For $k_0' = 1$ we have $\bar{k}_0' = \bar{\bar{k}}_0' = k_1' = \bar{k}_1' = 0$. Then minimizing s it holds $k_1^* = k_2^* = k_3 = k_4 = a = 0$ and hence we get

(361) $s \geq 68 - 1 = 67$.

For $k_0' = 0$ and $4 < \bar{k}_0' \leq 6$ we have $k_1' \leq 5$. Then minimizing s we have $a \geq k_1' + 5$, $5\bar{k}_0' - (5k_0^* + 4k_1^* + 3k_2^* + 2k_3 + k_4) \leq 20 - 2k_1'$ and hence

(3611) $s \geq 68 - (2 - \frac{1}{5}k_1') + (1 - \frac{1}{5}k_1') = 67$.

s=67 can be realized by (i) $k_0 = k_0' = 1, k_1 = k_1'' = 20, k_2 = k_2'' = 30, k_3 = k_4 = a = 0, k_5 = 16$ (described in [3], [4]) or by (ii) $k_0 = 7, \bar{k}_0' = 6, \bar{\bar{k}}_0' = 1$; $k_1 = k_1' + k_1'' = 5, k_2 = k_2'' = 35 - k_1', k_3 = 5 + k_1', k_4 = 0, k_5 = 15; a = k_1' + 5, 0 \leq k_1' \leq 5$.

To complete the proof we mention that the maximal simplex number is attained by decomposing 5 facets with one common vertex, say (11111), via standard triangulation. Then coning off to (00000) we receive 120 simplices (called standard triangulation of the C_5).

Similarly for d = 6 with corresponding simplex numbers k we get the important equation

(37) $s = 374 - \frac{1}{2}(3k_0' + 2\bar{k}_0' + \bar{\bar{k}}_0') - \frac{5}{12}(2\bar{\bar{k}}_1' + \bar{\bar{k}}_1'') - \frac{1}{12}(5k_1' + 4\bar{k}_2') + \frac{1}{6}a$
$+ \frac{1}{12}(6k_0^* + 5k_1^* + 4k_2^* + 3k_3^* + 2k_4 + k_5)$.

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SOME PROBLEMS OF TRIANGULATING POLYTOPES IN EUCLIDEAN d-SPACE

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Summary

For simplicial approximations of fixed points of continuous mappings more efficient algorithms are important for optimal work. This requires to find geometrical decompositions into simplices with a minimal number of tiles. This note is concerned with discussing triangulations of polytopes; our particular interest refers to minimal numbers of simplices triangulating important types of polytopes. Therefore here we first give some effective methods for triangulating a d-polytope. Secondly we show that the decomposition number of every vertex preserving triangulation of a d-octahedron is 2^{d-1} and that this number is the least decomposition number in triangulating a d-octahedron. Thirdly after presenting some known results about vertex preserving triangulations of a d-cube we analyse the triangulations of 3-cubes and 4-cubes in order to prepare methods and to give suggestions for higher dimensions. These investigations are illustrated by many examples important for further research. For vertex preserving triangulations of a 4-cube we establish formula (23) for the decomposition numbers and give examples which realize triangulations with special properties. Finally, for a 5-cube we can prove the conjectured property that every vertex preserving triangulation consists of at least 67 simplices.

In this note we consider total decompositions of convex polytopes into simplices in the Euclidean d-space E^d ($d \geq 2$). Let P^d be the set of convex d-polytopes in E^d and $P \in P^d$. First we give some definitions.

$\{S_1, S_2, \dots, S_n\}$ is called a triangulation of P : $\longleftrightarrow$

(i) n is a natural number and every S_j ($j=1,2,\dots,n$) is a d-simplex,

(ii) $\bigcup_{j=1}^{n} S_j = P$,

(iii) $\dim(S_j \cap S_k) < d$ for every $j \neq k$.

We call a triangulation $\{S_1, \dots, S_n\}$ of P vertex preserving (for shortness we introduce the notion: vertex-true): $\Longleftrightarrow$

For every vertex v of each $S_1, \dots, S_n$ is $v \in$ vert P.

For a triangulation $\{S_1, S_2, \dots, S_n\}$ of $P \in P^d$ many authors require the following property to be satisfied:

(iv) For each pair S_j, S_k ($j,k=1,2,\dots,n$; $j \neq k$) it holds:

$(\dim(S_j \cap S_k) = d-1) \Longrightarrow (S_j \cap S_k$ is a facet of S_j and a facet of $S_k)$.

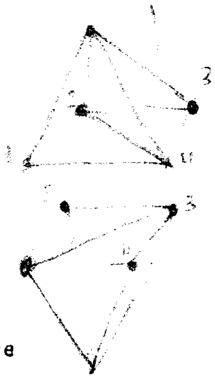
We don't require (iv).

Sometimes triangulations $\{S_1, \dots, S_n\}$ of $P \in P^d$ with the following property are considered:

(v) Let V be a given finite point set with vert $P \subseteq V \subset P$. Then it holds: vert $\{S_1, \dots, S_n\} = V$.

If V = vert P, the concerning triangulations are vertex-true.

n is called the decomposition number of the triangulation $\{S_1, S_2, \dots, S_n\}$. We are interested in the minimal decomposition



outside ($\hat{\hat{k}}_o'$)). Further let

$$\tilde{k}_o := 3k_o' + 2\bar{k}_o' + \bar{\bar{k}}_o' ,$$
$$k_o^* := k_o - k_o' - \bar{k}_o' - \bar{\bar{k}}_o' ,$$
$$k_1^* := k_1 - k_1' - \bar{k}_1' - k_1'' ,$$
$$k_2^* := k_2 - k_2'' .$$

Then we have

(30) $\quad (s =) \sum_{j=0}^{5} k_j = 120 - \tilde{k}_o - 2k_1' - \bar{k}_1' - k_1'' - k_2''$

(31) $\quad k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5 = 160 + a, \quad 0 \leq a \leq 80$,

(32) $\quad 2k_1' + \bar{k}_1' + k_1'' + 2k_2'' = 80 - a$,

(33) $\quad a - 2k_1' - \bar{k}_1' \geq 0$,

(34) $\quad 6k_o' + \bar{k}_o' \leq 6 \quad (\hat{k}_o' \leq 2, \hat{\hat{k}}_o' \leq 5)$.

Equation (30) follows by considering the volume of the simplices of T and by noticing that the 5-cube has volume 1. (31) is true since the 5-cube possesses 10 facets each one having at least 16 4-simplices, so that the sum of the exterior facets of the tiles of T is at least 160. We get (32) because of (24b). The inequality in (33) follows from (24c) and (25). Only (34) is a new fact concerning tiles of a 5-cube with no exterior facets. It is a consequence of the mutual position of the corresponding tiles.

The system (30),(31),(32) consists of three linear equations for k_1'', k_2'' and k_5 with system determinant -10 . Therefore we have a unique solution, especially

$$k_1'' + k_2'' = 52 - \tfrac{1}{2}(\tilde{k}_o + k_o) - \tfrac{1}{10}(4k_1^* + 3k_2^* + 2k_3 + k_4) - \tfrac{1}{5}a - \tfrac{8}{5}k_1' - \bar{k}_1' .$$

Then using this last result equation (30) becomes

(35) $\quad s = 68 - k_o' - \tfrac{1}{10}(5\bar{k}_o' - (5k_o^* + 4k_1^* + 3k_2^* + 2k_3 + k_4)) + \tfrac{1}{5}(a - 2k_1')$.

must be at least $24-2k_o-\bar{k}'_o-k'_1 \geq 20$. There really exists a triangulation of such a type with 20 tiles. Figure 14 shows the graph of an example for it. The nodes together with the numbers denote the corresponding simplices, the vertex matrix of which is given in table 3. Table 2' is a continuation of table 2 for this last fifth example in question.

	k_o	k_1	k_2	k_3	k_4	k'_o	$\bar{k}'_o$	k'_1	s
example 5	2	2	7	6	3	–	2	2	20

Table 2'

Using our knowledge about triangulations of 4-cubes we can establish for $d = 5$

Theorem 6. A vertex-true triangulation of a 5-cube consists of at least 67 and at most 120 simplices. There are such triangulations with 67 and 120 simplices.

We give a sketch of the proof. We consider a vertex-true triangulation T of a 5-cube. Let k_j $(j=0,1,2,3,4,5)$ be the number of simplices of T having j exterior facets and let k be the number of simplices of T with the properties, described in table 4.

k:=	k'_o	$\bar{k}'_o$	$\bar{\bar{k}}'_o$	k'_1	$\bar{k}'_1$	$k'_1{}'$	k''_2
number of exterior facets	0	0	0	1	1	1	2
volume, multiplied with 5!	4	3	2	3	2	2	2
type of the exterior facets	–	–	–	$W_o^{(4)}$	$\bar{W}_o^{(4)}$	$U^{(4)}$	$2xU^{(4)}$

Table 4

The number $\bar{k}'_o$ counts simplices of two different types, $\bar{k}'_o = \hat{k}'_o + \hat{\hat{k}}'_o$ (the centroid of the 5-cube belongs to the simplex $(\hat{k}'_o)$ or lies

number of all triangulations of a given polytope. Such a triangulation with minimal decomposition number is called an optimal triangulation. Following R.K. Guy, the minimal decomposition number is also called the "simplexity" of that polytope. Chapter 2 calculates the simplexity of a d-octahedron. A further problem not yet solved in general is **whether the minimal decomposition number of a polytope P coincides with the minimal decomposition number of all vertex-true triangulations of P.** Nevertheless in chapter 3 we simplify our problem, after refering to known results, by considering mostly vertex-true triangulations of a d-cube and describing optimal vertex-true triangulations of it for lower dimensions.

1. Some methods of decompositions

1.1. Cutting into polytopes

A first step for decOmposing a polytope $P \in P^d$ into simplices without extra vertices is partitioning P into two polytopes. This is possible if there exists a hyperplane h which contains

(i) an inner point of P and

(ii) no inner points of any edge k of P, if k does not totally belong to this hyperplane.

Then the intersection $P \cap h$ is a (d-1)-polytope and h dissects P into two polytopes $P_1, P_2 \in P^d$ with the common facet $P \cap h \in P^{d-1}$. The vertices of P_1 and P_2 are also vertices of P. Figure 1 shows an octahedron in E^3 with the vertices 0,1,2,3,4,5, which is cut by the plane h through the vertices 1,2,3,4 into two polyhedra P_1 and P_2 with the common facet $[1,2,3,4]$.

1.2. Cutting off a vertex

Specifying the first method we consider a vertex v_o of $P \in P^d$ and all the edges $k_o^1, k_o^2, \ldots, k_o^n$ of P having v_o as an endpoint. If all n endpoints of $k_o^1, \ldots, k_o^n$, which are unequal to v_o, lie in a hyperplane h_o, then h_o dissects P into two polytopes P_o and P_1 described in 1.1. P_o is a pyramid with apex v_o and its

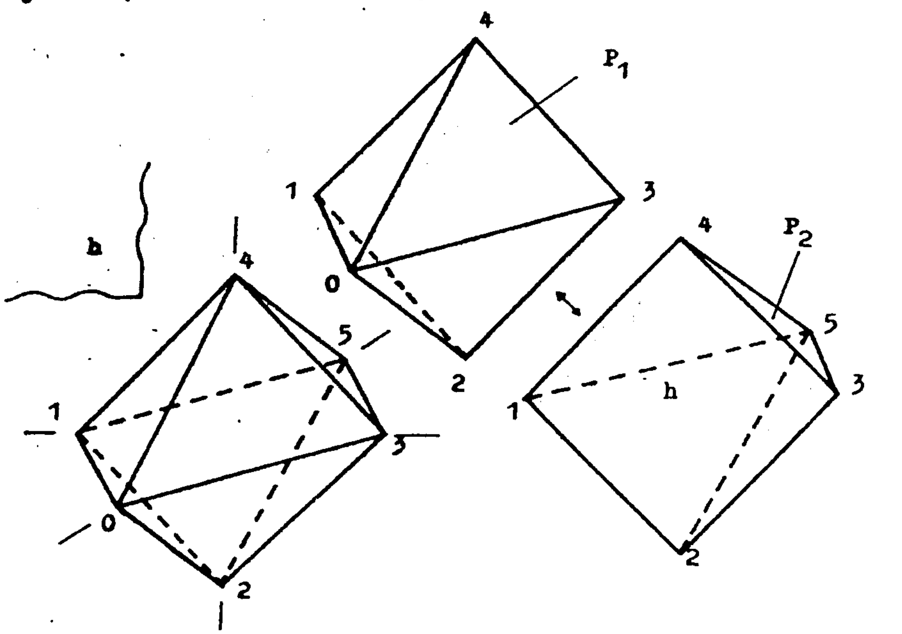


Fig. 1.

base lies in h_o. The other polytope P_1 does not contain the vertex v_o. We say in this case that the vertex v_o of P is cut off by the hyperplane h_o. This method works especially in the case where the vertex v_o has the degree $n = d$ (v_o has the degree n if exactly n facets of P contain the vertex v_o). Then the d endpoints of the d edges containing v_o trivially define a (d-1)-

$k_o = \bar{k}_o' = 2$:

1	2
0000	0000
0011	0011
1011	0111
1101	1101
1110	1110

$k_1 = k_1' = 2$:

3	4
0000	0000
1000	0100
1011	0111
1101	1101
1110	1110

$k_2 = 7$:

5	6	7	8	9	10	11
0000	0000	0000	0011	0000	0000	0000
0011	0011	0011	0111	0011	0011	0011
1010	1001	0110	1011	0101	0110	0101
1011	1011	1010	1101	1001	0111	0111
1110	1101	1110	1110	1101	1110	1101

$k_3 = 6$:

12	13	14	15	16	17
0000	0000	0000	0000	0000	0000
1000	1000	1000	0100	0100	0100
1100	1010	1001	1100	0110	0101
1101	1011	1011	1101	0111	0111
1110	1110	1101	1110	1110	1101

$k_4 = 3$:

18	19	20
0000	0111	0000
0010	1011	0001
0011	1101	0011
0110	1110	0101
1010	1111	1001

Table 3

__Remark 3.__ By elementary methods for convex bodies, which have already been used in the second consideration given above, we find that p_0 is an interior point of $W_0^{(4)}$ by the equation

$$p_0 = \sum_{i=1}^{4} c_i w_i'$$

$$w_1' = (1011),\ w_2' = w_2,\ w_3' = w_3,\ w_4' = w_4$$

with $c_i = \frac{1}{6}$ for $i = 1,2,3,4$.

__Remark 4.__ As noted above, if $\bar{k}_0' = 1$, then there is $k_0' = 0$ and $k_1' \neq 3$. The last inequality is true since $\bar{W}_0^{(4)}$ contains 5 edges of length $\sqrt{3}$, belonging to five different facets of the 4-cube. Therefore 3 facets remain. Only in these ones there can occur 3-simplices of type Z_3. – Now we consider the case $\bar{k}_0' = 2$. Then, as shown above, there is again $k_0' = 0$. More exactly, in this case there is $k_1' \leq 2$. This is true since in a triangulation of a 4-cube with two tiles, being congruent with $\bar{W}_0^{(4)}$, there exist altogether 6 edges of these two simplices which have length $\sqrt{3}$. They are body diagonals each being of six different facets of the 4-cube. Thus at most 2 facets contain a Z_3, and this means $k_1' \leq 2$. Therefore in this case the number of simplices

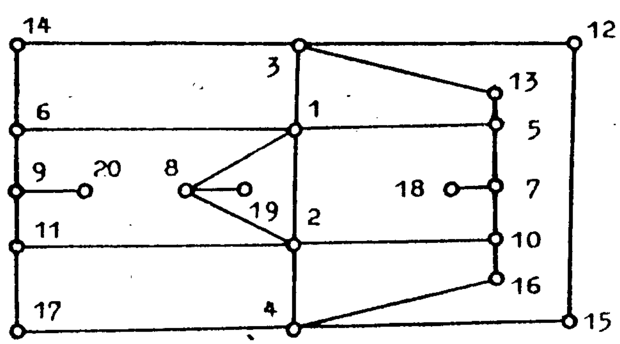


Fig. 14

dimensional hyperplane. Figure 2 shows a cube in E^3 with the vertices $1,2,\ldots,8$. If we cut off vertex 1 then P_1 remains. This is possible because vertex 1 has degree 3. Then the three edges with the common point 1 define a plane by their other endpoints 2,4,5. This plane cuts off vertex 1.–The decomposition shown in figure 1 may be interpreted as cutting off the vertex 0 of the octahedron so that the polyhedron $[1,2,3,4,5]$ remains.

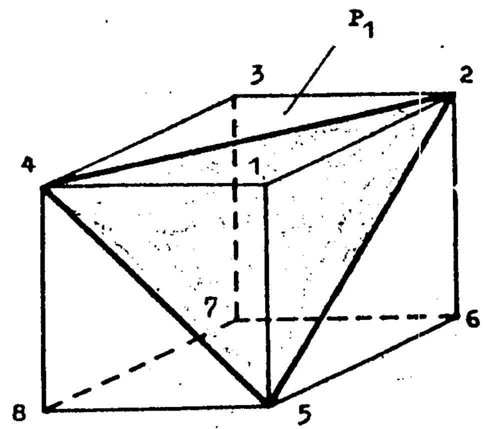


Fig. 2

1.3. Coning off a polytope to a vertex

Let $P \in P^d$ and v_0 be a vertex of P. P is said to be coned off to its vertex v_0, if P is totally decomposed into d-pyramids with apex v_0. The bases of these pyramids are the facets of P which are not adjacent to v_0.

If all facets of P are simplices (in this case P is called simplicial), then this decomposition of P is a dissection into simplices. So this decomposition into simplices is a triangulation. Of course, if only those facets of P are simplices which are not adjacent to the vertex v_0, then coning off P to v_0 already yields a triangulation of P. Coning off a convex simplicial polytope to a vertex does not yield an optimal triangulation in general. There even exist examples of simplicial polytopes for which an optimal triangulation is possible of course, but not by coning off.-Figure 3 shows a pyramid in R^3 with a

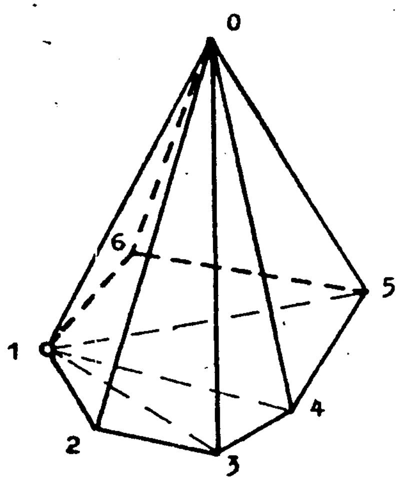


Fig. 3

hexagon $[1,2,3,4,5,6]$ as the base and the vertex 0 as the apex. Coning off this pyramid to the vertex 1, we get a decomposition into four pyramids with apex 1. The facets opposite 1 are triangles, and therefore this decomposition is a triangulation. The four simplices are $[1,2,3,0]$, $[1,3,4,0]$, $[1,4,5,0]$, and $[1,5,6,0]$.

sentation of p_0 in the manner

$$p_0 = \sum_{i=1}^{4} c_i w_i \quad \text{with} \quad c_1 = c_3 = c_4 = \frac{1}{4}\,, \quad c_2 = 0\,.$$

Since for all c_i $(i=1,2,3,4)$ one has $0 \leq c_i \leq 1$ and $\sum_{i=1}^{4} c_i = \frac{3}{4} < 1$, p_0 lies in $\overline{W}_0^{(4)}$. Because of $c_2 = 0$ and $c_i > 0$ for $i = 1,3,4$, p_0 lies in the interior of that facet of $\overline{W}_0^{(4)}$, which is opposite to w_2, and this is $\overline{S}^{(3)}$.

Since p_0 is an interior point of $W_0^{(4)}$ it holds $W_0^{(4)} \cap \overline{W}_0^{(4)} \neq \emptyset$. This means $\overline{W}_0^{(4)}$ cannot belong to T, if T contains $W_0^{(4)}$. Another tile W of C_4, being congruent with $\overline{W}_0^{(4)}$, has also a non-empty intersection with $W_0^{(4)}$, because p_0 lies also in the interior of a facet of W. This is right since, analogous to the mapping of $W_0^{(4)}$ above, there exists a mapping of the 4-cube, which also maps $\overline{W}_0^{(4)}$ onto W. Therefore p_0 is an interior point of a facet of W. Thus (30ii) is true.

Thirdly we propose for every vertex-true triangulation T of a 4-cube that

(30iii) $\qquad \overline{k}'_0 \leq 2.$

Let T be a vertex-true triangulation of C_4 which contains $\overline{W}_0^{(4)}$. Its facet $\overline{S}^{(3)}$ contains p_0 in its interior. Any other tile W of T, being congruent with $\overline{W}_0^{(4)}$, also contains p_0 as an interior point of a facet of it. Thus $\overline{W}_0^{(4)}$ and all possible W must have the same facet $\overline{S}^{(3)}$. This means, there exists besides $\overline{W}_0^{(4)}$ at most one further W so that $\overline{k}'_0 \leq 2$.

These three propositions (30i,ii,iii) give $0 \leq 2k'_0 + \overline{k}'_0 \leq 2$, and this is (24c). Thus lemma 6 is complatly proved.

a body diagonal of the 4-cube called e_o and e, respectively. Let f be a map, which maps the 4-cube in itself so that the body diagonal e is mapped onto e_o. Then by f D is mapped onto D_o and also W onto $W_o^{(4)}$.

The body diagonal $e_o = [(0000), (1111)]$ of the 4-cube is orthogonal to the 3-facet $S^{(3)}$ of $W_o^{(4)}$,

$$S^{(3)} = \begin{pmatrix} 0111 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix} .$$

Because of a symmetric position of the configuration e_o cuts $S^{(3)}$ in its centroid $(\frac{3}{4}\frac{3}{4}\frac{3}{4}\frac{3}{4})$. The length of $e_o \cap W_o^{(4)}$ is $\frac{3}{2}$. This can also be seen by considering the equation of the hyperplane through $S^{(3)}$, which is represented in Hesse's normal form by

$$\frac{1}{2}\cdot(3 - x - y - z - w) = 0$$

Therefore $p_o = (\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ lies in the interior of $W_o^{(4)}$.

Secondly we propose for every vertex-true triangulation of a 4-cube that

(3011) $\qquad k'_o = 1 \longrightarrow \bar{k}'_o = 0.$

Namely, the centroid p_o of C_4 is a boundary point of $\bar{W}_o^{(4)}$, and more exactly, p_o lies in the interior of the facet of $\bar{W}_o^{(4)}$ represented by

$$\bar{S}^{(3)} = \begin{pmatrix} 0000 \\ 0011 \\ 1101 \\ 1110 \end{pmatrix} .$$

We see this in the following way. Let w_i (i=0,1,2,3,4) be the vertices of $\bar{W}_o^{(4)}$, $w_o = (0000)$, $w_1 = (0011)$, $w_2 = (0111)$, $w_3 = (1101)$, $w_4 = (1110)$. Then there exists exactly one repre-

This is an optimal triangulation. - Figure 4 shows a simplicial polyhedron in R^3 where an optimal vertex-true triangulation can not be attained by coning off. This convex polyhedron P is constructed from a regular tetrahedron. To every facet of that tetrahedron a suitable pyramid is glued. The Schlegel diagram of P is drawn in figure 4 with 8 vertices, 12 facets, and 18 edges. There the numbers at the vertices denote the degree of the vertices. There are four vertices of degree 3 and four vertices of degree 6. To get a triangulation with as few simplices as possible, we choose a vertex with degree as high as possible, for example vertex v_1 (cf. figure 4) with the highest

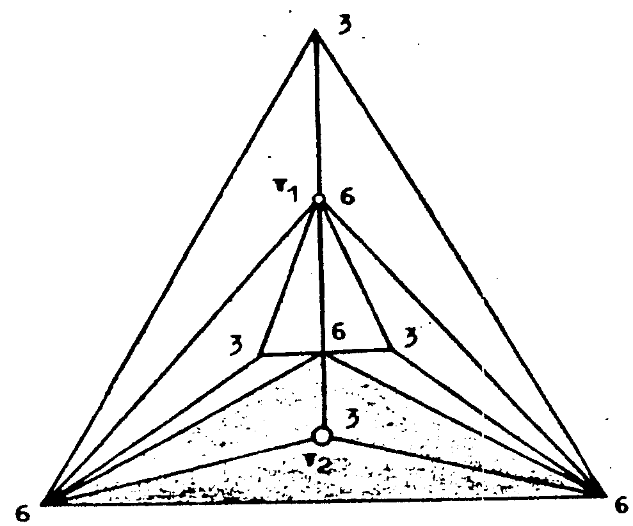


Fig. 4

degree. Then coning off to v_1 we get $12 - 6 = 6$ simplices. Coning off P to any other vertex, we get at least 6 simplices, since 6 is the highest degree of the vertices of P. But after first cutting off the vertex v_2, which yields the two polyhedra P_1 and P_2 (P_1 is a simplex, P_2 is a polyhedron with 7 vertices and 10 facets), and then coning off P_2 to v_1, we get $1 + (10-6) = 5$ simplices. This is a smaller number than in the case of the previous decomposition and therefore for every possible coning off of P. By the way 5 is the smallest possible decomposition number for a polyhedron in E^3 with 8 vertices.

1.4. Mixing the two last methods

As can be seen in the simple example of figure 4, we obtain a smaller decomposition number in certain cases than by only using coning off. For example, let v_0 be a vertex of degree r ($\geq d$) in a simplicial polytope $P \in P^d$. Cutting off v_0 by a hyperplane h_0 may be possible. Let p be the minimal decomposition number (vertex-true) of the (d-1)-polytope $P' = P \cap h_0$ and let $v_1 \neq v_0$ be a vertex of P but not in h_0. We consider the simplicial polytope $P_0 = conv(v_0, v_1, P')$. Coning off P_0 to v_1 we get r simplices. As the minimal decomposition number of P' is p we get not more than 2p simplices by cutting off v_0 and then coning off to v_0 and v_1, respectively. Since $p \leq r - (d-1)$ we can compare these two methods: In P first cutting off v_0 gives less simplices than coning off P to v_0 if $2p \leq 2 + 2r - 2d < r$. This yields $r < 2d-2$ for $d > 2$. – Especially, in a simple case we can establish.

The five vertices (0000), (0011), (0111), (1101), and (1110) of $\overline{W}_0^{(4)}$ cannot be cut off by simplices. $\overline{W}_0^{(4)}$ has five edges of length $\sqrt{3}$ being body diagonals of five different facets of the 4-cube. Therefore at least 5 facets of the 4-cube are not decomposed in the manner of (13). Since the 4-cube has 8 facets we have $k_1' \leq 8-5=3$. This gives $2k_0' + \overline{k}_0' + k_1' \leq 5$ and therefore all the more (24b) must be true. We note that for $\overline{k}_0' = 1$ or $\overline{k}_0' = 2$ and $k_0' = 0$ in both cases we even have $2k_0' + \overline{k}_0' + k_1' \leq 4$, since the tiles of T being congruent with $\overline{W}_0^{(4)}$ must have a special position. We shall show this below. If $k_0' = \overline{k}_0' = 0$, then all 8 facets of the 4-cube can be tiled in the manner of (13) so that $k_1' \leq 8$. Thus (24b) is absolutely true.

Finally we prove (24c). First we assert that for every vertex-true triangulation of a 4-cube it holds

$$(30\,1) \qquad k_0' \leq 1.$$

It suffices to show that the midpoint p_0 of the 4-cube is an interior point of $W_0^{(4)}$. For another tile W of C_4, being congruent with $W_0^{(4)}$, then contains p_0 as an interior point too, because there exists a mapping of the 4-cube in itself which maps W onto $W_0^{(4)}$. We can see this, if we compose $D_0 := W_0^{(4)} \cup S_0^{(4)}$ with

$$S_0^{(4)} = \begin{pmatrix} 0111 \\ 1011 \\ 1101 \\ 1110 \\ 1111 \end{pmatrix} .$$

W is to compose with a corresponding 4-simplex S, so that $D := W \cup S$ and D_0 are congruent. Both D_0 and D contain opposite facet (with $x_1 = 1$) has at most 2^{d-2} mutually non-neighbouring vertices. Vertices of the d-cube being mutually non-neighbouring and lying in a facet of the d-cube are also mutually non-neighbouring in this facet and vice versa. This is true in view of the definition of neighbouring vertices. Therefore the vertex set V of the d-cube has less than $2 \cdot 2^{d-2} = 2^{d-1}$ vertices. Only V_0 and V_1 are sets with 2^{d-1} mutually non-neighbouring vertices. This proves lemma 11.

By the way, for a d-cube the maximal number of vertices belonging to the set V of mutually neighbouring vertices which contains a vertex $w_0 \in V_0$ and a vertex $w_1 \in V_1$ is $2^d-(d-1)$. We get a corresponding vertex set by adding a vertex w_1 to V_0 and cancelling the d vertices of V_0 being neighbouring to w_1.

To prove (24b) we consider a vertex-true triangulation T of the 4-cube containing at least $W_0^{(4)}$ or $\bar{W}_0^{(4)}$(cf. (19)). First let $W_0^{(4)}$ be a simplex of T and $k_0' = 1 \wedge \bar{k}_0' = 0$. Then the five vertices (0000), (0111), (1011), (1101) and (1110) of $W_0^{(4)}$ cannot be cut off by simplices S_j. The simplex $W_0^{(4)}$ has four edges of length $\sqrt{3}$. Each of these edges is a body diagonal of a facet of the 4-cube, which is a 3-cube, having (0000) as a vertex. These facets are mutually different. Therefore at most 4 facets of the 4-cube (having (1111) as a vertex) are decomposed by T in the manner of (13). These facets contain a tile of volume $\frac{1}{3}$. Thus for this triangulation we get $k_1' \leq 4$. Together with the above values we have $2k_0' + \bar{k}_0' + k_1' \leq 6$.

Now let $\bar{W}_0^{(4)}$ be a simplex of T and $k_0' = 0 \wedge 1 \leq \bar{k}_0' \leq 2$.

Lemma 1. Let $P \in P^d$ be simplicial $(d > 2)$, v_0 be a vertex of P of degree d and v_1 be a vertex of P not connected with v_0 by an edge and in P of maximal degree $r_1 > d$. Then coning off P to any vertex of P does not give an optimal decomposition of P.

Proof. Let P have f facets ((d-1)-simplices). By coning off P to v_1 we get a decomposition into $f-r_1$ simplices. By coning off P to any other vertex the decomposition number can not be smaller because r_1 is the maximal degree of the vertices of P. By first cutting off v_0 and then coning off the rest polytope to v_1, we get $1 + (f-(d-1)) - r_1 = f - r_1 - (d-2) < f-r_1$, for $d > 2$. This proves lemma 1.

2. Optimal triangulation of a d-octahedron

A (regular) d-octahedron ($d \geq 2$) has 2d vertices and 2^d facets. Each facet is a (regular) (d-1)-simplex. Two of the vertices are called opposite if their midpoint is also the centre of the octahedron. Each vertex is touched by exactly 2^{d-1} facets. Therefore the degree of every vertex is 2^{d-1}. By coning off a d-octahedron to an arbitrary vertex we obtain $2^d-2^{d-1}=2^{d-1}$ simplices. All these simplices are congruent. Each of them contains one edge which is the link of two opposite vertices of the octahedron. Each other edge of this simplex coincides with an edge of the octahedron. As the d-octahedron is regular, these edges are of equal length, say a. The only edge that contains the midpoint of the d-octahedron then has length $a\sqrt{2}$. For a vertex-true triangulation we have

Lemma 2. Each vertex-true triangulation of a (regular) d-octahedron consists of exactly 2^{d-1} congruent simplices.

Proof. Let O be a d-octahedron. For d = 2, lemma 2 is trivial. Therefore let $d > 2$. For each vertex-true triangulation T of O we now can show that every simplex of T has exactly two facets in common with the surface of O:

A simplex of T contains exactly d+1 vertices of O. Conversely, arbitrary d+1 vertices of O which are not in a hyperplane form a d-simplex as their convex hull. Exactly two of these vertices, say v_1 and v_2, must be opposite vertices of O. Otherwise these d+1 vertices would lie in a hyperplane. Each of these opposite vertices v_1 and v_2 form together with the other d-1 vertices (unequal to v_1 and v_2) a facet of O and of course a facet of this simplex. Each other facet of that simplex contains both v_1 and v_2 and therefore the centre of O. It is not a facet of O, because the centre of O does not belong to a facet of O. Therefore the intersection of the simplex and the surface of O contains exactly two facets of O. Hence, every simplex of T has exactly two exterior facets.

In accordance with the above description these two exterior facets are neighbouring ones in O. All the simplices of T having two exterior facets, which are neighbouring ones in O, are mutually congruent, because O is regular. If the distance of two opposite vertices of O is 2 the volume of each such simplex is $\frac{2}{d!}$. For O we have the volume $\frac{2^d}{d!}$. Let k be the decomposition number of the vertex-true triangulation T. Then comparing the volume

This is trivial for d = 2. There do not exist two non-neighbouring vertices $w_0 \in V_0$ and $w_1 \in V_1$, because in this case two vertices, the one of V_0 and the other one of V_1, are always neighbouring. For dimension d = 3 the two vertices w_0 and w_1 must be opposite vertices (being the ends of a body diagonal of the 3-cube). Namely, the vertex w_0 has three neighbouring vertices belonging to V_1. There are exactly 4 vertices forming V_1 so that only that fourth vertex of V_1, being opposite to w_0, can be w_1. Then w_0 and w_1 have $2 \cdot 3 = 6$ neighbouring vertices and these are together with w_0 and w_1 all the 8 vertices of the 3-cube, so that such a vertex set contains exactly two vertices and does not have $2^2 = 4$.

Now we can prove lemma 11 by induction. Suppose $d > 3$. Since the d-cube has 2^d vertices and w_0, w_1 and their neighbouring vertices are altogether 2d+2 vertices there are

$$2^d - 2(d+1) \quad (> 0 \text{ for } d > 3)$$

vertices left. Therefore there exists a further vertex w_0' being non-neighbouring to w_0 and to w_1 and w.l.o.g. $w_0' \in V_0$. Assume w_0 and w_1 are non-opposite vertices (that means, there exists at least one coordinate which is the same in w_0 and in w_1). If they are opposite ones, choose instead of w_0 the vertex w_0'. Thus w.l.o.g. w_0 and w_1 are non-opposite vertices. Therefore assume w.l.o.g. $w_0 = (0\ldots)$ and $w_1 = (0\ldots)$. Then there exists a (d-1) facet of the d-cube containing w_0 and w_1; this is in our case for instance the facet with $x_1 = 0$. Assume inductively for this facet, that the set of mutually non-neighbouring vertices, containing also w_0 and w_1, has less than 2^{d-2} vertices. The

$\{v_{j_1}, v_{j_2}, \dots, v_{j_p}\}$ (the vertex v_{j_k} belongs to S_{j_k}) consists of vertices being mutually non-neighbouring. In view of lemma 7 every such simplex S_{j_k} has exactly d edges being also edges of C_d.

Hence, by lemma 7 and lemma 9 these p simplices have altogether $p \cdot d$ different edges with the d-cube in common. Since the d-cube contains $d \cdot 2^{d-1}$ edges, it holds

$$p \cdot d \leqq d \cdot 2^{d-1} \quad \text{or} \quad p \leqq 2^{d-1} ,$$

and this proves lemma 10.

Specifying lemma 10 to $d = 4$, we see the truth of (24a).

A more exactly assertion on the position of these simplices with d exterior facets being tiles of a possible triangulation of a d-cube will be given in

Lemma 11. In a vertex-true triangulation of a d-cube the maximal number $p = 2^{d-1}$ of simplices with d exterior facets is only reached, if the set of the corresponding vertices each belonging to these simplices is exactly one of the two sets

$$\text{(i) } V_0 = \left\{v:\ v = (x_1,\dots,x_d) \wedge \sum_{j=1}^{d} x_j \equiv 0 \pmod 2\right\}$$

(29)

$$\text{(ii) } V_1 = \left\{v:\ v = (x_1,\dots,x_d) \wedge \sum_{j=1}^{d} x_j \equiv 1 \pmod 2\right\}.$$

Of course, the vertices of V_0 are mutually non-neighbouring. The same is true for the set V_1. Further, we have $V_0 \cap V_1 = \emptyset$.

Proof. We show that a set of 2^{d-1} vertices of the d-cube cannot simultaneously contain a vertex $w_0 \in V_0$ and a vertex $w_1 \in V_1$.

of O and the congruent tiles of T we have

$$k \cdot \frac{2}{d!} = \frac{2^d}{d!} \quad \text{or} \quad k = 2^{d-1} ,$$

so our proof is complete.

Now consider an arbitrary (not necessary vertex-true) triangulation of O. Then from the 2^d facets of O at most two ones (or parts with interior points of them) belong to each simplex of this triangulation. To show this we consider an arbitrary simplex S_0 of this triangulation with at least two exterior facets. Then there exist two facets f_1 and f_2 of O each containing d vertices of S_0. Let $\{v_1, v_2, \dots, v_{d+1}\} = \text{vert } S_0$ and w.l.o.g. $v_j \notin f_j$ $(j=1,2)$ so that $\{v_2, v_3, \dots, v_{d+1}\} \subset f_1$ and $\{v_1, v_3, \dots, v_{d+1}\} \subset f_2$. Then we have $\{v_3, v_4, \dots, v_{d+1}\} \subset f_1 \cap f_2$. This shows, f_1 and f_2 are neighbouring facets of O, and the open segment $]v_1, v_2[$ lies in the interior of O. Then any facet of S_0 which does not lie in $f_1 \cup f_2$ contains the edge $[v_1, v_2]$ with inner points of O. Hence all facets of S_0 except those two lying in f_1 or f_2 are interior facets. This is the assertion given above.

Let k be the decomposition number of that triangulation. Then the property that each tile can only have at most two exterior facets implies $2k \geqq 2^d$. This gives $k \geqq 2^{d-1}$. Equality only holds if each simplex of the triangulation possesses exactly two (neighbouring) facets of the octahedron O. Then this triangulation is vertex-true. In the other case we have $k > 2^{d-1}$.

Thus we can establish

Theorem 1. The simplexity of a d-octahedron is 2^{d-1}.

In addition to a vertex-true triangulation of a d-octahedron in the last considerations we found the

Corollary. Let S be one of the 2^{d-1} congruent simplices which arises with the help of triangulating a d-octahedron O by coning off to a vertex. Each arbitrary vertex-true triangulation of the d-octahedron O can be attained by composing O with 2^{d-1} simplices S.

Remark 1. Theorem 1 remains true for a not necessarily regular d-polytope which is combinatorially equivalent to a d-octahedron.

Remark 2. The possible triangulations of a (regular) d-octahedron without extra vertices are not all isomorphic for $d > 2$. For $d = 2$, the 2-octahedron is a square. Only two triangulations are possible, namely, in either case, by one of the two diagonals of the square. But both triangulations are essentially the same triangulations. Triangulating a d-octahedron in dimension $d \geqq 3$, we get at least two kinds of non-isomorphic triangulations. The first kind is attained by coning off the d-octahedron to an arbitrary vertex. The second kind can be constructed in the following way: Let v_1 and v_2 be two opposite vertices of the octahedron O. The hyperplane h through the other 2d-2 vertices cuts O into two pyramids P_1, P_2 with v_1 or v_2 as apex, respectively. Let $v_3, v_4 \in h$ $(v_3 \neq v_4)$ be no opposite vertices of O. Coning off P_1 to v_3 and coning

non

$f(v_j) = v_o = (00\ldots0)$ and

$$f(v_l) = \bar{v}_l = (\underbrace{11\ldots11}_{r \text{ times}}00\ldots0) .$$

The simplices belonging to v_o and $\bar{v}_l$, respectively, are S_o and $\bar{S}_j$,

$$S_o = \begin{pmatrix} 0\ldots\ldots0 \\ 10\ldots\ldots0 \\ 010\ldots0 \\ \cdots\cdots \\ 0\ldots\ldots01 \end{pmatrix} , \qquad \bar{S}_j = \begin{pmatrix} 011\ldots1100\ldots00 \\ 101\ldots1100\ldots00 \\ \cdots\cdots\cdots\cdots \\ 111\ldots1000\ldots00 \\ 111\ldots1100\ldots00 \\ 111\ldots1110\ldots00 \\ \cdots\cdots\cdots\cdots \\ 111\ldots1100\ldots10 \\ 111\ldots1100\ldots01 \end{pmatrix} .$$

Hence only for $r = 2$ these two simplices have vertices in common, more exactly

$$(28) \qquad S_o \cap \bar{S}_j = \begin{cases} [\![(10\ldots0),(010\ldots0)]\!] , & \text{if } r = 2 \\ \emptyset , & \text{if } r > 2 . \end{cases}$$

But the edge $[\![(10\ldots0),(010\ldots0)]\!]$ is not an edge of the d-cube; it is a diagonal of a square. Therefore in any case $r \geqq 2$, these two simplices do not contain a common edge which is also an edge of the d-cube. Since also f^{-1} is a mapping of the d-cube in itself, also S_j and S_l have the same property, and this is lemma 9.

These lemmata lead us to the important

<u>Lemma 10.</u> In a vertex-true triangulation of C_d there are at most 2^{d-1} simplices, each with d exterior facets.

Proof. By virtue of lemma 8 and 9, a vertex-true triangulati[on] of C_d contains only simplices $S_{j_1}, S_{j_2}, \ldots, S_{j_p}$ for which the se[...]

$$S_0 = \begin{pmatrix} 0\ldots\ldots0 \\ 10\ldots\ldots0 \\ 010\ldots0 \\ \ldots\ldots\ldots \\ 0\ldots\ldots01 \end{pmatrix},\ S_1 = \begin{pmatrix} 00\ldots\ldots0 \\ 100\ldots\ldots0 \\ 1100\ldots0 \\ \ldots\ldots\ldots \\ 10\ldots\ldots01 \end{pmatrix},$$

belong to these vertices v_0 and v_1, respectively. Now we show $p_0 = (\frac{1}{2^1}, \frac{1}{2^2}, \ldots, \frac{1}{2^d})$ is an interior point of S_0 and S_1: The two hyperplanes cutting off the vertices v_k $(k=0,1)$ have the equations (represented by Hesse's normal form)

$$H_0 = \frac{1}{\sqrt{d}}(1 - \sum_{j=1}^{d} x_j) = 0 \text{ with respect to } v_0,$$

$$H_1 = \frac{1}{\sqrt{d}}(x_1 - \sum_{j=2}^{d} x_j) = 0 \text{ with respect to } v_1.$$

The interior of the d-cube is desribed by $0 < x_j < 1$ $(j=1,\ldots,d)$ and the interior of S_k is desribed by $0 < x_j < 1$ $(j=1,\ldots,d) \wedge H_k > 0$. The point p_0 lies in the interior of the d-cube, and from each of the two hyperplanes it has the oriented distance $\frac{1}{\sqrt{d}}(\frac{1}{2})^d > 0$. This proves lemma 8.

Further, for vertices being non-neighbouring we have

Lemma 9. If v_j and v_l are different vertices of C_d $(d \geqq 2)$ being non-neighbouring, then the intersection $S_j \cap S_l$ does not contain an edge of C_d.

Proof. A consequence of the definition of vertices being neighbouring is that the link e of the vertices v_j and v_l is not an edge of the d-cube. Furthermore, $e = [\![v_j, v_l]\!]$ is a body diagonal of a suitable r-cube. For these non-neighbouring vertices the length of e is $\sqrt{r}$, $r \in \mathbb{N} \wedge r > 1$. Then there exists a (bijective) mapping f of the d-cube in itself with

off P_2 to v_4 gives 2^{d-2} simplices in both cases. Taken together these are 2^{d-1} simplices and they represent a triangulation of O.

Similarly, we can interprete the triangulation in the previous case (first kind) by coning off P_1 and P_2 to the same vertex, for example v_3. In dimension 3 other essentially different cases than these ~~two ones~~ two are not possible. We can see this in the following way. The method used here is transferable into arbitrarily higher dimensions. Interpreting the 3-octahedron by its dual, the facets correspond to the vertices of a cube. In an arbitrary triangulation of the octahedron two exterior facets belonging to one simplex have exactly one common edge which is also an edge of the octahedron. In the cube the two vertices corresponding to these two facets are linked by an edge of the cube. Therefore we have to seek all possibilities to divide the 8 vertices of a cube (cf. figures 5a and 5b) into 4 pairs where every pair is linked by

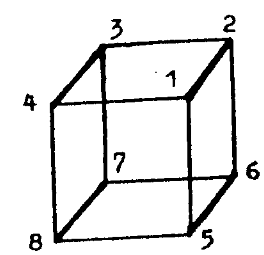


Fig. 5a

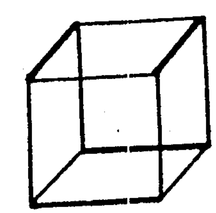


Fig. 5b

an (fat) edge of the cube. Let $[1,2]$ be the first pair. Then for vertex 3, only $[3,4]$ or $[3,7]$ are possible. In the first case, there can be $[5,6]$ and $[7,8]$ or $[5,8]$ and $[6,7]$. In the second case, only $[5,6]$ is possible and then $[4,8]$. But this is combinatorically the same as the last case. In this second case $[5,8]$ is not possible because the remaining vertices 4 and 6 are not linked by an edge of the cube. Therefore there are only the two essentially different possibilities drawn in figures 5a and 5b. The first one leads to the decomposition of the 3-octahedron by coning off to a vertex. The other one corresponds to the triangulation of the second kind. In dimension $d > 3$ there are more than two essentially different possibilities of vertex-true triangulations. In figure 6 we see all the six essentially different possibilities for the dimension 4, illustrated by the dual 4-cubes. There, every 4-cube describes a special triangulation of the 4-octahedron. In such a cube 8 fat edges link 8 pairs of vertices. Every pair corresponds to two facets of the octahedron in that triangulation. These two facets have the property that their intersection is a triangle. This means that these two facets generate a simplex of that decomposition.

3. Remarks on optimal triangulations of a d-cube

3.1. Known results

In this chapter we analyse known results and hope to give suggestions for further investigations. Essential results to

Conversely, if S is a simplex of a vertex-true triangulation of a d-cube with d exterior facets, then there exists a S_j (and a v_j) with $S = S_j$. Namely, a simplex with d exterior facets of a vertex-true triangulation possesses a vertex v of degree d, being the intersection of these exterior facets and being the endpoint of d (exterior) edges. Since all facets through v are exterior ones and since the triangulation is vertex-true, these d edges have length 1. Thus there exists a vertex v_j of the d-cube coinciding with v. Then the d exterior edges of the simplex having v_j in common must coincide with the edges corresponding with d of the d-cube. Hence S coincides with S_j. Since the other $\binom{d}{2}$ edges of S have length $\sqrt{2}$, they cannot be edges of the d-cube. Hence we have

<u>Lemma 7.</u> Let v_j be a vertex of C_d. Then C_d and S_j (belonging to v_j) have exactly d edges in common.

Continuing the proof of (24) we need further lemmata. For two simplices belonging to two vertices being neighbouring we have

<u>Lemma 8.</u> Let v_j and v_l be vertices of C_d being neighbouring. Then S_j and S_l, belonging to v_j, v_l, respectively, have interior points in common. •

Proof. It suffices to consider the two vertices $v_o = (0...0)$ and $v_1 = (10...0)$, since there exists a mapping f of the d-cube in itself with $f(v_j) = v_o$ and $f(v_l) = v_1$. v_o and v_1 are neighbouring vertices of the d-cube, because $[v_o, v_1]$ is an edge of C_d. The simplices S_o and S_1 with d exterior facets, four exterior facets and 8 have one exterior facet (containing a Z_3). Concerning the mutual position of these simplices there exist 6 essentially different optimal vertex-true triangulations of a 4-cube.

<u>Theorem 5.</u> A non-optimal vertex-true triangulation of a 4-cube consists of at least 17 and at most 24 simplices. There are such triangulations with 17 and 24 simplices.

Now it only remains to show the validity of lemma 6:

(24a) is a consequence of a more general result for the d-cube (cf. [2]). We will show this directly.

$v_k \in \mathrm{vert}(C_d)$ is called being neighbouring to $v_l \in \mathrm{vert}(C_d) \setminus \{v_k\}$:
$\Longleftrightarrow$ There exists an edge of C_d which links v_k and v_l .
Otherwise v_k and v_l are called non-neighbouring.

Let v_j $(j=0,1,\dots,2^d-1)$ be a vertex of the d-cube. Then the convex hull of v_j and the d vertices, being neighbouring to v_j, is a possible simplex of a vertex-true triangulation of the d-cube. This convex hull is a simplex, we call it S_j . S_j also arises by cutting off the vertex v_j in the d-cube by the hyperplane through the d vertices being neighbouring to v_j. S_j has d exterior facets since every vertex of the d-cube has degree d. This can also be seen by considering the coordinate matrix of S_j. Therefore S_j contains d edges of the d-cube having length 1. All the other edges of S_j have length $\sqrt{2}$. So in a S_j the vertex v_j is uniquely determined. Thus we say S_j and v_j belong together mutually.

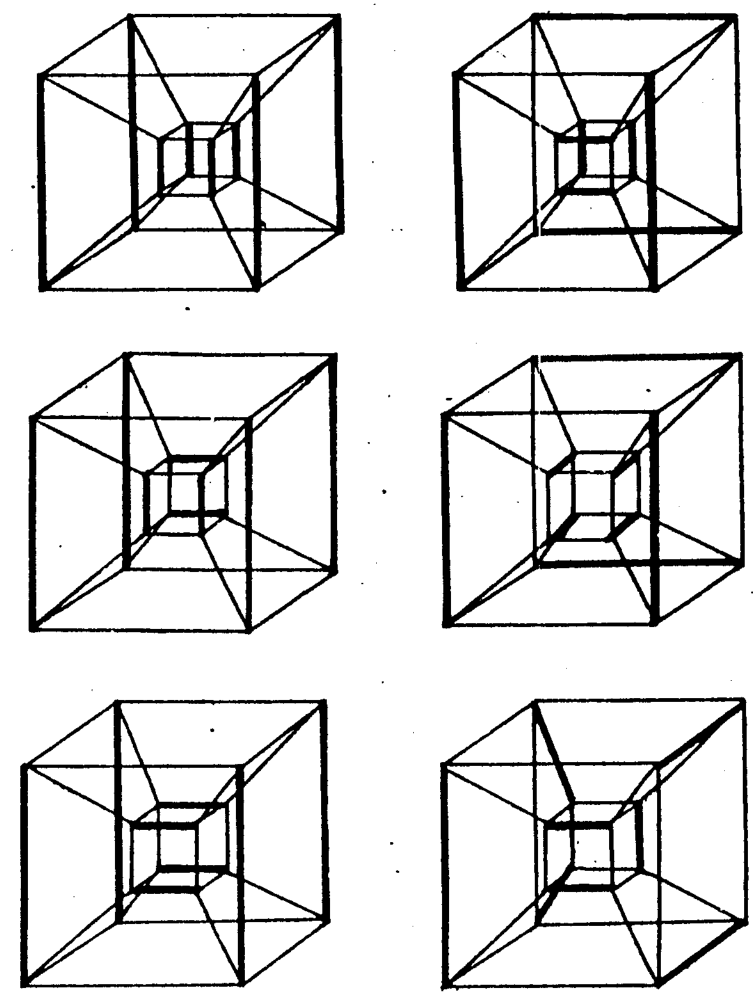


Fig. 6

this topic are from P.S. Mara [2] , J.F. Sallee [3] , and W.D. Smith [4] . Already for $d = 5$ the question for optimal vertex-true triangulations of a d-cube has been open up to now.

Let $T(d)$ be the decomposition number of an optimal triangulation of a d-cube. Then we have the following

Lemma 3. For $d \geq 1$ it holds

$$2^{d-1} \leq T(d) \leq P(d)$$

with

$$P(d) = 2^d - \frac{1}{2}d! + d!\left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!} + \frac{2^1-1}{2!} + \dots + \frac{2^{d-2}-1}{(d-1)!}\right) .$$

There exists a better lower bound attained by studying Hadamard's determinant inequality (cf. [3] , [4]). But for our purpose that bound in lemma 3 will do.

Proof. For the lower bound we first see that

(1) $T(d) \geq 2 \cdot T(d-1) \quad (d \geq 2)$ (cf. [4]):

A d-cube C_d $(d \geq 2)$ has 2^d vertices and $2d$ facets, each facet is a $(d-1)$-cube C_{d-1}. It is

(2) $T(1) = 1, \quad T(2) = 2$, and $T(3) > 4$.

The first value of $T(d)$ for $d = 1$ can be trivially obtained because a 1-cube and a 1-simplex are both segments. The second value for $d = 2$ follows from triangulating a square C_2 by drawing a diagonal. The last inequality for $d = 3$ is right because a 3-cube has 8 vertices. Namely, a decomposition of a d-polytope with n vertices without extra vertices into simplices as few as possible needs d+1 vertices for a first simplex.

3-simplices, described in (13). Thus the surface of the 4-cube contains $8 \cdot 5 = 40$ 3-simplices. Because of $k_1' = 8$, already 8 simplices (with exactly one exterior facet each) are known. Then there remain $40 - 8 = 32$ 3-simplices in the surface of the 4-cube; they belong to 4-simplices of the triangulation having at least one exterior facet. Therefore, it holds with $k_1^* := k_1 - k_1'$

(26) $k_1^* + 2k_2 + 3k_3 + 4k_4 = 32$.

Since (22), with $s = 16$, $k_0 = 0$, $k_1' = 8$ we have

(27) $k_1^* + k_2 + k_3 + k_4 = 8$.

Hence, multiplying equation (27) by 4 and subtracting equation (26) from this result we deduce

$$3k_1^* + 2k_2 + k_3 = 0 \quad \text{or} \quad k_1^* = k_2 = k_3 = 0 .$$

Thus we have the only value $k_4 = 8$, realized by example 1. That means, the minimal decomposition number of a vertex-true triangulation of a 4-cube is 16 with the only possible set of simplex numbers (unequal to zero) $k_1 = k_1' = k_4 = 8$. To determine the essentially different realizations of this solution we first cut off 8 mutually non-neighbouring vertices ($k_4 = 8$) of the 4-cube. This procedure generates a truncated 4-cube Z_4, which is a 4-octahedron. Since, as shown above, this octahedron can be optimally triangulated in 6 essentially different ways, we can establish (cf. also [1])

Theorem 4. An optimal vertex-true triangulation of a 4-cube consists of 16 simplices; always 8 of these ones have

As a consequence of lemma 5 we see that the maximal decomposition number of a vertex-true triangulation of a 4-cube is 24. Our example 4 is a realization of such a triangulation.

Now we ask for the minimal decomposition number. Table 2 shows that the minimal decomposition number cannot be greater than 16. Example 1 is a realization for this decomposition number. To find the minimal decomposition number we first establish assertions on the simplex numbers of (23) in

<u>Lemma 6.</u> For an arbitrary vertex-true triangulation of a 4-cube it holds

$$\text{(24)}\quad \begin{aligned} &\text{(a)}\ k_1' \leqq 8 \\ &\text{(b)}\ 2k_0'+\bar{k}_0'+k_1' \leqq \begin{cases} 8 & \text{for } 2k_0' + \bar{k}_0' = 0 \\ 6 & \text{for } 1 \leqq 2k_0' + \bar{k}_0' \leqq 2 \end{cases} \\ &\text{(c)}\ 0 \leqq 2k_0'+\bar{k}_0' \leqq 2\,. \end{aligned}$$

Before we prove lemma 6 we show that its results lead to an assertion on the minimal decomposition number of the 4-cube under vertex-true triangulations. Hence, from (23) and (24) we obtain

$$\text{(25)}\quad 24 \geqq s = 24-(2k_0'+\bar{k}_0'+k_1') \geqq \begin{cases} 24-8=16 & \text{for } 2k_0'+\bar{k}_0' = 0 \\ 24-6=18 & \text{for } 1 \leqq 2k_0'+\bar{k}_0' \leqq 2. \end{cases}$$

In the case of equality, s = 16 and s = 18 is realized by examples 1 and 2, respectively. Since (24c) and (25), the minimal decomposition number cannot be less than 16. If s = 16 is true, according to (25) and (24b) we see that $k_0' = \bar{k}_0' = 0$ and $k_1' = 8$. Therefore, in the surface of the 4-cube there exist 8 3-simplices of type Z_3.

Hence, each of the 8 facets of the 4-cube is decomposed into 5

Each of the remaining $n-(d+1)$ vertices gives at least one new polytope. Therefore the decomposition number for an optimal triangulation of this d-polytope is not less than

$$\text{(3)}\qquad 1 + n - (d+1) = n - d\,.$$

Using (3) for C_3 we get $T(3) \geqq 8 - 3 = 5$.

Triangulating the cube C_d every simplex of this decomposition has at least one side in the inner of the cube. Each of the 2d facets C_{d-1} of the cube is decomposed into at least $T(d-1)$ $(d-1)$-simplices. Then the surface of the cube is decomposed into at least $2dT(d-1)$ $(d-1)$-simplices. At most d of such $(d-1)$-simplices are exterior facets of a d-simplex which triangulates the d-cube C_d. Therefore every d-cube is decomposed into at least $\frac{1}{d}\cdot 2d\cdot T(d-1) = 2T(d-1)$ d-simplices; this establishes (1).

The general solution of (1) is $T(d) \geqq c_0 2^d$, $c_0 \in \mathbb{R}$. If for the initial values we require

$$1 = T(1) = c_0 \cdot 2^1 \quad\text{and}$$

$$2 = T(2) = c_0 \cdot 2^2$$

we get in either case $c_0 = \frac{1}{2}$. This yields

$$\text{(4)}\qquad T(d) \geqq 2^{d-1} \quad (d \geqq 1).$$

Comparing (4) and (2) equality in (4) is true only for d = 1 and d = 2.

For the upper bound P(d) J.F. Sallee gave a recursion formula

$$\text{(5)}\qquad P(d) = dP(d-1) - d\cdot 2^{d-2} + 2^d - d \quad (d > 1)$$

$$\text{(6)}\qquad P(1) = 1,\ P(2) = 2.$$

Solving the homogeneous part of the linear equation (5) we get

(7) $P_h(d) = c_c d!$.

A special solution of the given inhomogeneous equation (5) is

$$P_1(d) = 2^d + d! \cdot \left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!} + \frac{2^1-1}{2!} + \dots + \frac{2^{d-2}-1}{(d-1)!}\right).$$

Using the initial values (6), from the general solution $P_h(d) + P_1(d)$ we receive

$$1 = P(1) = c_0 + 2 + \frac{2^{-1}-1}{0!} \quad \text{and}$$

$$2 = P(2) = 2c_0 + 2^2 + 2\left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!}\right) .$$

Therefore in both cases holds $c_0 = -\frac{1}{2}$. This leads to

(8) $P(d) = -\frac{1}{2} d! + 2^d + d! \cdot \left(\frac{2^{-1}-1}{0!} + \frac{2^0-1}{1!} + \frac{2^1-1}{2!} + \dots + \frac{2^{d-2}-1}{(d-1)!}\right)$

given in lemma 3. From (8) we also obtain the asymptotic formula

(9) $P(d) \sim d! \cdot (\frac{1}{2}e^2 - e - \frac{1}{2})$ (cf. [4]).

The proof for $P(d) \geqq T(d)$ runs as follows, very clearly de-
scribed in [4] (cf. also [3]): Let C_d ($d \geqq 3$) be a d-cube
with edge length one. For every vertex v of C_d it holds that
exactly d edges end in v and exactly d facets touch v. The
triangulation of C_d , which is now described, may give P(d)
d-simplices. Of course, then $P(d) \geqq T(d)$ is true. Trunca-
ting the cube C_d in a special kind we get the d-polytope Z_d.
We obtain Z_d by cutting off $\frac{1}{2} \cdot 2^d = 2^{d-1}$ vertices of C_d in
the following way. We lay C_d into an Euclidean coordinate
system so that one vertex v_0 of C_d falls into the origin and
each of the d vertices, connected with v_0 by an edge (neigh-
bouring vertex to v_0) coincides with the unit point of a

that there are exactly(only)three types of simplices which do not have volume $\frac{1}{24}$:

K_0' (i) simplices of type $W_0^{(4)}$ (no exterior facet; volume $\frac{1}{8}$),

$\bar{K}_0'$ (ii) simplices of type $\bar{W}_0^{(4)}$ (no exterior facet; volume $\frac{1}{12}$),

K_1' (iii) simplices of type $U_1^{(4)}$ or $U_9^{(4)}$ (one exterior facet which is a Z_3; volume $\frac{1}{12}$) . different config classes

In an arbitrary triangulation T of C_4 let k_0', $\bar{k}_0'$, k_1' be the numbers of simplices belonging to the types (i), (ii), (iii), respectively. Further, for T let $k_j \geqq 0$ ($j=0,\dots,4$) be the number of tiles with j exterior facets. As the volume of the 4-cube is 1, then we can write

(21) $$\frac{1}{24}\left[(k_0+\bar{k}_0' + 2k_0') + (k_1+k_1') + k_2 + k_3 + k_4\right] = 1.$$

Let s be the decomposition number of T with

(22) $$s = \sum_{j=0}^{4} k_j \quad ,$$

then from (21) and (22) we obtain

<u>Lemma 5.</u> For an arbitrary vertex-true triangulation of a 4-cube it holds

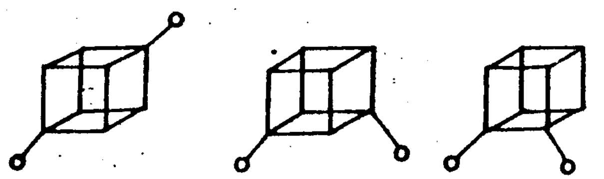
(23) $$s = 24 - 2k_0' - \bar{k}_0' - k_1' .$$

For our four examples considered above we find the corresponding simplex numbers in the following table 2:

	k_0	k_1	k_2	k_3	k_4	k_0'	$\bar{k}_0'$	k_1'	s
example 1	–	8	–	–	8	–	–	8	16
example 2	1	4	6	–	7	1	–	4	18
example 3	–	7	3	–	7	–	–	7	17
example 4	–	–	24	–	–	–	–	–	24

Table 2

second case together with the vertex (1100) there lie 36 simplices in facets of the 4-cube; 2, 4, and 2 are congruent with W', W''', and W'''', respectively. From the 12 remaining simplices there are 4 always of type $W_o^{(4)}$, $\overline{W}_1^{(4)}$, $\overline{W}_2^{(4)}$.– In the third case with vertex (1000) there arise no new types of sim-



case (1110) case (1100) case (1000)

Fig. 13

plices as combinatorial considerations show. 44 simplices have at least four vertices lying in a facet of the 4 cube, 6 simplices are congruent with W''. From the remaining ones 2 and 4 are of type $\overline{W}_1^{(4)}$ and $\overline{W}_2^{(4)}$, respectively (cf. table 1 and figure 13).

	in facets of the 4-cube	W'	W''	W'''	W''''	$W_o^{(4)}$	$\overline{W}_o^{(4)}$	$\overline{W}_1^{(4)}$	$\overline{W}_2^{(4)}$
case (1110)	24	6	6			2	12	6	
case (1100)	36	2		4	2		4	4	4
case (1000)	44		6					2	4

Table 1

Thus only simplices of types given in (19) have no exterior facets. This proves lemma 4.

Considering all vertex-true triangulations of a 4-cube we see

coordinate axis. Then the coordinates of every vertex of C_d are

(10) $(x_1,x_2,x_3,\dots,x_d)$, $x_j \in \{0,1\}$ $(j = 1,2,\dots,d)$.

There are 2^d possibilities and C_d has also 2^d vertices so that (10) describes all the vertices of C_d and every vertex corresponds to such a binary code of length d. Now we consider all the vertices of C_d with

(11) $\sum_{j=1}^{d} x_j \equiv 1 \pmod 2$.

Each vertex of C_d with the condition (11) is cut off by the hyperplane through the neighbouring vertices of v. The remaining body is called the truncated d-cube Z_d. Z_d has $\frac{1}{2}\cdot 2^d = 2^{d-1}$ vertices, 2d facets of type Z_{d-1} and 2^{d-1} simplicial facets (for $d \geq 3$). In particular, Z_3 is a regular simplex. Constructing Z_2 analogously we get an edge. Now Z_d can be triangulated into $P(d) - 2^{d-1}$ simplices. Each vertex of Z_d does not touch d facets (of type Z_{d-1}) of Z_d and $(2^{d-1}-d)$ simplical facets of Z_d. Coning off Z_d to an arbitrary vertex of Z_d, for example the vertex with the coordinates (0,0,...,0), we obtain

$$P(d) - 2^{d-1} = d(P(d-1) - 2^{d-2}) + (2^{d-1} - d) .$$

This gives (5) for $d \geq 3$. But since (6) holds, (5) is also valid for d = 2. Thus lemma 3 is proved.

As shown above we have

(12a) $T(1) = 1 = 2^0 = P(1)$ and

(12b) $T(2) = 2 = 2^1 = P(2)$.

Because of (2) we get from lemma 3

$$2^2 = 4 < T(3) \leqq P(3) = 5.$$

This yields

(12c) $\qquad T(3) = 5$.

Therefore an optimal vertex-true triangulation is obtained by dissecting C_3 into the 5 tetrahedra

(13)
$$Z_3 = \begin{pmatrix} 000 \\ 011 \\ 101 \\ 110 \end{pmatrix}$$
$$S_1^{(3)} = \begin{pmatrix} 000 \\ \underline{001} \\ 011 \\ 101 \end{pmatrix}, \quad S_2^{(3)} = \begin{pmatrix} 000 \\ \underline{010} \\ 011 \\ 110 \end{pmatrix}, \quad S_3^{(3)} = \begin{pmatrix} 000 \\ \underline{100} \\ 101 \\ 110 \end{pmatrix}, \quad S_4^{(3)} = \begin{pmatrix} 011 \\ 101 \\ 110 \\ \underline{111} \end{pmatrix},$$

where in these special matrices the lines denote the coordinates of a vertex of the tetrahedron. The underlined coordinates describe that vertex of the 3-cube, which is cut off by the corresponding tetrahedron.

3.2. Special cases

First let us study all vertex-true triangulations of a 3-cube C_3 (with edge length one). The optimal triangulation given in (13) is attained by cutting off the four vertices (001), (010), (100), and (111). The corresponding four tetrahedra $S_1^{(3)}$, $S_2^{(3)}$, $S_3^{(3)}$, $S_4^{(3)}$ are mutually congruent, exactly 3 facets of each $S_j^{(3)}$ ($j=1,2,3,4$) are exterior facets (e.g. they belong to the surface of the 3-cube), and the volume of $S_j^{(3)}$ is $\frac{1}{6}$. Z_3 is the remainder polyhedron lying in the interior of the 3-cube. Therefore no facet of Z_3 is an exterior one. Z_3 is a regular tetrahedron with edge length $\sqrt{2}$; its volume is $\frac{1}{3}$.

discuss these three cases we mention that the following four matrices W', W'', W''', and W'''' have rank 3:

$$(20)\quad W' = \begin{pmatrix} 0000 \\ 0001 \\ 0111 \\ 1001 \\ 1110 \end{pmatrix}, \quad W'' = \begin{pmatrix} 0000 \\ 0001 \\ 1001 \\ 1110 \\ 1111 \end{pmatrix}, \quad W''' = \begin{pmatrix} 0000 \\ 0001 \\ 0011 \\ 1100 \\ 1111 \end{pmatrix}, \quad W'''' = \begin{pmatrix} 0000 \\ 0011 \\ 1001 \\ 1100 \\ 1111 \end{pmatrix}$$

(cf. figure 12b). That means the five points decribed by W', W'', W''', or W'''' respectively, lie in a hyperplane and their convex hull has the volume zero. They represent degenerated 4-simplices. An innergeometric reason of these degenerations (20) is that the three point sets

$$M_1 = \{(0000),(0001),(0110),(0111),(1000),(1001),(1110),(1111)\}$$
$$M_2 = \{(0000),(0001),(0010),(0011),(1100)),(1101),(1110),(1111)\}$$
$$M_3 = \{(0000),(0011)),(0110),(1001),(1100),(1111)\}$$

each lie in hyperplanes. In these hyperplanes there lie 4,4, and 3 body diagonals of the 4-cube, respectively. For instance M_3 is the set of vertices of a 3-octahedron and the hyperplane through M_3 divides the corresponding 4-octahedron Z_4 of example 1 into two polyhedra, having M_3 in common.

Now we can consider the three cases.

In the first case with the vertex (1110) we consider the 56 possible simplices. 24 of these arising simplices have an exterior facet. 6 simplices are congruent with W' and 6 simplices are congruent with W''. Therefore only the remaining 20 simplices are of interest for us. From these ones there are 2, 12, and 6 simplices of type $W_0^{(4)}$, $\overline{W}_0^{(4)}$, $\overline{W}_1^{(4)}$ respectively. – In the

$$(a_{kj}) = \begin{pmatrix} a_{01} & a_{02} & a_{03} & a_{04} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

is the vector (A_1,A_2,A_3,A_4) with $A_j = \sum_{k=0}^{4} a_{kj}$ $(j = 1,2,3,4)$. S has no exterior facet, iff $1 < A_j < 4$ for all $j = 1,2,3,4$. W.l.o.g. we can assume $A_4 = 3$. Namely, if $A_4 = 2$, we map the 4-cube by replacing its vertices by their opposite ones; that means we change the values of the coordinates $0 \longmapsto 1$ and $1 \longmapsto 0$ (reflection on the middlepoint of the 4-cube). Then the image of S is congruent to S. – Further, we can assume that a special vertex belongs to S. Here we choose the vertex (0000). Because of $A_4 = 3$, in the facet with $x_4 = 0$ there can only lie one further vertex. For this vertex there are 3 essentially different cases, namely (1110), (1100) or (1000). Then in each case there are $\binom{8}{3}$ possibilities to distribute the other three vertices in the facet of the 4-cube with $x_4 = 1$.– Before we

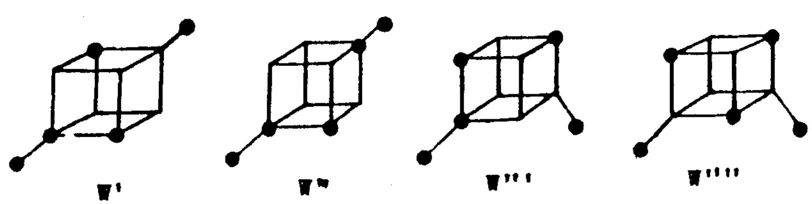


Fig. 12 b

Another triangulation of C_3 can be attained by cutting off the two opposite vertices (011) and (100) and then coning off the remaining polyhedron to (000). This yields the six tetrahedra $S_3^{(3)}, S_5^{(3)}, U_1^{(3)}, U_2^{(3)}, V_1^{(3)}, V_2^{(3)}$ with $S_3^{(3)}$ from (13) and

$$(14) \quad S_5^{(3)} = \begin{pmatrix} 001 \\ 010 \\ 011 \\ 111 \end{pmatrix}, \quad U_1^{(3)} = \begin{pmatrix} 000 \\ 001 \\ 010 \\ 111 \end{pmatrix}, \quad V_1^{(3)} = \begin{pmatrix} 000 \\ 001 \\ 101 \\ 111 \end{pmatrix},$$
$$U_2^{(3)} = \begin{pmatrix} 000 \\ 101 \\ 110 \\ 111 \end{pmatrix}, \quad V_2^{(3)} = \begin{pmatrix} 000 \\ 010 \\ 110 \\ 111 \end{pmatrix}.$$

$S_3^{(3)}$ and $S_5^{(3)}$ are congruent (e.g. isometric) each having three exterior facets. $U_1^{(3)}$ and $U_2^{(3)}$ are congruent (indirectly). Each of them contains exactly one exterior facet, the volume of $U_j^{(3)}$ is $\frac{1}{6}$. $V_1^{(3)}$ and $V_2^{(3)}$ are congruent (indirectly). Each of them contains exactly two exterior facets, the volume of $V_j^{(3)}$ is $\frac{1}{6}$. Each tetrahedron $U_1^{(3)}, U_2^{(3)}, V_1^{(3)}, V_2^{(3)}$ contains a body diagonal of C_3 as an edge. In the case of this triangulation all these edges coincide. The decomposition number of this triangulation is six.

For a vertex-true triangulation of C_3 there exist exactly 4 tiles of different shape, namely tetrahedra with j (j = 0,1,2,3) exterior facets, which are congruent with Z_3 (j=0), $U_1^{(3)}$(j=1), $V_1^{(3)}$(j=2), and $S_1^{(3)}$(j=3). Now let us determine all vertex-true triangulations of C_3 which are essentially different (e.g. there is no isomorphism, mapping one triangulation into another one).

For this purpose we classify the triangulations with respect to the numbers k_j of tiles having j exterior facets (j=0,1,2,3).

The 3-cube has 6 facets; each of them has to be dissected into 2 triangles. So the surface of the cube contains exactly 12 triangles. Then we have

$$\begin{aligned} &0\cdot k_o + 1\cdot k_1 + 2\cdot k_2 + 3\cdot k_3 = 12 \\ (15)\qquad &\tfrac{1}{3}\cdot k_o + \tfrac{1}{6}\cdot k_1 + \tfrac{1}{6}\cdot k_2 + \tfrac{1}{6}\cdot k_3 = 1 \\ &k_o \in \{0,1\},\ k_3 \in \{0,\dots,4\},\ k_1,k_2 \in \mathbb{N}. \end{aligned}$$

The first equation follows from the number of the exterior facets, the second one follows from calculating the volume of the tiles and of the whole cube. Because $0 \leqq k_o \leqq 1$ (got by combinatorical considerations) we have to distinguish two cases.

1.) $k_o = 1$. In this case we have

$$\begin{aligned} (16a)\qquad & k_1 + 2\cdot k_2 + 3\cdot k_3 = 12 \\ & k_1 + k_2 + k_3 = 4\ . \end{aligned}$$

From (16a) we get

$$k_3 = 4 + k_1\ .$$

Because $k_3 \leqq 4$ we have $k_1 \leqq 0$. Here there exists exactly one solution

(17a) $\qquad k_o = 1,\ k_1 = 0,\ k_2 = 0,\ k_3 = 4\ .$

2.) $k_o = 0$. In this case we have

$$\begin{aligned} (16b)\qquad & k_1 + 2\cdot k_2 + 3\cdot k_3 = 12 \\ & k_1 + k_2 + k_3 = 6\ . \end{aligned}$$

That means, a non-optimal vertex-true triangulation of a 3-cube consists of exactly 6 tetrahedra.

Lemma 4. If an arbitrary vertex-true triangulation of a 4-cube contains a simplex S which has no exterior facet, then S is congruent with $W_o^{(4)}$, $\bar{W}_o^{(4)}$, $\bar{W}_1^{(4)}$ or $\bar{W}_2^{(4)}$, having the vertex matrices

$$(19)\qquad \begin{aligned} &W_o^{(4)} = \begin{pmatrix} 0000 \\ 0111 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix}, \quad \bar{W}_o^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0111 \\ 1101 \\ 1110 \end{pmatrix}, \\ &\bar{W}_1^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 1001 \\ 1110 \\ 1111 \end{pmatrix}, \quad \bar{W}_2^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 1011 \\ 1100 \\ 1101 \end{pmatrix}. \end{aligned}$$

The volumes of these four simplices are $\frac{1}{8}, \frac{1}{12}, \frac{1}{24}, \frac{1}{24}$, respectively.

In figure 12a the simplices (19) are represented: Their vertices are embedded in a 4-cube and marked by a dot. Only in the first picture the totally edge skeleton of the 4-cube is drawn.

Proof. We say, in a vertex-true triangulation a simplex S is of type $W_o^{(4)}$ (of type $\bar{W}_j^{(4)}$), if S is congruent with $W_o^{(4)}$ (congruent with $\bar{W}_j^{(4)}$). The volume of these simplices can be calculated by considering the determinants of the corresponding matrices, divided by $\frac{1}{4!}$. These determinants have the values 3, 2, 1, 1, respectively (apart from the sign). We get the four types by elementary combinatorics:

The column characteristic of the simplex S described by the matrix (a_{kj}) consisting of the coordinates of the vertices of S, the 4-cube has-apart from one edge which is a body diagonal of the 4-cube (length 2),-two edges which are body diagonals of a 3-cube(length $\sqrt{3}$), and three edges which are diagonals of a square(length $\sqrt{2}$). The remaining four edges are of length 1 and generate the orthogonal edge chain of the 4-orthoscheme with the four essential dihedral angles, having meassure $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$. Hence the reflections on four suitable hyperplanes through one and the same body diagonal of the 4-cube generate the 24 4-orthoschemes being the simplices of a standard triangulation of a 4-cube.

Generally, calculating the volume of the tiles of a triangulation of a d-cube, we see that their volumes can be immediately inductively determined if such a simplex has at least one exterior facet and if we know the volume w of such an exterior facet. Then its volume is $\frac{1}{d} \cdot w$. Thus we see, that only new problems will arise by the simplices with no exterior facets. Now in our case d = 4 we consider all possible simplices with no exterior facets. Then we have

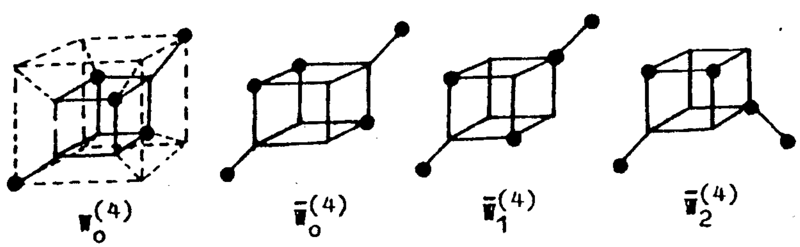


Fig. 12 a

From (16b) we get

$$k_1 = k_3$$
$$k_2 = 6 - 2k_3 .$$

Because $k_2 \geqq 0$ by means of the second equation we have

$$k_3 \leqq 3 .$$

Considering $k_3 = 3,2,1,0$ we see that there exist exactly the following 4 solutions

(17b) $k_o = 0,\ k_1 = 3,\ k_2 = 0,\ k_3 = 3$

(17c) $k_o = 0,\ k_1 = 2,\ k_2 = 2,\ k_3 = 2$

(17d) $k_o = 0,\ k_1 = 1,\ k_2 = 4,\ k_3 = 1$

(17e) $k_o = 0,\ k_1 = 0,\ k_2 = 6,\ k_3 = 0$

For every solution there exists at least one type of realization For (17a) and (17b) the realization is unique up to isomorphic mappings. For (17c) there are two essentially different cases, namely $(17c)_1$ cutting off two opposite vertices shown by (14), o $(17c)_2$ cutting off two not opposite vertices (for example (011) and (110) and then coning off to (000) in either case. In each case $(17c)_1$, $(17c)_2$, (17d), and (17e) there exist two subcases analogous to the triangulation of a 3-octahedron (cf. chapter 2, remark 2). In one subcase all tetrahedra, containing a body diagonal of the cube as an edge, have all the same body diagona in the other subcase these edges are different for at least two tiles. For example the second case corresponding to (14), completed by $S_3^{(3)}$, is attained by substituting $U_2^{(3)}$ and $V_2^{(3)}$ through $U_3^{(3)}$ and $V_3^{(3)}$ with

$$(18) \qquad U_3^{(3)} = \begin{pmatrix} 000 \\ 010 \\ 101 \\ 110 \end{pmatrix} , \quad V_3^{(3)} = \begin{pmatrix} 010 \\ 101 \\ 110 \\ 111 \end{pmatrix} .$$

A triangulation of C_3 belonging to (17e) where all tiles contain the same body diagonal of the cube as an edge is called a standard triangulation of the 3-cube.

A triangulation of the 3-cube causes a triangulation of the surface of the 3-cube. This triangulation of the cube sur-

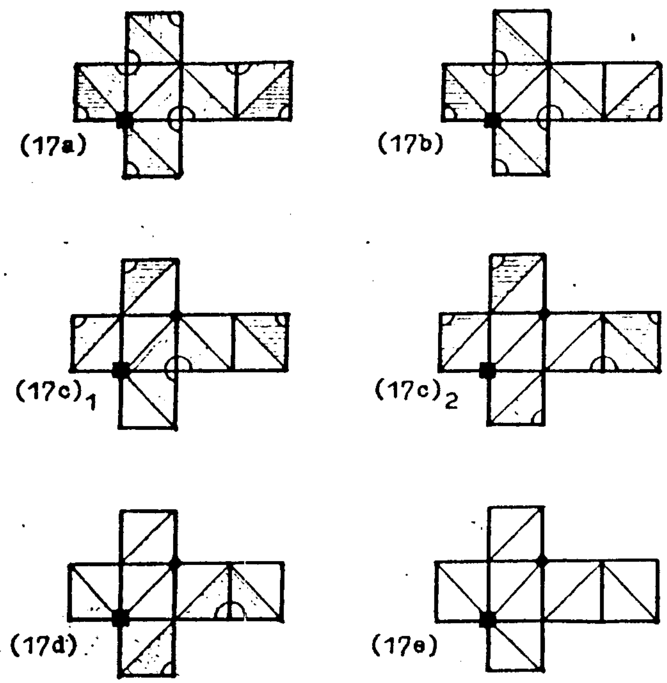


Fig. 7

for example

$$V_1^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 1001 \\ 1101 \\ 1111 \end{pmatrix} \quad \text{and} \quad V_2^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 0101 \\ 1101 \\ 1111 \end{pmatrix} .$$

$V_1^{(4)}$ and $V_2^{(4)}$ are (indirectly) congruent. By the way they are orthogonal-simplices (orthoschemes).

Every simplex of a standard triangulation is (directly or indirectly) congruent with $V_1^{(4)}$ and has exactly two exterior facets: All the 3-simplices in the triangulated four facets of the 4-cube via standard triangulation are congruent. They are orthogonal-tetrahedra (orthoschemes) with an edge chain of length 1,1,1 (for example between the vertices (1111), (1101), (1001),(0001)). The fifth vertex (0000) completes the 4-orthoscheme with an edge being orthogonal to the concerning orthogonal-tetrahedron and having length 1. (In our example this is the edge $\|(0000), (0001)\|$). Therefore every such 4-simplex is an orthoscheme with an edge chain of length 1,1,1,1. Thus this implies the congruence of the tiles of a standard triangulation. - The two exterior facets of the tiles are these ones opposite (0000) and (1111). The first one lies in a facet of the 4-cube containing (1111); it was used for coning off. The second exterior facet lies in a facet of the 4-cube containing (0000). We see this inductively because every simplex of a standard triangulation of a 3-cube has two exterior 2-facets, which, in this case, are posed in a special manner. The proof is a straightforward verification and is omitted.

We note that every simplex of a standard triangulation of

As a third example we note the result after cutting off the same 7 vertices v_j (j=9,...,15) of C_4 as in the previous example, but then coning off to the vertex (1110) instead to (0000) (cf. the graph in figure 11b). We get a more symmetric triangulation of the 4-cube into 17 simplices. From these ones 7 simplices have one exterior facet and are congruent with $U_1^{(4)}$, having volume $\frac{1}{12}$; 3 simplices have two exterior facets. They are

$$V_{16}^{(4)}=\begin{pmatrix}0000\\0001\\0010\\0100\\1110\end{pmatrix},\quad V_{17}^{(4)}=\begin{pmatrix}0000\\0001\\0010\\1000\\1110\end{pmatrix},\quad V_{18}^{(4)}=\begin{pmatrix}0000\\0001\\0100\\1000\\1110\end{pmatrix},$$

and are mutually congruent, but they are not congruent with $V_9^{(4)}$ (neither they are congruent with a later one which will be denoted by $V_1^{(4)}$); their volume is $\frac{1}{24}$; 7 simplices have 4 exterior facets and are congruent with $S_1^{(4)}$.

Now a fourth example of a triangulation of a 4-cube will be given. It is called a standard triangulation of the 4-cube. Triangulate the 4 facets of the C_4, which does not contain (0000), in the manner corresponding to (17e), first subcase (standard triangulation of the 3-cube). In each of these facets the common body diagonal of the tiles will always contain vertex (1111). Therefore the second endpoints of these body diagonals are (0001), (0010), (0100), (1000), respectively. Then cone off the 4-cube to (0000) with respect to the triangulated four facets. We get a triangulation of the 4-cube into 4·6 = 24 simplices each having volume $\frac{1}{24}$, because the tiles of the triangulated facets have volume $\frac{1}{6}$. Two possible 4-simplices of this triangulation are

face is essentially unique for the six cases (inclusive its subcases) (17a), (17b), $(17c)_1$, $(17c)_2$, (17d), (17e). The corresponding nets of the cube surface are also essentially unique. They are all drawn in figure 7. To get the corresponding triangulation first cut off the vertices (in figure 7 circumscribed by a small circle) and then cone off to one vertex (marked by a small black square). For the types $(17c)_1$, $(17c)_2$, (17d), and (17e) the other one of the two possible subcases can be constructed, if one suitable half of the remained truncated polyhedron is coned off to that vertex, in figure 7 marked by the small black square, and the other half of it is coned off to that vertex, in figure 7 marked by the small black circle. For example such two corresponding subcases of $(17c)_1$ are the two triangulations

$\{S_3^{(3)},S_5^{(3)},U_1^{(3)},U_2^{(3)},V_1^{(3)},V_2^{(3)}\}$ and $\{S_3^{(3)},S_5^{(3)},U_1^{(3)},U_3^{(3)},V_1^{(3)},V_3^{(3)}$

From this we obtain (cf. [2])

<u>Lemma 3.</u> An optimal vertex-true triangulation of a 3-cube is essentially unique. Its decomposition number is 5.

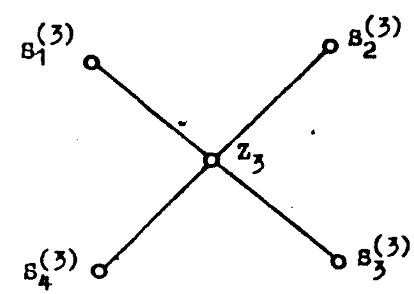


Fig. 8

An optimal vertex-true triangulation of a 3-cube is given by the five tetrahedra (13) (see also figure 9a). The graph in figure 8 illustrates the mutual position of the 5 tetrahedra tiling C_3. The nodes of this graph correspond to the tetrahedra. The edges of this graph always join two of the nodes iff the corresponding tetrahedra have one facet in common.

For a triangulation of C_3 which is not vertex-true the cardinality of the point set being the set of the vertices of the triangulation is greater than 8. Then because of (3) the concerning decomposition number must be greater than 8-3 = 5. A non-vertex-true triangulation of a 3-cube into 6 tetrahedra is possible. For example, decompose one of the tetrahedra of the triangulation (13) into two tetrahedra: $Z_3 = S' \cup S''$ with

$$S' = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} , \quad S'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

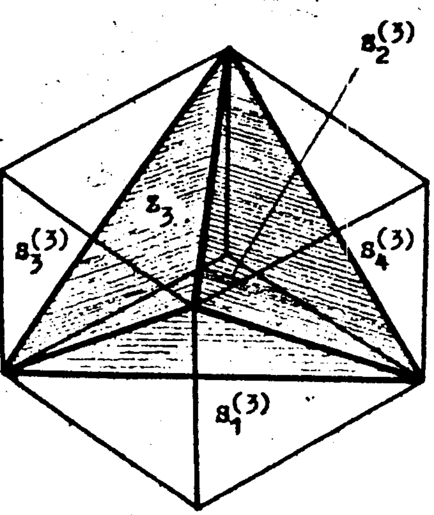


Fig. 9a

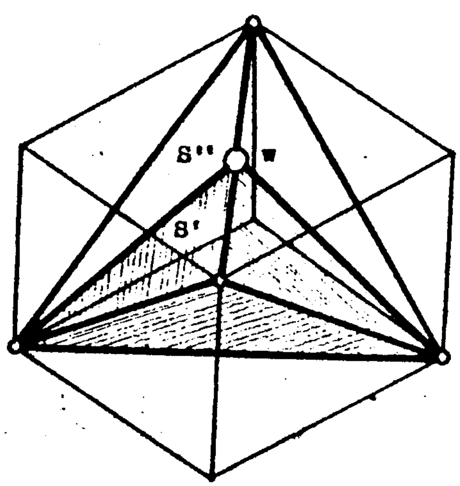


Fig. 9b

mutually congruent and each having two exterior facets, the volume of $V_j^{(4)}$ is $\frac{1}{24}$.

3.) $$W_o^{(4)} = \begin{pmatrix} 0000 \\ 0111 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix},$$

with no exterior facet, its volume is

$$\frac{1}{4!} \det \begin{pmatrix} 0111 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix} = \frac{1}{8} .$$

So, of course, the sum of that 18 simplices is nessesarily

$$7 \cdot \frac{1}{24} + 4 \cdot \frac{1}{12} + 6 \cdot \frac{1}{24} + 1 \cdot \frac{1}{8} = 1 ;$$

Figure 11a shows the graph of this triangulation.

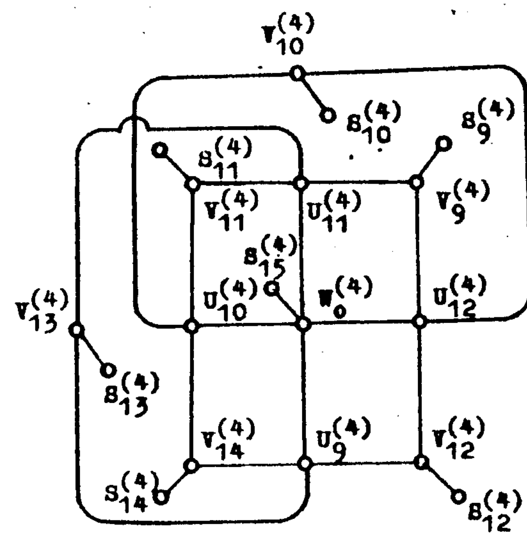


Fig. 11a

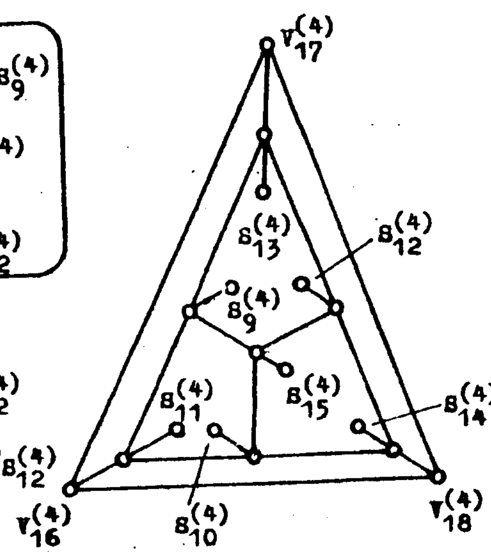


Fig. 11b

connect two nodes iff the facets, corresponding to these nodes, have interior points in common, but their intersection does not agree with the union of these facets.

A second triangulation of C_4 can be attained by cutting off the seven vertices v_j ($j = 9,10,\dots,15$) with

$$v_9 = (0011),\ v_{10} = (0101),\ v_{11} = (0110),\ v_{12} = (1001),$$
$$v_{13} = (1010),\ v_{14} = (1100),\ v_{15} = (1111).$$

This leads to the simplices $S_j^{(4)}$ ($j=9,10,\dots,15$) being congruent with $S_1^{(4)}$ and each having four exterior facets. Then the remainder polytope is coned off to (0000). That gives 11 simplices of three different types:

1.)
$$U_9^{(4)} = \begin{pmatrix} 0000 \\ 1000 \\ 1011 \\ 1101 \\ 1110 \end{pmatrix},\ U_{10}^{(4)} = \begin{pmatrix} 0000 \\ 0100 \\ 0111 \\ 1101 \\ 1110 \end{pmatrix},\ U_{11}^{(4)} = \begin{pmatrix} 0000 \\ 0010 \\ 0111 \\ 1011 \\ 1110 \end{pmatrix},\ U_{12}^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 0111 \\ 1011 \\ 1101 \end{pmatrix},$$

mutually congruent and each having one exterior facet. This facet is a Z_3. The volume of $U_9^{(4)}$ is $\frac{1}{12}$, but $U_9^{(4)}$ is not congruent with $U_1^{(4)}$. A congruent simplex is obtained iff in $U_1^{(4)}$ we would exchange the vertex (0000) with the vertex (1000) for example.

2.)
$$V_9^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 0010 \\ 0111 \\ 1011 \end{pmatrix},\quad V_{10}^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 0100 \\ 0111 \\ 1101 \end{pmatrix},\quad V_{11}^{(4)} = \begin{pmatrix} 0000 \\ 0010 \\ 0100 \\ 0111 \\ 1110 \end{pmatrix},$$
$$V_{12}^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 1000 \\ 1011 \\ 1101 \end{pmatrix},\quad V_{13}^{(4)} = \begin{pmatrix} 0000 \\ 0010 \\ 1000 \\ 1011 \\ 1110 \end{pmatrix},\quad V_{14}^{(4)} = \begin{pmatrix} 0000 \\ 0100 \\ 1000 \\ 1101 \\ 1110 \end{pmatrix},$$

$w = (\frac{1}{2}\ \frac{1}{2}\ 1)$ is a vertex of the triangulation which is not a vertex of the 3-cube (see figure 9b). Therefore 5 is the minimal decomposition number and it can only be attained by a vertex-true triangulation. Thus we have shown

<u>Theorem 2.</u> There exist exactly 10 essentially different vertex-true triangulations of a 3-cube. Exactly one of it is optimal. Every non-optimal vertex-true triangulation of a 3-cube consists of 6 tetrahedra. Every non-vertex-true triangulation of a 3-cube consists of at least 6 tetrahedra.

Hence we get

<u>Theorem 3.</u> The simplexity of a 3-cube is 5.

For $d = 4$ we only consider vertex-true triangulations. The decomposition of a 4-cube, generally described above, gives $P(4) = 16$ simplices. To construct this first triangulation we note that 8 vertices of the 4-cube have to be cut off. The remaining polytope Z_4 is a 4-octahedron. All the edges of Z_4 have length $\sqrt{2}$, all its 8 vertices have degree 8, and all its 16 facets are (regular) tetrahedra. As seen in chapter 2 the 4-octahedron can be optimally tiled into 8 simplices. But there are 6 topologically different types of such a triangulation of the 4-octahedron (cf. chapter 2, remark 2). Together with the 8 simplices, which originate from the vertices being cut off, we have the 16 simplices indicated by P(4). Here we have two sorts of tiles. The 8 simplices $S_j^{(4)}$ $(j=1,\dots,8)$, each having four exterior facets, belong to the first sort. They arise by cutting off the vertices v_j with

$$v_1 = (0001),\ v_2 = (1101),\ v_3 = (0100),\ v_4 = (0111),$$
$$v_5 = (1011),\ v_6 = (1000),\ v_7 = (1110),\ v_8 = (0010).$$

For example we have

$$S_1^{(4)} = \begin{pmatrix} 0000 \\ 0001 \\ 0011 \\ 0101 \\ 1001 \end{pmatrix} .$$

The volume of every $S_j^{(4)}$ is

$$\frac{1}{4!} \det \begin{vmatrix} 0001 \\ 0011 \\ 0101 \\ 1001 \end{vmatrix} = \frac{1}{24} .$$

The 8 simplices derived from the 4-octahedron belong to the second sort. They have one exterior facet. This facet is a Z_3 of a truncated facet of the 4-cube. Furthermore they are mutually congruent. Therefore we see that the volume of each one is $\frac{1}{12}$. In particular, these simplices are for example

$$U_1^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0101 \\ 1001 \\ 1111 \end{pmatrix},\ U_2^{(4)} = \begin{pmatrix} 0000 \\ 0101 \\ 1001 \\ 1100 \\ 1111 \end{pmatrix},\ U_3^{(4)} = \begin{pmatrix} 0000 \\ 0101 \\ 0110 \\ 1100 \\ 1111 \end{pmatrix},$$

$$U_4^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0101 \\ 0110 \\ 1111 \end{pmatrix},\ U_5^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 1001 \\ 1010 \\ 1111 \end{pmatrix},\ U_6^{(4)} = \begin{pmatrix} 0000 \\ 1001 \\ 1010 \\ 1100 \\ 1111 \end{pmatrix},$$

$$U_7^{(4)} = \begin{pmatrix} 0000 \\ 0110 \\ 1010 \\ 1100 \\ 1111 \end{pmatrix},\ U_8^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0110 \\ 1010 \\ 1111 \end{pmatrix}$$

(cf. also [2]).
Triangulating the 4-octahedron Z_4 in another manner (described in chapter 2; there are six different possibilities) for example we have to exchange $U_j^{(4)}$ by $\bar{U}_j^{(4)}$ $(j=1,2,3,4)$ with

$$\bar{U}_1^{(4)} = \begin{pmatrix} 0000 \\ 0101 \\ 0110 \\ 1001 \\ 1100 \end{pmatrix},\ \bar{U}_2^{(4)} = \begin{pmatrix} 0101 \\ 0110 \\ 1001 \\ 1100 \\ 1111 \end{pmatrix},\ \bar{U}_3^{(4)} = \begin{pmatrix} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1111 \end{pmatrix},\ \bar{U}_4^{(4)} = \begin{pmatrix} 0000 \\ 0011 \\ 0101 \\ 0110 \\ 1001 \end{pmatrix}$$

All the $\bar{U}_j^{(4)}$ are congruent with $U_1^{(4)}$ (one edge has length 2, all the other ones have length $\sqrt{2}$); of course, also each $\bar{U}_j^{(4)}$ has volume $\frac{1}{12}$. From the 6 essentially different positions of 16 simplices, triangulating C_4, figure 10 shows the graphs of the two ones which are described above. The dashed lines

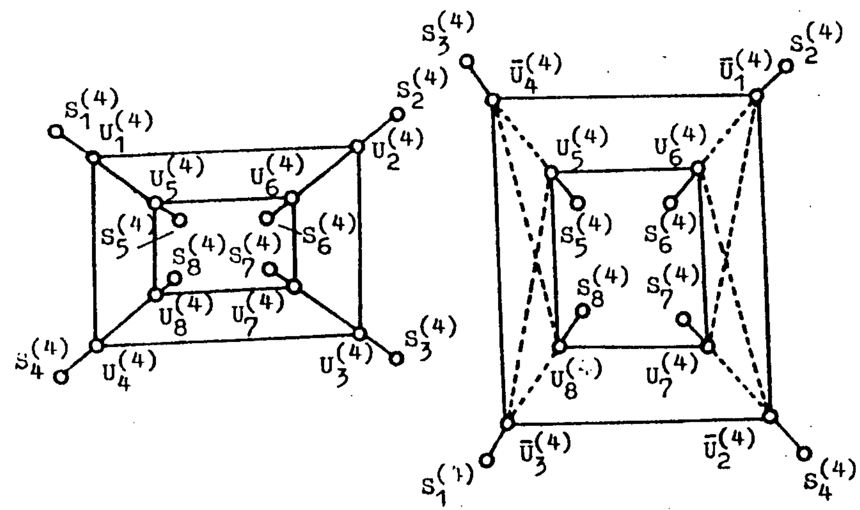


Fig. 10