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ON THE DISTRIBUTION OF POINT-CONFIGURATIONS

The Grassmannian and Probabilities of Order Types

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We have introduced oriented matroids as equivalence classes of ordered oriented hyperplanes or ordered point sets. They can be viewed as abstract equivalent classes of matrices. We have seen problems in computational synthetic geometry where oriented matroids play a fundamental role. Here we are going to apply geometric concepts around oriented matroids for finding probabilities of point configurations. The close connection of oriented matroid theory and properties of the grassmannian will be discussed first.

For this first part, the reader is referred to [1]. We cite the corresponding abstract:

Focusing on the interplay between properties of the Grassmann variety and properties of matroids and oriented matroids, this paper provides the opening of new algebraic methods in the theory of matroids and in the theory of oriented matroids; we introduce new algebraic varieties characterizing matroids and oriented matroids in the most general case.

This new concept admits a systematic study of matroids and oriented matroids by using additional methods from calculus, algebra and stochastics. An interesting new insight when using the matroid variety over $\mathbf{GF}_2$ *and the chirotope variety over* $\mathbf{GF}_3$ *shows that oriented matroids and matroids differ exactly by* *the underlying field.*

We investigate also a chirotope variety over $\mathbf{R}$, *its dimension and its relation to the Grassmann variety.*

In order to find efficient algorithms in computational synthetic geometry, a crucial step lies in finding a small number of conditions for defining oriented matroids. Our new algebraic framework yields new results and straight-forward proofs in this direction.

We consider the exterior algebra of an n-dimensional vectorspace $\mathbf{R}^n$, and the notion of Grassmann coordinates, simple (or decomposable) d-vectors in the exterior product space $\wedge_d \mathbf{R}^n$, as well as the close connection between simple d-vectors and d-dimensional vector subspaces of $\mathbf{R}^n$. We consider the Grassmann manifold $\mathscr{G}^{\mathbf{R}}_{n,d}$ of all d-dimensional (unoriented) subspaces of $\mathbf{R}^n$.

A point p on the Grassmann manifold, represented by a simple d-vector in $\wedge_d \mathbf{R}^n$, can be used to define a (realizable) oriented matroid or (realizable) chirotope $\chi : E^d \to \{-1, 0, +1\}$ of rank d on $E = \{1, 2, \ldots, n\}$. We denote the coordinates in $\mathbf{R}^{\binom{n}{d}}$ as formal expressions $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_d]$, $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$. These coordinates define similar expressions $[\pi(\lambda)] = [\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(d)}]$ via the alternating rule for any permutation π of the d numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$, $[\pi(\lambda)] = \mathrm{sign}(\pi) \cdot [\lambda]$.

These formal expressions turn into the bracket notation of N.White when the Grassmann Plücker relations are valid as well. In this case the points are called *simple.*

DEFINITION. A (realizable) *chirotope* χ of rank d with n points is given if and only if the coordinates $[\lambda_1, \lambda_2, \ldots, \lambda_d]$, $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$, form a simple d-vector and $\chi(\lambda_1, \lambda_2, \ldots, \lambda_d) = \mathrm{sign}\,[\lambda_1, \lambda_2, \ldots, \lambda_d]$ for all elements of E^d. The set of all realizable oriented matroids of given rank d and given number of points n is denoted by $\mathbf{ROM}_{n,d}$.

When comparing the definition of chirotopes with the definition of oriented matroids according to Bland and Las Vergnas, the equivalence of both structures, first completely proven by Lawrence, is not at all obvious. Therefore having names for both definitions is useful in many instances. Nevertheless, here we will use both names synonymously.

«Which bracket identities yield translations into basis exchange properties holding in all matroids and which do not. This is perhaps the central question in the study of basis-exchange properties. A reasonable answer to this question could be in the form of a decision procedure. Such a decision procedure would yield much insight into the relation between matroïd theory and linear algebra.» Kung.

Grassmann-Pl. syzygies $< \{C\vert A\vert B\} >$	2. Fund.Th. Inv.Theory $\ni$	Hom. Bracket syzygies g
$\downarrow$		$\downarrow$
weak basis exchange property $od(\{C\vert A\vert B\}) = 0$	Combin. proof $\Rightarrow$	basis exchange property $od(g) = 0$
$< od(\{C\vert A\vert B\}) >$	Algebr. proof $\ni$	$od(g)$

A remark about calculating probabilities for oriented matroids in the realizable case. For these questions and those of a similar nature, the embedding within a real variety is useful. The chirotope variety over $\mathbf{GF}_3$ is extended to a chirotope variety over $\mathbf{R}$, and the polynomials are replaced by homogeneous ones. This setting has more links with the Grassmann variety, and properties of the new chirotope variety over the reals are studied.

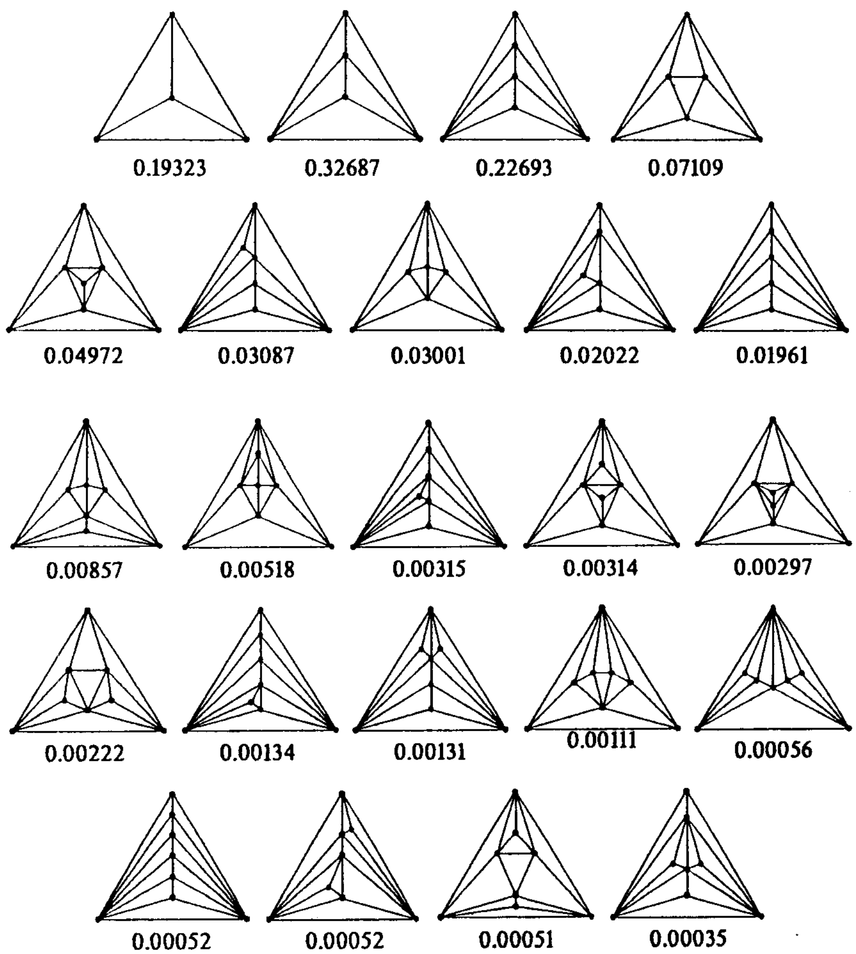
Whereas a point on the Grassmann variety leads to a realizable oriented matroid when the coordinates are replaced by their signs, a point on the new chirotope variety yields by the same replacement a general (not neccessarily realizable) oriented matroid. And moreover, all oriented matroids or chirotopes can be found in this way.

The second part is devoted to the following problem. Goodman and Pollack have asked to estimate the probabilities of order types (realizable oriented matroids) by using a uniformly distributed random generator on the unit interval. A recent paper [3] provides solutions to this question. By applying these methods, one can estimate the probability for various combinatorial types of polytopes with up to 8 points in dimension 3. This investigation also confirms a classification result in oriented matroid theory, [2].

There are at least two approaches to this question.

I. A first natural approach lies in using a uniform distribution on the unit interval and extending it to the n-cube with centre 0. d points picked at random in this cube (whereby those not lying in the insphere of the cube are discarded) determine together with the centre of the cube with probability 1 a linear d-space, i.e. a point $p \in \mathcal{G}^{\mathbf{R}}_{n,d}$, [J.E.Goodman, R.Pollack (1986)], [R.Schneider (1989)]. This provides us with a uniform distribution (invariant under rotations) on the Grassmann variety.

For n large we have at least the drawback that many points have to be discarded as the volume of the unit ball tends to zero as n tends to infinity.



II. A second approach lies in applying a result of Heiberger [4] and using uniform distributions on the orthogonal group $O(n)$. Via the action of $O(n)$ on $\mathcal{G}^{\mathbf{R}}_{n,d}$, we get a uniform distribution on $\mathcal{G}\mathbf{R}_{n,d}$.

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