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Abstract. *Goodman and Pollack have asked to estimate the probabilities of order types by using an equally distributed random generator on the unit interval. We provide solutions to this question, and we apply these methods for estimating the probability for various combinatorial types of polytopes with up to 8 points in dimension 3. Our investigation also confirms a classification result in oriented matroid theory [3].*

1. Introduction

The problem of calculating the probabilities for order types [9], i.e. realizable oriented matroids [1], or chirotopes [3], has been posed by Goodman and Pollack [8]. A formulation of their question in the literature can be found in [3, Problem 5.19] together with a conjecture concerning the maximum of all probabilities.

Problem 1.1. Compute or estimate the probability of some rank d oriented matroids with n points where $1 < d < n-1$.

Problem 1.2. Find an algorithm to generate random points (with respect to the Haar measure) on the Grassmann manifold $\mathcal{G}_{n,d}^{\mathbb{R}}$ over the reals using a random number generator for the unit interval.

Conjecture 1.3. The maximum among the probabilities of all rank d oriented matroids with n points is attained by the probability $p(\chi^{n,d})$ of the alternating oriented matroid corresponding to the cyclic polytope.

This paper discusses Problem 1.1, provides answers to Problem 1.2., and deals with applications of these methods concerning also Conjecture 1.3.

In our context our probability spaces under consideration appear to be very natural. The survey article of Buchta [6] refers to another classical approach which might be considered to be closely related to our investigations. It is well known in these cases that exact calculations of probabilities as required in Problem 1.1 are very hard or impossible. One reason for such difficulties can be viewed with respect to Mnev's result in [14]. Mnev has shown that the topological realization spaces of oriented matroids attain all possible topological types. There might be some hope when solvability sequences exist, see [2], in which the realization space is contractible.

Nevertheless, this situation emphasizes the question of Goodman and Pollack of finding an efficient random generator for realizable oriented matroids corresponding to the equal distribution on the grassmannian. Our answer will be given in so far as random generators for the equal distribution on the unit interval yield a corresponding equal distribution on the grassmannian.

Having a small number of such random generators is very essential for practical applications. In order to reach this goal as well, some parts of [16] were adjusted to our purposes. The corresponding solution to this problem presented in Section 3 and 4 was solved by the third author.

This method was applied with respect to investigations in [4] and [5] dealing with a complete overview and classification of all reorientation classes of oriented matroids in rank 4 with 8 points. This case is of particular interest since it is the smallest non-planar case in which non-realizable oriented matroids occur. The classification result in [4] was confirmed in so far as all the realizable

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oriented matroids occured during our simulation. Our results provide estimates for the probability of forming extreme points of the convex hull, or various combinatorial types of polytopes, respectively. Corresponding results can be found in Section 5.

Our methods can be applied in general. In the rank 4 case with 8 points pre-calculations were available and results were obtained in reasonable CPU-time limits.

2. Equal Distribution on the Grassmannian

Our main concern in this paper is a natural probability distribution on the set of all realizable oriented matroids or chirotopes. Stewart provided an algorithm for simulating the Haar measure on the orthogonal group $\mathrm{O}(n)$ of all $(n \times n)$-matrices, see [16]. For our purposes, we use some of his results in a slightly modified manner. In this section we recall these results and provide definitions as well as easy properties, thus introducing our notation and our probability distribution.

To each d-dimensional vector subspace W of $\mathbb{R}^n$, we assign in a canonical way a pair $\{\chi, -\chi\}$ of chirotopes of rank d with n points, see e.g. [3] for details about chirotopes. A d-dimensional vector subspace W can be represented by a $(n \times d)$-matrix. With $\mathrm{Mat}(n,d)$ we denote the set of all $(n \times d)$-matrices while $\mathrm{Mat}(n,d)_*$ denotes the set of all $(n \times d)$-matrices with rank d.

We identify each chirotope χ with its negative $-\chi$. In choosing the above vector subspace W of $\mathbb{R}^n$ at random due to the invariant distribution induced by the action of the orthogonal group $\mathrm{O}(n)$ on the Grassmann manifold $\mathcal{G}_{n,d}^{\mathbb{R}}$ of all d-dimensional subspaces of $\mathbb{R}^n$, we induce a probability distribution on the set of all realizable chirotopes. We consider this induced probability distribution on the set of all realizable oriented matroids or chirotopes as a suitable model. In this section our probability distribution can be generated by normal distributed matrices defined as follows.

Definition 2.1.
A random element $T = (T_{i,j})_{i=1,\dots,k, j=1,\dots,l}$ with values in the set of all $(k \times l)$–matrices is called *(k,l)–normal distributed* if all $T_{i,j}$ are independent and identically N(0,1) distributed.

Lemma 2.2.

(i) Let $n \geq d$ and $X: \Omega \to \mathrm{Mat}(k,l)$ be a (k,l)–normal distributed random element on $\mathrm{Mat}(k,l)$. Then $P(X \in \mathrm{Mat}(n,d)_*) = 1$ and

$$P\left(\omega \mid \det \begin{pmatrix} X_{j_1 1} & \dots & X_{j_1 l} \\ \vdots & & \vdots \\ X_{j_l 1} & \dots & X_{j_l l} \end{pmatrix} (\omega) \neq 0 \right) = 1$$

for all choices of indices satisfying $1 \leq j_1 < j_2 < \dots < j_l \leq k$.

(ii) Let T be a (k,l)–normal distributed random element and $O \in \mathrm{O}(k)$. Then OT is (k,l)–normal distributed, and $O' \in \mathrm{O}(l)$ implies TO' to be (k,l)–normal distributed.

(iii) Let V be (k,l)–normal distributed and let H be a random element with values in $\mathrm{O}(l)$. H and V being independent imply $W = VH$ is (k,l)–normal distributed and independent of H.

Proof of Lemma 2.2.

(i) We identify $\mathrm{Mat}(k,l)$ with the space $\mathbb{R}^{k \cdot l}$. The distribution of X is absolutely continuous with respect to the Lebesgue measure λ. For each non-constant polynomial $p: \mathbb{R}^k \to \mathbb{R}$, the equation $\lambda(\{x \in \mathbb{R}^k \mid p(x) = 0\}) = 0$ holds.

(ii) All components $(OT)_{ij} = (\sum_{\mu=1}^{k} O_{i\mu} T_{\mu j})_{ij}$ are normally distributed with mean 0 and variance 1 since the components of T are independent. Moreover, all components of OT are pairwise uncorrelated. This implies their independence because they are normally distributed. The second assertion follows in the same way.

(iii) (ii) tells us that for all $O' \in \mathrm{O}(l)$ and for each $B \in \mathcal{B}(\widetilde{M}_{k,l})$, a conditional probability for VH with respect to H is given by $P(VH \in B \mid H = O') = P(V \in B)$. We conclude $P(VH \in B) = P(V \in B)$ for all $B \in \mathcal{B}(\widetilde{M}_{k,l})$, i.e. VH is (k,l)–normal distributed. For each $C \in \mathcal{B}(\mathrm{O}(l))$, we have

$$P(VH \in B, H \in C) = \int_C P(VH \in B \mid H = O') P_H(dO') = P(VH \in B) P(H \in C)$$

This proves the independence of VH and H.

For $M \in \mathrm{GL}(n)$ there exists a unique (normalized) LQ-decomposition $M = LQ$ with $Q \in \mathrm{O}(n)$ and L is lower triangular matrix with positive diagonal elements. The following theorem is decisive for our investigations. For a proof we refer the reader to [16].

Theorem 2.3. (Stewart). Let V be a $(d \times d)$–normal distributed matrix valued random element. For a regular $V(\omega)$, we denote with $L(\omega)Q(\omega)$ its normalized LQ-decomposition, and we define $L(\omega) = Q(\omega) = \mathrm{id}$ if $V(\omega)$ is singular.
(i) Q is an equi-distributed random element on $\mathrm{O}(d)$ with respect to the Haar measure.
(ii) The random element Q and the entries of L are independent.
(iii) The diagonal entries l_{ii} of L are χ^2 distributed with $d+1-i$ degrees of freedom.
(iv) The subdiagonal elements of L are N(0,1) distributed.

3. On the Generation of Random Chirotopes

The orthogonal group $\mathrm{O}(n)$ acts transitively on $\mathcal{G}^{\mathbb{R}}_{n,d}$. This induces an invariant probability measure μ on $\mathcal{G}^{\mathbb{R}}_{n,d}$. We have to simulate an equi-distribution on $\mathcal{G}^{\mathbb{R}}_{n,d}$ with respect to μ. Let $GM_{n,d} = \{\{MA \mid A \in \mathrm{GL}(d)\} \mid M \in \mathrm{Mat}(n,d)_*\}$. The orthogonal group $\mathrm{O}(n)$ acts on $GM_{n,d}$ via $O\{MA \mid A \in \mathrm{GL}(d)\} = \{OMA \mid A \in \mathrm{GL}(d)\}$. In the following we will identify the Grassmann manifold with $GM_{n,d}$ since the columns of two $(n \times d)$ matrices M_1 and M_2 span the same d-dimensional subspace of $\mathbb{R}^n$ if and only if each column of M_2 can be expressed as a linear combination of the columns of M_1 (and vice versa). This gives us a natural projection.

$$\mathrm{pr}\colon \mathrm{Mat}(n,d)_* \to \mathcal{G}^{\mathbb{R}}_{n,d} \qquad M \mapsto \{MA \mid A \in \mathrm{GL}(d)\}.$$

Lemma 3.1. Let X be (n,d)–normal distributed and

$$\overline{\mathrm{pr}}\colon \mathrm{Mat}(n,d) \to \mathcal{G}^{\mathbb{R}}_{n,d}$$
$$M \mapsto \begin{cases} \mathrm{pr}(M) & \text{if } M \in \mathrm{Mat}(n,d)_* \\ \{(e_1, e_2, \ldots, e_d)A \mid A \in \mathrm{GL}(d)\} & \text{otherwise} \end{cases}$$

where e_i denotes the i-th unit vector in $\mathbb{R}^n$. Then the random element $Y = \overline{\mathrm{pr}} \circ X$ is equi-distributed on $\mathcal{G}^{\mathbb{R}}_{n,d}$ with respect to μ.

Proof. Let $O \in \mathrm{O}(n)$. For each Borel set $B \subseteq \mathcal{G}^{\mathbb{R}}_{n,d}$ the equation $P(OY \in B) = P(O\{XA \mid A \in GL(d)\} \in B) = P(\{OXA \mid A \in GL(d)\} \in B) = P(\{XA \mid A \in GL(d)\} \in B) = P(Y \in B)$ holds since OX is (n,d)–normal distributed (Lemma 2.2).

$[j_1, j_2, \ldots, j_d]_M$ denotes the determinant of the submatrix $M_{j_1,j_2,\ldots,j_d}$ of M which is formed by the j_1-th, j_2-th, ... and the j_d-th row of M. Let

$$\Psi\colon \mathrm{Mat}(n,d) \to GF_3^{\binom{n}{d}}$$
$$M \mapsto (\mathrm{sgn}[1,2,..,d]_M, .., \mathrm{sgn}[j_1,j_2,..,j_d]_M, .., \mathrm{sgn}[n-d+1, n-d+2, .., n]_M).$$

For $A \in \mathrm{GL}(d)$ the equation $\det(MA)_{j_1,j_2,\ldots,j_d} = (\det M_{j_1,j_2,\ldots,j_d}) \det A$ holds. Hence all corresponding subdeterminants of M and MA differ by the same scalar $\det A$. Therefore the mapping

$$\begin{aligned}\overline{\Psi}: \mathcal{G}_{n,d}^{\mathbb{R}} &\to PGF_3^{\binom{n}{d}} = GF_3^{\binom{n}{d}}/\{+1,-1\}\\ \{MA \mid A \in \mathrm{GL}(d)\} &\mapsto \Psi(M)\cdot\{+1,-1\} = \Psi(M)\cdot GF_3^*\end{aligned}$$

is well defined We are interested in the distribution of the random element $S = \overline{\Psi} \circ Y$. In the following we search for random elements with same distribution as S which can be simulated more easily.

The random elements S and $S' = (\Psi \circ X)\cdot GF_3^*$ have the same distribution since $P(X \notin \mathrm{Mat}(n,d)_*) = 0$. Lemma 2.2 shows $P(\omega \mid \text{one component of } S'(\omega) \text{ is zero}) = 0$, i.e. nonsimplicial chirotopes occur with probability 0.

In the following we will construct a measure η on $\mathrm{Mat}(n,d)$ whose distribution under the mapping $\overline{\Psi} \circ \overline{\mathrm{pr}}$ is equal to the distribution of the Haar measure under $\overline{\Psi}$. This measure η can be simulated more easily by pseudo-random elements than the Haar measure or the (n,d)–normal distribution. Let X_1 and X_2 be independent random elements which are (d,d)–normal and $(n-d,d)$–normal distributed. Let X be a random element on $\mathrm{Mat}(n,d)$ whose entries in the rows 1 up to d are given by X_1, and the entries in the rows (n-d+1) up to n are given by X_2. Then X is (n,d)–normal distributed. For all $\omega \in \Omega$ with invertible $X_1(\omega)$ the equation

$$\begin{aligned}\overline{\Psi}\{XA \mid A \in \mathrm{GL}(d)\}(\omega) &= \overline{\Psi}\Big\{ \begin{pmatrix} I \\ \cdots\cdots \\ X_2X_1^{-1} \end{pmatrix} X_1A \mid A \in \mathrm{GL}(d)\Big\}(\omega)\\ &= \overline{\Psi}\Big\{ \begin{pmatrix} I \\ \cdots\cdots\cdots \\ X_2Q^{-1}L^{-1} \end{pmatrix} A \mid A \in \mathrm{GL}(d)\Big\}(\omega)\end{aligned}$$

holds. By Theorem 2.3, Q and L are independent and therefore Q^{-1} and L^{-1} as well. By Lemma 2.2, X_2Q^{-1} is $(n-d,d)$–normal distributed. This proves the following theorem.

Theorem 3.2.

(i) T and L are independent random elements with values in $\mathrm{Mat}(n-d,d)$ and $\mathrm{Mat}(d,d)$, respectively,
(ii) the random element T is $(n-d,d)$–normal distributed,
(iii) the entries of L are independent,
(iv) the upper diagonal entries of L are zero,
(v) the lower diagonal entries are $N(0,1)$ distributed,
(vi) the i-th diagonal element of L is χ^2 distributed with $n+1-i$ degrees of freedom.

$$(i),\ldots,(vi)\ \mathit{imply}\quad S'' = \Psi \begin{pmatrix} I \\ \cdots\cdots \\ TL^{-1} \end{pmatrix} \cdot GF_3^* \text{ has the same distribution as } S.$$

In the $(8,4)$ case we may perform an additional step. The matrix elements of L are denoted l_{ij}. Let $\tilde{l} = l_{22}l_{33}l_{44}$ and

$$L' = \begin{pmatrix} \tilde{l} & 0 & 0 & 0 \\ -l_{21}l_{33}l_{44} & \tilde{l} & 0 & 0 \\ l_{44}(l_{32}l_{21} - l_{22}l_{31}) & -l_{22}l_{32}l_{44} & \tilde{l} & 0 \\ l_{33}(l_{21}l_{42} - l_{22}l_{41}) - l_{43}(l_{21}l_{32} - l_{22}l_{31}) & l_{22}(l_{32}l_{43} - l_{33}l_{42}) & -l_{22}l_{33}l_{43} & \tilde{l} \end{pmatrix}$$

We denote the diagonal matrix with the diagonal entries $(1/l_{11}\widetilde{l}, 1/l_{22}\widetilde{l}, 1/l_{33}\widetilde{l}, 1/l_{44}\widetilde{l})$ with D. Then $L^{-1} = L'D$ holds which leads to

$$\overline{\Psi}\{XA \mid A \in \mathrm{GL}(4)\}(\omega) = \overline{\Psi}\Big\{ \begin{pmatrix} I \\ \cdots\cdots\cdots \\ X_2Q^{-1}L'D \end{pmatrix} A \mid A \in \mathrm{GL}(4)\Big\}(\omega)$$
$$= \overline{\Psi}\Big\{ \begin{pmatrix} D^{-1} \\ \cdots\cdots \\ X_2Q^{-1}L' \end{pmatrix} DA \mid A \in \mathrm{GL}(4)\Big\}(\omega)$$
$$= \overline{\Psi}\Big\{ \begin{pmatrix} I \\ \cdots\cdots \\ X_2Q^{-1}L' \end{pmatrix} A \mid A \in \mathrm{GL}(4)\Big\}(\omega)$$

for all regular $X_1(\omega)$ since the multiplication of a row with a positive scalar does not change the sign of any subdeterminant. (Theorem 2.3 implies: the l_{ii} are χ^2 distributed.) The same arguments show the following theorem.

Theorem 3.3.
(i) T is $(4,4)$–normal distributed,
(ii) $L_{21}, L_{31}, L_{41}, L_{32}, L_{42}, L_{43}$ are N(0,1) distributed,
(iii) L_{ii} are χ^2 distributed with $5-i$ degrees of freedom ($i = 2,3,4$),
(iv) the random elements $T, L_{21}, \ldots, L_{44}$ are independent.
(v) $Z = Z(L_{21}, \ldots, L_{44})$ has the same shape as L' above the random element.

$$(i), \ldots, (v) \text{ imply} \quad S'' = \Psi \begin{pmatrix} I \\ \cdots\cdots \\ TZ \end{pmatrix} \cdot GF_3^* \text{ has the same distribution as } S.$$

4. Generating Random Chirotopes and Avoiding Local Defects.

Since the orthogonal group $\mathrm{O}(n)$ acts transitively on $\mathcal{G}^{\mathbb{R}}_{n,d}$, there exists a completely different approach to generate equi-distributed pseudo-random elements on $\mathcal{G}^{\mathbb{R}}_{n,d}$. Let $V_1, V_2, \ldots$ be a sequence of independent random elements on $\mathrm{O}(n)$ whose distribution is given by the Haar measure. For each element q of the Grassmann manifold, $V_1q, V_2q, \ldots$ is a sequence of independent and equi-distributed random elements on $\mathcal{G}^{\mathbb{R}}_{n,d}$. One can simulate the random elements V_j in several manners (see e.g. [11] and [16]). Since pseudo-random elements are generated from standard random numbers, one may obtain "transformation defects" which aggravate the defects of the standard random numbers. If one generates equi-distributed random elements on a group or on a homogeneous space, the pseudo-random elements sometimes distinguish certain regions. These effects were discussed in [15, pp. 58–62]. If one needs many standard random numbers for one pseudo-random element, this phenomenon is quite likely. Since the group $\mathrm{O}(n)$ is compact, one can attempt to reduce these effects. At first we proof the following lemma.

Lemma 4.1. Let G be a compact group with Haar measure μ_G and $V_1, V_2, \ldots$ denote a sequence of independent equi-distributed random elements. Then the sequence $W_1, W_2, \ldots$ defined by

$$W_1 = V_1, \qquad W_j = V_jW_{j-1} = V_jV_{j-1}\cdots V_1 \qquad \text{for } j \geq 2$$

is also independent and equi-distributed on G.

Proof. For all $A \subseteq G$ and all $j \geq 2$ a conditional probability is given by

$$P(W_j \in A \mid (W_1, \ldots, W_{j-1}) = (g_1, \ldots, g_{j-1})) = P(V_jg_{j-1} \in A) = \mu_G(Ag_{j-1}^{-1}) = \mu_G(A).$$

This suffices to prove

$$P((W_1, \ldots, W_k) \in (A_1 \times \cdots \times A_k)) = \prod_{i=1}^{k} \mu_G(A_j) = \prod_{i=1}^{k} P(W_j \in A_j)$$

from which the assertion follows. See the proof of Lemma 2.2 for further details.

The preceding lemma suggests the following method. Generate equi-distributed pseudo-random elements $\tilde{V}_1, \tilde{V}_2, \ldots$ and compute $\widetilde{W}_1, \widetilde{W}_2, \ldots$ where the $\widetilde{W}_j = \tilde{V}_j \widetilde{W}_{j-1}$.

One should intuitively expect that this method spreads local defects over the group and makes them smooth This intuitive idea is supported by the structure of the convolution semigroup $\mathcal{P}$ of the probability measures on a compact group G.

Let ν denote a probability measure on G and suppν its support. If $\liminf_{n\to\infty}[\text{supp}\nu]^n = G$, the sequence $\nu, \nu^{*2}, \ldots$ of convolution powers converges to the Haar measure μ_G in the topology of weak convergence. (By definition an element $g \in G$ is in $\liminf_{n\to\infty}[\text{supp}\nu]^n$ if and only if each neighborhood of g intersects all but finitely many of the $[\text{supp}\nu]^n$.) This fact and further information about $\mathcal{P}$ are given in [12, pp. 88–95].

If supp$\nu = G$, then ν^{*n} converges to μ_G as $n \to \infty$. The Haar measure μ_G is the unique minimal ideal in $\mathcal{P}$. In this sense we can view μ_G as a stable fixed point. We may regard the pseudo-random elements $\tilde{V}_j$ as realizations of independent random elements V_j' whose distribution ν is "close" to μ_G. If we assume that suppν fulfills the condition required above, the distributions of the random elements $V_1', V_2'V_1', \ldots$ converge to the Haar measure. Since this mechanism seems to be very robust, we have good reasons to hope that this mechanism will also work with pseudo-random elements instead of random elements. Of course, in general the random elements $V_1', V_2'V_1', \ldots$ are not independent but if ν is close to μ_G, the dependence should be "weak" and therefore neglectable. If the sequence $\tilde{V}_1', \widetilde{V_2'V_1'}, \ldots$ of pseudo-random elements is not too bad (in the sense of equi-distribution), this loss of independence is not a grave disadvantage of this method, since pseudo-random elements are never independent in the stochastic sense. Moreover, for our problem the property of equi-distribution should be much more important than the independence. Further details can be found in [15, pp. 91–95].

5. On the Efficiency of our Algorithms

In this section we restrict our attention to the $(8,4)$ case. If we use Theorem 3.3 to simulate S by pseudo-random elements $\tilde{S}_1, \tilde{S}_2, \ldots$, we have to generate 22 N(0,1) distributed random numbers and 3 χ^2 distributed random numbers with $1, 2$, and 3 degrees of freedom for each $\tilde{S}_k$.

If we want to unify the random number generation, we can simulate a χ^2 distributed random variable with i degrees of freedom by summing up the squares of i random numbers which are N(0,1) distributed. In this case we have to generate 28 $N(0,1)$ distributed random numbers for each pseudo-random element $\tilde{S}_k$. We may generate normally distributed random numbers with a method proposed by Marsaglia (see [7, 235-236]). For using Marsaglia's method, the average number of standard random numbers for each $\tilde{S}_k$ is about $28 \cdot 4/\pi \approx 36$. Furthermore, we have to compute one logarithm for each pair of normally distributed random numbers. Besides sparing the generation of some random numbers, there is another advantage in using Theorem 3.3 compared with the "brute force" simulation $S' = \Psi(X) \cdot GF_3^*$. The special structure of the random elements $(I \cdot TZ)$ is time saving when computing the subdeterminants. For each pseudo-random chirotope we have in fact just to compute one determinant of rank 4.

In using the method described in Section 4, we do not need to compute the matrix multiplications on the $O(n)$, explicitely. To compute $\widetilde{W}_{n+1}q$ it suffices to compute $\tilde{V}_{n+1}(\widetilde{W}_n q)$. If we use Stewarts algorithm (see [16]) for generating the $\tilde{V}_j$ we get $\tilde{V}_j$ as a product of dyadic matrices which simplifies the computation of $\widetilde{V}_{n+1}(\widetilde{W}_n q)$. In the 8×4 case we need about 35 normally distributed pseudo-random numbers for each $\widetilde{V}_j$ on average.

There exists a third possibility to generate random chirotopes. Let $||X_i||$ denote the norm of the i-th row of X which is (n,d)-normal distributed. Let us define another random element X' on $\mathrm{Mat}(n,d)$. For all $\omega \in \Omega$ for which none of the rows of $X(\omega)$ is equal to the zero vector, we set $X'_{ij}(\omega) = X_{ij}/||X_i||(\omega)$. For these ω we have $\Psi(X)(\omega) = \Psi(X')(\omega)$. Since the rows of X' are equi-distributed on S^{d-1}, this proves the following fact:
Let X''' be a random element on $\mathrm{Mat}(n,d)$ whose rows are equi-distributed on S^{d-1} and independent. Then $S''' = \Psi(X''') \cdot GF_3^*$. has the same distribution as S.
We may generate the rows with the well known rejection algorithm where we enclose the unit ball in a d-dimensional hypercube. In the 8×4 case on average we need about 104 standard random numbers for each pseudo-random element. In comparison with the method recommended at the beginning of this section, we spare to compute logarithms but we need a threefold of standard random variables. This could be troublesome because for this method one needs standard random numbers whose k-dimensional distributions are all right for at least all $k \leq 104$. Furthermore, the pseudo-random matrices $\widetilde{X}_j'''$ usually do not contain any zero entries. So we have to compute 70 subdeterminants of rank 4. For these reasons this method is only of theoretical interest.

6. On the Random Classification of Chirotopes

We have simulated the 8×4 case on a VAX 8530. The standard random numbers were generated with a linear congruential generator given by $u_{m+1} \equiv 134775813 * u_m + 1 \pmod{2^{32}}$. The normally distributed random numbers were generated with Marsaglia's method. Since one needs a large number of standard random numbers for each pseudo-random chirotope, it seems possible that defects of the linear congruential generator may affect the results of the simulation. Therefore, we generated about 800.000 pseudo-random chirotopes with an algorithm based on Theorem 2.3 and 300.000 with the algorithm suggested in Section 4. An attentive comparison showed that the results were quite similar. Especially, the order induced by the frequencies of the reorientation classes was essentially the same for both methods. Hence we view our results as reliable.

Starting point for our investigations was a classification result on 3-chirotopes with 8 points. We proved that in this case there are exactly 2628 reorientation classes [4]. Exactly 2604 of those classes turned out to be realizable. We wanted to confirm this result by generating all these reorientation classes by a random generator.

What is the connection between reorientation classes and chirotopes? In the $(8,4)$-case there are exacly $12,851,973,120$ different chirotopes (any pair $\{\chi, -\chi\}$ counted once). If all these chirotopes had a different probability, it would be hopeless to estimate these probabilities by a Monte-Carlo method. Fortunately, there are equivalence classes of chirotopes, namely the reorientation classes with the property that every member of the class has the same probability. This is a very natural fact, since the definition of chirotopes is completely symmetric with respect to any point, and it does not change if an arbitrary point is replaced by its negative. In the preceding paragraph we have seen that S''' has equal distribution as S. This also implies the equidistribution within a reorientation class. In the realizable case the symmetry with respect to any point corresponds to a permutation of the row vectors of the 8×4 realisation matrix. Multiplying an arbitrary row by -1 corresponds to replacing a point by its negative.

For a given d-chirotope χ with n points and a permutation $\pi \in S_n$, we define the permutation $\pi\chi$ of χ by

$$\pi\chi(\lambda_1, \ldots, \lambda_n) = \chi(\pi\lambda_1, \ldots, \pi\lambda_n),$$

and for a subset $A \in \{1, \ldots, n\}$, we define the reorientation χ^{-A} of χ by

$$\chi^{-A}(\lambda) = (-1)^{|A \cap \lambda|} \cdot \chi(\lambda).$$

If $p(\chi)$ denotes the probability of χ, we conclude:

for any χ; $A \in \{1,\ldots,n\}$; $\pi \in S_n$, *we have* $p(\chi) = p(\pi(\chi^{-A})$.

We define the reorientation class without renumbering the points of χ by

$$\overline{reor}(\chi) := \{\chi^{-A} \mid A \in \{1,\ldots,n\}\}.$$

We define the reorientation class with renumbering the points of χ by

$$reor(\chi) := \{\pi(\chi^{-A}) \mid A \in \{1,\ldots,n\}; \pi \in S_n\}.$$

All elements in a reorientation class have the same probability.

Given a chirotope χ the number of chirotopes in its reorientation class $reor(\chi)$ is $|reor(\chi)| = 2^n \cdot \frac{n!}{|Aut(\overline{reor}(\chi))|}$.

In our investigation we estimated probabilities for the 2604 realizable reorientation classes. It turned out that all these classes had a frequency greater than zero.

Remark 6.1 The classification result in [4] was confirmed in the sense that all realizable oriented matroids occured during our simulation.

Given the probability $p(\mathcal{R})$ of a reorientation class $\mathcal{R}$, one can compute the probability of a representative $\chi \in \mathcal{R}$ by

$$p(\chi) = \frac{1}{2^n} \cdot \frac{|Aut(\overline{reor}(\chi))|}{n!}.$$

Our estimates in the $(8,4)$–case can be seen as supporting Conjecture 1.3: The maximum among the probabilities of all rank d oriented matroids with n points is attained by the probability $p(\chi^{n,d})$ of the alternating oriented matroid.

7. On the Convex Hull of 8 Random Points in $\mathbf{R}^3$.

In this section we apply our estimations for the probabilty of the reorientation classes to give estimations for the probability of the combinatorial types of the convex hull of 8 random points in R^3. This (8,4) case is of particular interest since it is the smallest non-planar case in which non-realizable oriented matroids occur.

The question of determining the probability of the combinatorial structure of the convex hull of a random set of points has a long tradition. In the middle of the last century, Sylvester posed the question of determining the probability p_K, that the convex hull of four points chosen at random in a plane convex body K is a quadrangle. Some values of p_K for several convex bodies K are mentioned in [6]. We follow here the main ideas outlined in [3,Chapter 5.].

For the rest of this section we restrict our considerations to configurations in general position, since all other configurations have a probability of zero. So in the sequel it is sufficient for us to consider only uniform oriented matroids. Definitions will then become a bit easier as in the general case.

A uniform oriented matroid containing no positive circuit is called *cyclic* and *acyclic*, otherwise. A configuration in general position is called cyclic, or acyclic, if the underlying (uniform) oriented matroid is cyclic, or acyclic, respectively. An acyclic configuration in R^d in the realizable case can be identified with an affine configuration in R^{d-1}, see [1].

Given any generic point g on the Grassmann manifold $G^R_{n,d}$, its asociated configuration $C(g)$ is either cyclic or acyclic. If it is acyclic, one can determine the combinatorial type of the convex hull of the corresponding affine configuration $P(C(g))$. The probability that a certain combinatorial polytope P is the convex hull of n random points in dimension $d-1$ will be defined as

$$p_P = \frac{p(P(C(g)) = P; g \in G^R_{n,d})}{p(C(g) \text{ is acyclic }; g \in G^R_{n,d})}.$$

For given n and d we first want to calculate the probability $p(C(g)$ is acyclic $; g \in G^R_{n,d})$. The number of subsets A with the property that for a given oriented matroid M the reorientation M^{-A} is acyclic was shown by Las Vergnas in [13,Theorem 3.1.] to be exactly $t(M;2,0)$, where $t(M)$ denotes the tutte polynomial of M. So the number of acyclic reorientations of an oriented matroid only depends on the underlying matroid. In case of uniform rank-d oriented matroids with n points (having all the same underlying matroid), we can compute the number $R(d,n)$ of acyclic reorientations by the recursion

$$R(n,d) = R(n-1,d) + R(n-1,d-1); \quad \text{for } d \neq n \text{ and } d = 1,$$

$$R(d,d) = 2^d, \qquad R(n,1) = 2.$$

It is easy to prove that

$$R(n,d) = 2 \cdot \sum_{i=0}^{d-1} \binom{n-1}{i}.$$

We have $R(2d,d) = 2^{(d-1)}$. $R(n,d)$ corresponds to the number of full dimensional cells of the arrangement of (pseudo-)hyperspheres that belong to M. Since for any subset A and for any oriented matroid M we have $p(M) = p(M^{-A})$ (all reorientations of M have equal probability) the probability of a configuration being acyclic does only depend on n and d. We have:

$$p((C(g) \text{ is acyclic } ; g \in G^R_{n,d}) = \frac{R(n,d)}{2^d}.$$

And especially:

$$p((C(g) \text{ is acyclic } ; g \in G^R_{8,4}) = 0.5$$

This equation was in principle known to Wendel since 1962 [17] who proved that the probability $p_{n,d}$ that n points choosen at random on the boundary of the d-sphere is

$$p_{n,d} = 2^{-n+1} \sum_{i=0}^{d-1} \binom{n-1}{i}.$$

The concept of convex hulls has been generalized to oriented matroids in [13]. We will give a short outline of the main ideas. Let $\mathcal{O}(M)$ denote the set of cocircuits of an oriented matroid M. Two cocircuits $X, Y \in \mathcal{O}(M)$ will be called compatible if $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \emptyset$ The union $X \cup Y$ of two compatible cocircuits $X, Y \in \mathcal{O}(M)$ is defined by $(X \cup Y)^+ = X^+ \cup Y^+$ and $(X \cup Y)^- = X^- \cup Y^-$. The *cocircuit span* is the closure of $\mathcal{O}(M)$ under the following hull operator:

$$h(\mathcal{O}') = \mathcal{O}' \cup \bigcup_{X,Y \in \mathcal{O}'; X,Y \text{ compatible}} \{X \cup Y\}$$

So we have

$$span(\mathcal{O}(M)) = \mathcal{O}(M) \cup h(\mathcal{O}(M)) \cup h(h(\mathcal{O}(M))) \cup \ldots$$

The dimension $dim(X)$ of a signed support $X \in span(\mathcal{O}(M))$ is defined by $dim(X) = d - 1 - |X^0|$. The signed supports of dimension $d-1$ are called *full dimensional*. For $X, Y \in span(\mathcal{O}(M))$ we say that X conforms Y (written $X \leq Y$) if X and Y are compatible and $Y^0 \subseteq X^0$. The lattice $(span(\mathcal{O}(M)), \leq$ is isomorphic to the incidence structure of the cellcomplex of the arrangement of pseudo-hemispheres that is asociated to M. The full dimensional elements of the cocircuit span correspond to the full dimensional cells of the cellcomplex of the arrangement of pseudo-hemispheres. The cocircuits (0-demensional elements) correspond to the vertices of the arrangement. An acyclic uniform oriented matroid M contains the element $Z = (+,+,\ldots,+)$ in its cocircuit span. The lattice of all elements $X \in span(\mathcal{O}(M)$ compatible with Z ordered by the inclusion "$\leq$" is the dual face lattice of the convex hull of the oriented matroid M. The cocircuits correspond to the faces of the convex hull and the $d-2$-dimensional cells correspond to the vertices of the convex hull. In the realizable case this definition gives the usual convex hull of the affine configuration.

Using these relations one can compute the combinatorial type of the convex hull of an oriented matroid. Furthermore one can attain the set $acycl(M)$ of all acyclic reorientations of a given oriented matroid M by

$$acycl(M) = \{M^{-A} | A = X^-; X \text{ is a full dimensional element of } span(\mathcal{O}(M)\}.$$

For a given acyclic oriented matroid M we denote by $conv(M)$ the combinatorial structure of the convex hull of M. If M is not acyclic we define $conv(M) = \emptyset$. Let $rep(n,d)$ denote a set that contains exactly one representative of any reorientation class with n points in rank d. One can determine the probability $p_P^{8,4}$ that the convex hull of 8 random points in R^3 (generated by using an equal distribution on $G_{8,4}^R$ and taking the corresponding affine configurations) is of a certain type P by

$$p_P^{8,4} = \frac{1}{0.5} \sum_{M \in rep(8,4)} \sum_{\substack{A \subseteq E \\ conv(M^{-A}) = P}} \frac{1}{2^n} p(M).$$

Altogether there are 23 types of simplicial convex 3-polytopes of at most 8 points, see [10]. A complete list of the Schlegel diagrams of these types together with their estimated probability is given in the Appendix. Especially, one can derive the probability $p_k^{8,4}$ that the convex hull of 8 random points in R^3 contains exactly k points. We get:

$$p_4^{8,4} \approx 0.19323,\ p_5^{8,4} \approx 0.32687,\ p_6^{8,4} \approx 0.29802,\ p_7^{8,4} \approx 0.15043,\ p_8^{8,4} \approx 0.03145.$$

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APPENDIX 1

ESTIMATED PROBABILITIES FOR THE 23 SIMPLICIAL CONVEX 3-POLYTOPES WITH UP TO 8 VERTICES