $S_1 \subset U \cup V$. Then, according to the preceding paragraph, there can be no other point contained in $S_1 \cup U$, so U is adjacent to S_1. If S_1 and V are not adjacent, pick S_2, a subset of $S_1 \cup V$, similarly. Proceeding in this way, we get a chain $U, S_1, S_2, \ldots$, that must terminate in a point adjacent to V, since at each stage $|S_k \cup V|$ decreases. This gives a path from U to V in $G(\mathcal{O})$.

If, in the above, U and V are vertices of $G_p(\mathcal{O})$, so that p is in $U \cap V$, then all the sets S_k will also contain p. Therefore $G_p(\mathcal{O})$ is also connected.

Now suppose $\mathcal{O}$ has rank greater than 2, and that the theorem has been verified for oriented matroids of smaller rank.

Suppose U and V are points. Again we may assume that $U \neq V$ and $U \neq V^*$. If $\bar{U}$ and $\bar{V}$ both miss some element $\bar{s}$ of $\bar{E}$, then U and V are connected by a path in $G(\mathcal{O})$ because they are points of the oriented matroid derived from $\mathcal{O}$ by contracting at s and s^*.

Otherwise, we may find elements $\bar{s}$ and $\bar{t}$ of $\bar{U} \sim \bar{V}$ and $\bar{V} \sim \bar{U}$, respectively, and a point S with $\bar{S} \subset \bar{E} \sim \{\bar{s}, \bar{t}\}$, since the rank of $\mathcal{O}$ is bigger than 2. Then U and S are connected by a path since they both miss $\{t, t^*\}$, and S and V are connected by a path since they miss $\{s, s^*\}$, and S and V are connected by a path since they miss $\{s, s^*\}$. Therefore $G(\mathcal{O})$ is connected.

Now suppose U and V are vertices of $G_p(\mathcal{O})$. We may again assume $\bar{U} \cup \bar{V} = \bar{E}$. Pick $\bar{s} \in \bar{U} \sim \bar{V}$ and pick $\bar{t}$ in $\bar{V} \sim \bar{U}$. Choose $\bar{S}$ contained in $\bar{E} \sim \{\bar{s}, \bar{t}\}$. If $\bar{p} \in \bar{S}$, we may take S to that $p \in S$. Then there will be paths from U to S and from S to V in $G_p(\mathcal{O})$, as required. If $\bar{p} \notin \bar{S}$, then let $\bar{r}$ be an element of $\bar{S} \sim \bar{V}$. Since $\bar{U} \cup \bar{V} = \bar{E}$, $\bar{r} \in \bar{U} \cap \bar{S}$, $\bar{p} \in \bar{U} \sim \bar{S}$, so there is a point T with $\bar{p} \in \bar{T} \subset (\bar{U} \cup \bar{S}) \sim \{\bar{r}\}$. We may choose T so that $p \in T$. There is a path from U to T, since they both miss $\{t, t^*\}$, and there is a path from T to V, since they both miss $\{r, r^*\}$. Therefore $G_p(\mathcal{O})$ is connected. □

THEOREM 19. *Suppose $A \cup A^* \cup \{p, p^*\} = E$, $p \notin A \cup A^*$, and $h(A \cup \{p, p^*\}) \subset A \cup \{p, p^*\}$. Then one holds:*

(a) $h(A) = A \cup \{p, p^*\}$;

or

(b) *$h(A \cup \{p\}) = A \cup \{p\}$ and $h(A \cup \{p^*\}) = A \cup \{p^*\}$.*

□ Suppose A is not *closed*; i.e., that $A \neq h(A)$. Then p or p^* is in $h(A)$. We may assume that $p \in h(A)$. Suppose M is a minimal set with $M \subset A$ and $p \in h(M)$. If M is empty then p^* is also in $h(M)$, since $h(\varnothing) = h(\varnothing)^*$. Suppose $q \in M$. Then p is in $h(M)$, but not in $h(M \sim \{q\})$, so:

$$q^* \in h(M \cup \{p^*\}) \subset h(A \cup \{p^*\}) \subset A \cup \{p, p^*\}.$$

Therefore, $q^* \in A$, $M^* \subset A$, and $p^* \in h(M^*) \subset h(A)$. Therefore, $h(A) = A \cup \{p, p^*\}$.

We see that either A is closed or (a) holds. If A is closed then, since $h(A) \subset A \cup \{p, p^*\}$, both $A \cup \{p\}$ and $A \cup \{p^*\}$ are closed. □

COROLLARY. *Suppose D is a cell of $\mathcal{O}$ not containing p or p^*. Then if $D \cup \{p\}$ is a cell of $\mathcal{O}$, $D \cup \{p^*\}$ is also a cell of $\mathcal{O}$.*

□ This is dual to the above theorem. Let A be the set $E \sim (D \cup \{p, p^*\})$. The situation in the dual oriented matroid is that of the above theorem. □

Before we begin the construction of the complex (X, P, φ) we need still another notion.

Let K be a cell of $\mathcal{O} = (E, \mathscr{C}, *)$. Let $d(K) = d$, where $d + 1$ is the length of the longest chain:

$$\varnothing = K_0 \subset K_1 \subset \cdots \subset K_{d+1} = K,$$

where K_i is a cell, and $K_i \neq K_{i+1}$, for $0 \leqslant i \leqslant d$. If $d(K) = d$, then K will be called a *d-cell*. The following lemma relates the function d with the rank function r of the underlying matroid.

LEMMA. *Let K be a cell. Then $d(K) = r(\bar{E}) - r(\bar{E} \sim \bar{K}) - 1$.*

□ Clearly $d(\varnothing) = -1$, as required.

If C and D are cells with C properly contained in D, then the sets $\bar{E} \sim \bar{C}$ and $\bar{E} \sim \bar{D}$ are closed in the underlying matroid, and the first properly contains the second.

Suppose $r(\bar{E} \sim \bar{C}) - r(\bar{E} \sim \bar{D}) \geqslant 2$. $D \sim C$ is a cell of the oriented matroid obtained by deleting the set $C \cup C^*$. Therefore, $D \sim C$ contains a point P of this oriented matroid. Then if $K = C \cup P$, $C \subset K \subset D$, and $r(\bar{E} \sim \bar{K}) = r(\bar{E} \sim \bar{C}) - 1$, so that K is not the same as C or D.

We will show that K is a cell. Since P is a point of the oriented matroid obtained by deleting $C \cup C^*$, P may be written in the form $P = S \sim (C \cup C^*)$, where S is a point of $\mathcal{O}$. Pick S so that $S \cap C^*$ has a few elements as possible. Suppose x is an element of $S \cap C^*$. C, being a cell, is a union of points. One of these, say T, contains x^*. Let y be an element of P. Then $x \in S \cap T^*$ and $y \in S \sim T^*$. There is a circuit U of the dual oriented matroid with $y \in U \subset (S \cup T) \sim \{x, x^*\}$. Then $y \in U \sim (C \cup C^*) \subset P$, so $U \sim (C \cup C^*) = P$. But $U \cap C^*$ is a subset of $(S \cap C^*) \sim \{x\}$, contrary to the minimality of $S \cap C^*$. Therefore, $S \cap C^*$ must be empty, and $K = S \cup C$ is a cell of $\mathcal{O}$.

From this it follows that if we have the maximal chain of cells:

$$\varnothing = K_0 \subset K_1 \subset \cdots \subset K_{d+1} = K,$$

then $r(\bar{E} \sim \bar{K}_i) = r(\bar{E} \sim \bar{K}_{i+1}) + 1$, for $0 \leqslant i \leqslant d$. Therefore, $d(K) = r(\bar{E}) - r(\bar{E} \sim \bar{K}) - 1$. □

In an oriented matroid arising from an arrangement of pseudo-hemispheres, $d(K)$ is the dimension of the cell of the complex corresponding to the cell K of the oriented matroid.

Let $\hat{\mathscr{C}}$ be the set of points of the oriented matroid $\mathcal{O} = (E, \mathscr{C}, *)$. Suppose $p \in E$, and that deletion of $\{p, p^*\}$ yields an oriented matroid $\mathcal{O}_1$ whose rank r is the same as that of $\mathcal{O}$. Let $\mathcal{O}_2$ be the oriented matroid obtained by contraction of $\{p, p^*\}$.

Note that if $\hat{\mathscr{C}}_1$ and $\hat{\mathscr{C}}_2$ are the points of $\mathcal{O}_1$ and $\mathcal{O}_2$, then $\hat{\mathscr{C}}_1$ is the collection of minimal sets C contained in the set $\{D \sim \{p, p^*\} \mid D \in \hat{\mathscr{C}}\}$, and $\hat{\mathscr{C}}_2 = \{D \in \hat{\mathscr{C}} \mid D \subset E \sim \{p, p^*\}\}$.

Let P be the partially ordered set whose elements are cells of $\mathcal{O}$. Let P_1 and P_2 be the corresponding partially ordered sets for $\mathcal{O}_1$ and $\mathcal{O}_2$. Then:

$$P_1 = \{K \sim \{p, p^*\} \mid K \in p\},$$

and:

$$P_2 = \{K \in P \mid K \subset E \sim \{p, p^*\}\}.$$

Suppose that there are topological cell complexes (X, P_1, φ_1) and (Y, P_2, φ_2), where X is a sphere of dimension $d - r - 1$, and Y is a sphere of dimension $d - 1$. We will show that there is a cell complex (X, P, φ).

By the corollary to Theorem 19, cells of $\mathcal{C}$ are of four types:

(1) Cells D of $\mathcal{C}$ with $p \in D$ (or, $p^* \in D$), but for which $(D \sim \{p\}) \cup \{p^*\}$ (or, $(D \sim \{p^*\}) \cup \{p\}$) is not a cell;

(2) Cells D of $\mathcal{C}$ containing neither p nor p^*, for which neither $D \cup \{p\}$ nor $D \cup \{p^*\}$ are cells;

(3) Cells D of $\mathcal{C}$ containing neither p nor p^*, for which both $D \cup \{p\}$ and $D \cup \{p^*\}$ are cells;

and

(4) Cells D of $\mathcal{O}$ for which there is a cell D' of type (3) with $D = D' \cup \{p\}$ or $D = D' \cup \{p^*\}$.

We define φ on each type of cell in turn.

If D is of type (1) then $D_0 = D \sim \{p\}$ (or, $D_0 = D \sim \{p^*\}$) is a cell of $\mathcal{O}_1$. We define $\varphi(D) = \varphi_1(D_0)$.

If D is of type (2), then D is itself a cell of $\mathcal{O}_1$. We let $\varphi(D) = \varphi_1(D)$.

If D is of type (2) and if C is a cell of $\mathcal{O}$ with $C \subset D$, then C is also of type (2). Therefore $\varphi(C) = \varphi_1(C)$ is contained in the boundary of $\varphi(D) = \varphi_1(D)$, since (X, P_1, φ_1) is a topological cell complex.

If D is of type (1) and the cell C is contained in D then C is of type (1) or (2), and similar reasoning shows that $\varphi(C)$ is contained in the boundary of $\varphi(D)$.

We define $\varphi(D)$ for k-cells of type (3) by induction on k. The empty set is of type (2), so we begin by defining $\varphi(D)$ for 0-cells—points—of type (3) (if there are any). If D is a 0-cell of type (3) then D is a 1-cell of $\mathcal{O}_1$, by the lemma. We define $\varphi(D)$ to be any element of the 1-ball $\varphi_1(D)$. Note that $\varphi_1(D) \sim \varphi(D)$ has two connected components, each a 1-ball.

Assuming we have completed the definition of φ on m-cells of type (3) with $m < k$, we proceed as follows. Let D be a k-cell of type (3). Then D is a $(k+1)$-cell of $\mathcal{O}_1$. If D_0 is any cell of $\mathcal{O}$ contained properly in D then $\varphi(D_0)$ has already been defined, and lies in the boundary of $\varphi_1(D)$. Furthermore, D is a k-cell of $\mathcal{O}_2$, and any cell D_0 of $\mathcal{O}$ properly contained in it is a cell of $\mathcal{O}_2$. If Q is the partially ordered set consisting of cells D_0 of $\mathcal{O}$ properly contained in D, then Q is a subset of P_2, and there is a subcomplex (Z, Q, τ) of the complex (Y, P_2, φ_2), where Z is the boundary of $\varphi_2(D)$, a $(k-1)$-sphere. Therefore, the union Z' of the sets $\varphi(D_0)$, for $D_0 \in Q$, is a $(k-1)$-sphere. It lies in the boundary of the $(k+1)$-ball $\varphi_1(D)$.

To show that Z' is the boundary of a ball contained in $\varphi_1(D)$, we may use a result of Newman [18] on "star spheres." See also Brown [17]. Indeed, (Z, Q, τ) is a star sphere; that is, if D_0 is in Q then the partially ordered set Q' of cells in Q containing D_0 is the partially ordered set of a complex whose underlying space is a sphere. To see this, note that the partially ordered set Q' is isomorphic to that of cells properly contained in the cell $D \sim D_0$ of the oriented matroid determined from $\mathcal{C}$ by deleting $D_0 \cup D_0^*$. (The isomorphism identifies the cell C of Q' with the cell $C \sim D_0$.) It follows from Newman's Theorem 3 that there is a k-ball in $\varphi_1(D)$ whose boundary is Z' and which cuts $\varphi_1(D)$ into two connected components, each a $(k + 1)$-ball. (Newman's result is stated for simplicial complexes. However, it holds for our complexes as well, as can be seen by considering barycentric subdivision.) Let $\varphi(D)$ be this k-ball in $\varphi_1(D)$.

Having defined φ on cells of types (1), (2), and (3), we note that the union of sets of the form $\varphi(U)$, where U is of type (2) or (3), is a $(d - 1)$-sphere, since the sets of this form comprise the partially ordered set P_2.

This $(d - 1)$-sphere cuts X into two components. By Theorem 18, the cells V of type (1) with $p \in V$ are in one of the components, and those with $p^* \in V$ are in the other. We refer to that component containing cells V with $p \in V$ as the *p-component*, and to the other as the *p^*-component*.

Let D be a cell of type (4). Let D_0 be the corresponding cell of type (3). The cell D_0 is also a cell of $\mathcal{O}_1$, and, as we have seen, $\varphi_1(D_0) \sim \varphi(D_0)$ is the union of two connected components. One of them is a ball B in the p-component. The other is a ball B^* in the p^*-component. If $p \in D$ we define $\varphi(D)$ to be B, and, otherwise, $\varphi(D) = B^*$. Thus, for any cell D with $p \in D$, $\varphi(D)$ is in the p-component.

We must show that the boundary of $\varphi(D)$ is the union of the sets of the form $\varphi(C)$, where C is a cell of $\mathcal{O}$ and C is properly contained in D. We may suppose that $p \in D$. Any point x of the boundary of $\varphi(D)$ is contained in $\varphi(D_0)$ or is in the boundary of $\varphi_1(D_0)$. If it is in the boundary of $\varphi_1(D_0)$, then it is in a set of the form $\varphi_1(C_0)$, where C_0 is a cell of $\mathcal{O}_1$.

C_0 must be of one of the following types:

(a) $C_0 \cup \{p\}$ (or $C_0 \cup \{p^*\}$) is a cell of type (1);

(b) C_0 is a cell of $\mathcal{O}$ of type (2);

(c) C_0 is a cell of $\mathcal{O}$ of type (3), and $C_0 \cup \{p\}$ and $C_0 \cup \{p^*\}$ are cells of $\mathcal{O}$ of type (4).

If (a) holds, then it cannot be the case that $C_0 \cup \{p^*\}$ is the cell of $\mathcal{O}$, for then $x \in \varphi_1(C_0) = \varphi(C_0 \cup \{p^*\})$; but x is not in the p^*-component. Therefore, $C_0 \cup \{p\}$ is a cell of $\mathcal{O}$; $C_0 \cup \{p\} \subset D$, and $x \in \varphi_1(C_0) = \varphi(C_0 \cup \{p\})$.

If (b) holds then $x \in \varphi_1(C_0) = \varphi(C_0)$, and $C_0 \subset D$.

If (c) holds then:

$$\varphi_1(C_0) = \varphi(C_0) \cup \varphi(C_0 \cup \{p\}) \cup \varphi(C_0 \cup \{p^*\}).$$

Therefore, since x is not in the p^*-component, x is in one of $\varphi(C_0)$ and $\varphi(C_0 \cup \{p\})$. C_0 and $C_0 \cup \{p\}$ are both subsets of D, so we have the desired conclusion.

It is clear that $\varphi(C) \cap \varphi(D) = \varnothing$ if $C \neq D$, and that the union of sets of the form $\varphi(C)$, for C in P, is the same as that of those sets of the form $\varphi_1(C)$, for C in P_1; i.e., it is X.

Therefore (X, P, φ) is a topological cell complex.

With this construction in mind, the remainder of the proof of the following theorem will be reasonably simple.

THEOREM 20. (1) *Let $\mathcal{O} = (E, \mathscr{C}, *)$ be an oriented matroid. Let P be the set of cells of $\mathcal{O}$, ordered by inclusion. Then there is a topological cell complex (X, P, φ); X is a sphere whose dimension is $r - 1$, where r is the rank of the oriented matroid.*

(2) *Let $*$ be an involution of X that takes the set $\varphi(D)$ to the set $\varphi(D^*)$, for each cell D in P. If q is an element of E let P_q be the set of cells of $\mathcal{O}$ not containing q. Let $\sigma(q)$ be the union of the sets $\varphi(K)$, where K is in P_q. Let:*

$$\xi = \{\sigma(q) \mid q \in E \quad \text{and} \quad \{q\} \notin \mathscr{C}\}.$$

Then ξ is an arrangement of pseudo-hemispheres.

(3) *Let $\mathscr{D}$ be the collection of minimal subsets C of ξ with $C \cap C^* = \varnothing$ and $\bigcup_{s \in C} s = X$. Then, if U contains more than one element, U is in $\mathscr{C}$ if and only if either:*

(a) *$\{\sigma(q) \mid q \in U\}$ is in $\mathscr{D}$, and $\sigma(u) \neq \sigma(v)$ if u and v are distinct elements of U;*

or:

(b) *$U = \{u, v\}$, where $u \neq v^*$, but $\sigma(u) = \sigma(v)^*$.*

□ (1) The first part of this theorem may be proven by induction on the cardinality of E, for oriented matroids of fixed rank r. The smallest such oriented matroid is the one for which $|E| = 2r$ and $\mathscr{C} = \varnothing$. In this case each subset K of E with $K \cap K^* = \varnothing$ is a cell. The required complex may be derived in an obvious manner from the boundary of the dual of the r-cube. For any larger oriented matroid of rank r, there is an element p of E such that the oriented matroid obtained by deletion of p and p^* has the same rank, and the construction above may be used to derive the required complex from smaller ones.

(2) Clearly the required homeomorphism $*$ may be found. It will be fixed-point free, since if D is a cell, $\varphi(D)$ and $\varphi(D^*)$ have empty intersection.

$P_q \cap P_{q^*}$ consists precisely of the cells of $\mathcal{C}$ that are also cells of the oriented matroid obtained by contraction of $\{q, q^*\}$. If $\{q\}$ is not a circuit, then the closed subcomplex determined by this subset of P is, by part (1), a sphere of dimension $r - 2$, since in this case the rank of this oriented matroid is $r - 1$. This sphere cuts the bigger sphere X into two connected components. One of these contains, by Theorem 19, the cells of the complex corresponding to elements of $P_q \sim P_{q^*}$, and the other must contain those corresponding to the elements of $P_{q^*} \sim P_q$. Therefore, those corresponding to elements of P_q form a closed ball.

If S is a subset of ξ with $S = S^*$, then there is a set $S_0 \subset E$ with $S = \{\sigma(q) \mid q \in S_0\}$, and with $S_0 = S_0^*$. $\bigcap_{q \in S_0} \sigma(q)$ consists precisely of the union of sets of the form $\varphi(K)$, where K is a cell missing all of S_0; i.e., where K is a cell of the oriented matroid obtained from $\mathcal{O}$ by contracting S_0. Therefore, $\bigcap_{q \in S_0} \sigma(q)$ is a sphere of dimension $r_0 - 1$, where r_0 is the rank of this contraction.

It follows that ξ is an arrangement of pseudo-hemispheres.

(3) Suppose C is a circuit of $\mathcal{O}$. We must show that $\bigcup_{q \in C} \sigma(q) = X$. That is, if K is a cell of $\mathcal{O}$, we must show that $\varphi(K)$ is a subset of $\sigma(q)$, for some q in C. To do this we will show that $K \in P_q$ for some q in C; i.e., that C is not contained in K.

If h is the hull function for $\mathcal{O}$, then since K is a cell, $h(E \sim K^*) = E \sim K^*$. If $C \subset K$, then $C \subset E \sim K^*$. Then:

$$C^* \subset h(C) \subset E \sim K^*.$$

By symmetry, $C \subset E \sim K$. Then C, being contained in K and in $E \sim K$, is empty. This cannot be the case, so C is not contained in K.

Now suppose V is a subset of E with $V \cap V^* = \varnothing$, and that V contains no circuit. Let F be a maximal set with:

(a) $V \subset F$;

(b) $F \cap F^* = \varnothing$;

and

(c) F contains no circuit.

We will show that F is a cell.

Suppose $p \in E$ and neither p nor p^* is in F. Then, by the maximality of F, there must be circuits S and T with:

$$p \in S \subset F \cup \{p\},$$

and:

$$p^* \in T \subset F \cup \{p^*\}.$$

Then $p \in S \cap T^*$. If $S \neq T^*$, there is a circuit contained in the set $(S \cup T) \sim \{p, p^*\}$. But this is contained in F, so there can be no such circuit, and S and T^* must be the same. Then $S = \{p\}$, and $T = \{p^*\}$. $F \cup F^*$ contains all elements of E except those elements p for which $\{p\}$ is a circuit.

If $q \in h(F) \sim F$ then there is a circuit contained in $F \cup \{q^*\}$; therefore q^* is not in F, and $\{q\}$ is in $\mathscr{C}$. Therefore, $h(F)$ is the union of F and the elements of E which form singleton circuits. That is, $h(F) = E \sim F^*$. It follows that F^* is a cell, so F is a cell.

$V \subset F$, so, since it fails to contain $\varphi(F)$, $\bigcup_{v \in V} \sigma(v) \neq X$.

Therefore, if $V \cap V^* = \varnothing$, then V contains a circuit if and only if $\bigcup_{v \in V} \sigma(v) = X$. The required conclusion is immediate from this. □

Corollary. *There is a natural one-to-one correspondence between the arrangements of pseudo-hemispheres and the oriented matroids in which each circuit has at least three elements.*

□ For such an oriented matroid, (b) of part (3) of the theorem cannot hold, and U is in $\mathscr{C}$ precisely when $\{\sigma(q) \mid q \in U\}$ is in $\mathscr{D}$. Therefore, no two such oriented matroids can yield the same arrangement. □

This correspondence can be useful in visualizing properties of oriented matroids, particularly when the rank is small. If the rank is 3, the corresponding arrangements are in the 2-sphere, and they determine "arrangements of pseudolines" in the projective plane. A good discussion of these has been written by Grünbaum [6].

There are "non-stretchable" arrangements of pseudolines. These correspond to oriented matroids of rank 3 which are not realizable. It is easy to find pseudolines in the plane which fail to satisfy, say, the Pappus configuration, whereas it is impossible to find genuine lines for which the Pappus configuration fails. The oriented matroid to which such an arrangement of pseudolines corresponds is not realizable.

One might notice that, in this example, the underlying matroid is not realizable. Then, one might conjecture that, given an oriented matroid for which the underlying matroid is realizable, the oriented matroid itself will be realizable. This is not true. A *simple* oriented matroid of rank r is one such that *any* set of cardinality r in the underlying matroid is a basis. A simple oriented matroid of rank 3 corresponds to an arrangement of pseudolines of which any three have empty interesection. Ringel [11] has exhibited such an arrangement which is not realizable. Of course, the underlying matroid of the oriented matroid corresponding to this arrangement is realizable.

Folkman, without the aid of the correspondence we have developed here, discovered another example of a simple oriented matroid which is not realizable.

The duality for oriented matroids developed above and the correspondence established here between oriented matroids and arrangements of pseudo-hemispheres, taken together, yield a correspondence between arrangements of $2n$ pseudo-hemispheres on the $(r - 1)$-sphere and arrangements of $2n$ other pseudo-hemispheres on the $(n - r - 1)$-sphere. This may be viewed as an extension of a similar correspondence described by McMullen in [9] for arrangements of genuine hyperplanes. (See, also, Shannon [13].)

This correspondence is also an extension of that intended for "simple arrangements of pseudohyperplanes" by Halsey in [7]. One may obtain such an arrangement from an arrangement of pseudo-hemispheres by taking for the "pseudo-hyperplanes" the sets of the form $p \cap p^*$, where p is a pseudo-hemisphere. Topologically, these pseudohyperplanes are, then, spheres.

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V. Simple Oriented Matroids

Recall that an oriented matroid $(E, \mathscr{C}, *)$ is *simple* of rank r if $|E| \geqslant r$ and the underlying matroid $(\bar{E}, \bar{\mathscr{C}})$ has the property that any subset A of $\bar{E}$ is independent if an only if $|A| \leqslant r$. Obviously this will be the case if and only if each circuit $C \in \mathscr{C}$ has cardinality $r + 1$, and each set $B \subset E$ with $B = B^*$ and $|B| = 2(r + 1)$ contains a circuit.

Our object here is to give an interesting alternative characterization of these simple oriented matroids.

Let h be the hull function for the simple oriented matroid $(E, \mathscr{C}, *)$. If $C \in \mathscr{C}$, $h(C) = h(C)^*$, so $\overline{h(C)}$ is a closed set in the underlying matroid. Since $|\overline{h(C)}| > r$, $\overline{h(C)} = \bar{E}$. It follows that $h(C) = E$; i.e., if $p \notin C$ then there is a circuit D with $p^* \in D \subset C \cup \{p^*\}$.

We have seen that if a triple $(E, \mathscr{C}, *)$ is a simple oriented matroid of rank r, it satisfies:

(a) $|E| \geqslant 2r$; if $|E| > 2r$ then $\mathscr{C} \neq \varnothing$;

(b) If $A \in \mathscr{C}$ then $A^* \in \mathscr{C}$ and $A \cap A^* = \varnothing$;

(c) If A and B are in $\mathscr{C}$ and $A \subset B \cup B^*$, then $A = B$ or $A = B^*$.

(d) If $A \in \mathscr{C}$ then $|A| = r + 1$;

(e) If $A \in \mathscr{C}$ and $p^* \notin A$ then there is an element B of $\mathscr{C}$ with $p \in B \subset A \cup \{p\}$.

It will be proven that these conditions characterize the simple oriented matroids of rank r. Until this is proven we will call any triple satisfying these conditions a *positivity system of rank r*.

Lemma. *Suppose $(E, \mathscr{C}, *)$ is a positivity system of rank r where r is positive, and suppose p is in E. Let:*

$$\mathscr{C}_p = \{C \subset E \sim \{p, p^*\} \mid C \cup \{p\} \text{ or } C \cup \{p^*\} \text{ is in } \mathscr{C}\}.$$

Then $(E \sim \{p, p^*\}, \mathscr{C}_p, *)$ is a positivity system of rank $r - 1$.

□ If $|E| = 2r$ the assertion is obviously correct. If E has more than $2r$ elements, then $\mathscr{C} \neq \varnothing$. If $C \in \mathscr{C}$ then, applying (e), we find an element D of $\mathscr{C}$ with $D \subset C \cup \{p, p^*\}$ and $D \cap \{p, p^*\} \neq \varnothing$. Then $D \sim \{p, p^*\}$ is in $\mathscr{C}_p$, so $\mathscr{C}_p \neq \varnothing$. Therefore (a) is satisfied.

Suppose A and B are in $\mathscr{C}_p$ with $A \subset B \cup B^*$. Then $A \cup \{p\}$, say, is in $\mathscr{C}$, as is one of $B \cup \{p\}$ and $B \cup \{p^*\}$. From the fact that $A \cup \{p\} \subset B \cup B^* \cup \{p, p^*\}$ and application of (c) it follows immediately that $A = B$ or $A = B^*$, and (c) is satisfied.

Finally, suppose $A \in \mathscr{C}_p$ and $q^* \notin A$. We may suppose $A \cup \{p\} \in \mathscr{C}$. There is an element B_0 of $\mathscr{C}$ with $q \in B_0$ and $B_0 \subset A \cup \{p, q\}$. If p is in B_0 then $B_0 \sim \{p\}$ is the element of $\mathscr{C}_p$ required by (e). If not, then $B_0 = A \cup \{q\}$. There is an element U of $\mathscr{C}$ with $p^* \in U \subset B_0 \cup \{p^*\} = A \cup \{p^*, q\}$. If q is not in U then $U = A \cup \{p^*\}$. This cannot be the case, since $A \cup \{p\} \in \mathscr{C}$. Therefore, $q \in U$, and $U \sim \{p^*\}$ is the required element of $\mathscr{C}_p$. □

Lemma. *Suppose $S \subset E$, $S^* = S$, and $|S| = 2(r + 1)$. Then there is a set $A \subset S$ with $A \in \mathscr{C}$.*

□ We proceed by induction on the rank, the assertion being trivial for $r = 0$.

Suppose r is positive and that the assertion is valid for smaller values of r. Suppose $S \subset E$, $S^* = S$, and S has cardinality $2(r + 1)$. Let p be an element of S. Let $(E \sim \{p, p^*\}, \mathscr{C}_p, *)$ be as in the preceding lemma. By the inductive assumption there is a set $C \subset S \sim \{p, p^*\}$ with C in $\mathscr{C}_p$. Then $C \cup \{p\}$ or $C \cup \{p^*\}$ is an element of $\mathscr{C}$ contained in S, as required. □

Note that, from the preceding lemma, it follows that if the positivity system $(E, \mathscr{C}, *)$ is an oriented matroid, then its rank is r.

Lemma. *Suppose S and T are in $\mathscr{C}$, $\{p\} = S \sim T^*$, and $S \cup T$ has exactly $r + 3$ elements. Then $(S \cup T) \sim \{p, p^*\}$ is in $\mathscr{C}$.*

□ We proceed by induction on r. The situation cannot arise for $r = 0$. If $r = 1$ then $S = \{x, p\}$ and $T = \{y, p^*\}$ for some elements x and y of $E \sim \{p, p^*\}$, with $x \neq y$ and $x \neq y^*$. Since $\{p, y\} \notin \mathscr{C}$ and $x^* \notin \{p, y\}$, there is a circuit C such that $x \in C \subset \{p, y, x\}$. $C \neq \{p, y\}$ since $\{p^*, y\} \in \mathscr{C}$. Therefore $C = \{x, y\} \in \mathscr{C}$, as required.

Suppose $r = 2$, so that $S = \{x, y, p\}$ and $T = \{w, y, p^*\}$. Since $w^* \notin S$, there is a circuit C with $w \in C \subset S \cup \{w\}$. Suppose $C \neq \{w, x, y\}$. Then $C = \{w, x, p\}$, since $\{w, y, p\}$ is contained in $T \cup T^*$. Since $x^* \notin T$ there is an element D of $\mathscr{C}$ with $x \in D \subset T \cup \{x\}$. If $D \neq \{w, x, y\}$ then $D = \{w, p^*, x\}$. We cannot have both $C = \{w, p, x\}$ and $D = \{w, p^*, x\}$. Therefore, either C or D is $\{w, x, y\}$, which must then be in $\mathscr{C}$.

Now suppose r is bigger than 2 and the result has been established for positivity systems of rank $r - 1$. Then $|S \cap T| = r - 1 \geqslant 2$. Let x and y be in $S \cap T$, with $x \neq y$. Let:

$$\mathscr{O}_1 = (E \sim \{x, x^*\}, \mathscr{C}_x, *)$$

and:

$$\mathscr{O}_2 = (E \sim \{y, y^*\}, \mathscr{C}_y, *).$$

$S \sim \{x\}$ and $T \sim \{x\}$ are in $\mathscr{C}_x$; $S \sim \{y\}$ and $T \sim \{y\}$ are in $\mathscr{C}_y$. Each of these pairs satisfies our assumption, so, since the ranks of $\mathscr{O}_1$ and $\mathscr{O}_2$ are each $r - 1 < r$, there is an element C of $\mathscr{C}_x$ and an element D of $\mathscr{C}_y$ with $C = (S \cup T) \sim \{p, p^*, x\}$ and $D = (S \cup T) \sim \{p, p^*, y\}$. Utilizing (c), we see that it must be the case that $C \cup \{x\} = D \cup \{y\}$ is in $\mathscr{C}$, as required. □

Let $\mathscr{O} = (E, \mathscr{C}, *)$ be a positivity system of rank r. Let X be a subset of E. Let $H(\mathscr{O}, X)$ be the graph whose vertices are the elements C of $\mathscr{C}$ with $C \subset X$, with two vertices A and B adjacent provided $A \cap B^* = \varnothing$ and $|A \cup B| = r + 2$. If A and B are adjacent then $|A \sim B| = |B \sim A| = 1$.

LEMMA. *Suppose $X \subset E$ and that X is the union of elements of $\mathscr{C}$, but may not be represented in the form $S \cup S^*$, where $S \in \mathscr{C}$. Then $H(\mathscr{O}, X)$ is connected.*

□ Suppose this fails. Pick $\mathscr{O}$ and X with $|E|$ as small as possible with $H(\mathscr{O}, X)$ not connected. Let A and B be vertices in different components. We may assume $A \neq B^*$.

Let a be in $A \sim (B \cup B^*)$. Let C be such that $a \in C \subset B \cup \{a\}$. Then B and C are adjacent, so A and C are in different components of $H(\mathscr{O}, X)$.

Let $U = A \sim \{a\}$ and $V = C \sim \{a\}$. Then U and V are vertices of $H(\mathscr{O}', X \sim \{a, a^*\})$, where $\mathscr{O}'$ is the positivity system $(E \sim \{a, a^*\}, \mathscr{C}_a, *)$. Let $U = W_1, W_2, \ldots, W_m = V$ be a path from U to V in $H(\mathscr{O}', X \sim \{a, a^*\})$. Let $a_i \in \{a, a^*\}$ be such that $W_i \cup \{a_i\} \in \mathscr{C}$, for $0 \leqslant i \leqslant m$. If $a_i = a$ for $0 \leqslant i \leqslant m$, then $W_0 \cup \{a\}, W_1 \cup \{a\}, \ldots, W_m \cup \{a\}$ is a path from A to C in $H(\mathscr{O}, X)$. Such a path does not exist, so $a_i = a^*$ for some i. Let i be the least positive integer with $a_i = a^*$, and let j be the largest with $a_j = a^*$. Then the sets $W_{i-1} \cup W_i$ and $W_j \cup W_{j+1}$ are in $\mathscr{C}$ by the preceding lemma. Then $W_{i-1} \cup W_i$ is in the same component of $H(\mathscr{O}, X)$ as A, while $W_j \cup W_{j+1}$ is in that of C. Therefore they are in different components.

If $\mathscr{C}' = \{C \in \mathscr{C} \mid C \subset E \sim \{a, a^*\}\}$ then:

$$\mathscr{O}'' = (E \sim \{a, a^*\}, \mathscr{C}', *)$$

is obviously a positivity system, and $H(\mathscr{O}'', X \sim \{a, a^*\})$ is a connected subgraph of $H(\mathscr{O}, X)$. But $W_{i-1} \cup W_i$ and $W_j \cup W_{j+1}$ are both in this subgraph. This contradicts our assumption, so $H(\mathscr{O}, X)$ is connected. □

THEOREM 21. *$\mathscr{O} = (E, \mathscr{C}, *)$ is a simple oriented matroid of rank r.*

□ We have left to show that if S and T are in $\mathscr{C}$ with $p \in S \cap T^*$ and $S \neq T^*$, then there is an element C of $\mathscr{C}$ with $C \subset (S \cup T) \sim \{p, p^*\}$. Let $X = S \cup T$. Then $H(\mathscr{O}, X)$ is connected, so there is a path from S to T in $H(\mathscr{O}, X)$. The first element on this path not containing p is what is required. □

VI. ORIENTED MATROIDS AND LINEAR PROGRAMMING

Here we are interested in establishing Rockafellar's Theorem 7 in [12], which may be viewed as a thorem concerning realizable oriented matroids, in the setting of oriented matroids. We will make use of some of his terminology here.

Let $\bar{E} = \{e_1, e_2, \ldots, e_N\}$, and, let:

$$E = \{e_1^+, e_1^-, e_2^+, \ldots, e_N^-\}.$$

E has an obvious involution *. The real-valued functions on $\bar{E}$ form a vector space R^N. Let K be a linear subspace of R^N, and let $K^\perp$ be its orthogonal complement. If $X \in R^N$, then its *support* S is the set of e_i's in $\bar{E}$ on which X is non-zero. Its *signed support* may be viewed in an obvious way as a subset T of E, with $T \cap T^* = \varnothing$ and $\bar{T} = S$.

If $X \in K$ then the signed support S of X is called a *signed support* of K. If S contains no other non-empty signed support of K, and if $S \neq \varnothing$, then S is an *elementary* signed support. If $\mathscr{C}$ is the set of elementary signed supports of K then it follows easily, as in the proof of Theorem 1, that $(E, \mathscr{C}, *)$ is an oriented matroid. If $\hat{\mathscr{C}}$ is the set of elementary signed supports of $K^\perp$ then $(E, \mathscr{C}, *)$ and $(E, \hat{\mathscr{C}}, *)$ are dual. This is equivalent to Rockafellar's Theorem 4.

Theorems 2, 4, 5, 6, and 7 of Rockafellar's paper may be viewed as statements about oriented matroids, or about dual pairs of oriented matroids. Indeed, Bland has viewed them in this way in [1], and he has proven all but Theorem 7 in this setting.

Before we proceed to the proof, we need the following lemmas.

LEMMA. *Suppose* $(E, \mathscr{C}, *)$ *is an oriented matroid with hull function* h. *Suppose* $q \in E$, $A \subset E$, *and* $p \in h(A \cap \{q\}) \sim h(A)$. *Then there is a set* $S \subset A$ *with* $p \in h(S \cup \{q\})$ *and* $p \notin h(A \cup S^*)$.

□ We proceed by induction on $|A|$. For $|A| = 0$ the result holds, with $S = \varnothing$. Suppose that $A \neq \varnothing$ and that the result holds for smaller sets.

If for each x in A, $p \notin h((A \sim \{x\}) \cup \{q\})$, then $A \cup \{q, p^*\}$ is itself an element of $\mathscr{C}$. Then $p \notin h(A \cup A^*)$ since $A \cup A^* \cup \{p^*\}$ can contain no circuit. Therefore the lemma holds, with $S = A$.

We may suppose that there is an element x of A with $p \in h((A \sim \{x\}) \cup \{q\})$. There is a set $S \subset A \sim \{x\}$ with:

(a) $p \notin h((A \sim \{x\}) \cup S^*)$;

and

(b) $p \in h(S \cup \{q\})$.

If $x \in h((A \sim \{x\}) \cup S^*)$ then we have the desired result, since then $p \notin h(A \cup S^*)$, so we may assume this is not the case.

Suppose $p \in h((A \sim \{x\}) \cup S^* \cup \{x^*\})$. Then if also $p \in h(A \cup S^*)$, $p \in h((A \sim \{x\}) \cup S^*)$, contrary to (a). S is again the set required. We may assume:

$$p \notin h((A \sim \{x\}) \cup S^* \cup \{x^*\}).$$

Then p is in $h((A \sim \{x\}) \cup \{x, x^*, q\})$, but p is not in $h((A \sim \{x\}) \cup \{x, x^*\})$. Contracting at $\{x, x^*\}$, we see that there is a set $T \subset A \sim \{x\}$ with:

(c) $p \in h(T \cup \{q, x, x^*\})$;

and

(d) $p \notin h(A \cup T^* \cup \{x^*\})$.

If $x^* \in A$ then, since either $p \in h(T \cup \{q, x\})$ or $p \in h(T \cup \{q, x^*\})$, one of $T \cup \{x\}$ and $T \cup \{x^*\}$ is the set required. If $x^* \notin A$, we will show that $p \in h(T \cup \{q, x\})$, so that $T \cup \{x\}$ is the set we require.

Since (b) holds there is a circuit C with:

$$p^* \in C \subset S \cup \{p^*, q\}.$$

Since (c) holds, there is a circuit D with:

$$p^* \in D \subset T \cup \{p^*, q, x, x^*\}.$$

We need only show that $x^* \notin D$.

Suppose $x^* \in D$. Since $S \subset A \sim \{x\}$, neither x nor x^* is in C. $x \in D^* \sim C$, and $p \in D^* \cap C^*$, so there is a circuit U with:

$$x \in U \subset (D^* \cup C) \sim \{p, p^*\}.$$

If $q \in U$, then:

$$q^* \in U^* \subset (A \sim \{x\}) \cup S^* \cup \{q, q^*, x^*\},$$

so $q \in h((A \sim \{x\}) \cup S^* \cup \{x^*\})$. Then, since $S \subset A \sim \{x\}$ and $p \in h(S \cup \{q\})$, $p \in h((A \sim \{x\}) \cup S^* \cup \{x^*\})$ contrary to our assumption. If $q^* \in U$, then:

$$q^* \in U \subset (A \sim \{x\}) \cup T^* \cup \{q, q^*, x\},$$

and $q \in h(A \cup T^*)$. Then, by (c), we have:

$$p \in h(T \cup \{q, x, x^*\}) \subset h(A \cup T^* \cup \{x^*\}),$$

contradicting (d). Therefore $q^* \notin U$. Then:

$$x^* \in U^* \subset (A \sim \{x\}) \cup S^* \cup \{x^*\},$$

so $x \in h((A \sim \{x\}) \cup S^*)$, contrary to our assumption. $x^* \notin D$. □

LEMMA. *Suppose* $(E, \mathscr{C}, *)$ *and* $(E, \mathring{\mathscr{C}}, *)$ *are dual, with hull functions* h *and* $\mathring{h}$. *Suppose:*

$$E = A \cup A^* \cup \{p, p^*, q, q^*\},$$

where the three sets are pair-wise disjoint. Furthermore, suppose that $p \in h(A \cup \{q, q^*\})$, *and* $q \in \mathring{h}(A \cup \{p, p^*\})$. *Then there are disjoint subsets* B *and* C *of* A *with* $p \in h(B \cup \{q, q^*\})$ *and* $q \in \mathring{h}(C \cup \{p, p^*\})$.

□ Suppose $p \in h(A \cup \{q^*\})$ and $q \in \mathring{h}(A \cup \{p^*\})$. Pick minimal subsets B and C of A with $p \in h(B \cup \{q^*\})$ and $q \in \mathring{h}(C \cup \{p^*\})$. If $a \in B$ then $a^* \in h(B \cup \{p^*, q^*\})$. Then $a \notin \mathring{h}(C^* \cup \{p, q\})$, since $C^* \cup \{p, q\} \subset E \sim (B \cup \{a, p^*, q^*\})$; so $a^* \notin \mathring{h}(C \cup \{p^*, q^*\})$, and $a \notin C$. Therefore, $B \cap C = \varnothing$, as required.

Now we may suppose that, say, $p \notin h(A \cup \{q^*\})$, so $p \in h(A \cup \{q\}) \sim h(A)$. By the preceding lemma, there is a set $S \subset A$ with $p \in h(S \cup \{q\})$ and $p \notin h(A \cup S^*)$. $p \notin h(A \cup S^*)$, so since $S \subset A$ and $p \in h(S \cup \{q\})$, $q \notin h(A \cup S^*)$. Therefore:

$$q^* \in \mathring{h}(E \sim (A \cup S^* \cup \{q, q^*\})) = h((A^* \sim S^*) \cup \{p, p^*\}).$$

Then $q \in \hat{h}((A \sim S) \cup \{p, p^*\})$. Letting $B = S$ and $C = A \sim S$, we have the required conclusion. □

Finally, we have Rockafellar's theorem.

Write $E = P \cup P^*$, where $P \cup P^* = \varnothing$. If $S \subset E$ and $x \in \bar{E}$, we say x is contained *positively* (*negatively*) in S if there is an element p of P (P^*) with $x = \bar{p}$.

Let $\mathcal{O} = (E, \mathscr{C}, {}^*)$ and $\hat{\mathcal{O}} = (E, \hat{\mathscr{C}}, {}^*)$ be dual, with hull functions h and $\hat{h}$.

THEOREM 22. *Let one of the elements of $\bar{E}$ be painted black and one grey. Let each of the remaining elements be painted white, green, or red. Then one of the following alternatives holds, but not both:*

(a) *There exists a circuit C of $\mathcal{O}$ containing the black element positively and no red elements, and a circuit D of $\hat{\mathcal{O}}$ containing the grey element positively and no green elements, such that no white element belongs negatively to C or D;*

(b) *There exists a circuit of $\mathcal{O}$ containing the grey element and otherwise only green or white elements, with the grey and white elements contained positively; or there exists a circuit of $\hat{\mathcal{O}}$ containing the black element and positively; or there exists a circuit of $\hat{\mathcal{O}}$ containing the black element and otherwise only red and white elements, with the black and white elements contained positively; or both.*

Furthermore, if (a) *holds then C and D may be chosen so that they have no white elements in common.*

□ Let $\bar{p}$ be the black element and $\bar{q}$ be the grey element, where $p, q \in P$. Let W, G, and R be such that $W = W^*$, $G = G^*$, and $R = R^*$, with $\bar{W}$, $\bar{G}$, and $\bar{R}$ being the white, green, and red sets. Then E is the disjoint union of W, G, R, and the set $\{p, p^*, q, q^*\}$.

(b) is obviously equivalent to:

(b′) $q^* \in h(G \cup (W \cap P))$, or $p^* \in \hat{h}(R \cup (W \cap P))$, or both.

(b′) fails if and only if $q^* \notin h(G \cup (W \cap P))$ and $p^* \notin \hat{h}(R \cup (W \cap P))$; that is, if and only if:

(a′) $q \in \hat{h}(R \cup (W \cap P^*) \cup \{p, p^*\})$

and

$$p \in h(G \cup (W \cap P^*) \cup \{q, q^*\}).$$

But this is equivalent to (a).

It remains to be shown that if (a) holds then the sets C and D of (a) can be chosen so that $C \cap D \cap W = \varnothing$.

Let $\mathcal{O}'$ be the oriented matroid derived from $\mathcal{O}$ by contracting G and deleting R. Let g be its hull function. Let $\hat{g}$ be the dual hull function. Then:

$$g(A) = h(G \cup A) \sim (G \cup R),$$

and:

$$\hat{g}(A) = \hat{h}(R \cup A) \sim (G \cup R).$$

Then, by (a′), $p \in g((W \cap P^*) \cup \{q, q^*\})$, and $q \in \hat{g}((W \cap P^*) \cup \{p, p^*\})$. By the lemma there are disjoint subsets U and V of $(W \cap P^*)$ with $p \in g(U \cup \{q, q^*\})$, and $q \in \hat{g}(V \cup \{p, p^*\})$. Then $p \in h(G \cup U \cup \{q, q^*\})$, and $q \in \hat{h}(R \cup V \cup \{p, p^*\})$. There is a circuit C^* of $\mathcal{O}$ with:

$$p^* \in C^* \subset G \cup U \cup \{q, q^*, p^*\},$$

and a circuit D^* of $\hat{\mathcal{O}}$ with:

$$q^* \in D^* \subset R \cup V \cup \{p, p^*, q^*\}.$$

Then C and D are the circuits needed, and the proof of the theorem is complete. □

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Note

The Rotor Effect Can Alter The Chromatic Polynomial

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Let G be a finite graph with vertex set $V(G)$, let θ be an automorphism of G, let $J \subseteq V(G)$ be an orbit of θ, let v be vertex in J, and let $P \subseteq \mathscr{P}(J)$ be a partition of J into disjoint nonempty sets. Then (G, θ, J, v, P) is called a rotor of order Card J.

Let $G(P)$ denote the graph obtained from G by contracting each block B of P, together with the edges joining vertices of B among themselves, to a single vertex. To the rotor (G, θ, J, v, P) we associate a function $\phi : J \to J$ called reflection, given by $\phi(\theta^i(v)) = \theta^{-i}(v)$. Then $\phi(P)$ is another partition of J, denoted P'. The rotor effect is the transformation that associates $G(P')$ to $G(P)$.

It is known that $G(P)$ and $G(P')$ have the same number of spanning trees [2, 4]. Moreover, for rotors of order at most 5, the dichromate is unaltered by the rotor effect [3]. It was hoped that this result could be extended to rotors of any order k, thereby implying a fortiori that not only the number of spanning trees but also the chromatic polynomial is unchanged by the rotor effect. We are going to give a counterexample for any $k > 5$.

Let us recall that if the chromatic polynomial $P(G, \lambda)$ of a graph having n vertices is not 0, then the coefficient of λ^{n-1} in $P(G, \lambda)$ is the number of adjacent pairs of vertices of G, id est

$$\text{Card}\,\{\{x, y\} \subseteq V(G) \mid \exists \text{ at least 1 edge joining } x \text{ and } y\}.$$

This number can be called the adjacency number of G. We denote it by $a(G)$.

For $k > 5$ let us define the graph G_k as follows. $V(G_k)$ has $2k$ elements, partitioned into 2 disjoint sets I and J, each containing k elements indexed by the integers modulo k: $I = \{x_i : i \in Z_k\}$, $J = \{v_i : i \in Z_k\}$. Each x_i is adjacent with exactly 3 vertices: v_i, v_{i+1} and v_{i+3}. No two vertices of J are adjacent. G_k has no loops or multiple edges. G_6 is pictured in Fig. 1.

Let θ be the automorphism of G_k defined by $\theta(x_i) = x_{i+1}$, $\theta(v_i) = v_{i+1}$.

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