CHAPTER 23

Extremal Graph Theory

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Introduction

In extremal graph theory one explores in the relations between various graph invariants like order, size, connectivity, chromatic number, diameter, radius, clique number, minimal and maximal degrees, the circumference, the genus. More generally, one is interested in the values of these invariants ensuring that a graph having a certain property has another given property as well. Let us give two examples. Given a graph F, determine $\mathrm{ex}(n; F)$, the maximal number of edges in a graph of order n that does not contain F, the *forbidden graph*, as a subgraph. Given two properties of graphs, $\mathcal{P}$ and $\mathcal{Q}$, say, a number of graph invariants $f_1, \ldots, f_k$, and a natural number n, determine the set $A(n) = \{(a_1, \ldots, a_k)$: if a graph G of order n with $f_i(G) = a_i$, $i = 1, \ldots, k$, has property $\mathcal{P}$ then it also has property $\mathcal{Q}\}$.

The first of these is the classical extremal problem which, though important, is rather narrow; the second problem, on the other hand, is perhaps too broad a problem to be rightly claimed as a genuine extremal problem, since most problems in graph theory could be formulated in this way. In practice, one stays away from both extremes by considering a problem in graph theory to be an extremal problem if its "natural" formulation asks for some best possible inequalities among various graph invariants. However, in this chapter we shall take a rather narrow view of extremal problems, mostly for lack of space and also because several problems belonging to extremal graph theory are considered in other chapters of this volume, in chapters on Ramsey theory, Hamilton cycles, colouring, connectivity, matching, etc.

In a typical extremal problem, given a property $\mathcal{P}$ and an invariant ϕ for a class $\mathcal{G}$ of graphs, we wish to determine the least value f for which every graph G in $\mathcal{G}$ with $\phi(G) > f$ has property $\mathcal{P}$. The graphs in $\mathcal{G}$ without property $\mathcal{P}$ and satisfying $\phi(G) = f$ are the *extremal graphs* for the problem. More often than not, $\mathcal{G}$ consists of graphs of the same order n, namely $\mathcal{G} = \{G \in \mathcal{H}: |G| = n\}$, where $\mathcal{H}$ is a class of graphs, and so f is considered to be a function of n, determined by ϕ and $\mathcal{H}$. This function $f(n)$ is the *extremal function* for the problem.

A short review like this is easily overcrowded with a host of results. In order to avoid this, in section 1 we shall study the classical extremal problem, the problem of forbidden subgraphs, at a leisurely pace, giving some of the simpler proofs. The other sections are considerably shorter and are intended to provide the reader with only glimpses of the topics. Our aim is to give the flavour of the subject rather than overwhelm the reader with results. This review is based mostly on Bollobás (1978a) and an update of that book, Bollobás (1986).

1. Forbidden subgraphs

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a family of graphs of order at most n: the family of *forbidden graphs*. Write $\mathrm{ex}(n; \mathcal{F}) = \mathrm{ex}(n; F_1, \ldots, F_k)$ for the maximal size of a graph of order n containing no forbidden graph F_i, i.e., containing no subgraph isomorphic to a forbidden graph F_i. In this section we shall take $\mathscr{F}$ to be a fixed family, independent of n, and we are mostly interested in the asymptotic value of $\mathrm{ex}(n; \mathscr{F})$ as $n \to \infty$.

1.1. *Turán's theorem and its extensions*

One of the earliest substantial theorems in graph theory is due to Turán (1941) and it concerns the function $\mathrm{ex}(n; K_r)$, where K_r is the complete graph of order r. Turán's theorem was not only the starting point of extremal graph theory but it also signalled the birth of graph theory as an active subject. Although Mantel (1907) proved that $\mathrm{ex}(n; K_3) = \lfloor n^2/4 \rfloor$, Turán was the first to study $\mathrm{ex}(n; K_r)$ for all r.

Given $1 \leq s \leq n$, denote by $T_s(n)$ the complete s-partite graph with $\lfloor n/s \rfloor$, $\lfloor (n+1)/s \rfloor, \ldots, \lfloor (n+s-1)/s \rfloor$ vertices in the various classes. Thus $T_s(n)$ is the unique complete s-partite graph of order n whose classes are as equal as possible. Equivalently, it is also the unique s-partite graph of order n whose size is as large as possible. The graph $T_s(n)$ is the *s-partite Turán graph of order n*. Denote the *size*, i.e., the number of edges, of $T_s(n)$ by $t_s(n)$:

$$t_s(n) = \binom{n}{2} - \sum_{i=1}^{s} \binom{n_i}{2} = \sum_{1 \leq i < j \leq s} \left\lfloor \frac{n+i-1}{s} \right\rfloor \left\lfloor \frac{n+j-1}{s} \right\rfloor,$$

where $n_i = \lfloor (n+i-1)/s \rfloor$ is the number of vertices in the ith smallest class. In particular, $t_2(n) = \lfloor n^2/4 \rfloor$.

An $(r-1)$-partite graph does not contain a K_r; in particular, $T_{r-1}(n)$ does not contain a K_r. Consequently, $\mathrm{ex}(n; K_r) \geq t_{r-1}(n)$. Turán (1941) (see also Turán 1954) proved that, in fact, we have equality, and $T_{r-1}(n)$ is the only extremal graph.

Theorem 1.1.1. *Let $r \geq 2$. Then $\mathrm{ex}(n; K_r) = t_{r-1}(n)$ and $T_{r-1}(n)$ is the only extremal graph: it is the only graph of order n and size $t_{r-1}(n)$ that contains no complete graph of order r.*

Proof. The graph $T_{r-1}(n)$ is a maximal K_r-free graph: it contains no K_r and if we join two vertices belonging to the same class of $T_{r-1}(n)$ then these two vertices, together with $r-2$ vertices, one from each of the other classes, form a K_r. Hence it suffices to prove the second assertion: if G has order n, size $t_{r-1}(n)$, and contains no K_r, then G is (isomorphic to) $T_{r-1}(n)$.

The structure of $T_{r-1}(n)$ is ideal for proving this by induction on n. Indeed, given that we have $t_{r-1}(n)$ edges, the vertices in $T_{r-1}(n)$ have as equal degrees as possible: the minimal degree is $\delta_{r-1}(n) = \lfloor 2t_{r-1}(n)/n \rfloor = n - \lfloor (n+r-2)/(r-1) \rfloor = n - \lceil n/(r-1) \rceil$ and the maximal degree is $\Delta_{r-1}(n) = \lceil 2t_{r-1}(n)/2 \rceil = n - \lfloor n/(r-1) \rfloor$. Furthermore, if we delete a vertex x of minimal degree from $T_{r-1}(n)$ then we obtain $T_{r-1}(n-1)$. In particular, $t_{r-1}(n) - \delta_{r-1}(n) = t_{r-1}(n-1)$. Finally,

as $\delta_{r-1}(n) = n - 1 - \lfloor (n-1)/(r-1) \rfloor$, the vertex x is joined to all vertices of $T_{r-1}(n-1)$ except to the vertices in a smallest class.

Let us see then the proof by induction on n. For $n \leq r-1$ there is nothing to prove so let us assume that $n \geq r$ and the assertion holds for smaller values of n. Let G be a graph of order n and size $t_{r-1}(n)$ that does not contain a K_r. Let $x \in G$ be a vertex of minimal degree: $d(x) = \delta(G) \leq \lfloor 2e(G)/n \rfloor = \lfloor 2t_{r-1}(n)/n \rfloor = \delta_{r-1}(n)$. Set $H = G - x$. Then $e(H) = e(G) - d(x) \geq t_{r-1}(n) - \delta_{r-1}(n) = t_{r-1}(n-1)$. Since H contains no K_r, by the induction hypothesis H is $T_{r-1}(n-1)$ and $d(x) = \delta_{r-1}(n)$. The vertex x cannot be joined to $r-1$ vertices in distinct classes of $H = T_{r-1}(n-1)$ because then these r vertices would form a K_r. Consequently $T_{r-1}(n-1)$ has a class, no vertex of which is joined to x. But then this has to be a smallest class and x has to be joined to all the vertices in all the other classes. Therefore G is precisely $T_{r-1}(n)$. □

The proof above is not so much about graphs not containing a complete graph of order r but about the unusual ease with which $T_{r-1}(n)$ can be produced from $T_{r-1}(n-1)$. Let us give a slightly different slant to the proof of the induction step above. Since the degrees of the vertices of $T_{r-1}(n)$ are as equal as possible, given the number of edges, and since $e(G) = t_{r-1}(n)$, there is a vertex x in G with $d(x) \leq \delta_{r-1}(n) = \delta(T_{r-1}(n))$. Then, by the induction hypothesis, $H = G - x$ must be $T_{r-1}(n-1)$ and $d(x) = \delta_{r-1}(n)$. If the vertices not joined to x form a (smallest) class of $T_{r-1}(n-1)$ then we are done. Otherwise pick a vertex y in $T_{r-1}(n-1)$ which is not joined to x. Then y has degree $\delta_{r-1}(n)$ in G so, by the induction hypothesis, $G - y$ is also $T_{r-1}(n-1)$. But that is clearly not the case because, for example, $G - y$ contains a K_r.

This version of the proof of the induction step implies the following extension of Theorem 1.1.1.

Theorem 1.1.2. *Let $F_1, \ldots, F_k$ be graphs of order at most t, and let s be such that no $T_s(n)$ contains any of the F_i. Suppose $n_0 \geq t$ is such that $\mathrm{ex}(n_0; F_1, \ldots, F_k) = t_s(n_0)$ and $T_s(n_0)$ is the only extremal graph. Then the same assertion holds for every $n \geq n_0$: $\mathrm{ex}(n; F_1, \ldots, F_k) = t_s(n)$ and $T_s(n)$ is the only extremal graph.*

If we do not care about the uniqueness of the extremal graph $T_{r-1}(n)$ in Theorem 1.1.1, then all we need for the proof is that every graph of order $n \geq r+1$ and size $t_{r-1}(n)+1$ has minimal degree at most $\delta_{r-1}(n)$. This observation shows that if G is a graph of order n and size $t_{r-1}(n)+1$ then for every n', $r+1 \leq n' \leq n$, the graph G contains a subgraph of order n' and size at least $t_{r-1}(n')+1$. In particular, as shown by Dirac (1963), every graph of order $n \geq r+1$ and size $t_{r-1}(n)+1$ contains not only a K_r but also a K_{r+1}^-, a complete graph of order $r+1$ from which an edge has been deleted.

This observation can be carried over to greater excess size over $t_s(n)$. A graph G of order $n \geq (2q-1)s+2$ and size $t_s(n)+q$ has minimal degree at most $\delta_s(n)$ so G has a subgraph of order $n-1$ and size $t_s(n-1)+q$. This implies the following result.

Theorem 1.1.3. *Let $s \geq 2$, $q \geq 1$, $n_0 \geq (2q-1)s+2$ and let $F_1, \ldots, F_k$ be graphs such that* $\mathrm{ex}(n_0; F_1, \ldots, F_k) \leq t_s(n_0) + q$. *Then* $\mathrm{ex}(n; F_1, \ldots, F_k) \leq t_s(n) + q$ *for all* $n \geq n_0$.

Let us return to Turán's Theorem 1.1.1. This result claims that the size of a graph G of order n not containing a K_r is dominated by the size of an $(r-1)$-partite graph H of order n. Erdős (1970) proved that we can guarantee that this domination holds at every vertex: the edges of G can be rearranged and, perhaps, some more edges can be added to the graph in such a way that the resulting graph H is $(r-1)$-partite and every vertex is incident with at least as many edges in H as in G. As so often in mathematics (especially in combinatorics), the achievement is the *discovery* of this beautiful fact: the proof is straightforward.

Theorem 1.1.4. *Let G be a graph not containing a K_r, $r \geq 2$. Then there is an $(r-1)$-partite graph H with vertex set $V(H) = V(G) = V$ such that $d_G(x) \leq d_H(x)$ for every $x \in V$. Furthermore, H can be chosen to satisfy $e(G) < e(H)$, i.e., $d_G(x) < d_H(x)$ for at least one vertex x, unless G is a complete $(r-1)$-partite graph with $r-1$ non-empty classes.*

Proof. We apply induction on r. The assertion is obvious for $r = 2$, so we pass to the induction step. Suppose $r > 2$ and the assertion holds for smaller values of r. Let $v \in V$ be a vertex of maximal degree in G: $d_G(z) = \Delta(G)$, and let $W = \Gamma(v)$ be the set of neighbours of v. Then $\tilde{G} = G[W]$, the graph induced by W, does not contain a K_{r-1}. Hence, by the induction hypothesis, there is an $(r-2)$-partite graph $\tilde{H}$ with vertex set W such that $d_{\tilde{G}}(w) \leq d_{\tilde{H}}(w)$ for every $w \in W$.

Let us construct an $(r-1)$-partite graph H with vertex set V from $\tilde{H}$ by joining all vertices in $V \setminus W$ to all vertices in W. It is easily seen that $d_G(x) \leq d_H(x)$ for every $x \in V$. Furthermore, it is easily seen that if $\tilde{G}$ is a complete $(r-2)$-partite graph and $d_G(x) = \Delta(G)$ for every $x \in V \setminus W$ then G is a complete $(r-1)$-partite graph. □

Since $T_{r-1}(n)$ is the *unique* $(r-1)$-partite graph of order n and maximal size, Theorem 1.1.4 implies Theorem 1.1.1.

Let us say a few words about a natural extension of the function $\mathrm{ex}(n; \mathcal{F})$. For a graph G and a family $\mathcal{F}$ of graphs, let $\mathrm{ex}(G; \mathcal{F})$ be the maximal number of edges in a subgraph of G that contains no element of $\mathcal{F}$ as a subgraph. Thus, $\mathrm{ex}(n; \mathcal{F}) = \mathrm{ex}(K_n; \mathcal{F})$. It would be unreasonable to expect precise results about the function $\mathrm{ex}(G; \mathcal{F})$ or even $\mathrm{ex}(G; K')$ but, somewhat surprisingly, sharp results can be obtained in the case when G is a random graph $G_{n,p}$ (see Bollobás 1985, and chapter 6). Among other results, Babai et al. (1990) proved that, for a fixed value of p, with probability tending to 1, $\mathrm{ex}(G_{n,p}; K')$ is the maximal number of edges in an $(r-1)$-partite subgraph of $G_{n,p}$. They also conjectured the following result which was proved, a little later, by Frankl and Pach (1988).

Let us say that a graph has *property $P(k, l)$* if any k vertices have at most l common neighbours.

Theorem 1.1.5. *Let t, $r \geq 2$ be fixed integers, and let $0 < c \leq 1 - 1/(r-1)$. Let G be a K_r-free graph with n vertices, having property $P(t, cn)$. Then*

$$e(G) \leq c^{1/t}\left(1 - \frac{1}{r-1}\right)^{1-1/t} n^2/2 + o(n^2).$$

As an easy consequence of this result, one finds that $\mathrm{ex}(G_{n,p}; K_r) = p(1 - 1/(r-1))n^2/2 + o(n^2)$ with probability tending to 1.

1.2. *The number of complete subgraphs*

We know from Turán's theorem that a graph of order greater than $t_{r-1}(n)$ contains at least one K_r, and we know also that it has to contain at least two K_r. Let us go further: given $m > t_{r-1}(n)$, at least how many K_r are in a graph of order n and size m? Even more, if we know that a graph of order n has many K_p subgraphs, what can we say about the minimal number of K_r subgraphs it has to contain?

To formulate this problem precisely, let us introduce some notation. Denote by $k_r(G)$ the number of K_r in a graph G. Thus $k_2(G)$ is just the size of G, the number of edges of G, and Turán's theorem tells us that if G has order n and $k_2(G) > t_{r-1}(n)$ then $k_r(G) \geq 1$. For natural numbers $2 \leq p < r \leq n$ and a real number $x \geq 0$ define

$$k_r(k_p^n \geq x) = \min\{k_r(G^n) \colon G^n \text{ is a graph of order } n \text{ and } k_p(G^n) \geq x\}.$$

What can we say about the function $k_r(k_p^n \geq x)$? As shown by Bollobás (1976a), this function is also closely connected with the Turán graphs $T_2(n), T_3(n), \ldots$. For simplicity, let us suppress the variable n and put $T_q = T_q(n)$. The graph $G = T_{r-1}$ contains no K_r but it has $k_p(T_{r-1})$ complete graphs of order p, so $k_r(k_p^n \geq x) = 0$ for $0 \leq x \leq k_p(T_{r-1})$.

Let $\psi(x)$ be the *maximal convex function* defined on the interval $k_p(T_{r-1}) \leq x \leq \binom{n}{p}$ such that

$$\psi(k_p(T_q)) \leq k_r(T_q) \tag{1}$$

for $q = r-1, r, \ldots, n$. It is easily seen that, in fact, equality holds in (1) for every q. Also, the Turán graph T_q shows that for $x = k_p(T_q)$ we have

$$k_r(k_p^n \geq x) \leq \psi(x). \tag{2}$$

It turns out that $\psi(x)$ is actually a lower bound for $k_r(k_p^n \geq x)$ for all values of x.

Theorem 1.2.1. *Let $2 \leq p < r \leq n$. For $k_p(T_{r-1}) \leq x \leq \binom{n}{p}$ we have*

$$k_r(k_p^n \geq x) \geq \psi(x).$$

In particular, if a graph of order n has at least as many K_p subgraphs as $T_q(n)$ then it also has at least as many K_r subgraphs as $T_q(n)$. Also, if a graph of order n has more K_p subgraphs than $T_{r-1}(n)$ then it contains a K_r.

The last assertion above was first proved by Erdős (1962) and it was rediscovered by Sauer (1971).

Let us state a weaker but more transparent version of Theorem 1.2.1. The bound on the number of triangles given below was conjectured by Nordhaus and Stewart (1963).

Theorem 1.2.2. (i) *Let $n^2/4 \leq m \leq n^2/3$. Then every graph of order n and size m contains at least $n(4m - n^2)/9$ triangles.*

(ii) *Every graph of order n and size m contains at least $n^{r-2}(2(r-1)m - (r-2)n^2)/r^{r-1}$ copies of K_r.*

The bound above on the minimal number of triangles is fairly good: it is certainly best possible for $n = 3n_0$ and $m = n^2/3 = 3n_0^2$. However, when m is not much greater than $t_2(n) = \lfloor n^2/4 \rfloor$ then the estimate is rather crude. How can we construct a graph of order n and size $m = \lfloor n^2/4 \rfloor + l$ which contains few triangles? For $l < n/2$ we can join a vertex in a larger class of $T_2(n)$ to l vertices of the same class to obtain a graph containing precisely $l\lfloor n/2 \rfloor$ triangles. Erdős (1962) conjectured that we can never do better and proved that this is indeed the case if $l < cn$ for some $c > 0$. This conjecture was proved by Lovász and Simonovits (1976, 1983), who also proved a number of results concerning $k_r(k_2^n \geq x)$, the minimal number of complete r-graphs in a graph of order n, with at least x edges.

Theorem 1.2.3. *For $0 < l < n/2$, a graph with n vertices and $t_2(n) + l$ edges contains at least $l\lfloor n/2 \rfloor$ triangles.*

There are a good many results concerning the covering of graphs by complete subgraphs. The first result in this area was proved by Erdős et al. (1966b); this was sharpened by Bollobás (1976a), Chung (1981) and Győri and Kostochka (1979). The following result was conjectured by Erdős and proved by Pyber (1986).

Theorem 1.2.4. *Let G be a graph with n vertices. Then G and its complement can be covered with at most $\lfloor n^2/4 \rfloor + 2$ complete subgraphs. The graph $T_2(n)$ shows that this bound is best possible.*

A considerable extension of the original theorem of Erdős et al. was conjectured by Winkler, and proved by McGuinness (1994).

Theorem 1.2.5. *If maximal cliques are removed one by one from a graph with n vertices, then the graph will be empty after at most $n^2/4$ steps.*

In fact, Winkler made a stronger conjecture as well, which is still open: if

maximal cliques are removed one by one from a graph with n vertices, then the graph will be empty after the sum of the number of vertices in the cliques has reached $n^2/2$.

1.3. *Complete bipartite graphs*

Let us turn to the analogue of the Turán problem for bipartite graphs. Given natural numbers m, n, s and t, what is the maximal size of an m by n *bipartite graph* not containing a $K(s, t)$, a complete s by t bipartite graph? Denote this maximum by $z(m, n; s, t)$ and put $z(n; t) = z(n, n; t, t)$. Zarankiewicz (1951) asked this question for $s = t = 3$ and $m = n = 4$, 5, 6 and the general problem has also become known as the *problem of Zarankiewicz*. The similarity with Turán's problem is, unfortunately, only superficial: for the general function $z(m, n; s, t)$ there is no beautiful extremal graph and we are far from being able to determine even the order of $z(n; t)$ for a fixed (but large) value of t.

It is worth reformulating the Zarankiewicz problem in terms of 0–1 matrices: At most how many 1s can a 0–1 matrix of m rows and n columns contain if it has no s by t submatrix all whose entries are 1s?

The following rather trivial lemma is just about the most one can say about the general function $z(m, n; s, t)$. As, trivially, $z(m, n; 1, t) = m(t-1)$ for $1 \leq t \leq n$, it is sufficient to consider the case $2 \leq s \leq m$, $2 \leq t \leq n$.

Lemma 1.3.1. *Let m, n, s, t, r and k be integers, $2 \leq s \leq m$, $2 \leq t \leq n$, $0 \leq r \leq m$ and let G be an m by n bipartite graph of size $z = my = km + r$ without a $K(s, t)$. Then*

$$m\binom{y}{t} \leq (m-r)\binom{k}{t} + r\binom{k+1}{t} \leq (s-1)\binom{n}{t}. \tag{1}$$

Proof. Let (V_1, V_2) be the bipartition of G and let $V_1 = \{x_1, \ldots, x_m\}$, $d(x_i) = d_i$. Let us call a set $\{xy_1, xy_2, \ldots, xy_t\}$ of t edges of G incident with the same vertex a *claw*; furthermore, x is the *centre* of the claw and the t-set $\{y_1, \ldots, y_t\}$ is the *base*.

The graph G has $\sum_{i=1}^m \binom{d_i}{t}$ claws since there are $\binom{d_i}{t}$ claws with centre x_i. On the other hand, each t-subset of V_2 is the base of at most $s-1$ claws since G contains no $K(s, t)$. Therefore G has at most $(s-1)\binom{n}{t}$ claws and so

$$\sum_{i=1}^{m} \binom{d_i}{t} \leq (s-1)\binom{n}{t}. \tag{2}$$

Since $\sum_{i=1}^m d_i = z = km + r$, $0 \leq r < m$, and $\binom{u}{t}$ is a convex function of u for $u \geq$ … inequality (2) implies (1). □

Theorem 1.3.2. *Let m, n, s, t be natural numbers, $2 \leq s \leq m$, $2 \leq t \leq n$. Then*

$$z(m, n; s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.$$

Proof. Let G be an extremal graph for $z(m, n; s, t)$. Set $y = z(m, n; s, t)/m$. Then, since $y < n$, by Lemma 1 we have

$$m(y-(t-1))^t < (s-1)(n-(t-1))^t . \qquad \square$$

For a fixed value of $t \geq 2$, Theorem 1.3.2 implies that

$$z(n;t) \leq (t-1)^{1/t} n^{2-t/t} + O(n) \tag{3}$$

and it is conjectured that (3) is essentially best possible. To be precise, it is conjectured that

$$\lim_{n\to\infty} z(n;t)/n^{2-1/t} = c_t > 0 \tag{4}$$

for every $t \geq 2$. So far, the only value of t for which (4) is known to hold is $t = 2$. In fact, Kővári et al. (1954) and Reiman (1958) determined $z(n;2)$ for infinitely many values of n, but there is no $t \geq 3$ for which $z(n;t)$ is known for infinitely many values of n.

Theorem 1.3.3. (i) $z(n;2) \leq (n/2)\{1 + \sqrt{4n-3}\}$ *for all* $n \geq 2$.

(ii) *Let q be a prime power and let $n = q^2 + q + 1$. Then*

$$z(n;2) = \frac{n}{2}\{1 + \sqrt{4n-3}\} = (q-1)(q^2+q+1) .$$

(iii) $\lim_{n\to\infty} z(n;2)/n^{3/2} = 1$.

Proof. (i) Let G be an extremal graph for $z(n;2)$ and let the notation be as in the proof of Lemma 1.3.1. By inequality (2),

$$\binom{n}{2} \geq \sum_{i=1}^{n} \binom{d_i}{2}$$

$$n^2 - n \geq \sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i \geq \Big(\sum_{i=1}^{n} d_i\Big)^2 / n - \sum_{i=1}^{n} d_i = z^2/n - z .$$

This implies the required inequality.

(ii) From the proof of part (i) we see that equality holds in (i) if and only if (1) every vertex in G has the same degree d, (2) for every two vertices in V_2 there is precisely one vertex in V_1 joined to both, and (3) for every two vertices in V_2 there is precisely one vertex in V_1 joined to both. This means that the graph G can be considered as a finite projective plane: V_1 is the set of points, V_2 is the set of lines and $x \in V_1$ is joined to $y \in V_2$ iff the point x is incident with the line y. Now if q is a prime power then there is a projective plane of order q, that is with $n = q^2 + q + 1$ points and lines.

(iii) Since for every sufficiently large natural number n, there is a prime between $n - n^{2/3}/10$ and n, the assertion follows from (i) and (ii). $\square$

A somewhat weaker form of conjecture (4) is that $\overline{\lim}_{n\to\infty} z(n;t)/n^{2-1/t} > 0$. In addition to $t = 2$, this is known for $t = 3$. Brown (1966) proved that $\overline{\lim}_{n\to\infty} z(n;3)/n^{2-1/3} \geq 1$ by making use of the 3-dimensional affine space $AG(3, p)$ over the finite field of order p. However, for a general $t \geq 4$ all we know is that

$$\underline{\lim}_{n\to\infty} z(n;t)/n^{2-2/(t+1)} \geq 1 - (t!)^{-2} . \tag{5}$$

This is proved by making use of random graphs (see Bollobás 1979, p. 127). The gap between the upper bound, $n^{2-1/t}$, and the lower bound, $n^{2-2/(t+1)}$, is alarmingly large; as stated above, it is very likely that the upper bound gives the correct value.

The functions $\mathrm{ex}(n; K(s,t))$ and $z(n, n; s, t)$ are intimately connected; ir particular, for fixed values of s and t they have the same order. It is easily seer that

$$2\,\mathrm{ex}(n; K(s,t)) \leq z(n, n; s, t) \leq \mathrm{ex}(2n; K(s,t)) . \qquad (6$$

Indeed, given a graph G of order n and size $m = \mathrm{ex}(n; K(s,t))$, construct an n b n bipartite graph H as follows. Take two disjoint copies of $V(G)$, say V_1 and V_2 and join $x' \in V_1$ to $y'' \in V_2$ iff $xy \in E(G)$, where x and y are the vertices in $V(G$ corresponding to x' and y''. Then H has $2m$ edges and contains no $K(s,t)$ (and n $K(t,s)$, for that matter) so the first inequality in (6) holds. The second inequalit is trivial.

Combining inequality (6) with Theorem 1.3.2, and noting the analogue of (5) we have the following assertion.

Theorem 1.3.4. *If* $2 \leq s < n$ *then*

$$\begin{aligned} \tfrac{1}{2}(1-(s!)^{-2})n^{2-2/(s+1)} &\leq \mathrm{ex}(n; K(s,s)) \\ &\leq \tfrac{1}{2}(s-1)^{1/s}(n-s+1)n^{1-1/s} + \tfrac{1}{2}(s-1)n \\ &< n^{2-1/s} + \frac{s-1}{2} n . \end{aligned}$$

As (6) holds and we do not know the order of $z(n, n; t, t)$ for $t \geq 4$, neither c we know the order of $\mathrm{ex}(n; K(s,s))$ for $s \geq 4$. However, we do know th $\mathrm{ex}(n; K(2,2))$ has order $n^{3/2}$ and $\mathrm{ex}(n; K(3,3))$ has order $n^{5/3}$. In the case $K(2,2)$ we can do considerably better. As in the problem of determini $\mathrm{ex}(n; K(2,2))$ we do not care where the classes of $K(2,2)$ are, it is more natui to write C_4 instead of $K(2,2)$, indicating that $K(2,2)$ is just a *4-cycle quadrilateral*.

Inequality (5) and Theorem 1.3.3 (ii) imply that

$$\mathrm{ex}(n; C_4) \leq \frac{n}{4}\{1 + \sqrt{4n-3}\} . \qquad ($$

Erdős et al. (1966a) noticed that certain graphs constructed by Erdős and Ré

(1962) show that (6) is asymptotically best possible. The same assertion was proved independently by Brown (1966).

Theorem 1.3.5. *Let q be a prime power. Then*

$$\tfrac{1}{2}q(q+1)^2 \le \mathrm{ex}(q^2+q+1; C_4) \le \tfrac{1}{2}q(q+1)^2 + \frac{q+1}{2}. \tag{8}$$

Furthermore,

$$\lim_{n\to\infty} \mathrm{ex}(n; C_4)/n^{3/2} = \tfrac{1}{2}. \tag{9}$$

Proof. The second inequality is precisely inequality (6) for $n = q^2+q+1$. Let us prove the first inequality by describing the graph G_q constructed by Erdős and Rényi (1962).

The vertex set $V(G_q)$ is the set of q^2+q+1 points of the finite projective plane $PG(2, q)$ over the finite field of order q. A point is joined to all the points on its *polar* with respect to the conic $x^2+y^2+z^2=0$. Thus two points (a, b, c) and (α, β, γ) are joined iff $a\alpha + b\beta + c\gamma = 0$. Then a point not on the conic is joined to $q+1$ points, i.e., to all the lines on its polar, while each of the $q+1$ points on the conic is joined to q points, namely to the points on its polar except itself. Hence G_q has $\frac{1}{2}\{q^2(q+1) + (q+1)q\} = \frac{1}{2}q(q+1)^2$ edges.

The graph G_q does not contain a quadrilateral since any two lines meet in exactly one point so every vertex is determined by any two of its neighbours.

Relation (9) follows as Theorem 1.3.3 (iii). □

The bounds in (7) are tantalizingly close. The only reason why the graph G_q is not ideal for the problem is that it has *absolute points*, i.e., points lying on their polars. These $q+1$ points are joined to only q points, instead of $q+1$, as all the others. If we could avoid these absolute points by choosing a more suitable polarity then we would achieve the upper bound in (7). However, this is not to be: Baer (1946) proved that every polarity of a finite projective plane of order q has at least $q+1$ absolute points. Thus the Erdős–Rényi graph G_q cannot be made to have more edges by choosing a different polarity.

In view of this fact it is not too surprising that the way to improve (8) is to reduce the upper bound. This was achieved by Füredi (1983) (see also the remarks at the end of that paper) who thereby determined $\mathrm{ex}(n; C_4)$ for infinitely many values of n.

Theorem 1.3.6. *For every natural number q we have*

$$\mathrm{ex}(q^2+q+1; C_4) \le \tfrac{1}{2}q(q+1)^2.$$

In particular, if q is a prime power then

$$\mathrm{ex}(q^2+q+1; C_4) = \tfrac{1}{2}q(q+1)^2.$$

What happens if we forbid not only C_4 but C_5 as well? The projective plane graph in Theorem 1.3.3 (ii) contains no C_4, and as it is bipartite, it contains no C_5 either. Hence if $n = 2(q^2+q+1)$ for some prime power q then $\mathrm{ex}(n; C_4, C_5) \ge (q-1)(q^2+q+1)$, so $\mathrm{ex}(n; C_4, C_5) \ge (n/2)^{3/2} + o(n^{3/2})$ for all n. Erdős and Simonovits (1982) proved that this inequality is, in fact, an equality.

Theorem 1.3.7. $\mathrm{ex}(n; C_4, C_5) = (n/2)^{3/2} + o(n^{3/2})$.

It would be of interest to decide whether $\mathrm{ex}(n; C_4, C_5) = (q-1)(q^2+q+1)$ if q is a prime power and $n = 2(q^2+q+1)$.

1.4. The fundamental theorem of extremal graph theory

For $r \ge 3$, the Turán graph $T_{r-1}(n)$ has $t_{r-1}(n) = (r-2/2(r-1))n^2 + O(n)$ edges and contains no K_r. On the other hand, every graph of order n and size $t_{r-1}(n)+1$ has a K_r, in fact, several K_r. Furthermore, Theorem 1.2.2 implies that if $0 < \varepsilon < 1/2r(r-1)$ then every graph of order n and size $((r-2)/2(r-1)+\varepsilon)n^2$ contains at least $(2(r-1)\varepsilon/r^{r-1})n^r$ copies of K_r. Thus there is a sudden jump when the size reaches $t_{r-1}(n)$.

Although this sudden jump is quite startling, Erdős and Stone (1946) proved that a considerably more important change takes place when the size becomes significantly greater than $t_{r-1}(n)$. This result, which deserves to be called the *fundamental theorem of extremal graph theory*, states that for every $r \ge 3$ and $\varepsilon > 0$, there is a function $s = s(n)$ such that $s(n) \to \infty$ as $n \to \infty$, and every graph of order n and size $(((r-2)/2(r-1)) + \varepsilon)n^2$ contains a $K_r(s) = K(s, s, \ldots, s) = T_r(rs)$, a complete r-partite graph with s vertices in each of the classes. Thus we not only get a complete r-partite graph with one vertex in each class, as claimed by Turán's theorem, but we can guarantee even a complete r-partite graph with $s(n)$ vertices in each class, where $s(n) \to \infty$ as $n \to \infty$.

The assertion above does make sense for $r = 2$ as well although in that case Turán's theorem is completely trivial: every graph of order n and size at least εn^2, $0 < \varepsilon < \frac{1}{2}$, contains a complete bipartite graph with at least $s(n)$ vertices in each class, where $s(n) \to \infty$ as $n \to \infty$. This assertion is immediate from Theorem 1.3.4: if $0 < \varepsilon < \frac{1}{2}$ and $0 < c < \log 1/2\varepsilon$ are fixed then the assertion is true with $s(n) = \lceil c \log n \rceil$, provided n is sufficiently large.

Let us state then the fundamental theorem of extremal graph theory, proved by Erdős and Stone (1946).

Theorem 1.4.1. *Let $r \ge 2$ and $\varepsilon > 0$ be fixed. Then there is a function $s = s(n)$, with $\lim_{n\to\infty} s(n) = \infty$, such that every graph of order n and size at least $((r-2)/2(r-1)+\varepsilon)n^2$ contains a $K_r(s)$.*

As we are interested in the growth of $s(n)$, let us introduce the following notation. For $r \geq 2$ and $0 < \varepsilon < 1/2(r-1)$ define

$$s_{r,\varepsilon}(n) = \min\left\{t\colon \text{every graph of order } n \text{ and size at least } \left(\frac{r-2}{2(r-1)} + \varepsilon\right)n^2 \text{ contains a } K_r(t)\right\}.$$

Erdős and Stone (1946) proved that $s_{r,\varepsilon}(n) \geq (l_{r-1}(n))^{1/2}$ if n is sufficiently large, where $l_{r-1}(n)$ is the $r-1$ times iterated logarithm of n. Furthermore, Erdős and Stone conjectured that the order of $s_{r,\varepsilon}(n)$ is $l_{r-1}(n)$. Later Erdős (1967) announced that $s_{r,\varepsilon}(n) > c(\log n)^{1/(r-1)}$ for some constant $c > 0$ and sufficiently large n.

Rather unexpectedly, $s_{r,\varepsilon}(n)$ turns out to be much larger than these lower bounds. The true order of $s_{r,\varepsilon}(n)$ was determined by Bollobás and Erdős (1973).

Theorem 1.4.2. *Let $r \geq 2$ and $0 < \varepsilon < 1/2(r-1)$. Then there are positive constants $c_1 = c_1(r, \varepsilon)$ and $c_2 = c_2(r, \varepsilon)$ such that*

$$c_1 \log n < s_{r,\varepsilon}(n) < c_2 \log n\,. \tag{1}$$

In particular, every graph of order n and size at least $((r-2)/2(r-1) + \varepsilon)n^2$ contains a complete r-patite graph with at least $c_1 \log n$ vertices in each class.

How do c_1 and c_2 depend on r and ε? As pointed out by Bollobás and Erdős (1973), the constant c_2 can be chosen to be $5/\log(1/\varepsilon)$, provided n is sufficiently large. This can be seen by a simple application of random graphs. What about c_1? Improving inequality (1), Bollobás et al. (1976) proved that one can take $c_1 = c/r \log(1/\varepsilon)$ for some absolute constant $c > 0$, provided n is sufficiently large. Finally, Chvátal and Szemerédi (1981) showed that this is true without the factor r.

Theorem 1.4.3. *There is an absolute constant $c > 0$ such that*

$$\frac{c}{\log(1/\varepsilon)} \log n < s_{r,\varepsilon}(n) < \frac{5}{\log(1/\varepsilon)} \log n$$

if $r \geq 2$, $0 < \varepsilon < 1/2(r-1)$ and n is sufficiently large

First we shall sketch a proof of Theorem 1.4.2 and then we shall return to Theorem 1.4.3. As we remarked above, the upper bound in (1) is very easy: it follows from a straightforward application of random graphs. To prove the lower bound, we shall need the following lemma.

Lemma 1.4.4. *Let G be a graph of order n that contains no $K_{r+1}(s)$ but contains a $K_r(q)$, say $\tilde{K}$. Then G has at most*

$$((r-1)q + s)n + 2qn^{1-1/s}$$

edges joining $\tilde{K}$ to $G - \tilde{K}$.

Proof. As in the proof of Lemma 1.3.1, we define a *claw with centre $x \in G - \tilde{K}$* as the set of r edges incident with x such that precisely s of these edges join x to each of the r classes of $\tilde{K}$. It is easily checked that if $x \in G - \tilde{K}$ is joined to $(r-1)q + d$ vertices in $\tilde{K}$ then there are at least $\binom{q}{s}^{r-1}\binom{d}{s}$ claws with centre x. Hence if there are $(r-1)qn + D > (r-1)qn + sn$ edges joining $G - \tilde{K}$ to $\tilde{K}$ then there are at least $n\binom{q}{s}^{r-1}\binom{D/n}{s}$ claws in G.

Since G contains no $K_{r+1}(s)$, there are at most $s-1$ claws with the same *base*, the same set of vertices joined to the centre. As there are $\binom{q}{s}^r$ possible bases, G contains at most $(s-1)\binom{q}{s}^r$ claws. Consequently,

$$n\binom{D/n}{s} \leq (s-q)\binom{q}{s}.$$

Hence

$$D \leq n^{1-1/s}(s-1)^{1/s}q \leq 2n^{1-1/s}q\,,$$

proving the lemma. □

Armed with this lemma, we shall prove the main part of Theorem 1.4.2, the lower bound on $s_{r,\varepsilon}(n)$. To be precise, we shall prove the following assertion.

Theorem 1.4.2′. *Let $r \geq 2$, $0 < \varepsilon < 1/2(r-1)$ and $0 < \gamma_r < (r-1)!\varepsilon^{r-1}/\log(8/\varepsilon)$. Then if n is sufficiently large, every graph of order n and size at least*

$$\left(\frac{r-2}{2(r-1)} + \varepsilon\right)n^2$$

contains a $K_r(s)$ where $s = \lfloor \gamma_r \log n \rfloor$.

Proof. Let us add to Theorem 1.4.2 a trivial assertion concerning the case $r = 1$: for $\varepsilon > 0$, every graph of sufficiently large order contains a $K_1(s)$ for $s = \lfloor \gamma_1 \log n \rfloor$ where $\gamma_1 = 2/\varepsilon$.

Suppose then that the result is true for $r \geq 1$ but fails for $r+1$: there is a constant γ'_{r+1}, $0 < \gamma'_{r+1} < r!\varepsilon^r/\log(8/\varepsilon)$, such that for every n_0 there is a graph G_1 of order $n_1 \geq n_0$ and size at least $((r-1)/2r + \varepsilon)n_1^2$ without a $K_{r+1}(s_1)$, where $s_1 = \lfloor \gamma'_{r+1} \log n_1 \rfloor$. Such a graph G_1 has average degree $((r-1)/r + 2\varepsilon)n_1$ so it contains a subgraph G with $n \geq \frac{1}{2}\varepsilon n_1$ vertices and minimal degree at least $((r-1)/2r + \frac{3}{2}\varepsilon)n$. Let $\gamma'_{r+1} < \gamma_{r+1} < \varepsilon r \gamma_r < r!\varepsilon^r/\log(8/\varepsilon)$. Then, if n is sufficiently large (and that is the case if n_0 is sufficiently large), the graph G contains no $K_{r+1}(s)$, where $s = \lfloor \gamma_{r+1} \log n \rfloor$. However, it does contain a $K_r(q)$, say $\tilde{K}$, where $q = \lfloor \gamma_r \log n \rfloor$. By Lemma 1.4.4 there are at most $((r-1)q + s)n + 2qn^{1-1/s}$ edges joining $\tilde{K}$ to $G - \tilde{K}$, so some vertex of $\tilde{K}$ has degree at most

$$rq + \{((r-1)q + s)n + 2qn^{1-1/s}\}/rq\,.$$

Hence

$$\left(\frac{r-1}{r}+\frac{3}{2}\varepsilon\right)n \leq \delta(G) \leq \frac{r-1}{r}n + rq + \frac{sn}{rq} + \frac{2}{r}ns^{1-1/s}.$$

This inequality cannot hold if n is large enough since then $rq < \frac{1}{4}\varepsilon n$, $s/rq < \varepsilon$ and $(2/r)n^{-1/s} < \frac{1}{4}\varepsilon$. This contradiction completes the proof. □

The proof Theorem 1.4.3, given by Chvátal and Szemerédi (1981), is based on a deep and important lemma due to Szemerédi (1978). This result, to be stated below as Theorem 1.4.5 and usually called the *uniform density lemma* or *regularity lemma*, was one of the main tools in the proof of Szemerédi's (1975) theorem, one of the most difficult results in combinatorics, stating that every sequence of integers with positive upper density contains arbitrarily long arithmetic progressions.

For a graph G, and disjoint sets $U, W \subset V(G)$, denote by $e(U, W)$ the number of $U-W$ edges. The *density* of the edges between U and W is

$$d(U, W) = \frac{e(U, W)}{|U||W|}.$$

The pair (U, W) is *ε-uniform* or *ε-regular* if

$$|d(U', W') - d(U, W)| < \varepsilon$$

whenever $U' \subset U$, $W' \subset W$, $|U'| > \varepsilon|U|$ and $|W'| > \varepsilon|W|$.

Theorem 1.4.5. *Given $\varepsilon > 0$ and an integer m, there is an $M = M(\varepsilon, m)$ such that the vertices of every graph of order at least m can be partitioned into classes $V_0, V_1, \ldots, V_k$, where $m \leq k \leq M$, such that $|V_0| \leq |V_1| = |V_2| = \cdots = |V_k|$ and all but at most εk^2 of the pairs (V_i, V_j), $1 \leq i < j \leq k$, are ε-uniform.*

The following two immediate consequences of Theorem 1.4.1 show why the result is called the fundamental theorem of extremal graph theory. In the spirit of the notation used above, for a graph G and a set $U \subset V(G)$ define the *density* $d(U)$ of the subgraph $G[U]$ spanned by U as

$$d(U) = e(G[U]) \Big/ \binom{u}{2},$$

where $u = |U|$. Thus if U spans a complete graph then $d(U) = 1$, if U consists of independent vertices then $d(U) = 0$.

Let G be an infinite graph. Define the *upper density of* G to be

$$\bar{d}(G) = \sup\{\alpha: \text{for every } m > 0 \text{ there is a finite set } U \text{ satisfying } |U| > m \text{ and } d(U) > \alpha\}.$$

Putting it another way, if $\beta > \bar{d}(G)$ then there is an $m > 0$ such that whenever U has at least m vertices then $d(U) < \beta$, and $\bar{d}(G)$ is the smallest such number. Clearly, if G is the empty graph then $\bar{d}(G) = 0$, also, if G contains arbitrarily large complete graphs then $\bar{d}(G) = 1$. What are the possible values of the upper densities? It is rather natural to expect the closed interval to be the set of possible upper densities. Surprisingly, this is not the case.

Theorem 1.4.6. *The set of upper densities of infinite graphs is $\{1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$.*

Proof. Suppose $\bar{d}(G) > 1 - 1/r + \varepsilon$ for some $r \in \mathbb{N}$ and $\varepsilon > 0$. Then G contains a sequence of subgraphs, say $G_1, G_2, \ldots$ such that G_i has order n_i and size at least $((r-1)/r + \frac{3}{4}\varepsilon)\binom{n_i}{2} > ((r-1)/2r + \frac{1}{3}\varepsilon)n_i^2$, and $n_i \to \infty$. By Theorem 1.4.1, each G_i contains a $K_{r+1}(s_i)$, where $s_i \to \infty$. Now $d(K_{r+1}(s_i)) > r/(r+1)$ and the order of $K_{r+1}(s_i)$ tends to ∞, so $\bar{d}(G) \geq r/(r+1)$. □

The other immediate consequence of Theorem 1.4.1 concerns the approximate value of $\mathrm{ex}(n; F_1, F_2, \ldots, F_k)$. As observed by Erdős and Simonovits (1966), Theorem 1.4.1 implies that $\lim_{n\to\infty} \mathrm{ex}(n; F_1, \ldots, F_k)/\binom{n}{2}$ is a very simple function of the family $\{F_1, \ldots, F_k\}$.

Theorem 1.4.7. *Let $F_1, \ldots, F_k$ be fixed non-empty graphs. Set $r = \min_i \chi(F_i) - 1$, i.e., let $r+1$ be the smallest chromatic number of an F_i. Then*

$$\lim_{n\to\infty} \mathrm{ex}(n; F_1, \ldots, F_k) \Big/ \binom{n}{2} = 1 - \frac{1}{r}.$$

Proof. We may assume that $\chi(F_1) = r + 1$. The graph $T_r(n)$ contains no F_i so $\mathrm{ex}(n; F_1, \ldots, F_k) \geq t_r(n) = (1 - 1/r)\binom{n}{2} + (n)$. Hence $\lim_{n\to\infty} \mathrm{ex}(n; F_1, \ldots, F_k)/\binom{n}{2} \geq 1 - 1/r$.

On the other hand, if $\varepsilon > 0$ and n is sufficiently large, then by Theorem 1.4.1 every graph G of order n and size at least $(1 - 1/r + \varepsilon)\binom{n}{2}$ contains a $K_{r+1}(s)$ where $s > |F_1|$. But then $K_{r+1}(s)$ contains F_1 and, therefore, so does G. As this holds for every $\varepsilon > 0$, we have

$$\overline{\lim_{n\to\infty}}\, \mathrm{ex}(n; F_1, \ldots, F_k) \Big/ \binom{n}{2} \leq 1 - \frac{1}{r}. \quad \square$$

Although Theorem 1.4.7 is just an immediate corollary of Theorem 1.4.1, at the first sight it is, nevertheless, very surprising: the crude order of $\mathrm{ex}(n; F_1, \ldots, F_k)$ depends only on the minimal chromatic number of the F_i. In particular, the asymptotic value of $\mathrm{ex}(n; F_1, \ldots, F_k)$ is easily determined if no F_i is bipartite. Of course, this leaves several questions unanswered. What is the error term $\phi(n)$ in $\mathrm{ex}(n; F_1, \ldots, F_k) = ((r-1)/2r)n^2 + \phi(n)$? What is the asymptotic value of $\mathrm{ex}(n; F_1, \ldots, F_k)$ when some F_i is bipartite? We know from section 1. that we are far from being able to answer the third question for an arbitrary family, since we do not even know the asymptotic value of $\mathrm{ex}(n; K_{4,4})$, say, but we shall discuss the first two questions in section 1.5.

Let us note an easy application of Theorem 1.4.7, giving the rough solution of a seemingly intractable problem.

Theorem 1.4.8. *Let $\mathscr{F}$ be the family of graphs of order p and size q. Let* $r = \min\{s: t_{s+1}(p) \geq q\}$. *Then*

$$\lim_{n\to\infty} \mathrm{ex}(n;\mathscr{F})/n^2 = \frac{r-1}{2r}.$$

Proof. Note that $\min\{\chi(F): F \in \mathscr{F}\} = r+1$. □

To conclude this section, we state a weak form of Theorem 1.4.6, as it leads to some deep questions concerning *r*-graphs, i.e., *r*-uniform hypergraphs. Given $r \geq 2$ and $0 \leq \alpha < 1$, we say that α is a *jump value for r-graphs* if there is a $\beta > \alpha$ such that if $\varepsilon > 0$, $m \geq r$ and $n \geq n(\alpha, \varepsilon, m)$ then every *r*-graph with $n \geq n(\alpha, \varepsilon, m)$ vertices and at least $\alpha\binom{n}{r}$ hyperedges contains a subgraph with *m* vertices and at least $\beta\binom{m}{r}$ hyperedges. Note that α is a jump value for graphs if for some $\delta > 0$ the interval $(\alpha, \alpha+\delta)$ contains no upper density of an infinite graph. Hence the following result is immediate from either Theorem 1.4.1 or Theorem 1.4.6.

Theorem 1.4.9. *Every* $0 \leq \alpha < 1$ *is a jump value for graphs.*

Erdős posed the problem of deciding whether the same is true for *r*-graphs. The problem was open for several years and was eventually solved by Frankl and Rődl (1984).

Theorem 1.4.10. *Let* $r \geq 3$ *and* $s > 2r$ *be natural numbers. Then* $1 - s^{1-r}$ *is not a jump value for r-graphs.*

This beautiful and difficult problem leaves open a number of important questions. In particular, it would be interesting to determine the set of jump values for *r*-graphs and the set of upper densities for *r*-graphs.

1.5. The structure of extremal graphs

For a family $\mathscr{F} = \{F_1, \ldots, F_k\}$ of forbidden graphs, denote by $\mathrm{EX}(n;\mathscr{F}) = \mathrm{EX}(n; F_1, \ldots, F_k)$ the *set of extremal graphs of order n*. Thus a graph *G* belongs to $\mathrm{EX}(n;\mathscr{F})$ iff *G* has order *n*, size $\mathrm{ex}(n;\mathscr{F})$ and contains no forbidden graph, i.e., no member of $\mathscr{F}$. Turán's theorem, Theorem 1.1.1, tells us that $\mathrm{EX}(n; K_r) = \{T_{r-1}(n)\}$ for all *r* and *n*, $2 \leq r \leq n$. For a general family $\mathscr{F}$, Theorem 1.4.6, an immediate consequence of the Erdős–Stone theorem, the fundamental theorem of extremal graph theory, gives us the rough order of $\mathrm{ex}(n;\mathscr{F})$. But what is the more precise order of $\mathrm{ex}(n;\mathscr{F})$ for a general family $\mathscr{F}$ and what do extremal graphs look like?

These questions were answered, surprisingly precisely, by Erdős and

Simonovits (1966) and by Simonovits (1968); simpler proofs of the results can found in Bollobás (1978a, pp. 339–345). Here we shall state only of the resul

Theorem 1.5.1. *Let F be a graph with* $\chi(F) = r+1 \geq 3$, *and for* $n = 1, 2, \ldots$ *let be a graph of order n and size* $(1 - 1/r + o(1))\binom{n}{2}$ *not containing F. Then following assertions hold.*

(i) *There is a* $K(p_1, p_2, \ldots, p_r)$, $\sum_{i=1}^r p_i = n$, $p_i = (1+o(1))n/r$, *that can obtained from* G^n *by adding and subtracting* $o(n^2)$ *edges.*

(ii) G^n *contains an r-partite graph of size* $(1 - 1/r + o(1))\binom{n}{2}$.

(iii) G^n *contains an r-partite graph of minimal degree* $(1 - 1/r + o(1))n$.

The result above claims that if a graph G^n not containing *F* has *about* as ma edges as the Turán graph $T_r(n)$, which *trivially* fails to contain *F*, then G^n is *ve close* to the graph $T_r(n)$. For an extremal graph, considerably more is true.

Theorem 1.5.2. *Let* $\mathscr{F} = \{F_1, \ldots, F_k\}$ *be a fixed family of graphs, let* $r+1$ $\min_i \chi(F_i) \geq 2$ *and suppose that* F_1 *has an* $(r+1)$*-colouring in which one of t colour classes contains t vertices. Let* $G^n \in \mathrm{EX}(n;\mathscr{F})$. *Then, as* $n \to \infty$,

$$e(G^n) = \mathrm{ex}(n;\mathscr{F}) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + O(n^{2-1/t}),$$

$$\delta(G^n) = \left(1 - \frac{1}{r} + o(1)\right)n,$$

the vertices of G can be partitioned into r classes such that each vertex is joined to most as many vertices in its own class as in any other class, and for every $\varepsilon >$ *there are at most* $c(\varepsilon, \mathscr{F})$ *vertices joined to at least* εn *vertices of the same clas Furthermore, there are* $O(n^{2-1/t})$ *edges joining vertices belonging to the same clas and each class has* $n/r + O(n^{2-1/t})$ *vertices.*

This result gives us a very good hold on extremal graphs. In fact, the functi $O(n^{2-1/t})$ can be replaced by $O(\mathrm{ex}(n; K(s,t))$ where *s* and *t* are fixed. particular, we have the following better bound on $\mathrm{ex}(n;F)$ in terms of $\mathrm{ex}(m;F$ for some bipartite graph F_0.

Theorem 1.5.3. *Let* $F = F_0 + K_{r-1}(u)$ *where* F_0 *is a bipartite graph. Then*

$$\mathrm{ex}(n;F) \leq \left(1 - \frac{1}{r}\right)\binom{n}{2} + (r + o(1))\mathrm{ex}\left(\left\lfloor \frac{n}{r} \right\rfloor; F_0\right).$$

As an illustration of the power of Theorem 1.5.2, let us present a beautif theorem of Simonovits (1968) giving a complete solution to the forbidde subgraph problem for sK_{r+1}, i.e., for *s* disjoint copies of K_{r+1}, provided *n* sufficiently large.

What is a likely candidate for an extremal graph for sK_{r+1}? If we add $t-$ vertices to the Turán graph $T = T_r(n - t + 1)$ and join these vertices to each oth and to the vertices of T then the obtained graph, $K_{s-1}+T_r(n-t+1)$, has quite a few more (about $(t-1)n/r$ more) edges than $T_r(n)$, the extremal graph for one copy of K_{r+1}, and still fails to contain s disjoint copies of K_{r+1}. Indeed, every K_{r+1} in $K_{t-1}+T_r(n-t+1)$ must contain at least one of the $t-1$ vertices of K_{t-1}. The following theorem of Simonovits (1968) shows that our hunch is essentially correct.

Theorem 1.5.4. *Let r and s be fixed natural numbers, $r \geq 2$. If n is sufficiently large then $K_{t-1}+T_r(n-t+1)$ is the unique extremal graph for tK_{r+1}.*

Proof. Let us apply induction on t. The case $t=1$ is precisely Turán's theorem, so let us pass to the induction step.

Let $G=G^n$ be an extremal graph of order n for tK_{r+1}, and consider the partition $V=V_1 \cup V_2 \cup \cdots \cup V_r$ guaranteed by Theorem 1.5.2. Set $\varepsilon = 1/4tr$. Let us distinguish two cases.

Case (i) *Some vertex x is joined to at least εn vertices in its own class.* Let W_i be a set of $m=\lceil \varepsilon n \rceil$ neighbours of x in V_i. By Theorem 1.5.2 the r-partite subgraph of G spanned by $W_1 \cup W_2 \cup \cdots \cup W_r$ has $(1-1/r+o(1))r^2m^2/2$ edges so, rather trivially (or by Theorem 1.4.1, if we wish to conclude it instantly), it contains a $K_r(s)$ for $s=t(r+1)$, provided n is sufficiently large. But then $G-x$ cannot contain a $(t-1)K_{r+1}$ since any $(t-1)K_{r+1}$ could be extended to a tK_{r+1} so we are done by the induction hypothesis.

Case (ii) *Every vertex is joined to at most εn vertices in its own class.* In this case, our aim is to arrive at a contradiction. As $\delta(G) \geq (1-1/r+o(1))n$, we may assume that every vertex is joined to all but at most $2\varepsilon n$ vertices in the other classes. As in case (i), this implies that for every pair $\{x,y\}$ of vertices in the same class, in particular, for every edge xy joining vertices in the same class, the graph G contains a $K_{r-1}(s)$ for $s=t(r+1)$ such that both x and y are joined to all vertices of this $K_{r-1}(s)$. But then this implies that the graph H obtained from G by deleting all edges joining different classes contains at most $t-1$ independent edges.

Recall that the maximal degree of H is at most εn. Let $\{x_1y_1,\ldots,x_ky_k\}$, $k \leq t-1$, be a maximal set of independent edges in H. Since every edge of H meets the set $\{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$, we have $e(H) \leq 2k\varepsilon n < 2t\varepsilon n$. But then

$$t_r(n)+\frac{n}{r}-1 < \mathrm{ex}(n; tK_{r+1}) = e(G) < t_r(n)+2t\varepsilon n ,$$

contradicting the choice of ε, provided n is sufficiently large. □

A good many substantial general results concerning the structure of graphs in $\mathrm{EX}(n;F)$ were proved by Simonovits (1968, 1974).

Another result based on Theorem 1.5.2, a theorem of Bollobás et al. (1978), shows the surprisingly great difference one edge can make.

Let q be a prime power and let $n=q^2+q+1$. Let G be the graph obtained from $K(n,n)$ by placing an Erdős–Rényi graph G_q, described in the proof of Theorem 1.3.5, in each of the classes. Thus

$$e(G)=n^2+q(q+1)^2=q^4+3q^3+5q^2+3q+1 .$$

As G_q has maximal degree $q+1$ and contains no $C_4=K(2,2)$, the maximal t for which G contains a $K(2,2,t)$ is precisely $q+1 \sim \sqrt{n}$. *One more* edge guarantees the existence of a $K(2,2,\lfloor \gamma n \rfloor)$ where $\gamma > 0$ is an absolute constant.

Theorem 1.5.5. *There is a constant q_0 such that if $q \geq q_0$ is a prime power and $n=q^2+q+1$ then*

$$\mathrm{ex}(2n; K(2,2,q+2)) = n^2+q(q+1)^2 .$$

Furthermore, every graph of order $2n$ and size $n^2+q(q+1)^2+1$ contains a $K(2,2,t)$ with $t \geq 10^{-3}n$.

To conclude this section, we present a theorem of Erdős and Simonovits (1983). This result is related to Theorem 1.2.3: it concerns the number of $\mathcal{F}$-subgraphs of a graph with n vertices and *substantially* more than $\mathrm{ex}(n;\mathcal{F})$ edges. Similarly to the notation $k_r(G)$ used earlier, given a family $\mathcal{F}$ of graphs, denote by $k_{\mathcal{F}}(G)$ the number of subgraphs of a graph G isomorphic to elements of $\mathcal{F}$. Thus $\mathrm{ex}(n;\mathcal{F})=\max\{e(G)\colon G$ has n vertices and $k_{\mathcal{F}}(G)=0\}$. The following result is a special case of a theorem of Erdős and Simonovits (1983), proved for hypergraphs.

Theorem 1.5.6. *Let $\mathcal{F}$ be a finite family of graphs, with each $F \in \mathcal{F}$ having at least t vertices. Then for every constant $c>0$ there is a constant $c'>0$ such that if G is a graph with n vertices and at least $\mathrm{ex}(n;\mathcal{F})+cn^2$ edges then $k_{\mathcal{F}}(G) \geq c'n^t$.*

1.6. *The asymptotic number of graphs without forbidden subgraphs*

Given a forbidden graph F, denote by $f(n;F)$ the number of graphs on $[n]=\{1,2,\ldots,n\}$ not containing F. What can we say about $f(n;F)$ as $n \to \infty$? As always, we are particularly interested in the case $F=K_r$. Extending earlier results of Erdős et al. (1976), Kolaitis et al. (1987) proved the following beautiful and sharp theorem.

Theorem 1.6.1. *For $r \geq 3$, $f(n;K_r)$ is asymptotic to the number of $(r-1)$-partite graphs on $[n]$. In particular,*

$$f(n;K_r)=2^{\{(r-1)/2(r-1)+o(1)\}n^2} = 2^{(1+o(1))\mathrm{ex}(n;K_r)} .$$

As we shall see, Theorem 1.6.1 and a simple application of Szemerédi's uniformity lemma (Theorem 1.4.5) enable one to determine the asymptotic value of $\log f(n;F)$ for every F of chromatic number at least 3.

Let us start with a trivial lower bound for $f(n;F)$. *If a graph G on $[n]$ does not contain our forbidden graph F, then no subgraph of G contains F and so*

$$f(n;F) \geqslant 2^{e(G)}.$$

Since G can be chosen to have $\mathrm{ex}(n;F)$ edges, we find that

$$f(n;F) \geqslant 2^{\mathrm{ex}(n;F)}.$$

Erdős et al. (1986) showed that this trivial bound is not far from being best possible. The key to this result is the following property of ε-uniform and fairly dense pairs (see Theorem 1.4.5 and the paragraph preceding it).

Lemma 1.6.2. *Let $f \geqslant 1$, $r \geqslant 2$ and $0 < \varepsilon < (r-1)^{-1/2}$, and let $V_1, \ldots, V_r$ be disjoint subsets of the vertex set $V(G)$ of a graph G with $(1-\varepsilon)\varepsilon^{f-1}|V_i| \geqslant 1$, $i = 1, \ldots, r$. Suppose that each pair (V_i, V_j) is ε^f-uniform with density at least $\varepsilon + \varepsilon^2$. Then G contains every r-partite graph on f vertices.*

Proof. We apply induction on f. As for $f = 1$ there is nothing to prove we turn to the induction step: we assume that $f \geqslant 2$ and that the lemma holds for smaller values of f.

For every i, $2 \leqslant i \leqslant r$, the set V_1 has at most $\varepsilon^f|V_1|$ vertices joined to fewer than $(d(V_1, V_i) - \varepsilon^f)|V_i| \geqslant \varepsilon|V_i|$ vertices of V_i. Hence there are at least $(1-(r-1)\varepsilon^f)|V_1| > 0$ vertices in V_1, each of which is joined to at least $\varepsilon|V_i|$ vertices in V_i, $i = 2, \ldots, r$. Let x_1 be such a vertex and set $W_1 = V_1 \setminus \{x_1\}$ and $W_i = \Gamma(x_1) \cap V_i$, $i = 2, \ldots, r$. Then $|W_1| = |V_1| - 1 \geqslant \varepsilon|V_1|$ and $|W_i| \geqslant \varepsilon|V_i|$ for $i = 2, \ldots, r$. Hence, the sets $W_1, \ldots, W_r$ satisfy the conditions of the lemma with f replaced by $f-1$. As x_1 is joined to all vertices in $\bigcup_{i=2}^{r} W_i$, we are done by the induction hypothesis. □

We know from section 1.5 that the structure of an extremal graph for F is rather close to the structure of an extremal graph for K_r, where $r = \chi(F)$. The following theorem of Erdős et al. (1986) claims that *any* graph not containing F can be turned into a graph not containing K_r by the deletion of a few edges.

Theorem 1.6.3. *For every $\varepsilon > 0$ and graph F there is a constant $n_0 = n_0(\varepsilon, F)$ with the following property. Let G be a graph of order $n \geqslant n_0$ not containing F as a subgraph. Then G contains a set E' of at most εn^2 edges such that $G \setminus E'$ contains no K_r, where $r = \chi(F)$.*

Proof. We may assume that $r \geqslant 3$ and $\varepsilon < 2/(r-1)$. Set $f = |F|$, $m = \lceil 3/\varepsilon \rceil$ and $\varepsilon_0 = \varepsilon/4$. Let $M = M(\varepsilon_0^f, m)$ be the constant guaranteed by Szemerédi's uniformity lemma (Theorem 1.4.5).

We claim that $n_0 = n_0(\varepsilon, F) = \lceil (M+1)/\varepsilon_0^f \rceil$ will do. Indeed, let G be a graph of order $n \geqslant n_0$ not containing F. By Theorem 1.4.5 there is a partition $\bigcup_{i=0}^{k} V_i$ of $V(G)$ into disjoint sets such that $m \leqslant k \leqslant M$, $|V_0| \leqslant |V_1| = |V_2| = \cdots = |V_k|$, and all

but at most $\varepsilon_0^f k^2$ of the pairs (V_i, V_j), $1 \leqslant i < j \leqslant k$, are ε_0^f-uniform. Let E' be the union of the following sets of edges:

(1) the edges meeting V_0,
(2) the edges joining two vertices of V_i, $i = 1, \ldots, r$,
(3) the edges joining V_i to V_j for every pair (V_i, V_j) which is not ε_0^f-uniform,
(4) the edges joining V_i to V_j for every pair (V_i, V_j) of density less than $\varepsilon_0 + \varepsilon_0^2$.

By Lemma 1.6.2, the graph $G \setminus E'$ contains no K_r since otherwise it would contain F as well. Hence all we have to check is that E' is small enough. This is indeed the case:

$$\begin{aligned}|E'| &\leqslant \frac{n^2}{k+1} + k\binom{n/k}{2} + \varepsilon_0^f k^2 (n/k)^2 + (\varepsilon_0 + \varepsilon_0^2)\binom{n}{2} \\ &< n^2\left\{\frac{1}{k} + \frac{1}{2k} + \varepsilon_0^f + \varepsilon_0\right\} \\ &\leqslant n^2\left\{\frac{3}{2m} + \frac{\varepsilon}{2}\right\} \leqslant \varepsilon n^2 . \qquad \square\end{aligned}$$

From here it is a short step to the theorem of Erdős et al. (1976) concerning $f(n;F)$.

Theorem 1.6.4. *Let F be a graph with $r = \chi(F) \geqslant 3$. Then*

$$f(n;F) = 2^{(1+o(1))\mathrm{ex}(n;F)} = 2^{((r-2)/2(r-1)+o(1))n^2}.$$

Proof. Theorem 1.6.3 implies that if $\varepsilon > 0$ and n is sufficiently large then

$$f(n;F) \leqslant f(n;K_r)\binom{\binom{n}{2}}{\varepsilon n^2} \leqslant f(n;K_r)(2/\varepsilon)^{\varepsilon n^2}.$$

Hence, by Theorem 1.6.1,

$$f(n;F) \leqslant 2^{(1+o(1))\mathrm{ex}(N;K_r)+o(n^2)} = 2^{(1+o(1))\mathrm{ex}(n;F)}. \qquad \square$$

It is easily seen that Theorems 1.6.3 and 1.6.4 hold for *families* of forbidden graphs. Thus if $\mathcal{F} = \{F_1, \ldots, F_k\}$, with

$$\min_{1 \leqslant i \leqslant k} \chi(F_i) \geqslant 3,$$

then, with the obvious definition,

$$f(n;\mathcal{F}) = f(n;F_1, \ldots, F_k) = 2^{(1+o(1))\mathrm{ex}(n;\mathcal{F})}.$$

It is interesting to formulate the last assertion in terms of monotone properties. A *property* $\mathcal{P}$ of graphs is an infinite class of (finite) graphs which is closed under isomorphism. A property $\mathcal{P}$ is said to be *monotone* if every subgraph of every member of $\mathcal{P}$ is also in $\mathcal{P}$, and it is *hereditary* if every *induced* subgraph of every member of $\mathcal{P}$ is also in $\mathcal{P}$. Thus every monotone property is also hereditary;

furthermore, the intersection of a family of monotone hereditary properties is monotone, and the intersection of hereditary properties is hereditary.

Monotone properties are characterized by forbidden subgraphs. Indeed, given a family $\mathcal{F}$ of finite graphs, let $\mathcal{P}_{\mathcal{F}}$ be the class of graphs having no subgraph isomorphic to a member of $\mathcal{F}$. If $\mathcal{P}_{\mathcal{F}}$ is infinite then it is a monotone property; conversely, every monotone property is obtained in this way.

A monotone property is *principal* if it is obtained by forbidding a simple graph. Clearly, every property is the intersection of a (possibly infinite) family of principal properties.

Let us write $\mathcal{P}^n$ for the set of graphs in $\mathcal{P}$ with vertex set $[n]$. Thus $f(n;\mathcal{F}) = |\mathcal{P}^n_{\mathcal{F}}|$. The remarks above concerning $f(n;\mathcal{F})$ have the following reformulation.

Theorem 1.6.5. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots$ be monotone properties and set $\mathcal{P} = \bigcap \mathcal{P}_k$. Then*

$$|\mathcal{P}^n| = 2^{o(n^2)}|\mathcal{P}^n_k|$$

for some k. In particular,

$$|\mathcal{P}^n| = 2^{o(n^2)}|\mathcal{Q}^n|$$

for some principal monotone property $\mathcal{Q}$ containing $\mathcal{P}$.

Returning to $f(n;F)$, let us note that it is not known whether Theorem 1.6.4 holds for every bipartite F as well. In fact, it is not even known whether Theorem 1.6.4 holds for a 4-cycle C_4. Since, by Theorem 1.3.5, $\mathrm{ex}(n;C_4) \sim \frac{1}{2}n^{3/2}$, one would like to show that

$$f(n;C_4) = 2^{(1/2+o(1))n^{3/2}}.$$

While the right-hand side is a (trivial) lower bound for $f(n;C_4)$, the best upper bound, due to Kleitman and Winston (1980), is only $2^{cn^{3/2}}$, with c about 1.08.

1.7. The asymptotic number of graphs without forbidden induced subgraphs

Recently Prömel and Steger studied the structure and number of graphs without *induced* forbidden subgraphs. Given a graph F, let $f^*(n;F)$ be the number of graphs on $[n]$ containing no induced subgraph isomorphic to F (briefly, containing no induced F).

At least how large is $f^*(n;F)$? Suppose that there are integers k and l such that no k-partite graph, in which l of the classes have been replaced by complete graphs, contains an induced F. Then, clearly,

$$f^*(n;F) \geq 2^{((k-1)/2k+o(1))n^2},$$

since the classes can be chosen to be almost equal and the edges between the classes can be freely chosen.

Prömel and Steger (1992, 1993a,b) proved that this simple lower bound is essentially best possible. Let $\tau(F)$ be the maximal integer r such that for $k = r-1$ there is an l as above. This somewhat convoluted definition is explained by the fact that $\tau(F)$ is something like the chromatic number $\chi(F)$, which is the maximal integer r such that for $k = r-1$ no k-partite graph contains F. So the following result, whose proof is based on a generalization of Szemerédi's uniformity lemma to hypergraphs, is the exact analogue of Theorem 1.6.4.

Theorem 1.7.1. *Let F be a graph with $r = \tau(F) \geq 3$. Then*

$$f^*(n;F) = 2^{((r-2)/2(r-1)+o(1))n^2}.$$

For the case $F = C_4$, Prömel and Steger (1991) proved much more precise results. It is easily seen that $\tau(C_4) = 3$. Indeed, if $V(G)$ is the disjoint union of the sets V_1 and V_2, with $G[V_1]$ complete and V_2 an independent set (such graphs are known as *split graphs*), then G does not contain an induced C_4. Hence, by Theorem 1.7.1, we have $f^*(n;C_4) = 2^{(1/4+o(1))n^2}$. In fact, considerably more is true.

Theorem 1.7.2. (i) *Almost every graph containing no C_4 is a split graph: $f^*(n;C_4)$ is asymptotic to the number of split graphs on $[n]$.*

(ii) *There are positive constants c_1 and c_2 such that*

$$f^*(n;C_4) \sim c_j(2^{n^2/4+n})/n^{1/2},$$

where $j \equiv n \pmod 2$.

What happens if we forbid a family $\mathcal{F} = \{F_1, F_2, \ldots\}$ of finite graphs as induced subgraphs? Rather surprisingly, unlike the case of forbidden subgraphs, forbidding a *family* $\mathcal{F}$ induced subgraphs is very different from forbidding just one of them. Let $\mathcal{P} = \mathcal{P}_{\mathcal{F}^*}$ be the class of graphs containing an element of $\mathcal{F}$ as an *induced* subgraph. If $\mathcal{P}_{\mathcal{F}^*}$ is infinite then it is a hereditary property; conversely, every hereditary property is obtained in this way.

The growth of $|\mathcal{P}^n|$ for a hereditary property $\mathcal{P}$ depends on the colouring number $r(\mathcal{P})$ of $\mathcal{P}$, defined somewhat similarly to $\tau(F)$. An (r,s)-*colouring* of a graph H is a map $\psi: V(H) \to [r]$ such that $H[\psi^{-1}(i)]$ is complete for $1 \leq i \leq s$ and is empty for $s+1 \leq i \leq r$. Thus s of the colour classes induce complete graphs and $r-s$ of them induce empty graphs. The *colouring number* $r(\mathcal{P})$ of a property $\mathcal{P}$ of graphs is the maximal r for which there is an s, $0 \leq s \leq r$ such that every (r,s)-colourable graph has property $\mathcal{P}$. Equivalently, $r(\mathcal{P}_{\mathcal{F}^*}) = \max\{r$: for some s, $0 \leq s \leq r$, no $F \in \mathcal{F}$ is (r,s)-colourable$\}$. Note that

$$r(\mathcal{P}_{\mathcal{F}^*}) \geq \inf_{F \in \mathcal{F}} \{\tau(\mathcal{F}) - 1\},$$

with equality if $\mathcal{F} = \{F\}$ but, in general, the inequality may be strict.

Alekseev (1993) and Bollobás and Thomason (1994b) determined the asymptotic size of $\mathcal{P}^n$ for a hereditary property, thereby extending Theorems 1.6.4 and 1.7.1, concerning principal properties.

Theorem 1.7.3. *Let $\mathcal{P}$ be a hereditary property of graphs and let $\mathcal{P}^n$ be the set of graphs in $\mathcal{P}$ with vertex set $[n]$. Then*

$$|\mathcal{P}^n| = 2^{(1-1/r+o(1))n^2/2},$$

where $r = r(\mathcal{P})$ is the colouring number of $\mathcal{P}$.

This result implies that the analogue of Theorem 1.6.5 does not hold for hereditary properties: the intersection of two hereditary properties may be substantially smaller than either of the properties. For example, if $\mathcal{F}_1 = \{K_4\}$, $\mathcal{F}_2 = \{C_7\}$, $\mathcal{P}_i = \mathcal{P}_{\mathcal{F}_i}$, $i = 1, 2$, and $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$ then

$$|\mathcal{P}_i^n| = 2^{(1+o(1))n^2/3}$$

for $i = 1, 2$, but

$$|\mathcal{P}^n| = 2^{(1+o(1))n^2/4}.$$

In conclusion, let us note that the analogous problem for uniform hypergraphs is unsolved. If $\mathcal{P}$ is a property of k-graphs (k-uniform hypergraphs) then, as implied by some results of Alekseev (1982) and Bollobás and Thomason (1994a),

$$|\mathcal{P}^n| = 2^{(c+o(1))\binom{n}{k}}$$

for some constant c. However, for $r \geq 3$ the possible values for c are not known.

2. Cycles

In section 1 we discussed the forbidden subgraph problem for a *fixed family* of forbidden graphs $\mathcal{F}$ and found this problem to be fairly well understood, provided $\mathcal{F}$ contains no bipartite graph. What can we say about graphs of order n not containing any member of a family $\mathcal{F}_n$ of forbidden graphs, where $\mathcal{F}_n$ *depends on* n? The most frequently studied and best understood case of this problem is when $\mathcal{F}_n$ consists of cycles. In this section we shall discuss some of the results concerning this problem.

2.1. Hamilton cycles

What values of various graph parameters ensure that a graph has a Hamilton cycle? Let us start with the number of edges ensuring a Hamilton cycle: what is $\mathrm{ex}(n; C_n)$? Since a Hamiltonian graph has minimal degree at least 2, every graph of order n and size $\mathrm{ex}(n; C_n) + 1$ must have minimal degree at least 2. It is

immediate that the minimal number of edges ensuring that a graph of order n has minimal degree at least 2 is $\binom{n-1}{2} + 2$: adding a vertex x to K_{n-1} and joining x to one vertex of K_{n-1} we obtain the unique graph of order n, size $\binom{n-1}{2} + 1$, and minimal degree at most 1 (and so precisely 1). A moment's thought shows that the Hamilton cycle problem has the same solution: $\mathrm{ex}(n; C_n) = \binom{n-1}{2} + 2$, with the same extremal graph.

Although this seems somewhat disappointing, all it shows that the size in itself is not very effective in forcing a Hamilton cycle. The minimal degree is considerably better. (Contrast this with the remarks following Theorem 1.1.1 in the previous section.) Dirac (1952) proved that a graph of order n and minimal degree at least $n/2$ is Hamiltonian; the graph $K(\lfloor (n-1)/2 \rfloor, \lfloor (n+1)/2 \rfloor)$ shows that the result is best possible. This theorem of Dirac started the search for various degree conditions that, coupled with some other conditions, like a bound on the connectedness, imply that the graph is Hamiltonian.

As shown by Ore (1960), Dirac's theorem is implied by the following simple lemma, essentially due to Dirac.

Lemma 2.1.1. *Let x_1 and x_n be non-adjacent vertices in a graph G of order n such that $d(x_1) + d(x_n) \geq n$. Then G is Hamiltonian iff $G + x_1x_n$ is Hamiltonian.*

Proof. Suppose there is a Hamilton cycle in $G + x_1x_n$. If this cycle does not contain x_1x_n then G is Hamiltonian so we are done. Otherwise G contains a Hamilton path $x_1x_2 \cdots x_n$. Since $d(x_1) + d(x_n) \geq n$, there is an index i, $2 < i < n$, such that x_1 is joined to x_i and x_n is joined to x_{i-1}. But then $x_ix_1x_2 \cdots x_{i-1}x_nx_{n-1} \cdots x_i$ is a Hamilton cycle. □

Thus if a graph G is not Hamiltonian and x, y are non-adjacent vertices such that $d(x) + d(y) \geq n$ then $G' = G + xy$ is not Hamiltonian either. Of course, if in G' we can find non-adjacent vertices x', y' such that $d'(x') + d'(y') \geq n$, where d' denotes the degree in G', then $G'' = G' + x'y'$ is not Hamiltonian either, and so on. This led Bondy and Chvátal (1976) to introduce the k-closure of a graph. The *k-closure* $C_k(G)$ of a graph G is the minimal graph H containing G such that for any two non-adjacent vertices x, y of H we have $d_H(x) + d_H(y) \leq k - 1$. In other words, $C_k(G)$ is the unique graph obtained from G by successively joining all vertices the sum of whose degrees is at least k. Call a property P of graphs *k-stable* if whenever x, y are non-adjacent vertices of G such that $d(x) + d(y) \geq k$, and $G + xy$ has property P then so does G. By definition, if P is k-stable and $C_k(G)$ has P then G has P.

Lemma 2.1.1 states precisely that the property of being Hamiltonian (for graphs of order n) is n-stable. (In fact, the proof of Lemma 2.1.1 shows that the property of containing a cycle of length at least k is also n-stable; and it is easily seen that the property of containing a path of length at least l is $(n-1)$-stable.) Thus if $C_n(G)$ is Hamiltonian so is G. In particular, Lemma 1.1.1 implies Dirac's theorem, from whose proof the lemma was distilled.

Theorem 2.1.2. *Let G be a graph of order $n \geq 3$ and minimal degree at least $n/2$. Then G is Hamiltonian.*

Proof. Note that $C_n(G)$ is the complete graph K_n. Since K_n is Hamiltonian, so is G. □

The closure operation enables one to prove the theorem of Las Vergnas (1971) for the existence of a Hamilton cycle.

Theorem 2.1.3. *Let G be a graph with vertex set $\{x_1, x_2, \ldots, x_n\}$. Suppose there are no indices i and j such that x_ix_j is not an edge, $d(x_i) + d(x_j) \leq n-1$, $d(x_i) \leq i$, $d(x_j) \leq j-1$ and $j \geq \max\{i+1, n-i\}$. Then G is Hamiltonian.*

As an immediate consequence of this result, one obtains Chvátal's (1972) theorem answering a very natural extremal question concerning Hamilton cycles: what sequences $d_1, d_2, \ldots, d_n$ guarantee that if the ith vertex of a graph G of order n has degree at least d_i then G is Hamiltonian? By Dirac's theorem, $\lceil n/2 \rceil, \lceil n/2 \rceil, \ldots, \lceil n/2 \rceil$ is such a sequence.

Theorem 2.1.4. (i) *Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of a graph of order $n \geq 3$. Suppose*

$$d_k \leq k < \frac{n}{2} \text{ implies } d_{n-k} \geq n-k . \tag{1}$$

Then if G has vertex set $\{x_1, x_2, \ldots, x_n\}$ and $d(x_i) \geq d_i$ for every i, then G is Hamiltonian.

(ii) *If $(d_k)_1^n$ is the degree sequence of a graph and (1) fails then there is a non-Hamiltonian graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $d(x_i) \geq d_i$ for every i.*

Analogous results hold for Hamilton paths: if $C_{n-1}(G)$ has a Hamilton path then so does G, and condition (1) gets replaced by the condition that $d_k \leq k-1 < \frac{1}{2}(n-1)$ implies that $d_{n+1-k} \geq n-k$.

There are numerous other sufficient conditions for a graph to be Hamiltonian that do not demand that the vertices have very large degrees. The first notable result of this kind was proved by Nash-Williams (1971). Let us write $\alpha(G)$ for the *independence* (or *stability*) number of a graph G, i.e., for the maximal cardinality of an independent set of vertices.

Theorem 2.1.5. *Let G be a 2-connected graph of order n and minimal degree $\delta(G) \geq (n+2)/3$. If $\delta(G) \geq \alpha(G)$ then G is Hamiltonian.*

In proving Theorem 2.1.5, Nash-Williams made use of the following important lemma.

Lemma 2.1.6. *Let C be a longest cycle in a non-Hamiltonian graph G with n vertices. If $G - C$ has a component with at least 2 vertices then $\delta(G) \leq (n+1)/3$.*

This lemma has several extensions, including those by Jackson (1980) and Jung (1984).

Häggkvist (1980, 1989) proved the following deep and useful characterization of Hamiltonian graphs of fairly large minimal degree.

Theorem 2.1.7. *Every 2-connected non-Hamiltonian graph with n vertices and minimal degree $\delta \geq \frac{8}{17}(n-1)$ contains a set S of $m \geq 3\delta - n + 2 > \frac{7}{17}n$ vertices such that in the graph $G - S$ the vertex set cannot be covered by m paths.*

Note that in Theorem 2.1.4 one allows $d(x_i)$ to be strictly greater than d_i. As the following beautiful theorem of Jackson (1980) shows, if we demand that the graph is 2-connected and every vertex has degree *precisely* d, then a rather small value of d guarantees that the graph is Hamiltonian. The proof of this theorem is based on Jackson's extension of Lemma 2.1.6.

Theorem 2.1.8. *Let G be a 2-connected d-regular graph of order n. If $d \geq \frac{1}{3}n$ then G is Hamiltonian.*

The Petersen graph shows that, as stated, Theorem 2.1.8 is best possible, at least for $d = 3$. It is easily seen that it is close to being best possible for every $d \geq 3$.

What happens if our graph is not only 2-connected but also k-connected for some $k \geq 3$? At first sight it seems likely that a considerably smaller degree of regularity will suffice to imply that the graph is Hamiltonian. In particular, as conjectured by Bollobás (1978a, p. 167, Conjecture 36), it seems likely that if G is a d-regular k-connected graph with n vertices and $d \geq n/(k+1)$ then G is Hamiltonian. Jackson and Jung showed that this is false for $k \geq 4$.

The examples indicate that for a fixed value of k, k-connectedness is hardly any more use in finding Hamilton cycles in regular graphs than 3-connectedness. However, the conjecture may well be true for $k = 3$: if G is a 3-connected d-regular graph with n vertices and $d \geq n/4$ then G is Hamiltonian. This was conjectured by Häggkvist as well.

Recently Li Hao (1989a) took the first step towards proving this conjecture by showing that if we demand 3-connectedness then the degree of regularity can be allowed to drop substantially below the $n/3$ bound in Theorem 2.1.8.

Theorem 2.1.9. *Let G be a 3-connected d-regular graph of order n. If $d \geq \frac{7}{22}n$ then G is Hamiltonian.*

Note that Theorem 2.1.5 is another extension of Theorem 2.1.1. The following rather simple result in the vein of Theorem 2.1.5 is due to Chvátal and Erdős (1972).

Theorem 2.1.10. *Suppose G has at least three vertices and it is $\alpha(G)$-connected. Then it is Hamiltonian.*

Proof. Let $k = \alpha(G)$. Then $k \geq 2$ so G has a longest cycle C. Then $|C| \geq \delta(G) + 1 \geq k+1$. Assume that C is not a Hamilton cycle, i.e., there is a vertex $x \in G - C$. Since G is k-connected, there are k independent paths from x to C, i.e., there are $x - x_i$ paths ($i = 1, \ldots, k$) such that any two of them have only the vertex x in common, and any one of them has only the vertex x_i on C.

Giving C some orientation, let x_i^+ be the successor of x_i on C for $i = 1, \ldots, k$. Then, since C is a longest cycle, the set $S = \{x, x_1, x_2, \ldots, x_k\}$ is an independent set, contradicting our assumption that $\alpha(G) \leq k$. Hence C is a Hamilton cycle. □

Given a set S of vertices of a graph G, denote by $N(S)$ the set of neighbours of S: $N(S) = \{x \in G: xy \in E(G) \text{ for some } y \in S\}$. Fraisse (1986) proved the following essentially best possible condition for a k-connected graph to be Hamiltonian.

Theorem 2.1.11. *Let G be a k-connected graph of order n. Suppose that $|N(S)| > k(n-1)/(k+1)$ whenever S is an independent set of k vertices. Then G is Hamiltonian.*

The following graph constructed by Skupien (1979) shows that Theorem 2.1.11 is close to being best possible: let $n = (k+1)q + k$ and let G be obtained from the vertex-disjoint union of K_k and $k+1$ copies of K_q, by joining each vertex of K_k to every other vertex. Then G is a k-connected non-Hamiltonian graph of order n, in which any k independent vertices have $n - k - q = kq = k(n-k)/(k+1)$ neighbours.

Recently Häggkvist (1989) proved the following substantial extension of Theorem 2.1.5.

Theorem 2.1.12. *Let G be a non-Hamiltonian 2-connected graph of order n, independence number $\alpha \leq (n+1)/2$ and minimal degree $\delta \geq (n+2)/3$. Then, for every k, $1 \leq k \leq \delta + 1$, there exists an independent set S of k vertices such that*

$$|N(S)| \leq \max\{\alpha - 1, n - 2\delta + k - 2\}.$$

A consequence of Theorem 2.1.12 is that if G is a 2-connected non-Hamiltonian graph of order n with minimal degree $\delta \geq (n+2)/3$ then it contains an independent set of at least $(n+14)/6$ vertices with at most $(n-1)/2$ neighbours in total.

2.2. *Edge-disjoint Hamilton cycles*

Suppose the conditions on some set of graph parameters imply that our graph must contain a Hamilton cycle. Does our graph have to have many Hamilton

cycles? Does it have to have many edge-disjoint Hamilton cycles? The followin striking theorem of Nash-Williams (1971), whose proof is based on Theoren 2.1.6, shows that this is the case if the parameter is the minimal degree. To b precise, Nash-Williams proved the following substantial extension of Dirac' theorem, Theorem 2.1.2.

Theorem 2.2.1. *Let G be a graph of order n and minimal degree at least n/2. The G contains a set of $\lfloor 5(n+10)/224 \rfloor$ edge-disjoint Hamilton cycles.*

Once again, if we demand that our graph be regular then we can guarante considerably more edge-disjoint Hamilton cycles. Jackson (1979) made use of hi Theorem 2.1.8 to deduce the following result.

Theorem 2.2.2. *Let G be a d-regular graph of order $n \geq 14$. If $d \geq (n-1)/2$ the G contains a set of $\lfloor (n-1)/2 \rfloor$ edge-disjoint Hamilton cycles.*

Theorem 2.2.1 is rather far from being best possible. In the case when th minimal degree is a little larger than $n/2$, Häggkvist (1990) proved the followin deep results that are essentially best possible.

Theorem 2.2.3. *Let $\lambda > \frac{1}{2}$. If n is sufficiently large and G is a graph of order n an minimal degree at least λn, then G has a set of $\lfloor n/8 \rfloor$ edge-disjoint Hamilto cycles.*

Theorem 2.2.4. *Let $\lambda > \frac{1}{2}$. If n is sufficiently large and G is a d-regular graph order n, where d is an even integer not less than λn, then G has a Hamilto decomposition, i.e., the edge set of G can be partitioned into d/2 Hamilton cycle*

To see that, in some sense, Häggkvist's theorem 2.2.3 is essentially be possible, consider the following graph G given by Nash-Williams (1970). Take th complete bipartite graph with vertex sets $U = \{u_1, \ldots, u_{4k+1}\}$ and $W = \{w_1, \ldots, w_{4k-1}\}$, and add to it the edges u_1u_2, u_3u_4, $u_5u_6, \ldots, u_{4k-1}u_{4k}$ an $u_{4k}u_{4k+1}$. The obtained graph G has $n = 8k$ vertices and minimal degree $2k$. No that every Hamilton cycle in G has to contain two of the $2k+1$ edges in U, so has at most $\lfloor (2k+1)/2 \rfloor = k = n/8$ edge-disjoint Hamilton cycles.

Li Hao (1989b) proved a conjecture of Faudree and Schelp that if Ore condition in Lemma 2.1.1 is satisfied and the graph has *small* minimal degree the there are many edge disjoint cycles.

Theorem 2.2.5. *Let G be a graph with n vertices and minimal degree δ such th $n \geq 2\delta^2$ and the degree sum of any two non-adjacent vertices is at least n. Then t graph contains $k = \lfloor (\delta - 1)/2 \rfloor$ edge disjoint cycles of lengths $l_1, l_2, \ldots, l_k$, for $3 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq n$.*

2.3. Long cycles

For a graph G, let $C(G)$ be the set of lengths of cycles in G. The *circumference of* G is the length of a longest cycle: $c(G) = \max C(G)$, the *girth* of G is the length of a shortest cycle: $g(G) = \min C(G)$. What do various natural graph parameters (size, minimal degree, connectivity, etc.) tell us about $c(G)$, $g(G)$ and $C(G)$?

Let $x_1x_2 \cdots x_l$ be a longest path in a graph G, and let $k = \max\{i: x_1 \text{ is joined to } x_i\}$. Then $k \geq d(x_1) + 1 \geq \delta(G) + 1$ so, in particular, if $\delta(G) \geq 2$ then $c(G) \geq \delta(G) + 1$. This trivial observation was strengthened considerably by Alon (1986) to a result including Dirac's theorem (Theorem 2.1.2): if $\delta(G) \geq n/k$ then $c(G) \geq \lfloor n/(k-1) \rfloor$. The theorem was extended slightly by Egawa and Miyamoto (1989) and Bollobás and Häggkvist (1990) to the following best possible result.

Theorem 2.3.1. *Suppose $2 \leq k < n$ are integers and G is a graph of order n and minimal degree at least n/k. Then $c(G) \geq n/(k-1)$. Furthermore, for $2 \leq k < n$ there is a graph G of order n such that $\delta(G) = \lceil n/(k-1) \rceil - 1$ and $c(G) = \lceil n/(k-1) \rceil$.*

In fact, recently Bollobás and Brightwell (1993) extended Theorem 2.3.1 to the following result, whose proof turned out to be considerably easier than the proofs of Theorem 2.3.1.

Theorem 2.3.1′. *Let G be a graph of order n with a set W of $w \geq 3$ distinguished vertices. Suppose that every vertex of W has degree at least $d \geq 2$ and let $s = \lceil w/(\lceil n/d \rceil - 1) \rceil \geq 3$. Then there is a cycle in G containing at least s vertices of W.*

If we demand that our graph is 2-connected then we can guarantee a considerably longer cycle: as proved by Dirac (1952), if G is 2-connected then $c(G) \geq \min\{|G|, 2\delta(G)\}$. The following extension of a theorem of Pósa (1963) was proved by Bondy (1971a).

Theorem 2.3.2. *Let $3 \leq c \leq n$ and let G be a 2-connected graph of order n with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $2 \leq d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$. Suppose also that if $d_k \leq k < c/2$, $k < l$, $d_l < l$ and $x_kx_l \notin E(G)$ then $k + l \geq c + 1$. Then $c(G) \geq c$.*

Bondy proved also that if in a graph of order n the degree sum of any three independent vertices is at least $m \geq n + 2$ then $c(G) \geq \min\{n, 2m/3\}$, and conjectured the following much stronger result, proved by Fournier and Fraisse (1985) (cf. Theorem 2.1.8.).

Theorem 2.3.3. *Let G be a k-connected graph of order n, where $k \geq 2$, such that the degree sum of any $k + 1$ independent vertices is at least m. Then $c(G) \geq \min\{n, 2m/(k+1)\}$.*

Erdős and Gallai (1959) determined the minimal size of a graph of order n guaranteeing that the circumference is at least c.

Theorem 2.3.4. *Let $3 \leq c \leq n$. Then the circumference of a graph of order n and size $\lfloor (c-1)(n-1)/2 \rfloor + 1$ at least c.*

A graph G of order n is *pancyclic* if $C(G) = [3, n] = \{3, 4, \ldots, n\}$, i.e., if G contains a cycle of every possible length. We do know that $\lfloor n^2/4 \rfloor$ edges do not guarantee a triangle C_3, and many more edges are needed to guarantee a Hamilton cycle. However, as the following theorem of Bondy (1971b) shows, if a graph has more than $\lfloor n^2/4 \rfloor$ edges then a cycle of length $l > 3$ guarantees a cycle of length $l - 1$.

Theorem 2.3.5. *Let G be a graph of order n with more than $\lfloor n^2/4 \rfloor$ edges. Then $c(G) \geq \lfloor \frac{1}{2}(n+3) \rfloor$ and $C(G) = [3, c(G)]$. In particular, if G is also Hamiltonian then it is pancyclic.*

How large a minimal degree ensures that a graph G of order n is pancyclic? In view of Theorem 2.3.5 the answer is $\lfloor n/2 \rfloor + 1$, the degree ensuring the existence of a triangle. If G is not bipartite then, as proved by Häggkvist (1982), already $\delta(G) \geq (2n+1)/5$ ensures the existence of a triangle. Amar et al. (1983) proved that if G is also Hamiltonian, then the same condition guarantees that the graph is pancyclic, and Shi (1986) showed the following slight extension of this result.

Theorem 2.3.6. *Let G be a non-bipartite Hamiltonian graph of order n such that for any two non-adjacent vertices x and y we have $d(x) + d(y) \geq (4n+1)/5$. The G is pancyclic.*

It is easily seen that Theorem 2.3.6 is best possible. Indeed, let G be th $2k$-regular graph of order $n = 5k$ with vertex set $V = \bigcup_{i=1}^{5} V_i$ where $|V_1| = \cdots = |V_5| = k$ and with edges joining V_i to V_{i+1} for $1 = 1, \ldots, 5$, where $V_6 = V_1$. Then G is not pancyclic because it contains no 4-cycles.

Woodall (1972) determined the minimal number of edges ensuring that a grap G of order n and minimal degree δ satisfies $C(G) \supset [3, l]$. Here we state only consequence of this result.

Theorem 2.3.7. *Let $3 \leq (n+3)/2 \leq l \leq n$ and let G be a graph of order n and si*

$$\binom{l-1}{2} + \binom{n-l+2}{2} + 1 .$$

Then $C(G) \supset [3, l]$. The bound is best possible.

Although a graph with fewer than $\lfloor n^2/4 \rfloor$ edges cannot be guaranteed to ha *any* odd cycles, it *can* be guaranteed to have *even* cycles, both short and long. T following deep and almost best possible result was conjectured by Erdős (1965) and proved by Bondy and Simonovits (1974).

Theorem 2.3.8. *Let k be a natural number. Every graph of order n and size at least $90kn^{1+1/k}$ contains a cycle of length $2l$ for every integer l in the interval $k \leq l \leq kn^{1/k}$.*

2.4. *Girth and diameter*

What forces a graph to have small girth, i.e., short cycles? Many edges, or almost equivalently, large minimal degree. To study the connection between the minimal degree and the girth, for natural numbers $\delta \leq 2$ and $g \geq 3$ define

$$n(g, \delta) = \min\{|G|: g(G) \geq g \text{ and } \delta(G) \geq \delta\}.$$

A graph of minimal degree δ, girth at least g and order $n(g, \delta)$ is said to be a (δ, g)-*cage*.

It is not entirely immediate that $n(g, \delta) < \infty$, i.e., there are finite graphs of arbitrarily large girth and arbitrarily large minimal degree. However, this does follow from a simple argument using random graphs.

A cycle of length g shows that $n(g, 2) = g$ so we shall assume that $\delta \geq 3$. By estimating the number of vertices at distance d from a vertex or from an edge, one gets the following trivial lower bound on $n(g, \delta)$.

Theorem 2.4.1. *If $\delta \geq 3$ then*

$$n(g, \delta) \geq \begin{cases} 1 + \delta \dfrac{(\delta - 1)^{(g-1)/2} - 1}{\delta - 2} & \text{if } g \text{ is odd}, \\ \dfrac{2(\delta - 1)^{g/2} - 2}{\delta - 2} & \text{if } g \text{ is even}. \end{cases}$$

It is easily seen that in Theorem 2.3.1 equality holds for $\delta = 3$, $g = 3, 4, 5, 6$ and 8, and for $g = 4$ and all $\delta \geq 3$. For example, $n(5, 3) = 10$ is shown by the Petersen graph; the extremal graph for $z(7; 4) = 21$ (see Theorem 1.3.3) shows that $n(6, 3) = 14$ (thus the vertices are the 7 points and 7 lines of the projective plane $PG(2, 2)$, with a point joined to a line if they are incident); the graph $K(\delta, \delta)$ shows that $n(4, \delta) = 2\delta$.

Suppose that $g \geq 3$, $\delta \geq 3$ and G_0 is a graph showing that equality holds in Theorem 2.4.1. If g is odd, say $g = 2D + 1$, then G_0 is δ-regular and has diameter D; also $n(g, \delta)$ is the *maximal* order of a graph with *maximal* degree at most δ and *diameter* at most D. If $g = 2D + 2$ then G_0 is δ-regular and every vertex is within distance D of every edge (in fact, of every pair of vertices); also $n(g, \delta)$ is the *maximal* order of a graph with *maximal* degree at most δ in which *every* vertex is within distance D of *every* edge. Such a graph G_0 is called a *Moore graph of girth g and degree δ*. (If $g = 2D + 1$ then G_0 is also called a Moore graph of diameter D and degree δ.)

There are very few Moore graphs. Results of Hoffman and Singleton (1960), Kárteszi (1960), Feit and Higman (1964), Singleton (1966), Bannai and Ito (1973) and Damerell (1973) show that if there is a Moore graph of girth $g \geq 5$ and degree $\delta \geq 3$ then either $g = 5$ and $\delta = 3$, 7 or 57, or else $g = 6$, 8 or 12. For $g = 6$ and 8 there is a Moore graph for each finite projective geometry of order δ and dimension 2 and 3.

As there are so few graphs attaining the trivial lower bound in Theorem 2.4.1, what about graphs showing that $n(g, \delta)$ is not much larger than the trivial lower bound. Such graphs are not easy to come by either. The following theorem was proved by Erdős and Sachs (1963) without explicitly constructing a graph showing the inequality.

Theorem 2.4.2. *If $g \geq 3$ and $\delta \geq 3$ then*

$$n(g, \delta) \leq \begin{cases} \dfrac{\delta}{\delta - 2}\{(\delta - 1)^{g-1} - 1\} & \text{if } g \text{ is odd}, \\ \dfrac{4}{\delta - 2}\{(\delta - 1)^{g-2} - 1\} & \text{if } g \text{ is even}. \end{cases}$$

Note that for large values of g the upper bound given in Theorem 2.4.2 is about the *square* of the trivial lower bound in Theorem 2.4.1. This huge gap was narrowed by Margulis (1982) by an *explicit construction*: a most welcome success of constructive algebraic methods.

Let $p \geq 5$ be a prime and consider $SL_2(\mathbb{Z}_p)$, the multiplicative group of unimodular 2 by 2 matrices with entries from the field $\mathbb{Z}_p$. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ be elements of $SL_2(\mathbb{Z}_p)$. The *Margulis graph* $M(4, p)$ is the Cayley graph over $SL_2(\mathbb{Z}_p)$ with respect to the set $\{A, B, A^{-1}, B^{-1}\}$, i.e., $M(4, p)$ has vertex set $SL_2(\mathbb{Z}_p)$ with a matrix C joined to a matrix D iff $C^{-1}D \in \{A, B, A^{-1}, B^{-1}\}$. Margulis proved that the graph $M(4, p)$ has rather large girth.

Theorem 2.4.3. *Let $\alpha = 1 + \sqrt{2}$, $k \in \mathbb{N}$ and let $p \geq 2\alpha^k$ be a prime. Then the grapl $M(4, p)$ is a 4-regular graph of order $p(p^2 - 1)$ and girth at least $2k + 1$.*

Note that for large $n = p(p^2 - 1)$ the Margulis graph $M(4, p)$ has girth abou $(2/3 \log \alpha) \log n = \log_b n$, where $b = \alpha^{3/2} = 3.751\ldots$, while Theorem 2.4. guarantees only a graph of girth about $\log_3 n$.

Margulis (1982) used the same method to construct regular graphs of large girtl and arbitrary even degrees. Following Margulis, Imrich (1984) constructed Cayle graphs of factor groups of some subgroups of the modular group to improve th bound in Theorem 2.4.3.

Theorem 2.4.4. *For every $r > 2$ one can effectively construct infinitely many Cayle graphs with n vertices and girth at least*

$$0.4801\ldots(\log n)/\log(d - 1) - 2.$$

Furthermore, for $r = 3$ one can have girth at least

$$0.9601\ldots(\log n)/\log 2 - 5 .$$

It would be of interest to find other explicit constructions for graphs of large girth and large minimal degree.

2.5. The set of cycles in graphs of given minimal degree

A graph G of minimal degree $\delta \geq 2$ contains at least $\delta - 1$ cycles of different lengths, i.e., $|C(G)| \geq \delta - 1$. Indeed, let $x_1x_2 \cdots x_l$ be a longest path in G and let $x_2, x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ be the neighbours of x_1; then $k \geq \delta - 1$ and for every j, $1 \leq j \leq k$, the graph has a cycle of length i_j, namely $x_1x_2 \cdots x_{i_j}$. The graphs $K_{\delta+1}$ and $K(\delta, \delta)$ show that this trivial bound on $|C(G)|$ in terms of δ cannot be improved in general.

However, if $\delta(G) = \delta \geq 3$ and G has large girth then it is easily seen that $|C(G)|$ has to be (much) larger than $\delta - 1$. This suggests that if short cycle lengths are taken with large weights and long cycle lengths are taken with small weights, then the total weight of cycle lengths has to be large if the minimal degree is large. Erdős and Hajnal (see Erdős 1975) proposed taking a cycle of length r with *weight* $1/r$. For a graph G, let

$$S(G) = S(C(G)) = \sum \{1/r \colon r \in C(G)\} .$$

How large is then

$$f(k) = \inf\{S(G) \colon \delta(G) = k\} ?$$

The graph $K_{k,k}$, $k \geq 2$ has minimal degree k and its set of cycle lengths is $\{4, 6, \ldots, 2k\}$ so, as $k \to \infty$,

$$f(k) \leq S(K_{k,k}) = \tfrac{1}{2} \sum_{r=2}^{k} \frac{1}{r} = (\tfrac{1}{2} + o(1))\log k .$$

Erdős and Hajnal conjectured that $f(k)$ is of order $\log k$. To appreciate the difficulty in proving this conjecture, note that it seems to be difficult to prove that $f(k) \to \infty$ as $k \to \infty$.

This conjecture was proved by Gyárfás et al. (1984).

Theorem 2.5.1. *There are positive constants c and ε such that if $\delta(G) \geq c$ then $S(G) \geq \varepsilon \log \delta(G)$.*

The ingenious and beautiful proof makes good use of the so-called (k, α)-trees. Let T be a rooted tree of height h and levels $L_1, L_2, \ldots, L_h$, where the ith *level* L_i of T is the set of vertices at distance i from the root. This tree T is said to be a *(k, α)-tree* if for $i < h$ every vertex x at level i has at most k neighbours at level $i + 1$ and

$$\alpha |L_i| \leq |L_{i+1}| \leq k|L_i| .$$

An important stage on the way to proving Theorem 2.4.1 is the following assertion.

Theorem 2.5.2. *There are constants δ_0 and ε_0, $0 < \varepsilon_0 < 1$, such that if G is a bipartite graph and $\delta(G) \geq \delta_0$ then for some integer m the set $C(G)$ contains at least $4m/7$ even integers between 4 and $4m$.*

As an immediate consequence of Theorem 2.5.2, Gyárfás et al. (1984) proved another conjecture of Erdős and Hajnal.

Theorem 2.5.3. *If G is an infinite graph of infinite chromatic number then $C(G)$ has positive upper density in $\mathbb{N}$.*

Proof. By a simple compactness argument, for every k the graph G contains a finite subgraph H_k of minimal degree $2k$; every bipartite subgraph G_k of H_k with maximal size has minimal degree at least k. $\square$

Theorem 2.5.1 can easily be turned into a result connecting $S(G)$ with the *average degree* of G. For $\alpha > 0$ define

$$h(\alpha) = \inf\{S(G) \colon e(G) \geq \alpha|G|\} .$$

Since every graph G satisfying $e(G) \geq \alpha|G|$, i.e., having average degree at least 2α, has a subgraph of minimal degree at least α, Theorem 2.5.1 implies the following result.

Theorem 2.5.1′. *There are positive constants c and ε such that if $\alpha \geq c$ then $h(\alpha) \geq \varepsilon \log \alpha$.*

This result gives no information about $h(\alpha)$ for small values of α. Trivially, $h(\alpha) = 0$ for $\alpha \leq 1$ but a priori it is not clear that there is no $\alpha_0 > 1$ such that $h(\alpha) = 0$ for $\alpha \leq \alpha_0$. Gyárfás et al. (1985) proved that, in fact, $f(\alpha) > 0$ for every $\alpha > 1$.

Theorem 2.5.4. *If k is sufficiently large then $h(1 + 1/k) \geq (300k \log k)^{-1}$.*

3. Saturated graphs

A property P of graphs is *monotone increasing* if whenever a graph G has P, so does every graph obtained from G by the addition of some edges. Clearly, if $\mathcal{F}_n$ is the set of minimal graphs of order n having property P then P is determined by the sequence $(\mathscr{F}_n)_{n=1}^{\infty}$, and conversely, if $\mathscr{F}_n$ is a family of graphs for order n then $(\mathscr{F}_n)_{n=1}^{\infty}$ determines a monotone increasing property P: a graph G of order n has P if and only if it contains at least one element of $\mathscr{F}_n$. Using the terminology of the previous section, a graph of order n fails to have property P if it contains no *forbidden subgraph*, i.e., no element of $\mathscr{F}_n$.

A graph G is *P-saturated* or *saturated for P* if G does not have P but any graph obtained from P by the addition of an edge has P. In the first two sections we studied P-saturated graphs with maximal number of edges. Here we shall turn to the lower bound: at least how many edges does a P-saturated graph of order n have? Usually one writes $\mathrm{sat}(n; P)$ for this minimum, i.e., $\mathrm{sat}(n; P) = \min\{e(G): |G| = n$ and G is P-saturated$\}$. Also, the set of extremal graphs is $\mathrm{SAT}(n; P) = \{G: |G| = n, e(G) = \mathrm{sat}(n; P)$ and G is P-saturated$\}$. If P is given by the sequence $(\mathscr{F}_n)_{n=1}^{\infty}$ then we may write $\mathrm{sat}(n, \mathscr{F}_n)$ and $\mathrm{SAT}(n; \mathscr{F}_n)$ for $\mathrm{sat}(n; P)$ and $\mathrm{SAT}(n; P)$. Also, if $\mathscr{F}_n = \{F_1, \ldots, F_k\}$ then we may write $\mathrm{sat}(n; F_1, \ldots, F_k)$ and $\mathrm{SAT}(n; F_1, \ldots, F_k)$.

3.1. *Complete graphs*

Erdős et al. (1964) proved the following analogue of Turán's theorem for saturated graphs.

Theorem 3.1.1. *If* $2 \le r \le n$ *then* $\mathrm{sat}(n; K_r) = (r-2)(n-1) - \binom{r-2}{2} = (r-2)n - \binom{r-1}{2}$ *and* $\mathrm{SAT}(n; K_r) = \{K_{r-2} + \bar{K}_{n-r+2}\}$, *i.e., the edge set of the unique extremal graph for* $\mathrm{sat}(n; K_r)$ *is the set of all edges incident with a fixed set of* $r-2$ *vertices.*

Proof. Call a graph *K_r-saturated* if it is saturated for the property of containing a K_r subgraph. Furthermore, writing $k_r(G)$ for the number of K_r subgraphs of G, we call G *strongly K_r-saturated* if $k_r(G) < k_r(G^+)$ whenever G^+ is obtained from G by the addition of an edge. Clearly ever K_r-saturated graph is strongly K_r-saturated but a strongly K_r-saturated graph need not be K_r-saturated because it may contain a K_r-subgraph. Note that if G is strongly K_r-saturated then so is every graph obtained from G by the addition of some edges.

The graph $G_n = K_{r-2} + \bar{K}_{n-r+2}$ has $(r-2)n - \binom{r-2}{2}$ edges and it is K_r-saturated. Instead of the claim of the theorem, we shall prove the stronger assertion that every strongly K_r-saturated graph of order n has at least $(r-2)n - \binom{r-1}{2}$ edges, and G_n is the only strongly K_r-saturated graph with n vertices and $(r-2)n - \binom{r-1}{2}$ edges. In fact, as the property of being strongly K_r-saturated is a monotone increasing property, it suffices to prove the latter assertion. We shall do this by induction on $n + r$.

The assertion is trivial if $r = 2$ or $n = r$. Assume then that $3 \le r < n$ and the result is true for smaller values of $n + r$. Let G be a strongly K_r-saturated graph with n vertices and $(r-2)n - \binom{r-1}{2}$ edges. Let x_1 and x_2 be non-adjacent vertices of G. As G is strongly K_r-saturated, there are vertices $x_3, \ldots, x_r$ such that in the set $\{x_1, x_2, \ldots, x_r\}$ any two vertices are joined to each other, with the exception of x_1 and x_2. Let $H = G/\{x_1, x_2\}$ be the graph obtained from G by identifying x_1 and x_2. Thus $V(H) = \{\bar{x}_2, x_3, \ldots, x_n\}$, for $3 \le i < j \le n$ two vertices x_i, x_j are joined in H if and only if they are joined in G, and $\bar{x}_2$ is joined to x_j in H if and only if at least one of x_1 and x_2 joined to x_j in G. Clearly,

$$e(G) \ge e(H) + r - 2 .$$

Also, as G is strongly K_r-saturated, so is H. Hence, by the induction hypothesis,

$$e(H) \ge (r-2)(n-1) - \binom{r-1}{2},$$

with equality if and only if $H = G_{n-1}$. Therefore

$$e(G) \ge (r-2)n - \binom{r-1}{2}$$

and if equality holds then $H = G_{n-1}$ and for $i = 1$ and 2 the vertices $x_3, \ldots, x_r$ are the only neighbours of x_i in G. It is easily checked that this implies that $G = G_n$, as claimed. □

Let us give another proof of the fact that every strongly K_r-saturated graph of order n has at least $(r-2)n - \binom{r-1}{2}$ edges and so, in particular,

$$\mathrm{sat}(n; K_r) \ge (r-2)n - \binom{r-1}{2} = \binom{n}{2} - \binom{n-r+2}{2}.$$

Let G be a strongly K_r-saturated graph with n vertices. Let $A_1, A_2, \ldots, A_l$ be the (unordered) pairs of vertices not joined to each other. We have to prove that $l \le \binom{n-r+2}{2}$. For each set A_i there is an r-set $C_i \subset V(G)$ such that $A_i \subset C_i$ and the only two vertices of C_i not joined to each other are the vertices of A_i. Set $B_i = V(G) - C_i$.

Note that $|A_i| = 2$, $|B_i| = n - r$ and $A_i \cap B_i = \emptyset$. Furthermore, if $i \ne j$ then $A_i \cap B_j \ne \emptyset$. Indeed, if we had $A_i \cap B_j = \emptyset$ then the set $C_j = V(G) - B_j$ would contain at least two pairs of non-adjacent vertices, namely A_i and A_j. Hence $A_i \cap B_j = \emptyset$ if and only if $i = j$. Thus the required inequality is an immediate consequence of the following theorem of Bollobás (1965).

Theorem 3.1.2. *For two non-negative integers a and b write* $w(a, b) = \binom{a+b}{a}^{-1}$. *Let* $\{(A_i, B_i): i \in I\}$ *be a finite collection of finite sets such that* $A_i \cap B_j = \emptyset$ *if and only if* $i = j$. *For* $i \in I$ *set* $a_i = |A_i|$ *and* $b_i = |B_i|$. *Then*

$$\sum_{i \in I} w(a_i, b_i) \le 1$$

with equality if and only if there is a set Y and non-negative integers a and b, such that $|Y| = a + b$ *and* $\{(A_i, B_i): i \in I\}$ *is the collection of all ordered pairs of disjoint subsets of Y with* $|A_i| = a$ *and* $|B_i| = b$ *(and so* $B_i = Y - A_i$*).*

In particular, if $a_i = a$ *and* $b_i = b$ *for all* $i \in I$ *then* $|I| \le \binom{a+b}{a}$. *If* $a_i = 2$ *and* $b_i = n - r$ *for all* $i \in I$ *then* $|I| \le \binom{n-r+2}{2}$.

Proof. We shall prove the inequality; the case of equality requires a little more work.

We may assume that the sets A_i, B_i are subsets of $[n]$. Call a permutation $\pi = x_1, x_2, \ldots, x_n$ *compatible* with a set-pair (A_i, B_i) if in π every element of A_i precedes every element of B_i. Let N be the number of compatible pairs $(\pi, (A_i, B_i))$. Clearly each set-pair (A_i, B_i) is compatible with

$$\binom{n}{a_i + b_i} a_i! b_i! (n - a_i - b_i)! = n! w(a_i, b_i)$$

permutations π, so

$$N + n! \sum_{i \in I} w(a_i, b_i) .$$

On the other hand, no permutation π is compatible with two set-pairs, say (A_i, B_i) and (A_j, B_j). Indeed, otherwise we may assume that $\max\{k: x_k \in A_i\} \leqslant \max\{k: x_k \in A_j\}$. Then $\max\{k: x_k \in A_i\} \leqslant \max\{k: x_k \in A_j\} < \min\{k: x_k \in B_j\}$ so $A_i \cap B_j = \emptyset$, contradicting our assumption. Hence $N \leqslant n!$, so

$$N = \sum_{i \in I} w(a_i, b_i) n! \leqslant n! ,$$

implying the required inequality. □

In fact, Theorem 3.1.2 is an extension of the LYM inequality of Lubell (1966) Yamamoto (1954) and Meshalkin (1963), which, in turn, is an extension of Sperner's (1928) lemma, and the proof given above is just a variant of Lubell's proof of the LYM inequality. To be precise, the LYM inequality is simply the case $B_i = X - A_i$ of Theorem 3.1.2 where X is the ground set.

The original reason for proving Theorem 3.1.2 was to extend Theorem 3.1.1 to hypergraphs: with the appropriate definitions, every k-uniform hypergraph of order n which is saturated for a complete graph with r vertices has at least $\binom{n}{k} - \binom{n-r+k}{k}$ hyperedges.

The proof of Theorem 3.1.2 can be adapted to give us the bipartite version of Theorem 3.1.1, first proved by Bollobás (1967a,b) and Wessel (1966, 1967). An m by n bipartite graph with classes V_1 and V_2 is *strongly saturated for* $K(s, t)$ if the addition of any edge joining V_1 to V_2 creates at least one *new* complete bipartite subgraph with s vertices in V_1 and t vertices in V_2.

Theorem 3.1.3. *Let $2 \leqslant s \leqslant m$ and $2 \leqslant t \leqslant n$. An m by n bipartite graph which is strongly saturated for $K(s, t)$ has at least $mn - (m - s + 1)(n - t + 1)$ edges. There is only one extremal graph, the m by n bipartite graph containing all edges joining the two classes except those that join a fixed set of $n - t + 1$ vertices in the first class to a fixed set of $n - t + 1$ vertices in the second class.*

Duffus and Hanson (1986) studied refinements of the problem of determining $\mathrm{sat}(n; K_r)$. Let $\mathrm{sat}(n; K_r, \delta)$ be the minimal number of edges in a K_r-saturated graph with n vertices and minimal degree at least δ.

Theorem 3.1.4. *If $n \geqslant 5$ then $\mathrm{sat}(n; K_3, 2) = 2n - 5$ and if $n \geqslant 10$ then $\mathrm{sat}(n; K_3, 3) = 3n - 15$.*

It is easily seen that for $\delta = 2, 3$ the value of $\mathrm{sat}(n; K_3, \delta)$ is at most as large as claimed. Given a graph H and a vertex x of H, construct a graph G from H by adding to H a vertex and joining it to the neighbours of x. This graph G is said to have been obtained from H by *duplicating* the vertex x. Note that if H is K_3-saturated then so is G. As the 5-cycle C_5 and the Petersen graph P are K_3-saturated, so are the graphs with n vertices obtained from C_5 and P by repeated duplications of their vertices; these graphs have minimal degrees 2 and 3, and $2n - 5$ and $3n - 15$ edges.

Perhaps for every fixed $\delta \geqslant 1$ one has $\mathrm{sat}(n; K_3, \delta) = \delta n - O(1)$.

3.2. *General families*

Let us turn to the problem of determining or estimating $\mathrm{sat}(n; \mathcal{F})$ for a general family $\mathcal{F}$ of graphs. We know that if no member of $\mathcal{F}$ is bipartite then $\mathrm{ex}(n; \mathcal{F}) \geqslant \lfloor n^2/4 \rfloor$, i.e., there are (maximal) graphs of order n not containing any forbidden graphs which have at least $\lfloor n^2/4 \rfloor$ edges. On the other hand, as the following easy estimate shows, $\mathrm{sat}(n; \mathcal{F}) = O(n)$ for every fixed finite family $\mathcal{F}$.

Theorem 3.2.1. *Let $\mathcal{F}$ be a (non-empty) finite family of non-empty graphs and let $r = \max\{|F|: F \in \mathcal{F}\}$. Then for $n \geqslant r$ we have*

$$\mathrm{sat}(n; \mathcal{F}) \leqslant (r - 2)n - \binom{r-1}{2} .$$

Proof. Let us apply induction on r. For $r = 2$ the assertion is trivial because $K_2 \in \mathcal{F}$ so the empty graph $\bar{K}_n$ is $\mathcal{F}$-saturated. Suppose that $r \geqslant 3$ and the result holds for smaller values of r. If $\mathcal{F}$ contains a star $K_{1,s}$, $s \leqslant r = 1$, then a graph containing no member of $\mathcal{F}$ must have maximal degree at most $s - 1$ so

$$\mathrm{sat}(n; \mathcal{F}) \leqslant \frac{s-1}{n} \leqslant \frac{r-2}{2} n \leqslant (r - 2)n - \binom{r-1}{2} .$$

Suppose then that no member of $\mathcal{F}$ is a star. Set $\mathcal{F}' = \{F - \{x\}: F \in \mathcal{F}, x \in V(F)\}$. Then $\mathcal{F}'$ is a finite family of non-empty graphs, each with at most $r - 1$ vertices, so by the induction hypothesis,

$$\mathrm{sat}(n - 1; \mathcal{F}') \leqslant (r - 3)(n - 1) - \binom{r-2}{2} .$$

Let H be an extremal graph for $\mathrm{sat}(n - 1; \mathcal{F}')$ and let G be obtained from H by adding to it a vertex x and joining x to all $n - 1$ vertices in H. It is trivial that G is

Proof. We shall prove the inequality; the case of equality requires a little more work.

We may assume that the sets A_i, B_i are subsets of $[n]$. Call a permutation $\pi = x_1, x_2, \ldots, x_n$ *compatible* with a set-pair (A_i, B_i) if in π every element of A_i precedes every element of B_i. Let N be the number of compatible pairs $(\pi, (A_i, B_i))$. Clearly each set-pair (A_i, B_i) is compatible with

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In fact, Theorem 3.1.2 is an extension of the LYM inequality of Lubell (1966) Yamamoto (1954) and Meshalkin (1963), which, in turn, is an extension of Sperner's (1928) lemma, and the proof given above is just a variant of Lubell's proof of the LYM inequality. To be precise, the LYM inequality is simply the case $B_i = X - A_i$ of Theorem 3.1.2 where X is the ground set.

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Theorem 3.2.1. *Let $\mathcal{F}$ be a (non-empty) finite family of non-empty graphs and let $r = \max\{|F|\colon F \in \mathcal{F}\}$. Then for $n \geq r$ we have*

$$\mathrm{sat}(n; \mathcal{F}) \leq (r - 2)n - \binom{r-1}{2} .$$

Proof. Let us apply induction on r. For $r = 2$ the assertion is trivial because $K_2 \in \mathcal{F}$ so the empty graph $\bar{K}_n$ is $\mathcal{F}$-saturated. Suppose that $r \geq 3$ and the result holds for smaller values of r. If $\mathcal{F}$ contains a star $K_{1,s}$, $s \leq r = 1$, then a graph containing no member of $\mathcal{F}$ must have maximal degree at most $s - 1$ so

$$\mathrm{sat}(n; \mathcal{F}) \leq \frac{s-1}{n} \leq \frac{r-2}{2} n \leq (r-2)n - \binom{r-1}{2} .$$

Suppose then that no member of $\mathcal{F}$ is a star. Set $\mathcal{F}' = \{F - \{x\}\colon F \in \mathcal{F}, x \in V(F)\}$. Then $\mathcal{F}'$ is a finite family of non-empty graphs, each with at most $r - 1$ vertices, so by the induction hypothesis,

$$\mathrm{sat}(n - 1; \mathcal{F}') \leq (r - 3)(n - 1) - \binom{r-2}{2} .$$

Let H be an extremal graph for $\mathrm{sat}(n - 1; \mathcal{F}')$ and let G be obtained from H by adding to it a vertex x and joining x to all $n - 1$ vertices in H. It is trivial that G is

$\mathscr{F}$-saturated, so

$$\mathrm{sat}(n; F) \leqslant e(G) = \mathrm{sat}(n-1; \mathscr{F}') + n - 1 = (r-2)n - \binom{r-1}{2}. \qquad \square$$

Note that for $\mathscr{F} = \{K_r\}$ the simple inequality above is, in fact, an equality. Kászonyi and Tuza (1986) proved the following sharper upper bound for $\mathrm{sat}(n; \mathscr{F})$.

Theorem 3.2.2. *Let $\mathscr{F}$ be a family of non-empty graphs. Set*

$$u = \min\{|U|:\ F \in \mathscr{F},\ U \subset V(F),\ F - U \text{ is a star}\}$$

and

$$s = \min\{e(F-U):\ F \in \mathscr{F},\ U \subset V(F),\ F - U \text{ is a star and } |U| = u\}.$$

Furthermore, let p be the minimal number of vertices in a graph $F \in \mathscr{F}$ for which the minimum s is attained. If $n \geqslant p$ then

$$\mathrm{sat}(n; \mathscr{F}) \leqslant \left(u + \frac{s-1}{2}\right)n - \frac{u(s+u)}{2}.$$

Proof. We proceed as in proof of Theorem 3.2.1 but this time we apply induction on u. It is again trivial to start the induction: if $u = 0$ then $K_{1,s} \cup \bar{K}_{p-s-1} \in \mathscr{F}$, i.e., $\mathscr{F}$ contains the union of a star with s edges and $p - s - 1$ isolated vertices. Hence, if an $\mathscr{F}$-saturated graph G has an $n \geqslant p$ vertices than its maximal degree is at most $s - 1$ and so $e(G) \leqslant (s-1)n/2$. The induction step is as before: the family $\mathscr{F}'$ has parameters $u - 1$, s and $p - 1$ instead of u, s and p, so

$$\begin{aligned}\mathrm{sat}(n; \mathscr{F}) &\leqslant n - 1 + \left(u - 1 + \frac{s-1}{2}\right)(n-1) - \frac{(u-1)(s+u-1)}{2} \\ &= \left(u + \frac{s-1}{2}\right)n - \frac{u(s+u)}{2}. \qquad \square\end{aligned}$$

In the proof above, the star $K_{1,s}$ played a major role; In fact, as pointed out by Kászonyi and Tuza (1986), it is very easy to find the exact value of $\mathrm{sat}(n; K_{1,s})$. Indeed, if G is $K_{1,s}$-saturated, i.e., if G is a maximal graph with maximal degree at most $s - 1$, then any two vertices of degree less than s are joined. This remark and simple calculations imply the exact value of $\mathrm{sat}(n; K_{1,s})$.

Theorem 3.2.3. *If $s + 1 \leqslant n \leqslant s + s/2$ then*

$$\mathrm{sat}(n; K_{1,s}) = \binom{s}{2} + \binom{n-s}{2}$$

and if $n > s + s/2$ then

$$\mathrm{sat}(n; K_{1,s}) = \lceil (s-1)n/2 - s^2/8 \rceil.$$

Note that in Theorem 3.2.2 one also has equality for $\mathscr{F} = \{K_r\}$ since tl $u = r - s$ and $s = 1$. Furthermore, if $\mathscr{F} = \{C_4\}$, i.e., if the only forbidden grap the 4-cycle, then $u = 1$ and $s = 2$ so, by Theorem 3.2.2, $\mathrm{sat}(n; C_4) \leqslant 3(n-1)$ This bound is also close to being best possible. To obtain a slightly better up bound, given $n \geqslant 5$, take $t = \lfloor (n-3)/2 \rfloor$ triangles sharing a vertex, and $p = n - 2t - 1$ new vertices to p vertices of one of these triangles, using dependent edges. The obtained graph is C_4-saturated and has n-vertices $3t + p = \lfloor (3n-5)/2 \rfloor$ edges. As proved by Ollman (1972), this is the best one do.

Theorem 3.2.4. *If $n \geqslant 5$ then* $\mathrm{sat}(n; C_4) = \lfloor (3n-5)/2 \rfloor$.

In conclusion, it is worth remarking that, as noted by Kászonyi and T (1986), the function $\mathrm{sat}(n; \mathscr{F})$ lacks the expected regularity properties. Namel $\mathscr{F} \subset \mathscr{F}'$ and $F' \subset F$ then we need not have any of the relations $\mathrm{sat}(n; \mathscr{F}$ $\mathrm{sat}(n; \mathscr{F})$, $\mathrm{sat}(n; F') \leqslant \mathrm{sat}(n; F)$ and $\mathrm{sat}(n; \mathscr{F}) \leqslant \mathrm{sat}(n+1; \mathscr{F})$. Indeed, let F K_3 and a K_4 sharing a vertex and let $F' = K_4$. The graph consisting a K_5 and n edges incident with one of the vertices of the K_5 is F-saturated so $\mathrm{sat}(n; F) \leqslant$ 5. On the other hand, $\mathrm{sat}(n, F') = 2n - 3$ so if $n > 8$ then $\mathrm{sat}(n; F) < \mathrm{sat}(n;$ Also, with $\mathscr{F}' = \{F', F\}$ and $\mathscr{F} = \{F\}$ we have $\mathrm{sat}(n; \mathscr{F}) \leqslant n + 5 < \mathrm{sat}(n; \mathscr{F}$ $2n - 3$ for $n > 8$.

3.3. Weakly saturated graphs

Given a family $\mathscr{F}$ of graphs and a graph G, write $k_{\mathscr{F}}(G)$ for the numbe subgraphs of G that are isomorphic to members of $\mathscr{F}$. If $\mathscr{F} = \{K_r\}$ ther before, we write $k_r(G)$ instead of $k_{K_r}(G)$. Call a graph G *weakly $\mathscr{F}$-saturat* there is a sequence of graphs $G_0 = G \subset G_1 \subset \cdots \subset G_m$ such that $V(G_i) = V$ $e(G_i) = e(G_{i-1}) + 1$ and $k_{\mathscr{F}}(G_i) > k_{\mathscr{F}}(G_{i-1})$ for every i, $1 \leqslant i \leqslant m$, and G_m i complete graph on $V(G)$. Thus G is weakly $\mathscr{F}$-saturated if we can add to it e one by one in such a way that with each edge we strictly increase the numb $\mathscr{F}$-subgraphs and we stop the process only when our graph is complete. De by w-sat$(n; \mathscr{F})$ the minimal number of edges in a weakly $\mathscr{F}$-saturated graph n vertices.

Since an $\mathscr{F}$-saturated graph is also weakly $\mathscr{F}$-saturated, we have w-sat$($ $\leqslant \mathrm{sat}(n; \mathscr{F})$. As one would expect, w-sat$(n; \mathscr{F})$ can be much smaller $\mathrm{sat}(n; \mathscr{F})$. For example, let $F = kK_2$, i.e., let F be a set of k independent e Tutte's (1947) 1-factor theorem implies easily that every maximal graph w vertices without k independent edges is of the form $K_s + \bigcup_{i=1}^{q} K_{2n_i-1}$, $s \geqslant 0$, $n_i \geqslant 0$, $q = n + s - 2k + 2$ and $\sum_{i=1}^{q} (2n_i + 1) = n - s$ (see Bollobás 1 Corollary 1.9, p. 58). This implies that if $n \geqslant (5k-2)/2$ then the maximal nu of edges in an F-saturated graph with n vertices is

$$\mathrm{ex}(n; kK_2) = \binom{k-1}{2} + (k-1)(n-k+1).$$

as proved by Erdős and Gallai (1961), and if $n \geqslant 3k-3$ then the minimal number of edges in an F-saturated graph with n vertices is

$$\mathrm{sat}(n; kK_2) = 3(k-1),$$

the minimum being given by $k-1$ independent triangles. On the other hand, a graph with $n \geqslant 2k+1$ vertices and $k-1$ independent edges is weakly kK_2-saturated so

$$\text{w-sat}(n; kK_2) = k-1.$$

Also, it is easily seen that for $n \geqslant 4$ we have w-sat$(n; C_4) = n$ while Theorem 3.2.4 tells us that sat$(n; C_4) = \lfloor (3n-5)/2 \rfloor$ for $n \geqslant 5$.

It is fascinating that for $F = K_r$ a weakly F-saturated graph must have at least as many edges as an F-saturated graph: w-sat$(n; K_r) =$ sat$(n; K_r) = (r-2)n - \binom{r-1}{2}$. For very small values of r this is easily seen. For example, a weakly K_3-saturated graph must be connected so w-sat$(n; K_3) \geqslant n-1$ and hence w-sat$(n; K_3) =$ sat$(n; K_3) = n-1$. However, while for sat$(n; K_3)$ there is *just one* extremal graph, the extremal graphs for w-sat$(n; k_3)$ are precisely the *trees*. The large size of the family of extremal graphs even in this trivial case indicates that it is considerably harder to determine w-sat$(n; K_r)$ than sat$(n; K_r)$. This task was accomplished almost twenty years after the original results of Erdős et al. (1964) and Bollobás (1965), by Frankl (1982), Kalai (1984) and Alon (1985).

Theorem 3.3.1. *If* $2 \leqslant r \leqslant n$ *then* w-sat$(n; K_r) = (r-2)n - \binom{r-1}{2}$.

To see what is needed to obtain this result, let us return to the proof of Theorem 3.1.1 that led us to Theorem 3.1.2. Let G be a weakly K_r-saturated graph with n vertices and let $G_0 = G \subset G_1 \subset \cdots \subset G_l$ be the sequence showing this. Let A_i be the pair of vertices joined in G_i but not in G_{i-1}. Let C_i be the vertex set of a K_r contained in G_i but not in G_{i-1}, and let $B_i = V(G) - C_i$. Then $|A_i| = 2$ and $|B_i| = n-r$. As $A_i \subset C_i$, we have $A_i \cap B_i$. Furthermore, none of the pairs $A_{i+1}, A_{i+2}, \ldots, A_l$ can be contained in C_i since the vertices in A_i were the last two vertices to be joined in C_i. Hence for $j > i$ we have $A_j \cap B_i \neq \emptyset$. It turns out that these two conditions imply that $l \geqslant \binom{n-r+2}{2}$ which is the content of Theorem 3.3.1. In fact, Frankl (1982), Kalai (1984) and Alon (1985) proved the appropriate result for all values of $|A_i| = a$ and $|B_i| = b$, which implies the extension of Theorem 3.3.1 for uniform hypergraphs.

Theorem 3.3.2. *Let* $(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$ *be pairs of finite sets such that* $|A_i| = a$, $|B_i| = b$ *and* $A_i \cap B_i = \emptyset$ *for all i. Suppose furthermore that* $A_i \cap B_j = \emptyset$ *if* $i > j$. *Then* $l \leqslant \binom{a+b}{a}$.

The proofs of Theorem 3.3.2, given by Frankl, Kalai and Alon are all rather similar, very beautiful and very unexpected: they make use of exterior powers of

algebras. ***With hindsight*** this is perhaps not too unexpected since $\binom{n}{a}$ and $\binom{a+}{a}$ are clearly dimensions of exterior powers. Furthermore, some years earlic Lovász (1977) had used exterior algebras for a similar purpose. As it happer Theorem 3.3.2 is tailor-made for a proof by exterior powers. For the details, s chapter 24 by Frankl.

The following extension of Theorem 3.3.2 was conjectured by Frankl a Stečkin (1982) and proved by Füredi (1984).

Theorem 3.3.3. *Let* $(A_1, B_1), \ldots, (A_l, B_l)$ *be pairs of finite sets such that* $|A_i| \leqslant$ $|B_i| \leqslant b$ *and* $|A_i \cap B_i| \leqslant c$ *for all i. Suppose furthermore that* $|A_i \cap B_j| > c$ *if* $i >$ *Then* $l \leqslant \binom{a+b-2c}{a-c}$.

3.4. *Hamilton cycles*

So far we have considered only the function sat$(n; \mathscr{F})$, i.e., we have consider only the case when our forbidden family $\mathscr{F}$ does not depend on n. This section devoted to the problem of determining sat$(n; \mathscr{F}_n)$ for the prime example of family $\mathscr{F}_n$ depending on n, namely $\mathscr{F}_n = \{C_n\}$. A graph with n vertices C_n-saturated if it is a maximal non-Hamiltonian graph, i.e., if it is non-Hamil nian but the addition of any edge creates a Hamilton cycle. The following resu were proved by Bondy (1972).

Theorem 3.4.1. *Let G be a maximal non-Hamiltonian graph of order* $n \geqslant 7$ *with vertices of degree* 2. *Then G has at least* $(3n+m)/2$ *edges*.

Corollary 3.4.2. *If* $n \geqslant 7$ *then* sat$(n; C_n) \geqslant \lceil 3n/2 \rceil$.

When studying sat$(n; \mathscr{F})$ for a fixed family $\mathscr{F}$, it is usually easy to give an up bound for sat$(n; \mathscr{F})$ and the difficulty lies in proving that the function is at least large as claimed. Rather curiously, the situation is quite different for sat$(n; C$ the results above are fairly simple, and, as it happens, the lower bound is actual value of the function, but it is difficult to construct examples showing t sat$(n; C_n)$ is indeed $\lceil 3n/2 \rceil$ if n is not too small.

If n is even then sat$(n; C_n) = \lceil 3n/2 \rceil = 3n/2$ if there is a cubic graph satura for Hamilton cycles. Since a Hamiltonian cubic graph is 3-edge-colourable, need a C_n-saturated 4-edge-chromatic cubic graph of order n. In fact, 4-ed chromatic cubic graphs are not easy to come by: Isaacs (1975) was the first construct an infinite family of such graphs. By making use of this family, Cl and Entringer (1983) and Clark et al. (1988) proved that sat$(n; C_n) = \lceil 3n/2 \rceil$ most values of n.

In view of the difficulties with sat$(n; C_n)$, it is unlikely that one could determ even sat$(n; C_k)$ for every pair (k, n). However, getting good bounds on function may not be hopeless.

4. Packing graphs

Given graphs $G, G_1, G_2, \ldots, G_l$, we say that $G_1, G_2, \ldots, G_l$ *can be packed into* G if there are inclusions $V(G_i) \subset V(G)$, $i = 1, \ldots, l$, such that $E(G_i) \subset E(G) - \bigcup_{j \neq i} E(G_j)$. The inclusions above are said to form a *packing* of $G_1, G_2, \ldots, G_l$ into G. We may and shall assume that each G_i has exactly as many vertices as G since if G_i has k fewer vertices than G then we may add k isolated vertices to G_i without altering the existence of a packing. If G is the complete graph then we say simply that there is a *packing* of $G_1, G_2, \ldots, G_l$. Thus there is a packing of $G_1, G_2, \ldots, G_l$ into G if and only if there is a packing of $\bar{G}, G_1, \ldots, G_l$.

Note that Turán's theorem states that the complete graph K_r can be packed into every graph with $n \geq r \geq 2$ vertices and $t_{r-1}(n) + 1$ edges. Equivalently, if G is a graph with n vertices and $\binom{n}{2} - t_{r-1}(n) - 1 = \sum_{i=0}^{r-2} \binom{\lfloor (n+i)/(r-1) \rfloor}{2} - 1$ edges then there is a packing of K_r (or, equivalently, $K_r \cup \bar{K}_{n-r}$) and G. Of course, in this instance the terminology we have just introduced is rather cumbersome: it is more natural to use the *subgraph* terminology, as it was done in section 1. However, many results are natural to formulate in terms of packing, even when they concern only two graphs: these are the results that we shall be concerned with in this section.

Ideally, one would like to find large classes $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_l$, consisting of graphs with n vertices each such that if $G_i \in \mathcal{G}_i$ then there is a packing of $G_1, G_2, \ldots, G_l$. Needless to say, one cannot expect a sensible characterization of such classes. Therefore our aim is to describe large classes $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_l$ in terms of standard graph parameters such that if $G_i \in \mathcal{G}_i$ then there is a packing of $G_1, G_2, \ldots, G_l$.

Throughout the section, our graphs to be packed are assumed to have n vertices each.

4.1. *Packing graphs with few edges*

Let us start with the following simple result of Sauer and Spencer (1978).

Theorem 4.1.1. *Let G_1 and G_2 be graphs with n vertices each such that $e(G_1)e(G_2) < \binom{n}{2}$. Then there is a packing of G_1 and G_2.*

Proof. Let Φ be the set of all $n!$ bijections $\phi: V(G_1) \to V(G_2)$. For a bijection $\phi \in \Phi$, call the number of edges of G_1 mapped by ϕ (to be precise, by the map induced by ϕ) into edges of G_2 the *deficiency* of ϕ, and denote it by $d(\phi)$. Note that ϕ is a packing if and only if $d(\phi) = 0$.

Clearly each edge e of G_1 contributes $e(G_2)2(n-2)!$ to the sum $\sum_{\phi \in \Phi} d(\phi)$ since having specified the edge of G_2 into which we map e by ϕ, we have $2(n-2)!$ choices for ϕ. Hence

$$\sum_{\phi \in \Phi} d(\phi) = e(G_1)e(G_2)2(n-2)!\,.$$

As, by assumption, the right-hand side is less than $n!$, at least one of the $n!$ ter on the left-hand side is 0. Hence there is a $\phi \in \Phi$ with $d(\phi) = 0$. □

The simple proof above can be reformulated as follows. The expec deficiency of a random bijection is less than 1 so there is a bijection w deficiency 0, i.e., there is a bijection giving a packing of G_1 and G_2.

If one of the graphs has substantially fewer than $n/2$ edges then a packing ex even if the other graph has fairly many edges, as shown by the following resul Bollobás and Eldridge (1978).

Theorem 4.1.2. *Let $0 < \alpha < \frac{1}{2}$. If n is sufficiently large and G_1, G_2 are graphs w n vertices each such that*

$$e(G_1) \leq \alpha n \quad \text{and} \quad e(G_2) \leq \tfrac{1}{2}(1 - 2\alpha)n^{3/2}$$

then there is a packing of G_1 and G_2.

The result is not too far from being best possible in the sense that the expor in $n^{3/2}$ is correct. Indeed, let $\alpha > 0$ be fixed and set $s = \lfloor (2\alpha n)^1$ $G_1 = K_s \cup \bar{K}_{n-s}$ and $G_2 = \overline{T_{s-1}(n)}$. Then, $e(G_1) < \alpha n$ and, rather crud $e(G_2) < n^{3/2}/2\alpha^{1/2}$ if n is sufficiently large. Since G_2 is the union of $s - 1$ comp graphs, there is no packing of G_1 and G_2.

Theorem 4.1.1 implies that if $e(G_1) + e(G_2) < (2n(n-1))^{1/2} \sim \sqrt{2}n$ then tl is a packing of G_1 and G_2. On the other hand, if $\Delta(G_1) = n - 1$ and $\delta(G_2)$ then there is no packing of G_1 and G_2: a vertex of G_1 having degree $n - 1$ car be placed on any vertex of G_2. In particular, if G_1 is the star $K_{1,n-1}$ and consists of $\lceil n/2 \rceil$ edges, covering all vertices, then $e(G_1) + e(G_2) = n - \lceil n/2 \rceil = \lfloor (3n-1)/2 \lfloor$ and there is no packing of G_1 and G_2. Sauer and Spe (1978) proved that this example is worst possible: if $e(G_1) + e(G$ $\lfloor (3n-1)/2 \rfloor - 1 = \lfloor 3(n-1)/2 \rfloor$ then there is a packing of G_1 and G_2.

If neither G_1 nor G_2 has maximal degree $n - 1$ then we need more edge prevent the existence of a packing. Let G_1 be the union of a star with $n - 2$ e and an isolated vertex, and let G_2 be 2-regular, i.e., a disjoint union of cy Then for $n > 3$ neither G_1 nor G_2 has a vertex of degree $n - 1$. $e(G_1) + e(G$ $2n - 2$ and there is no packing of G_1 and G_2. Bollobás and Eldridge (1 proved that this example is essentially worst possible.

Theorem 4.1.3. *Let G_1 and G_2 be graphs with n vertices and maximal degre most $n - 2$. If $e(G_1) + e(G_2) \leq 2n - 3$ and $\{G_1, G_2\}$ is not one of seven except pairs of graphs, each with at most nine vertices, then there is a packing of G_1 G_2. In particular, if $n \geq 10$ and $e(G_1) + e(G_2) \leq 2n - 2$ then there is a packi G_1 and G_2.*

The original proof of the result above was slightly simplified by Teo (1985) also Yap 1986 and Teo and Yap 1987).

Corollary 4.1.4. *Let G_1 and G_2 be graphs with n vertices such that if one has maximal degree $n-1$ then the other has an isolated vertex. If $e(G_1)+e(G_2) \leq 2n-3$ then there is a packing of G_1 and G_2.*

Proof. If the maximal degrees are at most $n-2$ then the result follows from Theorem 4.1.3. Otherwise we may assume that G_1 has a vertex x of degree $n-1$ and G_2 has an isolated vertex y. Placing x on y, there remains to pack $G_1' = G_1 - x$, with $e(G_1)-n+1$ edges, and $G_2' = G_2 - y$, with $e(G_2)$ edges. Since $e(G_1') + e(G_2') \leq n-2$, it is trivial that there is such a packing, for example, by Theorem 4.1.1. □

This corollary implies immediately the result of Sauer and Spencer mentioned above: if $e(G_1)+e(G_2) \leq 3(n-1)/2$ then there is a packing of G_1 and G_2. Teo (see Yap 1988) extended Theorem 4.1.3 to graphs having a total of $2n-2$ edges. As expected, the number of exceptional pairs increases substantially. For simplicity, we state the result only for $n \geq 13$.

Theorem 4.1.5. *Let G_1 and G_2 be graphs with $n \geq 13$ vertices each such that $\Delta(G_i) \leq n-2$ and $e(G_1)+e(G_2) \leq 2n-2$. For $i=1, 2, 3$, let H_i be the disjoint union of a star with $n-i-1$ edges and a K_i: $H_i = K_{1,n-i-1} \cup K_i$, let H_4 be a disjoint union of cycles, i.e., a 2-regular graph of order n, and for $n=3k$ let H_5 be the disjoint union of k triangles: $H_5 = kK_3 = T_k(n)$. If $\{G_1, G_2\}$ is not one of the pairs $\{H_1, H_1\}$, $\{H_2, H_4\}$ and $\{H_3, H_3\}$ then there is a packing of G_1 and G_2.*

If one of the graphs to be packed is a tree then one can do considerably better. Extending various earlier results, Slater et al. (1985), proved that if T is a tree of order n, G is a graph of order n and size $n-1$, and neither T nor G is a star then there is a packing of T and G. Furthermore, by making use of Theorem 4.1.3 and this result, Teo and Yap (1987) characterized the graphs of order n and size n which can be packed into the complement of any tree of order n.

It is very likely that, in turn, Theorem 4.1.5 can be extended to graphs with a total of $2n-1$ edges at the expense of a further increase in the set of exceptional pairs but the proof is likely to be forbiddingly cumbersome. However, for the case when the maximal degree is restricted even more, Eldridge (1976) proved the following result. The bound cannot be improved in general.

Theorem 4.1.6. *Let $r \geq 4$ and let G_1 and G_2 be graphs with $n \geq 9r^{3/2}$ vertices and maximal degrees at most $n-r$. If $e(G_1)+e(G_2) < 2n + r(\sqrt{r}-2) - \sqrt{r}$ then there is a packing of G_1 and G_2.*

Rather little is known about packing many graphs with few edges. In particular, if true, the following conjecture of Bollobás and Eldridge (1978) is unlikely to be easy to prove.

Conjecture 4.1.7. For every $k \geq 1$ there is an $n(k)$ such that if $n \geq n(k)$ and G_1, $G_2, \ldots, G_k$ are graphs with n vertices such that $e(G_i) \leq n-i$ and $\Delta(G_i) \leq n-$ for every i, $i=1, 2, \ldots, k$, then there is a packing of $G_1, G_2, \ldots, G_k$.

4.2. *Graphs of small maximal degree*

The results above show that a trivial obstruction to packing is the existence vertices of very large degrees. If the maximal degrees are known to be small th the existence of a packing follows from much weaker bounds on the total numb of edges. Now we shall look for restrictions on the maximal degree only implyi the existence of a matching.

The following simple result was announced by Catlin (1974) and prov independently by Sauer and Spencer (1978).

Theorem 4.2.1. *Let G_1 and G_2 be graphs with n vertices such that $\Delta(G_1)\Delta(G_2)$ $n/2$. Then there is a packing of G_1 and G_2.*

Proof. As for $\Delta(G_1)\Delta(G_2) \leq 1$ there is nothing to prove, we may assume t $\Delta(G_1)\Delta(G_2) \geq 2$ and so $n \geq 5$. Choose an identification of the vertex sets $V($ and $V(G_2)$ in which G_1 and G_2 have a minimal number of edges in comm Suppose $V(G_1) = \{x_1, \ldots, x_n\}$, $V(G_2) = \{y_1, \ldots, y_n\}$ and x_i is identified with

Assume that, contrary to the assertion, G_1 and G_2 share an edge in t identification, say $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$. Let L be the set of indic such that either $x_2x_j \in E(G_1)$ and $y_jy_l \in E(G_2)$ or else $y_2y_j \in E(G_2)$ and x_jx $E(G_l)$. Since $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$, we have

$$|L| \leq (\Delta(G_2)-1)\Delta(G_1) + (\Delta(G_1)-1)\Delta(G_2) < n-2\,.$$

Hence there is a natural number k, $3 \leq k \leq n$, such that $k \notin L$. If we flip x_2 x_k, i.e., if we identify x_2 with y_k and x_k with y_2, then the number of ed common to G_1 and G_2 decreases, contradicting our assumption. □

How far is this result from being best possible? Let $d_1 \leq d_2 < n$ be nat numbers such that $n \leq (d_1+1)(d_2+1)-2$. Let G_1 be a graph such that d_2 o components are complete graphs of order d_1+1; similarly, let G_2 have components that are complete graphs of order d_1+1. For example, let G $d_2K_{d_1+1} \cup K_{d_1-1}$ and $G_2 = d_1K_{d_2+1} \cup K_{d_2-1}$. Note that $\Delta(G_1) = d_1$ and $\Delta(G$ d_2. Suppose that there is a packing of G_1 and G_2. Then every K_{d_1-1} componer G_1 has at least one vertex outside the K_{d_2+1} components of G_2. As there ar components of the form K_{d_1+1} in G_1 but only d_2-1 vertices of G_2 in the K components, this is impossible. Hence there is no packing of G_1 and G_2.

Bollobás, Eldridge and Catlin conjectured (see Bollobás 1978b) that example above is worst possible, i.e., $n/2$ in Theorem 4.2.1 can almos replaced by n.

Conjecture 4.2.2. Let G_1 and G_2 be graphs with n vertices such that $(\Delta(G$ $1)(\Delta(G_2)+1) \leq n+1$. Then there is a packing of G_1 and G_2.

At the moment we are very far from a proof of the above conjecture. The following difficult theorem of Hajnal and Szemerédi (1970) provides some evidence for the truth of the conjecture.

Theorem 4.2.3. *Every graph with maximal degree Δ has a $(\Delta+1)$-colouring in which the cardinalities of any two colour classes differ by at most* 1.

Note that the Hajnal–Szemerédi theorem implies Conjecture 4.2.2. in the case when G_2 is of the form $T_r(n)$; in fact, the theorem is more or less equivalent to the conjecture in this case. Indeed, if $G_2 = T_r(n)$ then $\Delta(G_2) = \lceil n/r \rceil - 1$ so if $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$ then $\Delta(G_2)+1 = \lceil n/r \rceil \leq (n+1)/(\Delta(G_1)+1)$. Therefore $r \geq \Delta(G_1)+1$ so Theorem 4.2.3 implies that there is a packing of G_1 and G_2.

One should emphasize that Theorem 4.2.3 itself is a substantial result; various special cases of the theorem had been proved earlier by Dirac (1952), Corrádi and Hajnal (1963), Zelinka (1966), Grünbaum (1968) and Sumner (1969).

Catlin (1977, 1980) proved some special cases of Conjecture 4.2.2, including the following result.

Theorem 4.2.4. *There is a function $f(n) = O(n^{2/3})$ such that if G_1 and G_2 are graphs with n vertices such that $\Delta(G_1) \leq 2$ and $\Delta(G_2) \leq n/3 - f(n)$, then there is a packing of G_1 and G_2.*

4.3. *Packing trees*

Very little is known about the possibility of packing more than two graphs. The only exception is the case when all the graphs to be packed are trees. In fact A. Gyárfás made the following beautiful conjecture (see Gyárfás and Lehel 1978).

Conjecture 4.3.1. Any sequence of trees $T_2, T_3, \ldots, T_n$ with T_i having i vertices, can be packed into K_n.

Note that the total number of edges of $T_2, T_3, \ldots, T_n$ is $\sum_{i=1}^{n-1} i = \binom{n}{2}$ so in a packing claimed by the conjecture every edge of K_n must belong to precisely one of the trees.

This conjecture, which has come to be known as the tree packing conjecture, is unlikely to be solved in the affirmative in the near future. At the moment the truth of the conjecture is known only in some very special cases. Here we shall give three examples: the first two are due to Gyárfás and Lehel (1978) and the third to Hobbs (1981). Recall that a *star* is a tree of the form $K_{1,m}$, i.e., a tree of diameter 2.

Theorem 4.3.2. *Let $T_2, T_3, \ldots, T_n$ be trees with T_i having i vertices, such that each T_i is a path or a star. Then there is a packing of $T_2, T_3, \ldots, T_n$ into K_n.*

Theorem 4.3.3. *Let $T_2, T_3, \ldots, T_n$ be trees with T_i having i vertices, such that all but at most two of them are stars. Then there is a packing of $T_2, T_3, \ldots, T_n$ into K_n.*

Theorem 4.3.4. *Let $T_2, T_3, \ldots, T_n$ be trees of diameter at most 3 such that T_i has i vertices. Then there is a packing of $T_2, T_3, \ldots, T_n$ into K_n.*

The first two results were extended by Straight (1979). In particular, extending Theorem 4.3.3, he proved the existence of a packing if $\Delta(T_i) \geq i-2$ with at most two exceptions. (Note that T_i is a star if $\Delta(T_i) = i-1$.) Furthermore, Straight (1979) verified the tree packing conjecture for $n \leq 7$, and Fishburn (1983) proved it for $n \leq 9$. Theorem 4.3.4 was also considerably extended by Fishburn (1983). These results indicate that even a disproof of Conjecture 4.3.1 is likely to be difficult.

Packing a family $(T_i)_2^k$ of trees of arbitrary shapes is fairly easy if k is not too large. The following easy result of Bollobás (1983) shows that here we can take $k = \lfloor cn \rfloor$ for some $c > 0$.

Theorem 4.3.5. *Let $(T_i)_2^k$ be a sequence of trees where $k = \lfloor \sqrt{2}n/2 \rfloor$ and T_i has i vertices. Then $T_2, T_3, \ldots, T_k$ can be packed into K_n.*

In fact this result has very little to do with packing, because under the conditions the trees can be packed into K_n one after the other: first we pack T_k, then T_{k-1}, then T_{k-2}, etc.; when we choose a packing of T_i we do not take into account the trees $T_{i-1}, T_{i-2}, \ldots, T_2$. A packing of T_i exists because the graph into which T_i is packed has fairly many edges. In fact, the bound $\lfloor \sqrt{2}n/2 \rfloor$ could be replaced by $\lfloor \sqrt{3}n/2 \rfloor$ if one could prove the following fascinating conjecture proposed by Erdős and Sós in 1963. As it happens, this conjecture was one of the motivations for the conjecture of Gyárfás.

Conjecture 4.3.6. Every graph with n vertices and more than $(k-1)n/2$ edges contains every tree with k edges.

Note that the number of edges is just sufficient to guarantee that the graph contains a path with k edges and a star with k edges.

Rather than strengthen Theorem 4.3.5, perhaps one could prove the following conjecture which is considerably weaker than the tree packing conjecture.

Conjecture 4.3.7. For every $k \geq 1$ there is an $n(k)$ such that if $n \geq n(k)$ and T_{n-k}, $T_{n-k+1}, \ldots, T_n$ are trees, with T_i having i vertices, then they can be packed into K_n.

4.4. *Packing bipartite graphs*

In this section, we shall prove an attractive result of Hajnal and Szegedy about a special type of packing of bipartite graphs.

Let G_1 and G_2 be n by m bipartite graphs, with bipartitions (U_1, W_1) and $(U_2,$

W_2). We say that there is a *bipartite packing* or simply *packing* of G_1 and G_2 if the n by m complete bipartite graph $K(n,m)$, with bipartition (U,W) contains edge-disjoint subgraphs H_2 and H_2 such that, for $i=1, 2$, the graph H_i is isomorphic to G_i, with U_i corresponding to U. (Note that, unless $n=m$ *and* G_1 and G_2 are rather sparse, a bipartite packing is just a packing of G_1 and G_2 as *bipartite graphs*, i.e., into $K(n,m)$. This justifies the abbreviated terminology.) Equivalently, G_1 and G_2 have a packing if there are one-to-one maps $f: U_1 \to U_2$ and $g: W_1 \to W_2$ such that if xy is an edge of G_1, with $y \in W_1$, then $f(x)g(y)$ is not an edge of G_2. We shall call the *pair* (f, g) a *packing* of G_1 and G_2.

In the proof of the theorem below, we shall need the following simple consequence of Hall's theorem (see chapter 3) about *matchings* in n by n bipartite graphs.

Lemma 4.4.1. *If the minimal degree of G is at least n/2 then G has a matching.*

To keep the notation we need self-explanatory and manageable, for $i=1, 2$, we denote by $d(U_i)$ the *average* of the degrees of the vertices of G_i belonging to U_i, and by $\Delta(U_i)$ the *maximum* of these degrees. Define $d(W_i)$ and $\Delta(W_i)$ analogously. We are ready to state and prove the promised result of Hajnal and Szegedy (1992).

Theorem 4.4.2. *Suppose that the n by m bipartite graphs G_1, G_2 with bipartition (U_1, W_1), (U_2, W_2), are such that*

$$60 \leq \Delta(W_1) < m/20d(U_2) ,$$

$$60 \leq \Delta(W_2) < m/20d(U_1) ,$$

and, for $i = 1, 2$,

$$\Delta(U_i) \leq m/2 \log(4m) ,$$

then there is a bipartite packing of G_1 and G_2.

Proof. Let $f: U_1 \to U_2$ be a one-to-one map. As we shall see in a moment, there is a one-to-one map $g: W_1 \to W_2$ such that (f, g) is a *packing* of G_1 and G_2 if and only if a certain m by n bipartite graph B_f has a matching.

Indeed, define a bipartite graph B_f with bipartition (W_1, W_2) by making y_1y_2 $(y_1 \in W_1, y_2 \in W_2)$ an edge of B_f if $g(y_1) = y_2$ does not violate the condition that if $xy \in E(G_1)$ then $f(x)g(y) \notin E(G_2)$. In other words, let $y_1y_2 \in E(B_f)$ if and only if $f(\Gamma(y_1)) \cap \Gamma(y_2) = \emptyset$, i.e., if $y_2 \notin \Gamma(f(\Gamma(y_1)))$, where $\Gamma(x)$ denotes the set of neighbours of a vertex x in the appropriate graph.

In view of Lemma 4.4.1, the theorem follows if we show that for some map f the minimal degree $\delta(B_f)$ of B_f is at least $m/2$. Hence it suffices to show that the *probability* that $\delta(B_f) \geq m/2$ for a *random* map f is *strictly positive*. In turn, it suffices to show that the probability, that the degree of a *particular* vertex of B_f is less than $m/2$, is less than $1/2m$. Our aim is then to prove this.

By symmetry it suffices to consider a fixed vertex $y_1 \in W_1$. For simplicity, let $U_2 = [n] = \{1, 2, \ldots, n\}$ and let d_i be the degree of vertex i in G_2. Then

$$d_{B_f}(y_1) = m - |\Gamma(f(\Gamma(y_1)))| \geq m - \sum_{i \in f(\Gamma(y_1))} d_i .$$

Hence, if $d(y_1) = |\Gamma(y_1)| = r$, i.e., y_1 has r neighbours in G_1, then

$$\mathbb{P}\Big(f_{B_f}(y_1) < \frac{m}{2}\Big) \leq \mathbb{P}_r\Big(\sum_{i \in \tau} d_i > \frac{m}{2}\Big) , \tag{1}$$

where $\mathbb{P}_r$ denotes the probability taken in $[n]^{(r)}$, the space of all r-subsets of $\{1, 2, \ldots, n\}$, and τ is a random element of $[n]^{(r)}$.

With the monotone increasing set system $\mathcal{A} = \{A \in \mathbb{P}(n): \sum_{i \in A} d_i > n/2\}$, inequality (1) becomes

$$\mathbb{P}\Big(d_{B_f}(y_1) < \frac{m}{2}\Big) \leq \mathbb{P}_r(\mathcal{A}) . \tag{2}$$

Setting $p = 5\Delta(W_1)/4n$, we see that with $q = 1-p$ we have $pqn \geq 3$ and $r \leq pn - (3pqn)^{1/2}$. A martingale-type inequality implies that, under these conditions,

$$\mathbb{P}_p(\mathcal{A}) \geq \Big(1 - \frac{1}{e}\Big)\mathbb{P}_r(\mathcal{A}) \geq \tfrac{1}{2}\mathbb{P}_r(\mathcal{A}) , \tag{3}$$

where $\mathbb{P}_p(\mathcal{A})$ is the binomial probability with probability p:

$$\mathbb{P}_p(\mathcal{A}) = \sum_{A \in \mathcal{A}} p^{|A|} q^{n-|A|} .$$

Furthermore, by a standard estimate of the probability in the tail of the binomial distribution,

$$\mathbb{P}_p(\mathcal{A}) < \frac{1}{4m} .$$

Combining this with (1), (2) and (3), we find that

$$\mathbb{P}\Big(d_{B_f}(y_1) < \frac{m}{2}\Big) < \frac{1}{2m} ,$$

as desired. □

The conditions in Theorem 4.4.2 are fairly tight: there are many ways o showing this with the aid of random graphs, but we do not go into the details Note also that in the theorem we proved more than we claimed: for ever $f: U_1 \to U_2$ there is a $g: W_1 \to W_2$ such that (f, g) is a bipartite packing.

4.5. The complexity of graph properties

The *complexity* $c(\mathcal{P})$ of a graph property $\mathcal{P}$ is the minimal number of entries in the adjacency matrix of a graph that must be examined in the worst case in order to decide whether the graph has the property or not. It is convenient to spell out this definition in terms of a *game* $\mathcal{P}$ between two players, called the *Constructor and Algy* (or *Hider* and *Seeker*). Denote by $\mathcal{G}^n$ the set of all graphs with a fixed set V of n vertices, say $V = \{1, 2, \dots, n\}$. Then a *property* $\mathcal{P}$ of graphs on V is a subset of $\mathcal{G}^n$ such that $G \in \mathcal{P}$ whenever a graph isomorphic to G belongs to $\mathcal{P}$. In the game $\mathcal{P}$ Algy asks questions from the Constructor about a graph G on V. Each question is of the form: "Is ab an edge of G?", and each question is answered by the Constructor. When posing a question, Algy takes into account all the information he has received up to that point. The Constructor need not have any particular graph in mind: he may change his choice of graph he is constructing edge by edge according to the questions asked by Algy. The game is over when Algy can decide whether or not the graph the Constructor has been defining will have property $\mathcal{P}$ or not. The aim of the Constructor is to keep Algy guessing for as long as possible. On the other hand, Algy tries to pose as pertinent questions as possible: he would like to decide as soon as possible whether the graph has $\mathcal{P}$ or not. The number of moves of Algy (i.e., the number of questions) in this game, assuming that both players play optimally, is the complexity $c(\mathcal{P})$ of the game $\mathcal{P}$.

Needless to say, the complexity of a digraph property is defined analogously. Moreover, the definition easily carries over to properties of subsets. Given a finite set X, a set system $\mathcal{F}$ on X, i.e., a subset $\mathcal{F}$ of the power set $\mathcal{P}(X)$, is said to be a *property of the subsets of* X. Thus a subset of X has property $\mathcal{F}$ if it belongs to $\mathcal{F}$. Algy's questions are of the form: "Is x an element of our subset $\mathcal{F}$?".

Note that a property of graphs on V is precisely a property of the subsets of $V^{(2)}$, the set of all unordered pairs of elements of V, which is invariant under the permutations (of $V^{(2)}$ induced by the permutations) of V.

A property $\mathcal{F} \subset \mathcal{P}(X)$ is *trivial* if either $\mathcal{F} = 0$ or $\mathcal{F} = \mathcal{P}(X)$; needless to say, one is not interested in trivial properties. As shown by Bollobás and Eldridge (1978), Theorem 4.1.3 concerning the packings of graphs implies a lower bound on the complexity of a non-trivial property of graphs.

Theorem 4.5.1. *The complexity of a non-trivial property of graphs of order n is at least $2n - 4$*

The bound given in this theorem is unlikely to be best possible although, as the following example due to Best et al. (1974) shows, it does give the correct order of magnitude. A *scorpion graph with n vertices* is a graph containing a path bmt such that b (the *body* vertex) has degree $n - 2$, m has degree 2 and t (the *tail* vertex) has degree 1. Note that the graph spanned by the $n - 3$ neighbours of b different from m is entirely arbitrary.

Theorem 4.5.2. *The graph property of containing a scorpion graph has complexity at most $6n$.*

For lack of space, in the rest of the section we shall concentrate on elusi properties. A property $\mathcal{F}$ of the subsets of X is *elusive* if $c(\mathcal{F}) = |X|$, i.e., if eve element of X must be examined in order to decide whether a subset of X belon to $\mathcal{F}$ or not. Thus a property $\mathcal{P}$ of graphs of order n is elusive if $c(\mathcal{P}) = \binom{n}{2}$ and property $\mathcal{Q}$ of digraphs of order n (containing at most one loop at each vertex) elusive if $c(\mathcal{Q}) = n^2$. Best et al. (1974), Kirkpatrick (1974), Milner and Wel (1976), Bollobás (1976b) and Yap (1986) have shown that a good many properti of graphs with n vertices are elusive. These properties include the property being planar (for $n \geq 5$), the property of containing a complete graph with vertices (for $2 \leq r \leq n$), the property of having chromatic number k (for $2 \leq k$ n), the property of being 2-connected, the property of being connected a Eulerian, and the property of being connected and containing a vertex of degr 1.

A property $\mathcal{F}$ of the subsets of a set X is *monotone increasing* if $A \in \mathcal{F}$ a $A \subset B \subset X$ imply that $B \in \mathcal{F}$; a *monotone decreasing* property is defined similarl A property is *monotone* if it is either monotone increasing or monoto decreasing. After some initial difficulties, Aanderaa, Rosenberg, Lipton a Snyder (see Rosenberg 1973 and Lipton and Snyder 1974) advanced t conjecture that every non-trivial monotone property of graphs is close to bei elusive in the sense that $c(\mathcal{P}) \geq \varepsilon n^2$ for some constant $\varepsilon > 0$. A little later, Best al. (1974) advanced a sharper form of this conjecture: every non-trivial monoto graph property is elusive. The weaker form of the conjecture was proved Rivest and Vuillemin (1976).

Theorem 4.5.3. *If $\mathcal{P}$ is a non-trivial property of graphs of order n then $c(\mathcal{P})$ $n^2/16$.*

In fact, Rivest and Vuillemin deduced this result from a theorem claiming th certain set properties are elusive. Given a property $\mathcal{F}$ of subsets of X (i.e., a s system $\mathcal{F} \subset \mathcal{P}(X)$, let $\mathrm{Aut}(\mathcal{F})$ be the group of automorphisms of $\mathcal{F}$, i.e., t group of permutations of X leaving $\mathcal{F}$ invariant: $\mathrm{Aut}(\mathcal{F}) = \{\pi\colon \pi$ is a permu tion of X such that if $A \in \mathcal{F}$ then $\pi(A) \in \mathcal{F}\}$.

Theorem 4.5.4. *Let X be a set with p^r elements, where p is a prime, and let $\mathcal{F}$ b property of subsets of X. If $\mathrm{Aut}(\mathcal{F})$ is transitive on X, $\emptyset \in \mathcal{F}$ and $X \notin \mathcal{F}$ then $\mathcal{F}$ elusive.*

Encouraged by this beautiful result, Rivest and Vuillemin conjectured th Theorem 4.5.4 was true without any restriction on the number of elements of This conjecture has turned out to be false: a counterexample was given by Illi (1978). However, Kahn et al. (1984) proved the exact form of the Best et conjecture for prime power values of n.

Theorem 4.5.5. *Let $n = p^r$ where p is a prime Then every non-trivial monoto property of graphs with n vertices is elusive.*

Kahn et al. used techniques from algebraic topology to prove their beautiful theorem. The crucial step in the proof is that if $\mathscr{F}$ is a non-elusive monotone decreasing property of subsets of X then the abstract simplicial complex of X formed by the elements of $\mathscr{F}$ is collapsible.

The bound $n^2/16$ in Theorem 4.5.3 was improved by Kleitman and Kwiatkowski (1980) to $n^2/9$; Kahn et al. used their theorem to give the even better lower bound $n^2/4 + o(n^2)$.

Let us close with a fascinating conjecture of Kahn et al. (1984) claiming that the analogue of the Best et al. conjecture holds for properties of subsets.

Conjecture 4.5.6. Let $\mathscr{F}$ be a non-trivial monotone property of subsets of X. If $\mathrm{Aut}(\mathscr{F})$ is transitive on X then $\mathscr{F}$ is elusive.

References

Alekseev, V.E.
[1982] Hereditary classes and coding of graphs, *Probl. Cybern.* **39**, 151–164 (in Russian).
[1993] On the entropy values of hereditary classes of graphs, *Discrete Math. Appl.* **3**, 191–199.

Alon, N.
[1985] An extremal problem for sets with applications to graph theory, *J. Combin. Theory A* **40**, 82–89.
[1986] The longest cycle of a graph with a large minimal degree, *J. Graph Theory* **10**, 123–127.

Amar, D., E. Flandrin, I. Fournier and A. Germa
[1983] Hamiltonian pancyclic graphs, *Discrete Math.* **46**, 327.

Babai, L., M. Simonovits and J. Spencer
[1990] Extremal subgraphs of random graphs, *J. Graph Theory* **14**, 599–622.

Baer, R.
[1946] Polarities in finite projective planes, *Bull. Amer. Math. Soc.* **52**, 77–93.

Best, M.R., P. van Emde Boas and H.W. Lenstra
[1974] *A sharpened version of the Aanderaa–Rosenberg Conjecture* (Mathematisch Centrum, Amsterdam).

Bollobás, B.
[1965] On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16**, 447–452.
[1967a] On a conjecture of Erdős, Hajnal and Moon, *Amer. Math. Monthly* **74**, 178–179.
[1967b] Determination of extremal graphs by using weights, *Wiss. Z. Hochsch. Ilmenau* **13**, 4194, 21.
[1976a] On complete subgraphs of different orders, *Math. Proc. Cambridge Philos. Soc.* **79**, 19–24.
[1976b] Complete subgraphs are elusive, *J. Combin. Theory B* **21**, 1–7.
[1978a] *Extremal Graph Theory, London Mathematical Society Monographs*, Vol. 11 (Academic Press, London) xx + 488pp.
[1978b] Problème, in: *Problèmes Combinatoires et Théorie des Graphes*, eds. J.-C. Bermond, J.-C. Fournier, M. Las Vergnas and D. Sotteau (Editions du CNRS, Paris) p. 437.
[1979] *Graph Theory – An Introductory Course, Graduate Texts in Mathematics*, Vol. 63 (Springer, Berlin) x + 180pp.
[1983] Some remarks on packing trees, *Discrete Math.* **46**, 203–204.
[1985] *Random Graphs* (Academic Press, London) xii + 447pp.
[1986] *Extremal Graph Theory with Emphasis on Probabilistic Methods, CBMS Regional Conference Series in Mathematics*, Vol. 62 (American Mathematical Society, Providence, RI) vii + 64pp.

Bollobás, B., and G.R. Brightwell
[1993] Cycles through specified vertices, *Combinatorica* **13**, 147–155.

Bollobás, B., and S.E. Eldridge
[1978] Packing of graphs and applications to computational complexity, *J. Combin. Theory B* **25**, 105–124.

Bollobás, B., and P. Erdős
[1973] On the structure of edge graphs, *Bull. London Math. Soc.* **5**, 317–321.

Bollobás, B., and R. Häggkvist
[1990] The circumference of a graph with a given minimal degree, in: *A Tribute to Paul Erdős*, eds. A. Baker, B. Bollobás and A. Hajnal (Cambridge University Press, Cambridge) pp. 99–106.

Bollobás, B., and A.G. Thomason
[1994a] Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.*, to appear.
[1994b] Hereditary and monotone properties of graphs, to appear.

Bollobás, B., P. Erdős and M. Simonovits
[1976] On the structure of edge graphs II, *J. London Math. Soc.* **12**(2), 219–224.

Bollobás, B., M. Simonovits and E. Szemerédi
[1978] Extremal graphs without large forbidden subgraphs, in: *Advances in Graph Theory*, ed. B. Bollobás (North-Holland, Amsterdam) pp. 29–41.

Bondy, J.A.
[1971a] Large cycles in graphs, *Discrete Math.* **1**, 121–132.
[1971b] Pancyclic graphs I, *J. Combin. Theory B* **11**, 80–84.
[1972] Variations on the hamiltonian theme, *Canad. Math. Bull.* **15**, 57–62.

Bondy, J.A., and V. Chvátal
[1976] A method in graph theory, *Discrete Math.* **15**, 111–135.

Bondy, J.A., and M. Simonovits
[1974] On an upper bound of the graph's chromatic number depending on the graph's degree and density, *J. Combin. Theory B* **23**, 247–250.

Brown, W.G.
[1966] On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9**, 281–285.

Catlin, P.A.
[1974] Subgraphs of graphs, I, *Discrete Math.* **10**, 225–233.
[1977] Embedding subgraphs under extremal degree conditions, in: *Proc. 8th South-Eastern Conf. on Combinatorics, Graph Theory and Optimization, Congress. Numerantium* **XIX**, 139–145.
[1980] On the Hajnal–Szemerédi theorem on disjoint cliques, *Utilitas Math.* **17**, 163–177.

Chung, F.R.K.
[1981] On the decomposition of graphs, *SIAM J. Alg. Discr. Math.* **2**, 1–12.

Chvátal, V.
[1972] On Hamilton's ideals, *J. Combin. Theory B* **12**, 163–168.

Chvátal, V., and P. Erdős
[1972] A note on Hamiltonian circuits, *Discrete Math.* **2**, 111–113.

Chvátal, V., and E. Szemerédi
[1981] On the Erdős–Stone theorem, *J. London Math. Soc.* **23**, 207–214.

Clark, L.H., and R.C. Entringer
[1983] Smallest maximally non-hamiltonian graphs, *Period. Math. Hungar.* **14**, 57–68.

Clark, L.H., R.P. Crane, R.C. Entsinger and H.D. Shapiro
[1986] On smallest maximally nonhamiltonian graphs, in: *Combinatorics, Graph Theory and Computing, Proc. 17th Southeast. Conf., Boca Raton, 1986, Congress. Numerantium* **53**, 215–220.

Corradi, K., and A. Hajnal
[1963] On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14**, 423–439.

Dirac, G.A.
[1952] Some theorems on abstract graphs, *Proc. London Math. Soc.* **2**(3), 69–81.
[1963] Extensions of Turán's theorem on graphs, *Acta Math. Sci. Hungar.* **14**, 418–422.

Duffus, D.A., and D. Hanson
[1986] Minimal k-saturated and color critical graphs of prescribed minimum degree, *J. Graph Theory* **10**, 55–67.

Egawa, Y., and T. Miyamoto
[1989] The longest cycles in a graph G with minimum degree at least $|G|/k$, *J. Combin. Theory B* **46**, 356–362.

Eldridge, S.E.
[1976] *Packings of graphs*, Ph.D. Thesis (University of Cambridge, Cambridge).

Erdős, P.
[1962] On the number of complete subgraphs contained in certain graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **7**, 459–464.
[1965] Extremal problems in graph theory, in: *Theory of Graphs and its Applications*, ed. M. Fiedler (Academic Press, New York) pp. 29–36.
[1967] Extremal problems in graph theory, in: *A Seminar in Graph Theory*, ed. F. Harary (Holt, Rinehart & Winston, New York) pp. 54–64.
[1970] On the graph theorem of Turán (in Hungarian) *Mat. Lapok* **21**, 249–251.
[1975] Some recent progress on extremal problems in graph theory, in: *Proc. Sixth Southeastern Conf. on Combinatorics, Graph Theory and Computing* (Utilitas Math., Winnipeg) pp. 3–14.

Erdős, P., and T. Gallai
[1959] On maximal paths and circuits of graph, *Acta Math. Acad. Sci. Hungar* **10**, 337–356.
[1961] On the minimal number of vertices representing the edges of a graph, *Publ. Math. Inst. Hungar. Acad. Sci.* **6**, 181–203.

Erdős, P., and A. Rényi
[1962] On a problem in the theory of graphs (in Hungarian), *Publ. Math. Inst. Hungar. Acad. Sci.* **7**.

Erdős, P., and H. Sachs
[1963] Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg Math.-Natur. Reihe* **12**, 251.

Erdős, P., and M. Simonovits
[1966] A limit theorem in graph theory, *Studia Sci. Hungar.* **1**, 51–57.
[1982] Compactness results in extremal graph theory, *Combinatorica* **2**, 275–288.
[1983] Supersaturated graphs and hypergraphs, *Combinatorica* **3**, 181–192.

Erdős, P., and A.H. Stone
[1946] On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52**, 1087–1091.

Erdős, P., A. Hajnal and J.W. Moon
[1964] A problem in graph theory, *Amer. Math. Monthly* **71**, 1107–1110.

Erdős, P., A. Rényi and V.T. Sós
[1966a] On a problem of graph theory, *Studia Sci. Math. Hungar.* **1**, 215–235.

Erdős, P., A. Goodman and L. Pósa
[1966b] The representation of graphs by set intersections, *Canad. J. Math.* **18**, 106–112.

Erdős, P., D.J. Kleitman and B.L. Rothschild
[1976] Asymptotic enumeration of K_n-free graphs, in: *Int. Colloq. on Combinatorics, Atti dei Convegni Licei* **17**, Vol. **2**, pp. 19–27.

Erdős, P., P. Frankl and V. Rödl
[1986] The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, *Graphs and Combinatorics* **2**, 113–121.

Fishburn, P.C.
[1983] Packing graphs with odd and even trees, *J. Graph Theory* **7**, 369–383.

Fournier, I., and P. Fraisse
[1985] On a conjecture of Bondy, *J. Combin. Theory B* **39**, 17–26.

Fraisse, P.
[1986] Circuits including a given set of vertices, *J. Graph Theory* **10**, 553–557.

Frankl, P.
[1982] An extremal problem for two families of sets, *European J. Combin.* **3**, 125–127.

Frankl, P., and J. Pach
[1988] An extremal problem on K_r-free graphs, *J. Graph Theory* **12**, 519–523.

Frankl, P., and V. Rödl
[1984] Hypergraphs do not jump, *Combinatorica* **4**, 149–159.

Frankl, P., and B.S. Stečkin
[1982] Problem 6.41, in: *Combinatorial Analysis, Problems and Exercises* (in Russian) (Nauka, Moscow).

Füredi, Z.
[1983] Graphs without quadrilaterals, *J. Combin. Theory B* **34**, 187–190.
[1984] Geometrical solution of an intersection problem for two hypergraphs, *European J. Combin.* **5**, 133–136.

Grünbaum, B.
[1968] A result in graph-coloring, *Michigan Math. J.* **13**, 381–383.

Gyárfás, A., and J. Lehel
[1978] Packing trees of different orders into K_n, *Colloq. Math. Soc. János Bolyai* **18**, 463–469.

Gyárfás, A., J. Komlós and E. Szemerédi
[1984] On the distribution of cycle lengths in graphs, *J. Graph Theory* **8**, 441–462.

Gyárfás, A., H.J. Prömel, E. Szemerédi and B. Voigt
[1985] On the sum of reciprocals of cycle lengths in sparse graphs, *Combinatorica* **5**, 41–52.

Győri, E., and A.V. Kostochka
[1979] On a problem of G.O.H. Katona and T. Tarján, *Acta Math. Acad. Sci. Hungar.* **34**, 321–327.

Häggkvist, R.
[1980] A characterization of non-hamiltonian graphs with large degrees, in: *Combinatorics 79, Part II*, ed M. Deza, and I.G. Rosenberg, *Ann. Discrete Math.* **9**, 259.
[1982] Odd cycles of specified length in non-bipartite graphs, in: *Advances in Graph Theory*, ed. B. Bollobá *Ann. Discrete Math.* **13**, 89–100.
[1989] *On the structure of non-hamiltonian graphs I, II*, Manuscript.
[1990] *Divide and conquer I–V*, Manuscripts.

Hajnal, A., and E. Szemerédi
[1970] Proof of a conjecture of Erdős, in: *Combinatorial Theory and its Applications II*, eds. P. Erd A. Rényi V.T. Sós, *Colloq. Math. Soc. János Bolyai* **4**, 601–623.

Hajnal, P., and M. Szegedy
[1992] Packing bipartite graphs, *Combinatorica* **12**, 295–302.

Hobbs, A.M.
[1981] Packing trees, in: *Proc. 12th South-Eastern Conf. on Combinatorics, Graph Theory* & *Computing* (Utilitas Math., Winnipeg) pp. 63–73.

Illies, N.
[1978] A counter-example to the Generalized Aanderaa-Rosenberg Conjecture, *Inform. Process. Lett.* 154–155.

Imrich, W.
[1984] Explicit construction of regular graphs without small cycles, *Combinatorica* **4**, 53–59.

Isaacs, R.
[1975] Infinite families of nontrivial trivalent graphs which are not Tait colorable, *Amer. Math. Monthly* 221–239.

Jackson, B.
[1979] Edge-disjoint cycles in regular graphs of large degree, *J. London Math. Soc. (2)* **19**, 13–16.
[1980] Hamilton cycles in regular 2-connected graphs, *J. Combin. Theory B* **29**, 27–46.

Jung, H.A.
[1984] Longest circuits in 3-connected graphs, in: *Finite and Infinite Sets*, eds. A. Hajnal, L. Lovász V.T. Sós, *Colloq. Math. Soc. János Bolyai* **37**, 403–438.

Kahn, J., M. Saks and D. Sturtevant
[1984] A topological approach to evasiveness, *Combinatorica* **4**, 297–306.

Kalai, G.
[1984] Intersection patterns of convex sets, *Israel J. Math.* **48**, 161–174.

Kászonyi, L., and Zs. Tuza
[1986] Saturated graphs with minimal number of edges, *J. Graph Theory* **10**, 203–210.
Kirkpatrick, D.
[1974] Determining graph properties from matrix representations, in: *Proc. Sixth SIGACT Conf., Seattle,* pp. 84–90.
Kleitman, D., and D.J. Kwiatkowski
[1980] Further results on the Aanderaa–Rosenberg conjecture, *J. Combin. Theory Ser B* **28**, 85–95.
Kleitman, D.J., and K.J. Winston
[1980] The asymptotic number of lattices, in: *Combinatorial Mathematics, Optimal Designs and their Applications*, ed. S. Shrivastani, *Ann. Discrete Math.* **6**, 243–249.
Kolaitis, P.H., H.J. Prömel and B.L. Rothschild
[1987] K_{l+1}-free graphs: asymptotic structure and a 0–1 law, *Trans. Amer. Math. Soc.* **303**, 637–671.
Kővári, P., V.T. Sós and P. Turán
[1954] On a problem of K. Zarankiewicz, *Colloq. Math.* **3**, 50–57.
Las Vergnas, M.
[1971] Sur une propriété des arbres maximaux dans un graphe, *C.R. Acad. Sci. Paris Ser. A-B* **272**, 1297–1300.
Li, Hao
[1989a] Hamiltonian cycles in regular graphs, *Chinese Sci. Bull.* **34**, 267–268.
[1989b] Edge disjoint cycles in graphs, *J. Graph Theory* **13**, 313–322.
Lipton, R.J., and L. Snyder
[1974] On the Aanderaa–Rosenberg conjecture, *SIGACT News* **6**, 30–31.
Lovász, L.
[1977] Flats in matroids and geometric graphs, in: *Proc. 6th British Combinatorial Conf.*, ed. P.J. Cameron (Academic Press, New York) pp. 45–86.
Lovász, L., and M. Simonovits
[1976] On the number of complete subgraphs of a graph, in: *Proc. 5th British Combinatorial Conf.*, eds. C.St.J.A. Nash-Williams and J. Sheehan (Utilitas Math., Winnipeg) pp. 431–441.
[1983] On the number of complete subgraphs of a graph, II, in: *Studies in Pure Mathematics*, ed. P. Erdős (Birkhäuser, Basel) pp. 459–495.
Lubell, D.
[1966] A short proof of Sperner's lemma, *J. Combin. Theory* **1**, 299.
Mantel, W.
[1907] Problem 28, solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W.A. Wythoff, *Wiskundige Opgaven* **10**, 60–61.
Margulis, G.A.
[1982] Explicit constructions of graphs without short cycles and low density codes, *Combinatorica* **2**, 71–78.
McGuinness, S.
[1994] The greedy clique decomposition of a graph, *J. Graph Theory* **18**, 427–430.
Meshalkin, L.D.
[1963] A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Probab. Ver. Primen.* **8**, 219–220 [1964, *Theory Probab. Appl.* **8**, 204–205].
Milner, E.C., and D.J.A. Welsh
[1976] On the computational complexity of graph theoretical properties, in: *Proc. 5th British Combinatorial Conf.*, eds. C.St.J.A. Nash-Williams and J. Sheehan (Utilitas Math., Winnipeg) pp. 471–487.
Nash-Williams, C.St.J.A.
[1970] Hamilton lines in graphs whose vertices have sufficiently large valencies, in: *Combinatorial Theory and Its Applications*, eds. P. Erdős, A. Rényi and V.T. Sós (North-Holland, Amsterdam) pp. 813–819.
[1971] Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency, in: *Studies in Pure Mathematics*, eds. R. Rado and L. Mirsky (Academic Press, London) pp. 157–183.
Nordhaus, L.A., and B.M. Stewart
[1963] Triangles in an ordinary graph, *Canad. J. Math.* **15**, 33–41.

Ollman, L.T.
[1972] $K_{2,2}$-saturated graphs with a minimal number of edges, in: *Proc. 3rd South-Eastern Conf. on Combinatorics, Graph Theory and Computing* (Utilitas Math., Winnipeg) pp. 367–392.
Ore, O.
[1960] Note on Hamiltonian circuits, *Amer. Math. Monthly* **67**, 55.
Pósa, L.
[1963] On the circuits of finite graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **8**, 355–361.
Prömel, H.J., and A. Steger
[1991] Excluding induced subgraphs: quadrilaterals, *Random Structures and Algorithms* **2**, 55–71.
[1992] Excluding induced subgraphs III: a general asymptotic, *Random Structures and Algorithms* **3**, 19–31.
[1993a] Excluding induced subgraphs II: Extremal graphs, *Discrete Appl. Math.* **44**, 283–294.
[1993b] The assymptotic structure of H-free graphs, in: *Graph Structure Theory, Contemporary Mathematics*, Vol. 147, eds. N. Robertson and P. Seymour (AMS, Providence, RI) pp. 167–178.
Pyber, L.
[1986] Clique coverings of graphs, *Combinatorica* **6**, 393–401.
Reiman, I.
[1958] Über ein Problem von K. Zarankiewicz, *Acta Math. Acad. Sci. Hungar.* **9**, 269–279.
Rivest, R.L., and J. Vuillemin
[1975] A generalization and proof of the Aanderaa–Rosenberg conjecture, in: *Proc. 7th Annu. ACM Symp. on Theory of Computing, Albuquerque, NM* (Assoc. Comput. Mach., New York) pp. 6–11.
Rosenberg, A.G.
[1973] On the time required to recognize properties of graphs: a problem, *SIGACT News* **5**, 15–16.
Sauer, N.
[1971] A generalization of a theorem of Turán, *J. Combin. Theory B* **10/11**, 109–112.
Sauer, N., and J. Spencer
[1978] Edge disjoint placement of graphs, *J. Combin. Theory B* **25**, 295–302.
Shi, Ronghua
[1986] *Zentralblatt* **566**, 30–31.
Simonovits, M.
[1968] A method for solving extremal problems in graph theory, in: *Theory of Graphs*, eds. P. Erdős and T. Gallai (Academic Press, New York) pp. 279–310.
[1974] Extremal graph problems with symmetrical extremal graphs. Additional chromatic conditions, *Discr. Math.* **7**, 349–376.
Skupien, Z.
[1979] On maximal non-Hamiltonian graphs, *Rostock Math. Kolloq.* **11**, 97–106.
Slater, P.J., S.K. Teo and H.P. Yap
[1985] Packing a tree with a graph of the same size, *J. Graph Theory* **9**, 213–216.
Sperner, E.
[1928] Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27**, 544–548.
Straight, H.J.
[1979] Packing trees of different sizes into a complete graph, in: *Topics in Graph Theory*, ed. F. Harary, *Ann. New York Acad. Sci.* **328**, 190–192.
Sumner, D.P.
[1969] On a problem of Erdős, in: *Recent Progress in Combinatorics*, ed. W.T. Tutte (Academic Press, New York) pp. 319–322.
Szemerédi, E.
[1975] On a set containing no k elements in arithmetic progression, *Acta Arithm.* **27**, 199–245.
[1978] Regular partitions of graphs, in: *Problèmes Combinatoires et Théorie des Graphes*, eds. J. C. Bermond, J.-C. Fournier, M. Las Vergnas and D. Sotteau (Editions du CNRS, Paris) pp. 399–40
Teo, S.K.
[1985] *Packing of graphs*, M.Sc. Thesis (Department of Mathematics, National University of Singapore)
Teo, S.K., and H.P. Yap
[1987] Two theorems on packing of graphs, *European J. Combin.* **8**, 199–207

Turán, P.
[1941] On an extremal problem in graph theory (in Hungarian), *Mat. Fiz. Lapok* **48**, 436–452.
[1954] On the theory of graphs, *Colloq. Math.* **3**, 19–30.

Tutte, W.T.
[1947] The factorisation of linear graphs, *J. London Math. Soc.* **22**, 107–111.

Wessel, W.
[1966] Über eine Klasse paarer Graphen, I: Beweis einer Vermutung von Erdős, Hajnal and Moon, *Wiss. Z. Techn. Hochsch. Ilmenau* **12**, 253–256.
[1967] Über eine Klasse paarer Graphen, II: Bestimmung der Minimalgraphen, *Wiss. Z. Techn. Hochsch. Ilmenau* **13**, 423–426.

Woodall, D.R.
[1972] Sufficient conditions for circuits in graphs, *Proc. London Math. Soc.* **24**(3), 739–755.

Yamamoto, K.
[1954] Logarithmic order of free distributive lattices, *J. Math. Soc. Jpn.* **6**, 343–353.

Yap, H.P.
[1986] *Some Topics in Graph Theory, London Mathematical Society Lecture Note Series*, Vol. 108 (Cambridge University Press, Cambridge) vi+230pp.
[1988] Packing graphs – a survey, in: *Proc. First Japan Conf. on Graph Theory and Applications, Hakone, 1986, Discrete Math.* **72**, 395–404.

Zarankiewicz, K.
[1951] Problem P 101, *Colloq. Math.* **2**, 116–131.

Zelinka, B.
[1966] On the number of independent complete subgraphs, *Publ. Math. Debrecen* **13**, 95–97.

CHAPTER 24

Extremal Set Systems

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Contents

HANDBOOK OF COMBINATORICS
Edited by R. Graham, M. Grötschel and L. Lovász

1. Introduction

Let X be an n-element set and $\mathcal{F} \subset 2^X$ a family of *distinct* subsets of X. Suppose that the members of $\mathcal{F}$ satisfy some given conditions. What is the maximum (minimum) value of $|\mathcal{F}|$? This is the generic problem in extremal set theory and we shall try to give an overview of the existing results and methods. Here is the simplest result:

Theorem 1.1. *If $F \cap F' \neq \emptyset$ holds for all F, $F' \in \mathcal{F} \subset 2^X$, then $|\mathcal{F}| \leq 2^{n-1}$.*

Proof. For each $A \subset X$ either A or $X \backslash A$ (or both) are absent from $\mathcal{F}$. Thus $|\mathcal{F}| \leq \frac{1}{2}2^n = 2^{n-1}$. □

2. Basic definitions and conventions

For s, t positive integers, $s \geq 2$, a family $\mathcal{F}$ is called *s-wise t-intersecting* if $|F_1 \cap \cdots \cap F_s| \geq t$ holds for all $F_1, \ldots, F_s \in \mathcal{F}$. If $t = 1$, then t is omitted. Also if $s = 2$, then s-wise is omitted. Thus, "intersecting" means 2-wise 1-intersecting.

A family $\mathcal{F}$ is called *k-uniform* or a *k*-graph if $|F| = k$ for all $F \in \mathcal{F}$.

The *size* of a family $\mathcal{F}$ is $|\mathcal{F}|$ and it is often denoted simply by m. The members of $\mathcal{F}$ are also called *edges*. Let $\binom{X}{k}$ denote the family of all k-element subsets of X.

For $\mathcal{F} \subseteq 2^X$, set $\mathcal{F}^{(i)} = \{F \in \mathcal{F}\colon |F| = i\}$, and $f_i = |\mathcal{F}^{(i)}|$. In this case $f = (f_0, \ldots, f_n)$ is called the *f-vector* of $\mathcal{F}$.

Let $[n]$ denote $\{1, \ldots, n\}$, $[i, j] = \{l\colon i \leq l \leq j\}$. Usually we suppose $X = [n]$.

For $i \in X$, define $\mathcal{F}(i) = \{F \backslash \{i\}\colon i \in F \in \mathcal{F}\}$, the *link* of i; $\mathcal{F}(\bar{i}) = \{F \in \mathcal{F}\colon i \notin F\}$.

The *degree* $d_{\mathcal{F}}(i)$ is simply $|\mathcal{F}(i)|$; $\delta(\mathcal{F})$ and $\Delta(\mathcal{F})$ denote the minimum and maximum degree, respectively.

$\mathcal{F}^c = \{X \backslash F\colon F \in \mathcal{F}\}$ is the *complementary family* of $\mathcal{F}$.

$\mathcal{F} \subset 2^X$ is called *hereditary* if $E \subset F \in \mathcal{F}$ implies $E \in \mathcal{F}$. (Note that $\emptyset \in \mathcal{F}$.)

$\mathcal{F} \subset 2^X$ is called a *filter* if $\mathcal{F}^c$ is hereditary.

The lth *shadow* $\sigma_l(\mathcal{F})$ of a family $\mathcal{F}$ is defined by:

$$\sigma_l(\mathcal{F}) = \left\{G \in \binom{X}{l}\colon \exists F \in \mathcal{F}, G \subset F\right\}.$$

$\partial(\mathcal{F}) = \{G \subseteq X\colon G \notin \mathcal{F}; \exists F \in \mathcal{F}, |G \Delta F| = 1\}$ is called the *boundary* of $\mathcal{F}$.

$\nu(\mathcal{F})$, the *matching number* of $\mathcal{F}$, is the maximum number of pairwise disjoint edges in $\mathcal{F}$; $\nu(\mathcal{F}) = \infty$ if $\emptyset \in \mathcal{F}$.

$\tau(\mathcal{F})$, the *covering number* of $\mathcal{F}$, is the minimum cardinality of a set T with $T \cap F \neq \emptyset$ for all $F \in \mathcal{F}$; $\tau(\mathcal{F}) = \infty$ if $\emptyset \in \mathcal{F}$.

$\mathcal{F}$ is called *ν-critical* if $\nu(\mathcal{G}) > \nu(\mathcal{F})$ holds for every family obtained from $\mathcal{F}$ by replacing one of its edges by a proper subset of it.

$\mathcal{F}$ is called *τ-critical* if $\tau(\mathcal{G}) < \tau(\mathcal{F})$ for all $\mathcal{G} \subset \mathcal{F}$.

$\mathcal{F}$ is called an *antichain* if $F \not\subseteq F'$ holds for all $F, F' \in \mathcal{F}$.

Define the *reverse lexicographic order* $<_L$ on 2^X by $A <_L B$ if $A \subset B$ or $\max\{x \in A\setminus B\} < \max\{x \in B\setminus A\}$.

Let $\mathcal{L}(m, k)$ ($\mathcal{R}(m, k)$) be the largest (smallest) m members of $\binom{[n]}{k}$ in the reverse lexicographic order.

Note that $\mathcal{R}(\binom{s}{k}, k) = \binom{[s]}{k}$.

We call $\mathcal{F}$, $\mathcal{G}$ *cross-intersecting* if $F \in \mathcal{F}$ and $G \in \mathcal{G}$ implies $F \cap G \neq \emptyset$.

$\mathcal{F}$ is called a *sunflower* of size m and with *center* C if $F \cap F' = C$ for all distinct $F, F' \in \mathcal{F}$ and $|\mathcal{F}| = m$.

$\mathcal{F}$ is said to be *intersection-closed* if $F, F' \in \mathcal{F}$ implies $F \cap F' \in \mathcal{F}$.

We close this section with a conjecture of Frankl (1979).

Conjecture 2.1. If $\mathcal{F}$ is intersection-closed, $|\mathcal{F}| \geq 2$, then $\delta(\mathcal{F}) \leq |\mathcal{F}|/2$ holds.

3. Basic theorems

The oldest result in extremal set theory is Sperner's Theorem.

Theorem 3.1 (Sperner 1928). *If $\mathcal{F} \subset 2^X$ is an antichain, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ with equality if and only if $\mathcal{F} = \binom{X}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{X}{\lceil n/2 \rceil}$ holds.*

Recent research on antichains belongs to the theory of partially ordered sets. We refer the reader to chapter 8 or the book by Engel and Gronau (1985).

The maximum size of intersecting k-graphs was determined in 1938 by Erdős, Ko and Rado although they did not publish their result until much later.

Theorem 3.2 (Erdős et al. 1961). *If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting, $k > t \geq 1$, $n \geq n_0(k, t)$, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.*

From the work of Frankl (1978) and Wilson (1984) we know that the conclusion holds if and only if $n \geq (k - t + 1)(t + 1)$.

Another classical result is due to Erdős and Rado (1960).

Theorem 3.3. *If $\mathcal{F} \subset \binom{X}{k}$, $|\mathcal{F}| > k!(r-1)^k$, then $\mathcal{F}$ contains a sunflower of size r.*

Erdős (1981) offers \$1000 for a proof that the same holds for $|\mathcal{F}| > c(r)^k$, where $c(r)$ is an appropriate constant.

Probably the single most important result in finite set theory is the Kruskal–Katona Theorem, which was proved by Kruskal (1963) and Katona (1966) [see also Lindström (1967), where a somewhat weaker statement is proved].

Theorem 3.4. *If $\mathcal{F} \subset \binom{X}{k}$ is a family of size m, then for all $l < k$, $|\sigma_l(\mathcal{F})| \geq \sigma_l(\mathcal{R}(m, k))$.*

Evaluating $|\sigma_l(\mathcal{R}(m, k))|$ one can get explicit bounds, which, however, are often unsuitable for computations. The irregular behaviour of the Kruskal–Katona function is explained in Frankl et al. (1995c). Lovász (1979) gives the following weaker but more convenient version.

Theorem 3.5. *Let $\mathcal{F} \subset \binom{X}{k}$, $|\mathcal{F}| = m$, and define the real number $x \geq k$ by $m = \binom{x}{k}$. Then $|\sigma_l(\mathcal{F})| \geq \binom{x}{l}$ holds for all $l < k$.*

A simple common proof of Theorems 3.4 and 3.5 was given by Frankl (1984). The values of m and k for which $\mathcal{R}(m, k)$ is the unique optimal family in Theorem 3.4 were determined independently by Füredi and Griggs (1986) and Mörs (1985).

Hilton (1976) noticed that the Kruskal–Katona Theorem can be restated in the following form.

Theorem 3.6. *If $\mathcal{F} \subset \binom{X}{k}$ and $\mathcal{G} \subset \binom{X}{l}$ are cross-intersecting, then so are $\mathcal{L}(|\mathcal{F}|, k)$ and $\mathcal{L}(|\mathcal{G}|, l)$.*

Theorem 3.7 (Matsumoto and Tokushige 1989). *If $\mathcal{F} \subset \binom{X}{k}$ and $\mathcal{G} \subset \binom{X}{l}$ are cross-intersecting and $n \geq 2k \geq 2l$, then $|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}\binom{n-1}{l-1}$.*

Another important theorem on shadows is due to Katona (1964).

Theorem 3.8. *If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting, then for all $k - t \leq l \leq k$ one has*

$$|\sigma_l(\mathcal{F})|/|\mathcal{F}| \geq \binom{2k-t}{l} \Big/ \binom{2k-t}{k} \geq 1 .$$

Katona used this theorem to determine the maximum size of t-intersecting families $\mathcal{F} \subset 2^X$, which we will discuss in section 5. Katona showed also that the case $t = 1$ of the Erdős–Ko–Rado Theorem 3.2 is an easy consequence of Theorem 3.8.

The *discrete isoperimetric problem* can be stated as follows: given m, determine $\min\{|\partial\mathcal{F}|: \mathcal{F} \subset 2^X, |\mathcal{F}| = m\}$.

A *ball* with *center* A and *radius* r is the family $\mathcal{B}(A, r) = \{B \subseteq X: |A \,\Delta\, B| \leq r\}$.

If $\mathcal{B}(A, r) \subseteq \mathcal{F} \subseteq \mathcal{B}(A, r+1)$, then $\mathcal{F}$ is called a *generalized ball*. Harper (1966) shows that generalized balls have minimum boundary.

Theorem 3.9. *For every $\mathcal{F} \subset 2^X$ there exists a generalized ball $\mathcal{G} \subset 2^X$ of the same size with $|\partial(\mathcal{F})| \geq |\partial(\mathcal{G})|$.*

A short proof of this result was given by Frankl and Füredi (1981).

For $\mathcal{F} \subset \binom{X}{k}$ one defines its k-boundary $\kappa(\mathcal{F})$ by:

$$\kappa(\mathcal{F}) = \left\{ G \in \binom{X}{k}:\ G \notin \mathcal{F};\ \exists F \in \mathcal{F},\ |G \,\Delta\, F| = 2 \right\} .$$

One of the outstanding open problems is the isoperimetric problem for $\binom{X}{k}$.

Open Problem 3.10. Given m, determine $\min\{|\kappa(\mathcal{F})|: \mathcal{F} \subset \binom{X}{k}, |\mathcal{F}| = m\}$.

The next result is due to Kleitman (1966a).

Theorem 3.11. *Let $\mathcal{C}$, $\mathcal{D} \subset 2^X$ be hereditary. Then*

$$|\mathcal{C} \cap \mathcal{D}| \geq |\mathcal{C}|\,|\mathcal{D}|/2^n .$$

Proof. Apply induction on n, the case $n = 0$ being trivial. Set $c_0 = |\mathcal{C}(\bar{n})|$, $c_1 = |\mathcal{C}(n)|$, $d_0 = |\mathcal{D}(\bar{n})$, and $d_1 = |\mathcal{D}(n)|$. Then

$$\begin{aligned} |\mathcal{C} \cap \mathcal{D}| &= |\mathcal{C}(n) \cap \mathcal{D}(n)| + |\mathcal{C}(\bar{n}) \cap \mathcal{D}(\bar{n})| \\ &\geq (c_1 d_1 + c_0 d_0)/2^{n-1} \quad \text{(by induction)} \\ &= (c_0 + c_1)(d_0 + d_1)/2^n + (c_0 - c_1)(d_0 - d_1)/2^n . \end{aligned}$$

Using $\mathcal{C}(n) \subseteq \mathcal{C}(\bar{n})$ and $\mathcal{D}(n) \subseteq \mathcal{D}(\bar{n})$, $(c_0 - c_1)(d_0 - d_1) \geq 0$, which completes the proof. □

By now there are many generalizations of Theorem 3.11, some of which are discussed in chapter 8.

4. Basic tools

The most useful tool for investigating s-wise t-intersecting families is an operation called shifting, which was introduced by Erdős et al. (1961).

Definition 4.1. For $\mathcal{F} \subset 2^X$ and $1 \leq i < j \leq n$, define the *(i, j)-shift* S_{ij} by $S_{ij}(\mathcal{F}) = \{S_{ij}(F): F \in \mathcal{F}\}$, where

$$S_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} =: \tilde{F} & \text{if } j \in F, i \not\in F \text{ and } \tilde{F} \not\in \mathcal{F} , \\ F & \text{otherwise} . \end{cases}$$

Some of the useful properties of the (i, j)-shift are summarized by the next lemma.

Lemma 4.2.
(i) $|\mathcal{F}| = |S_{ij}(\mathcal{F})|$ *and* $|F| = |S_{ij}(F)|$;
(ii) $\sigma_l(S_{ij}(\mathcal{F})) \subseteq S_{ij}(\sigma_l(\mathcal{F}))$;
(iii) *if $\mathcal{F}$ is s-wise t-intersecting, then so is $S_{ij}(F)$*;
(iv) $\nu(S_{ij}(\mathcal{F})) \leq \nu(\mathcal{F})$.

Iterating the (i, j)-shift for all $1 \leq i < j \leq n$ will eventually produce a family $\mathcal{G}$ which is invariant with respect to the (i, j)-shift.

Definition 4.3. We call $\mathcal{G}$ *stable* if $S_{ij}(\mathcal{G}) = \mathcal{G}$ for all $1 \leq i < j \leq n$. The following result is straightforward to show.

Proposition 4.4. *$\mathcal{G}$ is stable if and only if for all $G \in \mathcal{G}$, $1 \leq i < j \leq n$, with $j \in G$ $i \not\in G$, $(G \setminus \{j\} \cup \{i\})$ is also in $\mathcal{G}$.*

A variation of the (i, j)-shift, called down-shift was defined by Kleitman (1966).

Definition 4.5. For $\mathcal{G} \subset 2^X$ and $i \in X$, define the *down-shift* D_i by $D_i(\mathcal{G}) = \{D_i(G): G \in \mathcal{G}\}$, where

$$D_i(G) = \begin{cases} G - \{i\} & \text{if } i \in G \in \mathcal{G} \text{ and } (G - \{i\}) \not\in \mathcal{G} , \\ G & \text{otherwise} . \end{cases}$$

Define the *trace* $\mathcal{F}|_Y = \{F \cap Y: F \in \mathcal{F}\}$.

Some important properties of the down-shift are summarized in the next lemma; property (ii) is due to Kleitman (1966), and (iii) to Frankl (1983).

Lemma 4.6.
(i) $|D_i(\mathcal{G})| = |\mathcal{G}|$;
(ii) *if $|F \Delta F'| \leq d$ holds for all F, $F' \in \mathcal{F}$, then the same holds for $D_i(\mathcal{F})$*;
(iii) $|D_i(\mathcal{G})|_Y| \leq |\mathcal{G}|_Y|$ *for all $i \in X$ and $Y \subset X$.*

Iterating the down-shift again produces an invariant family.

Proposition 4.7. *$D_i(\mathcal{G}) = \mathcal{G}$ holds for all $i \in X$ if and only if $\mathcal{G}$ is hereditary.*

Let us use this proposition to give a simple proof of the following result which was discovered independently by three sets of authors: Sauer; Shelah and Perles and Vapnik and Chervonenkis.

Theorem 4.8. *If $|\mathcal{F}| > \sum_{0 \leq i < r} \binom{n}{i}$, then there is some $R \in \binom{X}{r}$ with $\mathcal{F}|_R = 2^R$.*

Proof. Suppose that $|\mathcal{F}|_R| < 2^r$ for all $R \in \binom{X}{r}$. In view of Lemma 4.6 (iii) we may apply the down-shift to $\mathcal{F}$, and by Proposition 4.7 obtain a complex $\mathcal{G}$, still satisfying $|\mathcal{G}|_R| < 2^r$ for all $R \in \binom{X}{r}$. However, since $\mathcal{G}$ is hereditary, this implies $|G| < r$ for all $G \in \mathcal{G}$, whence $|\mathcal{G}| \leq \sum_{0 \leq i < r} \binom{n}{i}$ follows. □

We point out that the largest r such that there exists a set $R \in \binom{X}{r}$ with $\mathcal{F}|_R = 2^R$ is called the Vapnik–Chervonenkis dimension of $\mathcal{F}$. This concept has found interesting applications in combinatorial and computational geometry, and learnability theory (e.g., see Blumer et al. 1989, Clarkson et al. 1988, and Linial et al. 1991).

Another important tool for investigating families of finite sets is the inclusion matrices.

Definition 4.9. For $\mathcal{F} \subset 2^X$, the $|\sigma_j(\mathcal{F})|$ by $|\mathcal{F}|$ matrix $M(j, \mathcal{F})$ has its rows indexed by $G \in \sigma_j(\mathcal{F})$ and its columns by $F \in \mathcal{F}$, and its general entry is

$$m(G, F) = \begin{cases} 1 & \text{if } G \subseteq F, \\ 0 & \text{if } G \not\subseteq F. \end{cases}$$

Simple computation gives the next result.

Proposition 4.10. (i) *$M(j, \mathcal{F})^T M(j, \mathcal{F})$ is an $|\mathcal{F}|$ by $|\mathcal{F}|$ matrix with general entry*

$$n(F, F') = \binom{|F \cap F'|}{j};$$

(ii) *$M(j, \mathcal{F})^t M(j, \mathcal{F}^c)$ is an $|\mathcal{F}|$ by $|\mathcal{F}|$ matrix with general entry*

$$n(F, F') = \binom{|F \setminus F'|}{j}.$$

Definition 4.11. $\mathcal{F} \subset \binom{X}{k}$ is called *k-partite* if there exists a partition $X = X_1 \cup \cdots \cup X_k$ with $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$, $1 \leq i \leq k$.

A simple but useful result of Erdős and Kleitman (1968) is the following lemma.

Lemma 4.12. *Every k-graph $\mathcal{F}$ contains a k-partite k-graph $\mathcal{G}$ with $|\mathcal{G}|/|\mathcal{F}| \geq k!/k^k$.*

Definition 4.13. For a k-partite $\mathcal{F} \subset \binom{X}{k}$ and $F \in \mathcal{F}$, define $\Pi(F, \mathcal{F}) = \{\Pi(F \cap F')\colon F \neq F' \in \mathcal{F}\}$, where $\Pi(A) = \{i\colon A \cap X_i \neq \emptyset\}$. (Thus $\Pi(F, \mathcal{F}) \subset 2^{[k]}$.)

Definition 4.14. We call $\mathcal{F} \subset 2^X$ *r-complete* if for all distinct $F, F' \in \mathcal{F}$ there is a sunflower of size r and with center $F \cap F'$ formed by members of $\mathcal{F}$.

Füredi (1983) discovered the following lemma which has since proved very useful.

Lemma 4.15. *There exists a positive constant $c = c(k, l)$ such that every $\mathcal{F} \subset \binom{X}{k}$ has a k-partite subfamily $\mathcal{F}^*$ satisfying*

(i) $|\mathcal{F}^*| \geq c|\mathcal{F}|$;

(ii) *$\mathcal{F}^*$ is k-partite with $\Pi(\mathcal{F}^*) = \Pi(F, \mathcal{F}^*)$ being the same for all $F \in \mathcal{F}^*$;*

(iii) *$\mathcal{F}^*$ is l-complete.*

Proposition 4.16 (Deza). *If $l > k$ in Lemma 4.15, then $\Pi(\mathcal{F}^*)$ is intersection-closed.*

Proof. Take $D', D'' \in \Pi(\mathcal{F}^*)$, and choose $F, F', F'' \in \mathcal{F}^*$ with $D' = \Pi(F \cap F')$, $D'' = \Pi(F \cap F'')$. Let $G_1, \ldots, G_{k+1}$ and $H_1, \ldots, H_{k+1}$ be members of $\mathcal{F}^*$ forming sunflowers with centers $C' = F \cap F'$ and $C'' = F \cap F''$, respectively. The sets $G_1 \setminus C', G_2 \setminus C', \ldots, G_{k+1} \setminus C'$ are pairwise disjoint; thus one of them, say $G_1 \setminus C'$, is disjoint from C''. Similarly, $H_1 \setminus C'', \ldots, H_{k+1} \setminus C''$ are pairwise disjoint, implying that one of them, say $H_1 \setminus C''$, is disjoint from G_1. Now $G_1 \cap H_1 = C' \cap C''$, implying $\Pi(C' \cap C'') = D' \cap D'' \in \Pi(\mathcal{F}^*)$ (in the last step we used that $\mathcal{F}^*$ is k-partite). □

Having some information about $\Pi(\mathcal{F}^*)$, one can often use it to get upper bounds on $|\mathcal{F}^*|$ (and thus for $|\mathcal{F}|$).

Proposition 4.17.

$$|\mathcal{F}^*| \leq \binom{n}{\tau(\Pi(\mathcal{F}^*)^c)}.$$

Proof. Let $T \subset [1, k]$ be a minimal set with $T \cap ([1, k] \setminus P) \neq \emptyset$ for all $P \in \Pi(\mathcal{F}^*)$. That is, $|T| = \tau(\Pi(\mathcal{F}^*)^c)$ and $T \not\subseteq P$ for all $P \in \Pi(\mathcal{F}^*)$. For each $F \in \mathcal{F}^*$, let $T(F)$ be the unique subset of F with $\Pi(T(F)) = T$. Since $T \not\subseteq \Pi(F \cap F')$ for distinct $F, F' \in \mathcal{F}^*$, all the $T(F)$ are distinct subsets of X, which concludes the proof. □

5. Intersecting families

Let us define the family $\mathcal{K}(n, t)$ as follows:

$$\mathcal{K}(n, t) = \begin{cases} \{K \subseteq X\colon |K| \geq (n + t)/2\} & \text{if } n + t \text{ is even}, \\ \{K \subset X\colon |K \cap [2, n]| \geq ((n - 1) + t)/2\} & \text{if } n + t \text{ is odd}. \end{cases}$$

It is easy to check that $\mathcal{K}(n, t)$ is t-intersecting. Let us state and prove Katona's Theorem.

Theorem 5.1 (Katona 1964). *If $\mathcal{K} \subset 2^X$ is t-intersecting, then $|\mathcal{K}| \leq |\mathcal{K}(n, t)|$, and moreover, for $t \geq 2$, equality holds only if $\mathcal{K}$ is (isomorphic to) $\mathcal{K}(n, t)$.*

Proof. Let us start with a definition. $\mathcal{F} \subset 2^X$ has the *t-union property* if $|F \cup F'| \leq n - t$ for all $F, F' \in \mathcal{F}$.

Now $\mathcal{F} = \mathcal{K}^c$ has the t-union property.

We shall deal only with the case $n - t$ odd: the even case is slightly easier. Set $s = (n + 1 - t)/2$. Recall that f_i is the number of i-sets in $\mathcal{F}$.

Claim 5.2.

$$f_i + \frac{i+t-1}{i} f_{n-i-t+1} \le \binom{n}{i}, \quad 0 \le i \le s .$$

Proof. Let us consider $\sigma_i(\mathcal{H}^{(i+t-1)})$. If A is in this family, then $A \notin \mathcal{F}^{(i)}$ since otherwise $|A \cup B^c| = n - t + 1$ holds for $B \in \mathcal{H}^{(i+t-1)}$, $A \subset B$, violating the hypothesis. Thus, $f_i + |\sigma_i(\mathcal{H}^{(i+t-1)})| \le \binom{n}{i}$. Since $\mathcal{H}$ is t-intersecting we may apply Theorem 3.8 to get

$$|\sigma_i(\mathcal{H}^{(i+t-1)})| \ge f_{n-(i+t-1)}(i+t-1)/i ,$$

which yields Claim 5.2. □

Proof of Theorem 5.1 (continued). For $i = s$ one has $n - i - t + 1 = i$ and from Claim 5.2, $f_s \le \binom{n-1}{s-1}$ follows. Adding up this inequality, together with Claim 5.2 applied to $0 \le i < s$ and noting $f_i = 0$ for $i > n - t$, we obtain

$$|\mathcal{F}| \le \binom{n-1}{s-1} + \sum_{0 \le i < s} \binom{n}{i} = 2 \sum_{0 \le i < s} \binom{n-1}{i} = |\mathcal{H}(n,t)| .$$

If $t \ge 2$, then $(i+t-1)/i > 1$; thus in the case of equality $\mathcal{H}^{(i+t-1)} = \emptyset$ and consequently $\mathcal{F}^{(i)} = \binom{[n]}{i}$ for $i < s$, which gives already the bulk of the proof of uniqueness. To conclude the proof one notes that $\mathcal{F}^{(s)}$ is intersecting, and $f_s = \binom{n-1}{s-1}$, so by the uniqueness part of the Erdős–Ko–Rado Theorem (which we will discuss subsequently) $\mathcal{F}^{(s)} = \{F \in \binom{[n]}{s} : 1 \in F\}$. This implies $\mathcal{F} = \mathcal{H}(n,t)^c$. □

Theorem 5.3 (Kleitman 1966b). *Suppose that $\mathcal{F} \subset 2^X$ satisfies $|F \,\Delta F'| \le n - t$ for all $F, F' \in \mathcal{F}$. Then $|\mathcal{F}| \le |\mathcal{H}(n,t)|$.*

Proof. In view of Lemma 4.6 we may repeatedly replace $\mathcal{F}$ by $D_i(\mathcal{F})$. Thus by Proposition 4.7 we may suppose that $\mathcal{F}$ is hereditary. Since for arbitrary $G, G' \in \mathcal{F}$ we can take subsets $F, F' \in \mathcal{F}$ with $F \,\Delta F' = G \cup G'$, $\mathcal{F}$ has the t-union property. Thus Theorem 5.3 follows from Theorem 5.1. □

Let us define some intersecting families $\mathcal{H}(k,s)$ for $2 \le s \le k$:

$$\mathcal{H}(k,s) = \left\{ H \in \binom{[n]}{k} : 1 \in H \text{ and } [2, s+1] \cap H \ne \emptyset \right\} \cup \left\{ H \in \binom{[n]}{k} : [2, s+1] \subseteq H \right\} .$$

It is easy to check that for $n > 2k$, $|\mathcal{H}(k,3)| < \cdots < |\mathcal{H}(k,k)|$ holds. Checking the degrees one sees that

$$\Delta(\mathcal{H}(k,s)) = \binom{n-1}{k-1} - \binom{n-1-s}{k-1} .$$

Theorem 5.4 (Frankl 1987a). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting, $n > 2k$. If*

$$\Delta(\mathcal{F}) \le \binom{n-1}{k-1} - \binom{n-1-s}{k-1}$$

holds for some $2 \le s \le k$, then $|\mathcal{F}| \le |\mathcal{H}(k,s)|$. Moreover, equality holds onl[y] [if $\mathcal{F}$] is isomorphic to $\mathcal{H}(k,s)$, or $s = 3$ and $\mathcal{F}$ is isomorphic to $\mathcal{H}(k,2)$.

Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family in which the intersection of al[l members] satisfies $\bigcap \mathcal{F} = \emptyset$. That is, for each $i \in X$ there is some $F \in \mathcal{F}$ with $i \notin F$[...] implies

$$d_{\mathcal{F}}(i) \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} .$$

Thus Theorem 5.4 implies:

Theorem 5.5 (Hilton and Milner 1967). *If $\mathcal{F} \subset \binom{X}{k}$ is an intersecting famil[y,] $\bigcap \mathcal{F} = \emptyset$, then for $n > 2k$, $|\mathcal{F}| \le |\mathcal{H}(n,k)|$ with equality holding if and [only if] $\mathcal{F} \cong \mathcal{H}(n,k)$, or $k = 3$ and $\mathcal{F} \cong \mathcal{H}(n,2)$.*

Let us mention that the restriction $n > 2k$ is essential because for $n =$ [...] family $\mathcal{F} \subset \binom{[2k]}{k}$ is intersecting if and only if it contains no set together w[ith its] complement. Thus there are $2^{\binom{2k-1}{k-1}}$ distinct intersecting families with [...] members in $\binom{[2k]}{k}$. Can they be regular, i.e., $d_{\mathcal{F}}(i) = d$ for some d and all $i \in$ [...] Simple computation shows that $d = \frac{1}{2}\binom{2k-1}{k-1}$ which is an integer if and only [if k is] not a power of 2.

Theorem 5.6 (Brace and Daykin 1972). *There exists a regular intersecting [family] of maximum size $\binom{2k-1}{k-1}$ in $\binom{[2k]}{k}$ if and only if k is not a power of 2.*

Definition 5.7. Let A denote the set of all even integers $2k$ such that there [is] an intersecting family $\mathcal{F} \subset \binom{[2k]}{k}$ with $|\mathcal{F}| = \binom{2k-1}{k-1}$ and such that the [auto]morphism group $\mathrm{Aut}(\mathcal{F})$ is transitive on $[2k]$.

Theorem 5.8 (Cameron et al. 1989).

(i) *If $a \in A$ then $ab \in A$ for $b \in A$ and for b odd.*
(ii) *$4a + 2 \in A$ for all positive integers a*
(iii) *$3 \cdot 2^k \notin A$ for $k \ge 2$.*

Actually, an even number $2k \in A$ if and only if there is a transitive perm[utation] group on $[2k]$ in which every 2-element has a fixed point.

Conjecture 5.9. $a \cdot 2^d \notin A$ holds for every fixed a and $d > d_0(a)$.

The maximum size of t-intersecting families in $\binom{X}{k}$ is determined

Erdős–Ko–Rado Theorem for $n \geq n_0(k, t)$. However, for $t \geq 2$ this leaves open a whole range of cases $2k - t < n < (k - t + 1)(t + 1)$. Define the t-intersecting families $\mathcal{A}_i = \mathcal{A}_i(n, k, t)$ for $0 \leq i \leq k - t$ by:

$$\mathcal{A}_i = \left\{ A \in \binom{[n]}{k} : |A \cap [2i + t]| \geq i + t \right\}.$$

Conjecture 5.10 (Frankl 1978). If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting and $n \geq 2k - t$, $k \geq t \geq 2$, then

$$|\mathcal{F}| \leq \max_i |\mathcal{A}_i| .$$

Let us prove a weaker statement.

Proposition 5.11. *If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting and $n \geq 2k - t$, then*

$$|\mathcal{F}| \leq \binom{n}{k-t} .$$

Proof. In view of Lemma 4.2 we may assume that $\mathcal{F}$ is stable. The following lemma is often useful.

Lemma 5.12 (Frankl 1978). *If $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting and stable, then $|F \cap F' \cap [2k - t]| \geq t$, i.e., $\mathcal{F}_{[2k-t]}$ is t-intersecting.*

Proof. Suppose that Lemma 5.12 is not true and choose a counterexample (F, F') with $|F \cap [2k - t]|$ as large as possible. Fix $j \in F \cap F'$ with $j > 2k - t$. If $i \notin F \cup F'$ for some $i \in [2k - t]$, then replacing (by Proposition 4.4) F by $(F \setminus \{j\}) \cup \{i\}$ contradicts the maximality of $|F \cap [2k - t]|$. Thus $F \cup F' \supseteq [2k - t]$. However,

$$|(F \cup F') \cap [2k - t]| \leq |F| + |F'| - |F \cap F' \cap [2k - t]| < 2k - t ,$$

a contradiction. □

Proof of Proposition 5.11 (continued). Apply induction on k. The case $k = t$ is trivial. Also, in the case $n = 2k - t$ one has $|\mathcal{F}| \leq \binom{2k-t}{k} = \binom{2k-t}{k-t}$. Let $n > 2k - t$ and define

$$\mathcal{F}_i = \left\{ A \in \binom{[2k-t]}{i} : \exists F \in \mathcal{F}, A = F \cap [2k - t] \right\} .$$

Then by Lemma 5.12 and induction,

$$|\mathcal{F}_i| \leq \binom{2k-t}{i-t}$$

holds. This implies

$$|\mathcal{F}| \leq \sum_{i=t}^{k} \binom{2k-t}{i-t}\binom{n-2k+t}{k-i} = \binom{n}{k-t} .$$ □

Theorem 5.13 (Kleitman 1966a). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^X$ be intersecting. Th* $|\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r| \leq 2^n - 2^{n-r}$.

Proof. Apply induction on r; the case $r = 1$ is just Theorem 1.1. We can assu that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are filters. Consider $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{r-1}$. By the inducti hypothesis, $|\mathcal{F}| \leq 2^n - 2^{n-r+1}$. Also $|\mathcal{F}_r| \leq 2^{n-1}$ by Theorem 1.1. Since $\mathcal{F}$ and are both filters, using Theorem 3.11 we obtain $|\mathcal{F} \cap \mathcal{F}_r| \geq |\mathcal{F}| \cdot |\mathcal{F}_r|/2^n$. Sumn rizing,

$$|\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r| = |\mathcal{F}_r| + |\mathcal{F}| - |\mathcal{F} \cap \mathcal{F}_r| \leq |\mathcal{F}_r| + |\mathcal{F}| - \frac{|\mathcal{F}_r||\mathcal{F}|}{2^n} .$$

The right-hand side is monotone increasing in both $|\mathcal{F}|$ and $|\mathcal{F}_r|$. Thus we an upper bound by substituting $|\mathcal{F}_r| = 2^{n-1}$ and $|\mathcal{F}| = 2^n - 2^{n-r+1}$. This comple the proof.

Another application of Theorem 3.11 is the following result which was pro originally in a different way by Daykin and Lovász, and Schönheim.

Theorem 5.14. *If $\mathcal{F} \subset 2^X$ is intersecting and has the "union property" ($F \cup F' \neq$ for $F, F' \in \mathcal{F}$), then $|\mathcal{F}| \leq 2^{n-2}$.*

Proof. Define

$$\mathcal{F}^* = \{G \subseteq X : \exists F \in \mathcal{F}, F \subseteq G\} , \quad \text{and} \quad \mathcal{F}_* = \{G : \exists F \in \mathcal{F}, G \subseteq F\} .$$

Then $\mathcal{F}^*$ is an intersecting filter and $\mathcal{F}_*$ is hereditary and has the union prope Using Theorems 1.1 and 3.11 we deduce

$$|\mathcal{F}| \leq |\mathcal{F}^* \cap \mathcal{F}_*| \leq |\mathcal{F}^*||\mathcal{F}_*|/2^n \leq 2^{n-2}$$

It was shown by Frankl (1975) (proving a conjecture of Katona) that maximum size of an intersecting family having the t-union property is $|\mathcal{K}($ $1, t)|$.

Example. Let $t, t' \geq 1$ and suppose $X = Y \cup Y'$ is a partition with $|Y| \geq t$, $|Y'|$ Let $\mathcal{A} \subset 2^Y$ be a copy of $\mathcal{K}(|Y|, t)$ and $\mathcal{B} \subset 2^{Y'}$ be a copy of $\mathcal{K}(|Y'|, t')^c$. $\mathcal{C}(Y) = \{H \subset X : H \cap Y \in \mathcal{A}, H \cap Y' \in \mathcal{B}\}$. Then $\mathcal{C}$ is t-intersecting and has t'-union property.

Conjecture 5.15. If $\mathcal{F} \subset 2^X$ is t-intersecting and has the t'-union property, t $|\mathcal{F}| \leq |\mathcal{C}(Y)|$ for an appropriate $Y \subset X$.

This conjecture can be found in Frankl's dissertation of 1976 and first appe in English in Bang et al. (1981).

Let us close this section with the following important conjecture of Chvát

Conjecture 5.16. If $\mathscr{C}$ is hereditary, $\mathscr{F} \subset \mathscr{C}$, and $\mathscr{F}$ is intersecting, then $|\mathscr{F}| \leqslant \Delta(\mathscr{C})$.

For some partial results and references on this conjecture see Miklòs (1984).

6. Families with prescribed intersection sizes

Let $L = \{l_0, \ldots, l_{s-1}\} \subseteq [0, k-1]$ with $l_0 < l_1 < \cdots < l_{s-1}$.

Definition 6.1. A family $\mathscr{F} \subseteq \binom{X}{k}$ is called an *(n, k, L)-system*, or an *L-system* for short, if $|F \cap F'| \in L$ holds for all distinct $F, F' \in \mathscr{F}$. For example, a t-intersecting family is an L-system with $L = \{t, t+1, \ldots, k-1\}$.

Definition 6.2. Let $m(n, k, L)$ denote the maximum size of an (n, k, L)-system.

The next fundamental theorem was first proved by Deza.

Theorem 6.3 (Deza et al. 1978).

$$m(n, k, L) \leqslant \prod_{l \in L} (n-l)/(k-l) \quad \textit{for } n > n_0(k, L).$$

We remark that an L-system $\mathscr{F} \subset \binom{X}{k}$ with $L = \{0, 1, \ldots, t-1\}$, is called a partial t-design and clearly Theorem 6.3 holds for all $n \geqslant k$ in this case. A celebrated result of Rödl (1985) is the following.

Theorem 6.4.

$$m(n, k, \{0, 1, \ldots, t-1\}) = (1 - o(1))\binom{n}{t} \Big/ \binom{k}{t},$$

where $k \geqslant t > 0$ are fixed and $n \to \infty$.

Taking $L = \{t, t+1, \ldots, k-1\}$, one sees that for $n > n_0(k, L)$, Theorem 6.3 extends Theorem 3.2.

Definition 6.5. We say that Theorem 6.3 is *asymptotically exact* (respectively, gives the *correct exponent*) if

$$\limsup_{n \to \infty} m(n, k, L) \Big/ \prod_{l \in L} \frac{n-l}{k-l}$$

is equal to one (respectively, is positive). For example, Theorem 6.4 shows that Theorem 6.3 is asymptotically exact for all $k \geqslant t > 0$ and $L = \{0, 1, \ldots, t-1\}$.

Definition 6.6. $L - a = \{l - a: l \in L\}$.

In view of the following result of Deza et al. (1978) we may suppose in what follows that $0 \in L$.

Proposition 6.7. $m(n, k, L) = m(n - l_0, k - l_0, L - l_0)$ *for* $n > n_0(k, L)$.

The next result gives some values of k and L for which Theorem 6.3 is asymptotically exact.

Theorem 6.8 (Frankl and Rödl 1985). *Let $d \geqslant t > 0$ and let q be a prime power. Then Theorem 6.3 is asymptotically exact for*

$$k = q^d, \qquad L = \{0, 1, \ldots, q^{t-1}\}$$

and

$$k = (q^d - 1)/(q-1), \qquad L = \{(q^i - 1)/q - 1): i = 0, 1, \ldots, t-1\}.$$

Definition 6.9. $a(k, L) = \sup\{\alpha: \limsup_{n \to \infty} m(n, k, L) n^{-\alpha} > 0\}$.

That is, $a(k, L) \leqslant |L|$ with equality if and only if Theorem 6.3 gives the correct exponent. Clearly $a(k, L) \geqslant 1$ for all $\emptyset \neq L \subseteq [0, k-1]$.

Conjecture 6.10. There exist positive constants $c(k, L)$ and $\tilde{c}(k, L)$ for all k, L such that

$$c(k, L) n^{a(k,L)} < m(n, k, L) < \tilde{c}(k, L) n^{a(k,L)}.$$

Theorem 6.11 (Frankl 1986b). *For every rational number $\alpha \geqslant 1$ there are infinitely many choices of k and L for which Conjecture 6.10 holds with $a(k, L) = \alpha$.*

One can use Lemma 4.15 and Proposition 4.16 to get upper bounds on $a(k, L)$. Let $\mathscr{F}$ be an (n, k, L)-system and apply Lemma 4.15 with $l = k + 1$ to get the intersection-closed family $\mathscr{A} = \prod(\mathscr{F}^*) \subseteq 2^{[k]}$.

We call a set $B \subset [k]$ a *base* (for $\mathscr{A}$) if $B \not\subseteq A$ for all $A \in \mathscr{A}$ but no proper subset of B has this property. Also, $b(\mathscr{A}) = \min\{|B|: B \text{ is a base}\}$.

For $D \subseteq [k]$ define $\langle D \rangle = \bigcap \{A: D \subseteq A \in (\mathscr{A} \cup \{[k]\})\}$. That is, $\langle D \rangle = [k]$ if and only if D contains some base for $\mathscr{A}$.

Since $\mathscr{F}^*$ is an L-system, $|A| \in L$ for all $A \in \mathscr{A}$. By Proposition 4.16, there is at most one l_0-element set in $\mathscr{A}$ and one can prove easily that $b(\mathscr{A}) \leqslant |L|$. In fact, more is true. For elementary properties of matroids, we refer the reader to chapter 9.

Theorem 6.12 (Frankl 1982). $b(\mathscr{A}) \leqslant |L| - 1$ *unless $\mathscr{A} \cup [k]$ forms the flats of a matroid of rank $|L|$. In this case $b(\mathscr{A}) = |L|$.*

Proof. We apply induction on k; the case $k = 1$ is trivial. Suppose that $b(\mathscr{A})$

$|L|$. Define $\mathscr{A}_i = \{A \in \mathscr{A}: |A| = l_i\}$, $0 \le i < s = |L|$. We have to show that for every $A \in \mathscr{A}_i$ and $x \in [k] \setminus A$, there is a unique member of $\mathscr{A}_{i+1}$ containing both x and A. Define $\tilde{A} = \bigcap \{A': (A \cup \{x\}) \subseteq A' \in \mathscr{A}\}$. Then $\tilde{A} \in \mathscr{A}$. All we have to show is $\tilde{A} \in \mathscr{A}_{i+1}$. It is easy to see that there exists a set D with $\langle D \rangle = A$, $|D| \le i$. Also, if $\tilde{A} \in \mathscr{A}_j$, then one can find a set E with $|E| \le s - j$ and $\langle \tilde{A} \cup E \rangle = [k]$. Thus $\langle D \cup \{x\} \cup E \rangle = [k]$, giving $i + 1 + s - j \le s$, i.e., $j \le i + 1$. Since $|\tilde{A}| > l_i$, $j = i + 1$ follows. □

Definition 6.13. Define $b(k, L) = \max b(\mathscr{A})$, where the maximum is taken over all intersection-closed families $\mathscr{A} \subset 2^{[k]}$ with $|A| \in L$ for all $A \in \mathscr{A}$.

Conjecture 6.14 (Füredi 1983). *$a(k, L) > b(k, L) - 1$ for all k and L.*

Since $a(k, L) \le b(k, L)$ by Proposition 4.17, this conjecture would mean that $\lceil a(k, L) \rceil = b(k, L)$ holds.

The smallest open cases are $L = \{0, 1, 3\}$, $k \equiv 1$ or $3 \pmod 6$, $k \ge 13$ [$b(k, L) = 3$ in this case, but $a(k, L) > 2$ is unknown for $k \ne 3^d$ or $2^d - 1$], and $L = \{0, 1, 2, 3, 5\}$, $k = 11$ [$b(k, L) = 5$ in this case]. Recently, all exponents for $k \le 10$ were determined by Frankl et al. (1995b).

In Deza et al. (1985), an infinite family of cases where Theorem 6.3 gives the correct exponent is exhibited, e.g., $L = \{0, 1, 2, q + 1\}$, $k = q^2 + 1$, q a prime power.

For k and L with $b(k, L) = 1$, Conjecture 6.14 is obvious, since then $a(k, L) = 1$ follows from $a(k, L) \le b(k, L)$. If $b(k, L) = 2$, then $a(k, L) > 1$ follows using constructions due to Frankl (see Füredi 1983).

A general upper bound, extending earlier results of Ray-Chaudhuri and Wilson (1975) and Babai and Frankl (1980), is the following.

Theorem 6.15 (Frankl and Wilson 1981). *Suppose that p is a prime such that $k \not\equiv l \pmod p$ holds for all $l \in L$. Let r be the number of residue classes of L modulo p. Then*

$$m(n, k, L) \le \binom{n}{r}.$$

7. One missing intersection

An important special case of the problem treated in the preceding section is when $L = [0, k - 1] \setminus \{l\}$ for some $l \in [0, k - 1]$.

Set $m(n, k, \bar{l}) = m(n, k, [0, k - 1] \setminus \{l\})$.

There are two natural constructions for excluding the intersection size l. One is by taking all k-subsets of X containing a fixed $(l + 1)$-element subset. This gives

$$m(n, k, \bar{l}) \ge \binom{n - l - 1}{k - l - 1}.$$

The other is by taking a partial l-design. By Rödl's Theorem 6.4 this gives a lower bound of $(1 - o(1))\binom{n}{l}/\binom{k}{l}$. The next result of Frankl and Füredi (1985) shows that one of these constructions always gives the correct exponent.

Theorem 7.1. $m(n, k, \bar{l}) = O(n^{\max\{l, k-l-1\}})$.

Proof. Consider $\mathscr{A} = \prod(\mathscr{F}^*)$ from the preceding section. We have to show that $b(\mathscr{A}) \le \max\{l, k - l - 1\}$. Let B be a base for $\mathscr{A}$ and suppose that $|B| \ge l$. For $x \in B$ consider $A_x = \langle B \setminus \{x\} \rangle \in \mathscr{A}$. Note that $A_x \cap B = B \setminus \{x\}$. Define the family (of not necessarily distinct sets)

$$\mathscr{C} = \{A_x \setminus B: x \in B\} \subseteq 2^{[k] \setminus B}.$$

Claim 7.2. *The size of the intersection of r members of $\mathscr{C}$ is never $r - c$, $1 \le r \le |B| = |\mathscr{C}|$, where $c = |B| - l > 0$.*

Proof. Since for distinct elements $x_1, \ldots, x_r \in B$ one has $|A_{x_1} \cap \cdots \cap A_{x_r} \cap B| = |B| - r$, $|A_{x_1} \cap \cdots \cap A_{x_r}| \ne l$ implies the claim. □

Proof of Theorem 7.1 (continued). Now a simple result of Frankl and Katona (cf. Frankl and Füredi 1985) says that any family $\mathscr{C}$ of not necessarily distinct subsets of a b-element set and satisfying the assertion of Claim 7.2 has $|\mathscr{C}| \le b + c -$ Since in our case $b = k - |B|$, $c = |B| - l$, we infer that $|B| = |\mathscr{C}| \le k - l - 1$. Since B was an arbitrary base for $\mathscr{A}$, the result follows.

For the case $k > 2l + 1$, more is true.

Theorem 7.3 (Frankl and Füredi 1985). *$m(n, k, \bar{l}) = \binom{n - l - 1}{k - l - 1}$ holds for $k \ge 2l +$ and $n > n_0(k)$. Moreover, the only optimal family is $\mathscr{F} = \{F \in \binom{[n]}{k}: [l + 1] \subset F$*

For $k \le 2l + 1$ one can improve on the lower bound given by partial l-design

Proposition 7.4. *Let $\mathscr{P} \subset \binom{[n]}{2k - l - 1}$ be a partial l-design. Then $|F \cap F'| \ne l$ for $F, F' \in \sigma_k(\mathscr{P})$.*

Proof. Take $P, P' \in \mathscr{P}$ with $F \subset P$, $F' \subset P'$. If $P \ne P'$, then $|F \cap F'| \le |P \cap P'| <$ If $P = P'$ then $|F \cap F'| \ge |F| + |F'| - |P| = l + 1$.

Using Theorem 6.4 again one obtains

$$m(n, k, \bar{l}) \ge (1 - o(1))\binom{2k - l - 1}{k}\binom{n}{l} \Big/ \binom{2k - l - 1}{l}.$$

This inequality is partially complemented by the following result of Frankl (198 Recall that an $S(n, a, l)$ is a partial l-design $\mathscr{S} \subset \binom{[n]}{a}$ with $|\mathscr{S}| = \binom{n}{l}/\binom{a}{l}$.

Theorem 7.5.

$$m(n,k,\bar{l}) \leq \binom{2k-l-1}{k}\binom{n}{k}\Big/\binom{2k-l-1}{l}$$

holds if $k \geq 2l+1$ and $k-l$ is a prime power. Moreover, if $k-l$ is a prime, then equality is achieved only for $\sigma_k(\mathcal{S})$ where $\mathcal{S}$ is an $S(n, 2k-l-1, l)$.

Conjecture 7.6. Theorem 7.5 holds even if $k-l$ is not a prime power.

Settling a long-standing open problem of Erdős (cf. Erdős 1981), the following result was proved in Frankl and Rödl (1986).

Theorem 7.7. *Let $0<\alpha\leq\frac{1}{4}$ and l be an integer, $\alpha n \leq l \leq (\frac{1}{2}-\alpha)n$. Then there exists $\varepsilon=\varepsilon(\alpha)>0$ such that every family $\mathcal{F}\subset 2^{[n]}$ with $|\mathcal{F}|>(2-\varepsilon)^n$ contains two sets whose intersection has size exactly l.*

For l fixed and n sufficiently large the problem was solved exactly by Frankl and Füredi (1984a). To avoid intersections of size l one can take $\mathcal{K}(n, l+1)$ which is $(l+1)$-intersecting from Katona's Theorem 5.1 and adjoin all subsets of size less than l.

Theorem 7.8. *If $\mathcal{F}\subset 2^X$ satisfies $|F\cap F'|\neq l$ for all distinct F, $F'\in\mathcal{F}$, then*

$$|\mathcal{F}| \leq |\mathcal{K}(n, l+1)| + \sum_{i<l}\binom{n}{i}$$

for $n>n_0(l)$.

An important tool in the proof is the following result extending Theorem 3.8 on the shadow of t-intersecting families. Recalling the definition of M, we have:

Theorem 7.9. *Suppose that the columns of $M(j,\mathcal{F})$ are linearly independent over $\mathbb{R}$, where $\mathcal{F}\subset\binom{X}{k}$. Then $|\sigma_s(\mathcal{F})|/|\mathcal{F}| \geq \binom{k+j}{s}/\binom{k+j}{k}$ for all $j\leq s<k$.*

The following problem was raised by Larman and Rogers (1972). Determine

$$s(n)=\max\{|\mathcal{F}|: \mathcal{F}\subset 2^{[n]}, |F\,\Delta F'|\neq n/2 \text{ for all } F, F'\in\mathcal{F}\}.$$

It is easy to see that $s(n)=2^n$ if n is odd and that $s(n)=2^{n-1}$ if $n\equiv 2 \pmod 4$. Let $n=4l$ and consider the following family:

$$\mathcal{R}(l)=\{R, R^c: R\in 2^{[n]}, |R\cap[n-1]|\leq l-1\}.$$

Then $|\mathcal{R}(l)| = 4\sum_{i<l}\binom{4l-1}{i}$ and $|R\,\Delta R'|\neq 2l$ for all R, $R'\in\mathcal{R}(l)$.

Theorem 7.10 (Frankl 1986a). $s(4l)=4\sum_{i<l}\binom{4l-1}{i}$ *if l is the power of an odd prime.*

Conjecture 7.11. Theorem 7.9 holds for all positive integers l.

8. s-wise t-intersecting families

Let $q(n,s,t)$ denote the maximum size of an s-wise t-intersecting family $\mathcal{F}\subset 2^X$. For a more complete treatment we refer the reader to Frankl (1987b).

Proposition 8.1. *$q(n,s,t)/2^{-n}$ is monotone increasing and therefore $q(s,t)=\lim_{n\to\infty} q(n,s,t)/2^{-n}$ exists.*

Proof. If $\mathcal{F}\subset 2^X$ is s-wise t-intersecting, then so is $\mathcal{F}'=\mathcal{F}\cup\{F\cup\{n+1\}: F\in\mathcal{F}\}$, showing $q(n+1,s,t)\geq 2q(n,s,t)$, as desired. The second part of the proposition is a direct consequence of the first part. □

From the proposition we see that $q(s,t)\leq\frac{1}{2}$ for all $s\geq 2$, $t\geq 1$. Since $\lim_{n\to\infty}|\mathcal{K}(n,t)|/2^n=\frac{1}{2}$, $q(2,t)=\frac{1}{2}$ for all $t\geq 1$.

In view of Lemma 4.2 (iii), from now on $\mathcal{F}\subset 2^X$ will be a stable, s-wise t-intersecting family of maximum size. (Consequently, $\mathcal{F}$ is a filter.) Define the sets:

$$A_i=[n]\setminus\{t+i+ps: 0\leq p\leq(n-t-i)/s\}$$

for $0\leq i<s$ and note that

$$A_0\cap\cdots\cap A_{s-1}=[t-1]. \tag{8.2}$$

Lemma 8.3. (i) $A_0\notin\mathcal{F}$;
(ii) *for every $F\in\mathcal{F}$ there exists a $j\geq 0$ with $|F\cap[t+sp]|\geq t+(s-1)p$.*

Proof. Since $\mathcal{F}$ is a stable filter, $A_0\in\mathcal{F}$ would imply by repeated applications o[f] Proposition 4.4 that $A_i\in\mathcal{F}$, $1\leq i\leq s-1$. However, by (8.2) this is impossible which proves (i). To prove (ii), suppose that $F=[n]\setminus\{a_0,\ldots,a_l\}$ is in $\mathcal{F}$ [in particular, $l\geq(n-t)/p$] $1\leq a_0<\cdots<a_l$. If $a_p\leq t+ps$ for $0\leq p\leq(n-t)/p$ then $A_0\in\mathcal{F}$ follows from Proposition 4.4, contradicting (i). Thus for some p w[e] have $a_p>t+ps$, i.e., $|F\cap[t+ps]|\geq t+p(s-1)$, as desired. □

Let us consider the polynomial x^s-2x+1, for $s\geq 3$. It has exactly one root say $\beta(s)$, in the open interval $(\frac{1}{2},1)$. For example, $\beta(3)=(\sqrt{5}-1)/2$.

Theorem 8.4 (Frankl 1976). $q(n,s,t)<2^n\beta(s)^t$.

Proof (*sketch*). Consider the probability space of all infinite (0, 1)-sequences wit[h] the uniform distribution. Standard computation shows that the probability of the event {there exists $p \geq 0$ such that the number of 1's up to $t+ps$ is $\geq t+p(s-1)$} is $\beta(s)^t$. By Lemma 8.3 this is a (strict) upper bound on $|\mathcal{F}|/2^n$ [we associate with $F \in \mathcal{F}$ all the (0, 1)-sequences extending its characteristic vector]. □

Define the families:

$$\mathcal{B}_p = \mathcal{B}_p(n,s,t) = \{B \subseteq [n]: |B \cap [t+sp]| \geq t + (s-1)p\}, \quad p \leq (n-t)/s.$$

Then $\mathcal{B}_p$ is s-wise t-intersecting and $|\mathcal{B}_p|/2^n$ is independent of n. The following result combines Theorem 8.4 and some computation involving $|\mathcal{B}_p|/2^n$.

Corollary 8.5. *There exists a positive constant c such that $c\beta(s)^t/t < q(t,s) < \beta(s)^t$.*

Conjecture 8.6. $q(n,s,t) = \max\{|\mathcal{B}_p|: 0 \leq p \leq (n-t)/s\}$.

Let us mention that Conjecture 8.6 holds for $s = 2$ (Katona's Theorem) and in general for $t < s \cdot 2^s/150$ (Frankl 1979). It also holds for $s \geq t \geq 2$ with $q(n,s,t) = 2^{n-t}$. Next, we show how to use this last result to give a simple proof of an important theorem of Brace and Daykin (1971).

Theorem 8.7. *Let $\mathcal{F} \subset 2^{[n]}$ be s-wise intersecting with $\bigcap \mathcal{F} = \emptyset$. Then*

$$|\mathcal{F}| \leq |\mathcal{B}_1(n,s,1)| = (s+2)2^{n-s-1}$$

Proof. We may suppose that $\mathcal{F}$ is a filter and thus, since $\bigcap \mathcal{F} = \emptyset$, it contains $[n]\backslash\{i\}$ for all $1 \leq i \leq n$. This will not change by shifting. Therefore, we may assume that $\mathcal{F}$ is stable.

We apply induction on s. For $s = 2$, one has $|\mathcal{B}_1(n,2,1)| = 2^{n-1}$; thus the statement follows from Theorem 1.1. Let $s \geq 3$ and suppose that Theorem 8.7 has been proved for smaller values of s. Consider $\mathcal{F}(1)$ and $\mathcal{F}(\bar{1})$.

Claim 8.8. (i) $|\mathcal{F}(1)| \leq (s+1)2^{n-s-1}$;
(ii) $|\mathcal{F}(\bar{1})| \leq 2^{n-s-1}$.

Now the theorem follows from $|\mathcal{F}| = |\mathcal{F}(1)| + |\mathcal{F}(\bar{1})|$ once we prove the claim.

Proof of Claim 8.8. Note that $\mathcal{F}(1)$ is $(s-1)$-wise intersecting on $[2,n]$, since otherwise $F_1 \cap \cdots \cap F_{s-1} = \{1\}$ for some $F_1, \ldots, F_{s-1}$ implying $\{1\} \in \bigcap \mathcal{F}$. Also, $(n]\backslash\{i\}) \in \mathcal{F}$ implies $([2,n]\backslash\{i\}) \in \mathcal{F}(1)$ for $2 \leq i \leq n$. Thus $\bigcap \mathcal{F}(1) = \emptyset$. Hence, (i) follows from the induction assumption. To prove (ii), we only have to show that $\mathcal{F}(\bar{1})$ is s-wise s-intersecting (on $[2,n]$). Otherwise, since $\mathcal{F}(\bar{1})$ is a stable filter, we can find $F_1, \ldots, F_s \in \mathcal{F}(\bar{1}) \subset \mathcal{F}$ with $F_1 \cap \cdots \cap F_s = [2,s]$. Define $G_i = (F_i\backslash\{i\}) \cup \{1\}$ for $i = 2, \ldots, s$. Then $G_i \in \mathcal{F}$ by Proposition 4.4. However, $F_1 \cap G_2 \cap \cdots \cap G_s = \emptyset$, which is a contradiction. □

9. The covering number

Recall the definition of $\tau(\mathcal{F})$.

Theorem 9.1 (Gyárfás 1977). *A k-graph $\mathcal{F}$ has at most $k^{\tau(\mathcal{F})}$ covers T of size $\tau(\mathcal{F})$.*

Proof. Set $t = \tau(\mathcal{F})$. We prove by backward induction on $l \leq t$ that every l-element set is contained in at most k^{t-l} covers of $\mathcal{F}$. The case $l = t$ is trivial and the case $l = 0$ will prove the theorem.

Let $0 \leq l < t$ and consider an l-element set A. Since $l < t = \tau(\mathcal{F})$, there exists an $F \in \mathcal{F}$ with $A \cap F = \emptyset$. Every cover of $\mathcal{F}$ containing A must contain at least one of the $(l+1)$-element sets $A \cup \{x\}$, $x \in F$. Each of these sets is (by induction) in at most k^{t-l-1} covers of $\mathcal{F}$ of size t. This gives altogether $k \cdot k^{t-l-1} = k^{t-l}$. □

For a generalization see Tuza (1988).

Considering τ pairwise disjoint sets of size k shows that Theorem 9.1 is best possible. An important corollary of the theorem is the following.

Theorem 9.2 (Erdős and Lovász 1975). *Let $\mathcal{F}$ be an intersecting k-graph with $\tau(\mathcal{F}) = k$. Then $|\mathcal{F}| \leq k^k$.*

Proof. Every $F \in \mathcal{F}$ is a cover of size k.

Construction (Erdős and Lovász 1975). Let $X_1, \ldots, X_k$ be disjoint sets of size $1, \ldots, k$, respectively. Define

$$\mathcal{E}_i = \{E: |E| = k, X_i \subset E, X_j \cap E \neq \emptyset, i < j \leq k\}.$$

Set $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$.

Now $\mathcal{E}$ is intersecting with $\tau(\mathcal{E}) = k$ and $|\mathcal{E}| = \lfloor k!e \rfloor$. Lovász conjectured that no intersecting k-graph with covering number k has more edges, but this is disproved in Frankl et al. (1995b).

How few edges can such a k-graph have?

Let $g(k)$ denote the minimum size of a k-graph $\mathcal{F}$ with $\tau(\mathcal{F}) = k$. Erdős and Lovász (1975) show that $g(k) \geq 8k/3 - 3$ and they conjecture that $\lim_{k\to\infty} g(k)/k = \infty$. However, using an ingenious construction, Kahn (1992) proved that $g(k) = O(k)$ holds.

Let $\mathcal{P}$ be the set of lines of a projective plane of order $k-1$. Then $\mathcal{P}$ has the following strong property.

Claim 9.3. *If S is a cover of $\mathcal{P}$ with $|S| = k$, then $S \in \mathcal{P}$.*

Proof. Suppose that S is not a line and let $L \in \mathcal{P}$ be a line with $|L \cap S| \geq 2$. Choose $x \in L \setminus S$. Then there are $k-1$ lines besides L through x, and each of them has to intersect S. Thus $|S| \geq 2 + k - 1 > k$. □

Such an intersecting family is called a *maximal intersecting family*, i.e., the addition of any new k-set destroys the property of being intersecting.

Let $f(k)$ denote the minimum size of a maximal intersecting k-graph. Meyer (1974) conjectured that $f(k) \geq k^2 - k + 1$ with equality if a projective plane of order $k-1$ exists. This was disproved in Füredi (1980) by the following construction.

Example 9.4. Let $\mathcal{A}$ be the family of lines of an affine plane of order k and let $\mathcal{A} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{k+1}$ be the partition of the lines into parallel classes. Consider three vertex-disjoint copies $\mathcal{A}^1$, $\mathcal{A}^2$, and $\mathcal{A}^3$ of $\mathcal{A}$ and let $L_1^i, \ldots, L_k^i$ be the lines in $\mathcal{L}_{k+1}^i$. Define:

$$\mathcal{F} = \{L_j^i \cup L\colon\ L \in \mathcal{L}_j^{i+1}, i = 1, 2, 3, j = 1, \ldots, k\}.$$

Then $|\mathcal{F}| = 3k^2$ and $\mathcal{F}$ is a maximal intersecting family, showing $f(2k) \leq 3k^2$ if an affine plane of order k exists.

Theorem 9.5 (Boros et al. 1989). *$f(q+1) \leq q^2/2 + O(q)$ for $q \equiv -1 \pmod 6$, q a prime power.*

Theorem 9.6 (Blokhuis 1987). *$f(k) \leq k^5$ for all k.*

Thus, Theorem 9.6 gives a polynomial upper bound for all k. However, it is not even known whether $\lim_{k\to\infty} f(k)/k = \infty$.

10. τ-critical k-graphs

Let us start with the following result of Bollobás (1965).

Theorem 10.1. *Let $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ be two families of subsets of $[n]$ satisfying*

(i) $A_i \cap B_i = \emptyset$, $1 \leq i \leq m$;

(ii) $A_i \cap B_j \neq \emptyset$, $1 \leq i \neq j \leq m$.

Then

$$\sum_{1 \leq i \leq m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

Proof. Apply induction on n; the cases $n = 0, 1$ are trivial. For notational convenience we shall speak of the two families as a set-pair family $\{(A_i, B_i)\colon 1 \leq i \leq m\}$ satisfying (i) and (ii).

For each $x \in [n]$ consider the set-pair family $\mathcal{P} = (A_i, B_i \setminus \{x\})$, where i runs over i with $x \notin A_i$. Then $\mathcal{P}$ satisfies (i) and (ii). Applying the induction hypothesis to $\mathcal{P}$ on $[n]\setminus\{x\}$ and adding up the corresponding inequalities one notes that $\binom{|A_i|+|B_i|}{|A_i|}^{-1}$ occurs $n - |A_i| - |B_i|$ times, and $\binom{|A_i|+|B_i|-1}{|A_i|}^{-1}$ occurs $|B_i|$ times. Thus we have

$$\sum_{i \leq i \leq m} (n - |A_i| - |B_i|) \cdot \binom{|A_i| + |B_i|}{|A_i|}^{-1} + |B_i| \cdot \binom{|A_i| + |B_i| - 1}{|A_i|}^{-1} \leq n.$$

Dividing by n, the theorem follows. □

Tuza (1984) notes that the inequality of Yamamoto (1954) is a consequence of Theorem 10.1.

Corollary 10.2. *Let $\{A_1, \ldots, A_m\}$ be an antichain on X. Then*

$$\sum_{1 \leq i \leq m} \binom{n}{|A_i|}^{-1} \leq 1.$$

Proof (Tuza 1984). Set $B_i = X - A_i$ and note that the hypotheses of Theorem 10.1 are fulfilled. □

Recall the definition of τ-critical families.

Corollary 10.3. *If $\mathcal{A}$ is τ-critical with $\tau(\mathcal{A}) = t$, then*

$$\sum_{A \in \mathcal{A}} \binom{|A| + t - 1}{t - 1}^{-1} \leq 1.$$

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ and let B_i be a cover of size $t-1$ for $\mathcal{A} \setminus \{A_i\}$. Now, apply Theorem 10.1. □

Note that Corollary 10.3 implies that $|\mathcal{A}| \leq \binom{k+t-1}{k}$ for every τ-critical k-graph with $\tau(\mathcal{A}) = t$. Considering $\binom{[k+t-1]}{k}$ shows that this is best possible.

This result was re-proved and extended in several ways. We refer to the survey of Füredi (1988) for a full account. Here we mention only two related results.

Theorem 10.4 (Füredi 1984). *Let $(A_1, \ldots, A_m)$ be a collection of a-sets and $(B_1, \ldots, B_m)$ a collection of b-sets such that $|A_i \cap B_i| \leq t$ for all i and $|A_i \cap B_j| > t$ for $1 \leq i < j \leq m$. Then $m \leq \binom{a+b-2t}{a-t}$.*

Theorem 10.5 (Tuza 1985). *Let $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ be collections of sets with $A_i \cap B_i = \emptyset$ for all i and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for $i \neq j$. Then $\sum_{1 \leq i \leq m} p^{|A_i|} q^{|B_i|} \leq 1$ holds for all positive p and q with $p + q = 1$.*

Proof. Let $[n]$ be the union of all the sets $A_i \cup B_i$. Consider all subsets of $[n]$ with a weight function $w(E)=p^{|E|}q^{n-|E|}$. Define $\mathcal{A}_i=\{E\subset[n]:\ A_i\subseteq E,\ B_i\cap E=\emptyset\}$ and note that $\mathcal{A}_1,\ldots,\mathcal{A}_m$ are pairwise disjoint. Also, note that

$$\sum_{E\in\mathcal{A}_i} w(E)=p^{|A_i|}q^{|B_i|}.$$

Now we can deduce the result:

$$\sum_{1\le i\le m} p^{|A_i|}q^{|B_i|}=\sum_{1\le i\le m}\sum_{E\in\mathcal{A}_i} w(E)\le\sum_{E\subseteq[n]} p^{|E|}q^{n-|E|}=1. \qquad \square$$

For a more general result see Tuza (1988).

11. Matchings

Let $s\geq 2$ be fixed. How large can a family $\mathcal{F}\subset 2^X$ be if $\nu(\mathcal{F})<s$? For $s=2$, this means that $\mathcal{F}$ is intersecting and the answer 2^{n-1} was given in Theorem 1.1.

Let $\nu(n,s)$ denote max $|\mathcal{F}|$, where $\mathcal{F}\subset 2^X$, $\nu(\mathcal{F})<s$. Clearly, $\nu(n+1,s)\geq 2\nu(n,s)$ holds for all n. Considering $\mathcal{K}=\{K\subseteq X:\ |K|>n/s\}$ shows that

$$\nu(n,s)\geq\sum_{i>n/s}\binom{n}{i}.$$

Kleitman (1968a) showed that this is best possible for $n\equiv -1 \pmod s$.

Theorem 11.1.

$$\nu(bs-1,s)=\sum_{i=b}^{n}\binom{n}{i},$$
$$\nu(bs,s)=2\nu(bs-1,s).$$

For $n\neq 0,-1 \pmod s$, the value of $\nu(n,s)$ is unknown, except for $s=3$, where Quinn (1987) showed that for $n=3b+1$ the best construction is

$$\mathcal{Q}=\mathcal{K}\cup\left\{Q\in\binom{[n]}{b}:\ 1\in Q\right\}=\{Q\subset[n]:\ |Q|+|Q\cap[1]|\geq b+1\}.$$

Conjecture 11.2. For $n=bs+r$, $1\leq r<s$,

$$\nu(n,s)=|\{K\subseteq[n]:\ |K|+|K\cap[s-r-1]|\geq b+1\}|.$$

A problem with a similar flavor was solved by Kleitman (1968b) for $s=2$ and, using the same technique, by Frankl (1977) for all s.

Theorem 11.3. *Let $n=bs+s-1$ and suppose that $\mathcal{F}\subset 2^{[n]}$ contains no s pairwise disjoint sets along with their union. Then*

$$|\mathcal{F}|\leq|\{G\subset[n]:\ b\leq|G|<bs\}|.$$

Again, the maximum value is unknown for $n\not\equiv -1 \pmod s$. Let $\nu(n,s,k)$ denote max$|\mathcal{F}|$, where $\mathcal{F}\subset\binom{X}{k}$ and $\nu(\mathcal{F})<s$. To avoid trivialities, suppose that $n\geq sk$.

Example. $\mathcal{E}_0=\binom{[sk-1]}{k}$, $\mathcal{E}_1=\mathcal{E}_1(n)=\{E\subset\binom{[n]}{k}:\ E\cap[s-1]\neq\emptyset\}$.

Conjecture 11.4 (Erdős 1965). $\nu(n,s,k)=\max\{|\mathcal{E}_0|,|\mathcal{E}_1|\}$.

Erdős (1965) proved that for $n>n_0(s,k)$ the conjecture is true and $\mathcal{E}_1$ is the only extremal example. Bollobás et al. (1976) show that $n_0(s,k)\leq 2sk^3$ holds.

The next proposition is essentially due to Kleitman (1968a).

Proposition 11.5. $\nu(ks,s,k)=\binom{ks-1}{k}$ *and, for $s\geq 3$, the only optimal family is $\mathcal{E}_0$.*

Proof. Take $\mathcal{F}\subset\binom{[ks]}{k}$ with $\nu(\mathcal{F})\leq s-1$. Consider a random partition $P=(P_1,\ldots,P_s)$ of X. That is, $P_1\cup\cdots\cup P_s=X$, $|P_i|=k$ and all P have the same chance of being chosen. Then the probability of the event $P_i\in\mathcal{F}$ is $|\mathcal{F}|/\binom{ks}{k}$. Thus the expected number of i with $P_i\in\mathcal{F}$ is $s|\mathcal{F}|/\binom{ks}{k}$. On the other hand, $\nu(\mathcal{F})<s$ implies that this number is always less than s. Thus $s|\mathcal{F}|/\binom{ks}{k}\leq s-1$. [One can come to the same conclusion by the double-counting argument of Katona (1974).]

Rearranging gives $|\mathcal{F}|\leq\binom{ks-1}{k}$, with equality holding if and only if out of each partition P, exactly $s-1$ sets are in $\mathcal{F}$. That is, $\binom{X}{k}\setminus\mathcal{F}$ is an intersecting family of size $\binom{ks}{k}-\binom{ks-1}{k}=\binom{ks-1}{k-1}$. Now the uniqueness of $\mathcal{F}$ for $s\geq 3$ follows from the uniqueness part of the Erdős–Ko–Rado Theorem (see Theorem 5.3). $\square$

Proposition 11.6. $\nu(n,s,k)\leq(s-1)\binom{n-1}{k-1}$ *for all $n\geq sk$.*

Proof. Use induction on n. The case $n=sk$ is covered by Proposition 11.5. Let $\mathcal{F}\subset\binom{X}{k}$ be a family with $|\mathcal{F}|=\nu(n,s,k)$, $\nu(\mathcal{F})<s$. In view of Lemma 4.2 (iv) we may assume that $\mathcal{F}$ is stable. Consider the two families $\mathcal{F}(\bar{n})$, $\mathcal{F}(n)$.

Claim 11.7. $|\mathcal{F}(\bar{n})|\leq(s-1)\binom{n-2}{k-1}$, $|\mathcal{F}(n)|\leq(s-1)\binom{n-2}{k-2}$.

Since $|\mathcal{F}|=|\mathcal{F}(\bar{n})|+|\mathcal{F}(n)|$, this implies the theorem.

Proof of Claim 11.7. The first inequality is true by induction. To prove the second we have to show $\nu(\mathcal{F}(n))<s$. Suppose the contrary and let $G_1,\ldots,G_s$ be pairwise disjoint sets in $\mathcal{F}(n)$. Since $|G_1|+\cdots+|G_s|=(k-1)s$, we can find distinct elements $x_1,\ldots,x_s\in[n]\setminus(G_1\cup\cdots\cup G_s)$. Since $\mathcal{F}$ is stable, $G_i\cup\{x_i\}$ is in $\mathcal{F}$. That is, $\nu(\mathcal{F})\geq s$, which is a contradiction. $\square$

Formulating Proposition 11.5 for the complements $\mathcal{G}=\mathcal{F}^c$, we obtain that an s-wise intersecting family $\mathcal{G}\subset\binom{[ks]}{k(s-1)}$ can have at most $\binom{ks-1}{k(s-1)-1}$ members. This was generalized by Frankl (1976).

Theorem 11.8. *If $\mathscr{G} \subset \binom{[n]}{l}$ is s-wise intersecting, $n \geq sl/(s-1)$, then $|\mathscr{G}| \leq \binom{n-1}{l-1}$. Moreover, unless $s=2$, $n=2l$, equality is achieved only if $\mathscr{G} \cong \{G \in \binom{[n]}{l}: 1 \in G\}$.*

For a new proof see Frankl (1987b).

12. The number of vertices in τ- and ν-critical k-graphs

Following Tuza (1985), let us call $P=\{(A_i, B_i): 1 \leq i \leq m\}$ an (a,b)-system if $|A_i| = a$, $|B_i| = b$ for all i, and moreover, Theorem 10.1 (i) and (ii) hold.

Let $n(a,b)$ be $\max|\bigcup_{i=1}^{m}(A_i \cup B_i)|$, where the maximum is over all (a,b)-systems. Let $n_1(a,b)$ be $\max|\bigcup_{i=1}^{n} A_i|$, where the maximum is over all (a,b)-systems.

As we saw in the proof of Corollary 10.3, to every τ-critical k-graph $\mathscr{A}$ with $\tau(\mathscr{A}) = t$ one can associate a $(k, t-1)$-system. This implies:

$$|\bigcup \mathscr{A}| \leq n_1(k, t-1) \quad \text{if } \mathscr{A} \text{ is a } \tau\text{-critical } k\text{-graph with } \tau(\mathscr{A}) = t\,.$$

Obviously $n_1(a,b) \leq n(a,b) = n(b,a)$. The following surprising symmetry relation holds.

Theorem 12.1 (Tuza 1985). $n_1(a, b-1) = n_1(b, a-1)$ *for all* a, $b \geq 1$.

Proposition 12.2 (Tuza 1985).

$$n_1(a'+a'', b'+b'') \geq a' + b' + \binom{a'+b'}{a'} n_1(a'', b'')\,.$$

Proof. Let $\mathscr{P}$ (respectively, $\mathscr{Q}$) be an (a', b')-system ((a'', b'')-system). For each $(A_i, B_i) \in \mathscr{P}$, let $\mathscr{Q}_i$ be a copy of $\mathscr{Q}$, where $\mathscr{P}, \mathscr{Q}_1, \ldots, \mathscr{Q}_m$ are all vertex-disjoint. The general element of $\mathscr{Q}_i$ is denoted by $(C_j^{(i)}, D_j^{(i)})$. Define:

$$\mathscr{R} = \{(A_i \cup C_j^{(i)}, B_i \cup D_j^{(i)}): (A_i, B_i) \in \mathscr{P}, (C_j^{(i)}, D_j^{(i)}) \in \mathscr{Q}_i\}\,.$$

Then $\mathscr{R}$ is an $(a'+a'', b'+b'')$-system, which proves the theorem. □

Tuza (1985) proves the following surprisingly sharp bounds.

Theorem 12.3.

(i) $\frac{1}{4}\binom{a+b+1}{b+1} < n(a,b) < \binom{a+b+1}{b+1}$ *for* $a \geq b$, $a \geq 1$;

(ii) $\frac{1}{4}\binom{a+b+1}{b+1} < n_1(a,b) < \binom{a+b+1}{b+1}$ *for* $a \geq 1$, $b \geq 0$.

Let us mention that Tuza proves both the upper and lower bounds in a stronger form. In particular, applying Proposition 12.2 with $a' = \lfloor ab/(b+1) \rfloor$, $b' = b$, he

obtains

$$n_1(a,b) \geq \lceil a/(b+1) \rceil \binom{\lfloor ab/(b+1) \rfloor + b}{b} + \lfloor ab/(b+1) \rfloor + b$$

and he suggests that equality holds here for $a \geq b+2$. He also conjectures t
$n_1(a,b) = n(a,b)$ holds if and only if $a \geq b$.

Recall the definition of a ν-critical family $\mathscr{F}$. A family $\mathscr{F}$ is said to have ran
if $k = \max_{F \in \mathscr{F}} |F|$ holds.

Improving earlier bounds of Lovász (1975), Tuza (1985) shows:

Theorem 12.4. *If $\mathscr{F}$ is a ν-critical family of rank k, then it has fewer t
$\binom{\nu(\mathscr{F})k+k}{k}$ vertices.*

Proof. Set $\nu = \nu(\mathscr{F})$ and let $\mathscr{K}$ consist of those sets which are the union
pairwise disjoint edges in $\mathscr{F}$. Let $\mathscr{K}' = \{H_1, \ldots, H_m\} \subset K$ be minimal with res
to $\bigcup \mathscr{K}' = \bigcup \mathscr{K}$. Then for every $H_i \in \mathscr{K}'$ there is a vertex $x_i \in H_i$ such that x_i
for $i \neq j$. By ν-criticality there is some $F_i \in \mathscr{F}$ with $F_i \cap H_i = \{x_i\}$ and consequ
$(F_1 \setminus \{x_i\}) \cap H_j \neq \emptyset$ for all $i \neq j$. Now $\{(H_i, F_i \setminus \{x_i\}): 1 \leq i \leq m\}$ is a system sat
ing Theorem 10.1 (i) and (ii), and also $|H_i| \leq \nu k$, $|F_i \setminus \{x_i\}| \leq k-1$. Thus,

$$|\bigcup \mathscr{F}| = |\bigcup \mathscr{K}'| \leq n_1(\nu k, k-1) < \binom{\nu k + k}{k}\,.$$

In the case $\nu = 1$ we have the following sharper results.

Theorem 12.5 (Tuza 1985). *Let $\nu(k)$ denote the maximum order of a ν-cr
intersecting family $\mathscr{F}$ with rank k. Then*

$$2k - 4 + 2\binom{2k-4}{k-2} \leq \nu(k) \leq \binom{2k-1}{k-1} + \binom{2k-3}{k-2}\,.$$

Both bounds improve earlier results of Erdős and Lovász (1975).
conjectures that the lower bound – given by the following constructio
optimal for $k \geq 4$.

Example. For each partition $[2k-4] = F \cup F'$ with $|F| = |F'| = k-2$, take
new vertices x, x', y, y' and form the k-element sets $F \cup \{x, y\}$, $F \cup \{x'$
$F' \cup \{x, y'\}$ and $F' \cup \{x', y\}$. These sets form a ν-critical k-graph.

For k fixed and ν large we have the following.

Conjecture 12.6 (Lovász 1975). There exists a constant $c = c(k)$ such that
ν-critical family $\mathscr{F}$ of rank k has at most $c\nu(\mathscr{F})$ vertices. For $k = 2$, the
possible bound $3\nu(\mathscr{F})$ was shown by Gallai (1963).

13. Excluded configurations I

Let $\mathscr{C} = \{\mathscr{A}_1, \ldots, \mathscr{A}_r\}$ be a collection of k-graphs.

Set $\mathrm{ex}(n, \mathscr{C}) = \max|\mathscr{F}|$, where the maximum is taken over all $\mathscr{F} \subseteq \binom{X}{k}$, $\mathscr{F}$ containing no subfamily isomorphic to a family in $\mathscr{C}$. If $\mathscr{C} = \{\mathscr{A}\}$ then we also write $\mathrm{ex}(n, \mathscr{A})$ instead of $\mathrm{ex}(n, \mathscr{C})$. A classical result of Katona et al. (1964) is the following.

Theorem 13.1. $\mathrm{ex}(n, \mathscr{C})/\binom{n}{k}$ *is monotone decreasing, and therefore* $\mu(\mathscr{C}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathscr{C})/\binom{n}{k}$ *exists.*

Proof. Let $1 \le h < n$ and consider a family $\mathscr{F} \subset \binom{X}{k}$ without any subfamily isomorphic to some $\mathscr{A} \in \mathscr{C}$ and such that $|\mathscr{F}| = \mathrm{ex}(n, \mathscr{C})$. Choose a subset $H \in \binom{X}{h}$ at random, with uniform distribution. Then $|\binom{H}{k} \cap \mathscr{F}| \le \mathrm{ex}(h, \mathscr{C})$ holds for all H. On the other hand, the expectation of $|\binom{H}{k} \cap \mathscr{F}|$ is $|\mathscr{F}|$ times the probability that a fixed $F \in \binom{X}{k}$ is in $\binom{H}{k}$, i.e., $|\mathscr{F}|\binom{h}{k}/\binom{n}{k}$. Thus $|\mathscr{F}|\binom{h}{k}/\binom{n}{k} = \mathrm{ex}(n, \mathscr{C})\binom{h}{k}/\binom{n}{k} \le \mathrm{ex}(h, \mathscr{C})$.

Dividing by $\binom{h}{k}$ shows the desired result. □

It follows from a result of Erdős (1964) that $\mu(\mathscr{C}) = 0$ if $\mathscr{C}$ contains some k-partite k-graph. Actually, Erdős obtains an upper bound of the form $n^{k-e(\mathscr{C})}$ where $e(\mathscr{C})$ is a positive constant. The determination of the best possible value of $e(\mathscr{C})$ seems to be very difficult even in very simple cases. In this section we suppose that there are no k-partite k-graphs in $\mathscr{C}$. Let us first state Turán's well-known problem.

Example. Let $[n] = X_0 \cup X_1 \cup X_2$ be a partition with $|X_i| = \lfloor (n+i)/3 \rfloor$. Define:

$$\mathscr{T}(4,3) = \left\{ T \in \binom{X}{3} : |T \cap X_i| = 1, i = 0, 1, 2 \right\}$$
$$\cup \left\{ T \in \binom{X}{3} : |T \cap X_i| = 2, |T \cap X_{i+1}| = 1 \text{ for some } i = 0, 1, 2, \right.$$
$$\left. \text{where } X_3 \text{ denotes } X_0 \right\}.$$

It is conjectured by Turán that $t(n, 4, 3) = |\mathscr{T}(4,3)|$. Kostoschka (1982) has given exponentially many non-isomorphic 3-graphs with $|\mathscr{T}(4,3)|$ edges and without a $\binom{[4]}{3}$. This suggests, that if Turán's conjecture is correct, then it could be very hard to prove. Kalai (1985) has proposed a more general algebraic conjecture.

Example. Let $[n] = X_0 \cup X_1 \cup \cdots \cup X_{t-1}$ be a partition with $|X_i| = \lfloor (n+i)/t \rfloor$.

Define:

$$\mathscr{T}(n, t(k-1)+1, k) = \binom{[n]}{k} - \bigcup_{0 \le i < t} \binom{X_i}{k}.$$

Clearly, $\mathscr{T}(n, t(k-1)+1, k)$ contains no $\binom{[t(k-1)+1]}{k}$. It is conjectured tha $n > n_0(t,k)$ one has $|\mathscr{T}(n, t(k-1)+1, k)| = t(n, t(k-1)+1, k)$, although Br (1983) has produced other examples with the same cardinality.

The simplest non-3-partite 3-graph is $\mathscr{R}_3 = \binom{[4]}{3} \setminus [2,4]$. Even for this 3-gr $\mathrm{ex}(n, \mathscr{R}_3)$ is unknown.

Proposition 13.2. $\frac{2}{7} \le \mu(\mathscr{R}_3) \le \frac{1}{3}$.

Here, the upper bound is due to de Caen (1982), the lower bound to Fr and Füredi (1984b).

With every k-graph $\mathscr{F}$ let us associate a polynomial $q(\mathscr{F})$ as follows.

Definition 13.3. Define $q(\mathscr{F}, \boldsymbol{x}) = \sum_{F \in \mathscr{F}} \prod_{i \in F} x_i$.

Then $q(\mathscr{F})$ is a homogeneous polynomial of degree k which is linear in variable.

Define the *Lagrange function* $\lambda(\mathscr{F}) = \max q(\mathscr{F}, \boldsymbol{x})$, where the maximum is over all $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_i \ge 0$, $x_1 + \cdots + x_n = 1$.

Using the theory of Lagrange multipliers one obtains:

Lemma 13.4 (Frankl and Rödl 1984). *There exists an* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *with* x $x_1 + \cdots + x_n = 1$, *such that* (i)–(iii) (*following*) *hold. Set* $Y = \operatorname{supp} x = \{i : x_i$

(i) $\lambda(\mathscr{F}) = q(\mathscr{F}, \boldsymbol{x})$;
(ii) $\partial q(\mathscr{F}, \boldsymbol{x})/\partial x_i = k\lambda(\mathscr{F})$ *for all* $i \in Y$;
(iii) *every pair* $P \in \binom{Y}{2}$ *is contained in some edge* $F \in \mathscr{F}$ *with* $F \subseteq Y$.

Note that $\lambda(\mathscr{F}) \ge |\mathscr{F}|/n^k$. One can use this to show the following simple r

$$\mu(\mathscr{C}) = k! \sup\{\lambda(\mathscr{F}) : \mathscr{F} \text{ is a } k\text{-graph without a copy of any } \mathscr{A} \in$$
(

Katona (1974) asked for the determination of the maximum nu $\mathrm{symm}(n,k)$ of k-subsets of an n-set such that none of them contain symmetric difference of two others. This problem can be formulated in ter $\mathrm{ex}(n, \mathscr{C})$, but for k large $\mathscr{C}$ will contain many k-graphs (all with three edg

Conjecture 13.5 (Bollobás 1974). $\mathrm{symm}(n,k) = \prod \ldots \lfloor (n+i)/k \rfloor$ with eq holding for the complete equipartite k-graph.

Bollobás (1974) solves the case $k = 3$ (the case $k = 2$ is very easy an solved by Mantel in 1906). De Caen (1986) gives a new proof for $k=3$ and proposes a different problem.

Problem. Determine $\mathrm{ex}(n,\mathscr{C}_k)=c(n,k)$, where $\mathscr{C}_k=\{\mathscr{A}_2,\mathscr{A}_3,\ldots,\mathscr{A}_k\}$ with $\mathscr{A}_i=\{[1,k],[1,k-1]\cup\{k+1\},[i,k+i-1]\}$.

Clearly, $\mathrm{symm}(n,k)\le c(n,k)$ for all k and for $k=2,3$, the two problems are the same.

Sidorenko (1987) realized the relevance of Lemma 13.4 and proved the following.

Theorem 13.6. $c(n,k)=\prod_{0\le i<k}\lfloor (n+i)/k\rfloor$ *holds for* $k=2,3,4$.

Proof. To avoid technical difficulties we shall only prove $c(n,k)\le(n/k)^k$ (which is the same as the theorem if n is a multiple of k), i.e., $\lambda(\mathscr{F})\le 1/k^k$ if $\mathscr{F}$ contains no copy of $\mathscr{A}_i\in\mathscr{C}_k$.

In view of Lemma 13.4 in proving the above inequality we may suppose that $\sigma_2(\mathscr{F})=\binom{[n]}{2}$, i.e., every air $P\in\binom{[n]}{2}$ is contained in some $F\in\mathscr{F}$. Now if $F, F'\in\mathscr{F}$ with $|F\cap F'|=k-1$, then $|F\,\Delta F'|=2$ and therefore we find $F''\in\mathscr{F}$ with $F\,\Delta F'\subseteq F''$, i.e., $\{F,F',F''\}\in\mathscr{F}$, which is a contradiction. Thus $|F\cap F'|\le k-2$ for all $F,F'\in\mathscr{F}$. (Note that this is not true in general for $\mathscr{F}$ containing no copy of $\mathscr{A}_i\subset\mathscr{C}_k$; however, Lemma 13.4 ensures the existence of a subfamily with this property and the same value for the Lagrange function.) That is, $\binom{F}{k-1}\cap\binom{F'}{k-1}=\emptyset$ for distinct $F,F'\in\mathscr{F}$. In other words, $\partial q(\mathscr{F},\boldsymbol{x})/\partial x_i$ and $\partial q(\mathscr{F},\boldsymbol{x})/\partial x_j$ have no common term for $i\ne j$. Let

$$s_{k-1}(\boldsymbol{x})=\sum_{A\in\binom{[n]}{k-1}}\prod_{i\in A}x_i$$

be the $(k-1)$th elementary symmetric polynomial. Adding up Lemma 13.4 (ii) for $1\le i\le n$, we obtain

$$kn\lambda(\mathscr{F})\le s_{k-1}(\boldsymbol{x})\le\binom{n}{k-1}\left(\frac{1}{n}\right)^{k-1}.$$

Rearranging gives

$$\lambda(\mathscr{F})\le\frac{(n-1)\cdots(n-k+2)}{k!\,n^{k-1}}. \tag{13.7}$$

Now for $n\ge k$, the right-hand side of (13.7) is at most k^{-k}, both for $k=2$ and $k=3$, and also for $k=4$ unless $n=5$. However, the case $n=5$ is impossible, because any two 4-subsets of [5] overlap in three elements. This concludes the proof. □

Using the same approach, Frankl and Füredi (1989) determined $\mu(\mathscr{C}_k)$ for $k=5$ and $k=6$.

Let $\mathscr{W}_{11}$ ($\mathscr{W}_{12}$) be the (unique) (11, 5, 4) ((12, 6, 5)) Steiner-system. Tl $\mathscr{W}_{12}\subset\binom{[12]}{6}$ and for each $A\in\binom{[12]}{5}$ *there is a unique set* $B\in\mathscr{W}_{12}$ with $S\subset I$ $\mathscr{W}_{11}=\mathscr{W}_{12}$ (12).

Example. For $X=X_1\cup\cdots\cup X_{12}$, $|X_i|=n/12$, define:

$$\mathscr{B}_6=\left\{B\in\binom{X}{6}:\ \{i:\ B\cap X_i\ne\emptyset\}\in\mathscr{W}_{12}\right\};$$

$\mathscr{B}_5$ is defined analogously.

Theorem 13.8. (i) $\mathrm{ex}(n,\mathscr{C}_5)\le 66(n/11)^5$ *with equality iff* $11\,|\,n$, *in which c is the only optimal family.*

(ii) $\mathrm{ex}(n,\mathscr{C}_6)\le 132(n/12)^6$ *with equality iff* $12\,|\,n$, in which case $\mathscr{B}_6$ *is th optimal family.*

14. Excluded configurations II: k-partite k-graphs

Many of the problems treated earlier can be formulated in the form: det $\mathrm{ex}(n,\mathscr{C})$. For example, the determination of $m(n,k,L)$ is such a problem. start with three problems which come up in other contexts.

Call a family $\mathscr{F}\subset 2^X$ *barely overlapping if* $F\not\subseteq F'\cup F''$ holds for all dist $F',F''\in\mathscr{F}$. Let $h(n,k)$ denote the maximum size of a barely overlapping $\mathscr{F}\subset\binom{X}{k}$.

Theorem 14.1 (Erdős et al. 1982). (i) $h(n,2l-1)\le\binom{n}{l}/\binom{2l-1}{l}$ *with* ϵ *holding iff there exists an* $S(n,2l-1,l)$.

(ii) $h(n,2l)\le\binom{n-1}{l}/\binom{2l-1}{l}$ *with equality achieved for some* $\mathscr{F}$ *if and* $|\bigcap\mathscr{F}|=1$ (say $\bigcap\mathscr{F}=\{1\}$) and $\mathscr{F}(1)$ is an $S(n-1,2l-1,l)$.

Proof. We only prove (i) and even this only for $n\ge 3l$. Let $G\subset F\in\mathscr{F}$. We a *distinguished subset* of F if $G\not\subseteq F'$ for all $F\ne F'\in\mathscr{F}$. Let us define a function $w:\mathscr{F}\times\binom{[n]}{l}\to\mathbb{R}_+$ by:

$$w(F,G)=\begin{cases}1 & \text{if } G\in\binom{F}{l} \text{ and } G \text{ is an eigen-subset of } F,\\ 1/l & \text{if } G\cap F=:H\in\binom{F}{l-1} \text{ and } H \text{ is an eigen-subset of}\\ 0 & \text{otherwise}.\end{cases}$$

Claim. $\sum_F w(F,G)\le 1$, $\sum_G w(F,G)\ge\binom{2l-1}{l}$.

Proof. The first part follows by noting that if G is an eigen-subset of F.

subset of G can be an eigen-subset of some other $F' \in \mathscr{F}$ and $w(F, G) = 1/l$ can hold for a fixed G at most l times, once for each of its $(l-1)$-subsets

To prove the second part, note that if $F = A \cup B$, $|A| = l$, $|B| = l - 1$, then either A or B (or both) are eigen-subsets of F because $\mathscr{F}$ is barely overlapping. If A is an eigen-subset, it contributes 1; if B is, then B contributes $(n-(k-1))/l > 1$. Since there are $\binom{2l-1}{l}$ such partitions of F, the inequality follows. □

Proof of Theorem 14.1 (continued). Using the claim, it is easy to show that

$$|\mathscr{F}|\binom{2l-1}{l} \leq \sum_{F \in \mathscr{F}} \sum_{G \in \binom{X}{l}} w(F, G) = \sum_{G} \sum_{F} w(F, G) \leq \binom{n}{l},$$

i.e., $|\mathscr{F}| \leq \binom{n}{l}/\binom{2l-1}{l}$, as desired. In case of equality, equality must hold in (i). Thus all $G \in \binom{X}{l}$ are eigen-subsets. That is, $\mathscr{F}$ is a partial l-design. Consequently, $|\mathscr{F}| = \binom{n}{l}/\binom{2l-1}{l}$ if and only if $\mathscr{F}$ is an $S(n, 2l-1, l)$. □

For further results and problems on barely overlapping and related families we refer to Frankl (1988).

We call $\mathscr{F} \subset 2^X$ *union-free* if $F \cup F' = G \cup G'$ implies for $F, F', G, G' \in \mathscr{F}$ that $\{F, F'\} = \{G, G'\}$. Let $u(n, k)$ denote $\max|\mathscr{F}|$, where $\mathscr{F} \subset \binom{X}{k}$ is union-free.

Theorem 14.2 (Frankl and Füredi 1986a). *There are positive constants c_k, c'_k such that*

$$c_k n^{2k/3+\varepsilon(k)} < u(n, k) < c'_k n^{2k/3+\varepsilon(k)},$$

with $\varepsilon(k) = 0$, $\frac{1}{3}$ or $\frac{1}{6}$ according to whether $k \equiv 0$, 1 or 2 (mod 3).

Let us mention that the proof of the lower bound is rather involved. The acquired family $\mathscr{F}$ is defined via systems of nonlinear equations over finite fields.

Again, for more information on this and related problems we refer to Frankl (1988).

We call $\mathscr{F}$ disjoint-union-free if it contains no four sets F, G, H, K with $F \cup G = H \cup K$ and $F \cap G = H \cap K = \emptyset$. Let $u_d(n, k)$ denote the maximum size of $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ disjoint-union-free.

Clearly, $u_d(n, k) \geq \binom{n-1}{k-1} + 1$; (take $\{G \in \binom{[n]}{k}: 1 \in G\} \cup \{[2, k+1]\}$). It is possible that for $k \geq 4$, $n > n_0(k)$, equality holds. However, it was unknown for many years whether $u_d(n, k) = O(n^{k-1})$ held. Füredi (1983) gave an ingenious argument to show the following.

Theorem 14.3. *$u_d(n, k) < \frac{7}{2}\binom{n}{k-1}$ for all $n > k \geq 3$.*

Let us note that in the case of graphs ($k = 2$), the condition is that the graph is C_4-free and $u_d(n, 2)$ is of the order $n^{3/2}$ (cf. chapter 23).

The paper by Frankl and Füredi (1987) gives a rather general treatment (and

often solutions) of a class of excluded-configuration-type problems for k-graph

We mention just a few of the results of that paper.

Let $\phi(a, b)$ be the maximum size of an a-graph without sunflowers of size

Also, let $\phi(n, k, l, s)$ be the maximum size of $\mathscr{F} \subset \binom{X}{k}$, where $\mathscr{F}$ contains r

sunflower of size s whose center has size l.

Theorem 14.4. $\phi(n, k, l, s) = (\phi(l+1, s) + o(1))\binom{n-l-1}{k-l-1}$ *if $k > 2l + 1$.*

It is conjectured that the same holds for $k = 2l + 1$ as well; however, this h

only been proved (in Chung and Frankl 1987) for $l = 1$. For $k < 2l + 1$ it follo

from Theorem 7.1 via Lemma 4.15 that $\phi(n, k, l, s)$ has order n^l; however, tl

correct coefficient of n^l is unknown.

Conjecture 14.5.

$$\phi(n, k, l, s) = \left(\binom{l-1+s(k-l)}{k} + o(1)\right)\binom{n}{l}\Big/\binom{l-1+s(k-l)}{l}.$$

The construction is given by taking $\mathscr{F} = \sigma_k(\mathscr{S})$, where $\mathscr{S} \subset \binom{X}{l-1+s(k-l)}$ is (partial) l-design.

Let $\mathscr{A}$ be a k-graph. Set $p = |\bigcap \mathscr{A}|$ and let q be the number of vertices of $\mathscr{A}$

degree 2 or more.

Theorem 14.6. *If $2p + q + 1 < k$, then $ex(n, \mathscr{A}) = (\gamma(\mathscr{A}) - o(1))\binom{n}{k-p-1}$, wh*

$\gamma(\mathscr{A})$ is a positive integer depending only on $\mathscr{A}$.

In the case $p = 0$, one can define $\gamma(\mathscr{A})$ by taking $\gamma(\mathscr{A}) + 1$ to be the size of t

smallest set T satisfying $|T \cap A| = 1$ for all $A \in \mathscr{A}$. Note that such a T exists if

is k-partite, which – in turn – follows from $q < k$. In general, $\gamma(\mathscr{A}) \leq \phi(p + |\mathscr{A}|)$.

Theorem 14.7. *Set $\mathscr{A} = \{\{1,2,3,5,7\}, \{1,2,3,6,8\}, \{1,2,4,5,8\}\}$. Then $ex(n,$*

$= o(n^4)$. However, $\lim_{n\to\infty} ex(n, \mathscr{A})/n^\alpha = \infty$ for all $\alpha < 4$.

This result shows that $ex(n, \mathscr{A})$ does not always have a proper exponent. T

proof extends that of Ruzsa and Szemerédi (1978), where a similar phenomen

is described.

Another type of extremal problem, considered by Kászonyi and Tuza (1986)

the following.

Definition 14.8. For a k-graph $\mathscr{A}$, let $sat(n, \mathscr{A})$ denote $\min|\mathscr{F}|$, where $\mathscr{F} \subset ($

and $\mathscr{F}$ contains no copy of $\mathscr{A}$, but adding any new k-subset of $[n]$ produces a c

of $\mathscr{A}$.

Conjecture 14.9 (Tuza). $sat(n, \mathscr{A}) = O(n^{k-1})$ for every k-graph ...

References

Bang, C., H. Sharp and P. Winkler
[1984] On coverings of a finite set: depth and subcover, *Period. Math. Hungar.* **15**, 51–60.

Blokhuis, A.
[1987] More on maximal intersecting families of finite sets, *J. Combin. Theory A* **44**, 299–303.

Blumer, A., A. Ehrenfeucht, D. Hausler and M.K. Warmuth
[1989] Learnability and the Vapnik–Chervonenkis dimension, *J. ACM* **36**, 929–965.

Bollobás, B.
[1965] On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16**, 447–452.
[1974] Three graphs without two triples whose symmetric difference is contained in a third, *Discrete Math.* **8**, 21–24.

Bollobás, B., D.E. Daykin and P. Erdős
[1976] Sets of independent edges of a hypergraph, *Quart. J. Math. Oxford* **27**, 25–32.

Boros, E., Z. Füredi and J. Kahn
[1989] Maximal intersecting families and affine regular polygons in PG(2, q), *J. Combin. Theory A* **52**, 1–9.

Brace, A., and D.E. Daykin
[1971] Cover theorems for finite sets, *Bull. Aust. Math. Soc.* **5**, 197–202.
[1972] Sperner type theorems for finite sets, in: *Proc. British Combinatorial Conf., Oxford* (Institute of Mathematics and its Applications, Southend-on-Sea) pp. 18–37.

Brown, W.G.
[1983] On an open problem of P. Turán concerning 3-graphs, in: *Studies in Pure Mathematics to the Memory of P. Turán*, ed. P. Erdős, pp. 91–93.

Cameron, P.J., P. Frankl and W.M. Kantor
[1989] Intersecting families and permutation groups without fixed-point-free 2-elements, *European J. Combin.* **10**.

Chung, F.R.K., and P. Frankl
[1987] The maximum number of edges in a 3-graphs not containing a given star, *Graphs and Combin.* **3**, 111–126.

Clarkson, K., H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl
[1988] Combinatorial complexity bounds for arrangements of curves and surfaces, in: *Proc. 29th Annu. Symp. on Comput. Geom.*, pp. 568–599.

de Caen, D.
[1982] *On Turán's hypergraph problem*, Ph.D. Thesis (University of Toronto).

Deza, M., P. Erdős and P. Frankl
[1978] Intersection properties of systems of finite sets, *Proc. London Math. Soc.* (3) **36**, 368–384.

Deza, M., P. Frankl and J.W.P. Hirschfeld
[1985] Sections of varieties over finite fields as large intersection families, *Proc. London Math. Soc.* **50**, 405–425.

Engel, K., and H.D.O.F. Gronau
[1985] *Sperner Theory of partially ordered sets*, *Teubner Texte in Mathematik* (Teubner, Leipzig).

Erdős, P.
[1964] On extremal problems on graphs and generalized graphs, *Israel J. Math.* **2**, 183–190.
[1965] A problem of independent r-tuples, *Ann. Univ. Budapest* **8**, 93–95.
[1981] On the combinatorial problems which I would most like to see solved, *Combinatorica* **1**, 25–42.

Erdős, P., and D.J. Kleitman
[1968] On colorings of graphs to maximize the proportion of multicolored k-edges, *J. Combin. Theory* **5**, 164–169.

Erdős, P., and L. Lovász
[1975] Problems and results on 3-chromatic hypergraphs and some related questions, *Colloq. Math. Soc. János Bolyai*, pp. 609–627.

Erdős, P., and R. Rado
[1960] Intersection theorems for systems of sets, *J. London Math. Soc.* **35**, 95–90.

Erdős, P., C. Ko and R. Rado
[1961] Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford (2)* **12**, 313–320.

Erdős, P., P. Frankl and Z. Füredi
[1982] Families of finite sets in which no set is covered by the union of two others, *J. Combin. Thec A* **33**, 158–166.

Frankl, P.
[1975] The proof of a conjecture of Katona, *J. Combin. Theory A* **19**, 208–213.
[1977] Families of finite sets containing no k sets and their union, *Period. Math. Hungar.* **8**, 29–31.
[1978] The Erdős–Ko–Rado theorem is true for $n = ckt$, *Colloq. Math. Soc. János Bolyai* **18**, 365–375.
[1983] On the trace of finite sets, *J. Combin. Theory A* **34**, 41–45.
[1984] A new short proof for the Kruskal–Katona theorem, *Discrete Math.* **48**, 327–329.
[1986a] Orthogonal vectors and codes with missing distances, *Combinatorica* **6**, 279–286.
[1986b] All rationals occur as exponents, *J. Combin. Theory A* **42**, 200–206.
[1987a] Erdős–Ko–Rado theorem with conditions on the maximal degree, *J. Combin. Theory A* **46**, 252–2
[1987b] The shifting technique in extremal set theory, in: *Combinatorial Surveys 1987*, ed. C. Whiteh (Cambridge University Press, Cambridge) pp. 81–110.
[1988] Intersection and containment problems without quantitative restrictions, in: *Algebraic, Extremal (Metric Combinatorics* (Cambridge University Press) pp. 62–111.

Frankl, P., and Z. Füredi
[1981] A short proof for a theorem of Harper about Hamming spheres, *Discrete Math.* **34**, 311–313.
[1984a] On hypergraphs without two edges intersecting in a given number of vertices, *J. Combin. The A* **36**, 230–236.
[1984b] An exact result for 3-graphs, *Discrete Math.* **50**, 323–328.
[1985] Forbidding just one intersection, *J. Combin. Theory A* **39**, 160–176.
[1986] Union free families of sets and equations over fields, *J. Number Theory* **23**, 210–218.
[1987] Exact solution of some Turán-type problems, *J. Combin. Theory A* **45**, 226–262.
[1989] Extremal problems whose solutions are the blow-ups of the small Witt-designs, *J. Combin. The A* **52**, 129–147.

Frankl, P., and V. Rödl
[1984] Hypergraphs do not jump, *Combinatorica* **4**, 149–159.
[1985] Near perfect coverings in graphs and hypergraphs, *European J. Combin.* **6**, 317–326.
[1987] Forbidden intersections, *Trans. Amer. Math. Soc.* **300**, 259–286.

Frankl, P., and R.M. Wilson
[1981] Intersection theorems with geometric consequences, *Combinatorica* **1**, 357–368.

Frankl, P., K. Ota and N. Tokushige
[1995a] Exponents of uniform L-systems, *J. Combin. Theory A*, submitted.
[1995b] Uniform intersecting families with covering restrictions, *J. Combin. Theory A*, to appear.

Frankl, P., M. Matsumoto, I.Z. Ruzsa and N. Tokushige
[1995c] Minimum shadows in uniform hypergraphs and a generalization of the Takagi function, *J. Com Theory A* **69**, 125–148.

Füredi, Z.
[1980] On maximal intersecting families of finite sets, *J. Combin. Theory A* **28**, 282–289.
[1983] On finite systems whose every intersection is a kernel for a star, *Discrete Math* **47**, 129–132.
[1984] Geometrical solution of an intersection problem for two hypergraphs, *European J. Combin.* **5**, 136.
[1988] Extremal problems concerning the principal parameters of hypergraphs, *Graphs and Combi* 115–206.

Füredi, Z., and J.R. Griggs
[1986] Families of finite sets with minimum shadows, *Combinatorica* **6**, 393–402.

Gallai, T.
[1963] Neuer Beweis eines Tutteschen Satzes, *MTA Mat. Kut. Int. Közl.* 135–139.

Gyárfás, A.
[1977] Partition covers and blocking sets in hypergraphs (in Hungarian), *MTA SZTAKI Tanulmányok* 71, Budapest.

Harper, L.H.
[1966] Optimal numberings and isoperimetric problems on graphs, *J. Combin. Theory* **1**, 385–394.

Hilton, A.J.W.
[1976] unpublished Manuscript.

Hilton, A.J.W., and E.C. Milner
[1967] Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford (2)* **18**, 369–384.

Kahn, J.
[1992] On a problem of Erdős and Lovász, *Combinatorica* **12**, 417–423.

Kalai, G.
[1985] A new approach to Turán's conjecture, *Graphs and Combin.* **1**, 107–109.

Kászonyi, L., and Z. Tuza
[1986] Saturated graphs with minimal number of edges, *J. Graph Theory* **10**, 203–210.

Katona, G.O.H.
[1964] Intersection theorems for systems of finite sets, *Acta Math. Hungar.* **15**, 329–337.
[1966] A theorem of finite sets, in: *Theory of Graphs, Proc. Colloq., Tihany* (Akadémiai Kiadó, Budapest) pp. 187–207.
[1974] A simple proof of the Erdős–Ko–Rado theorem, *J. Combin. Theory B* **13**, 183–184.

Katona, G.O.H., T. Nemetz and M. Simonovits
[1964] On a problem of Turán in the theory of graphs, *Mat. Lapok* **15**, 228–238 (in Hungarian, English summary).

Kleitman, D.J.
[1966a] Families of non-disjoint subsets, *J. Combin. Theory* **1**, 153–155.
[1966b] On a combinatorial conjecture of Erdős, *J. Combin. Theory* **1**, 209–214.
[1968a] Maximal number of subsets of a finite set no k of which are pairwise disjoint, *J. Combin. Theory* **5**, 157–163.
[1968b] On families of subsets of a finite set containing no two disjoint sets and their union, *J. Combin. Theory* **5**, 235–237.

Kostoshka, A.V.
[1982] A class of constructions for Turán (3,4) problem, *Combinatorica* **2**, 187–192.

Kruskal, J.B.
[1963] The number of simplices in a complex, in: *Mathematical Optimization Techniques* (University of California Press, Berkeley, CA) pp. 251–278.

Larman, D.G., and C.A. Rogers
[1972] The realization of distances within sets in Euclidean space, *Mathematika* **19**, 1–24.

Linial, N., Y. Mansour and R.L. Rivest
[1991] Results on learnability and the Vapnik–Chervonenkis dimension, *Information and Computation* **90**, 33–49.

Lovász, L.
[1975] On minimax theorems of combinatorics (in Hungarian), *Matematikai Lapok* **26**, 209–264.
[1979] *Combinatorial Problems and Exercises* (Akadémiai Kiadó/North-Holland, Budapest/Amsterdam).

Matsumoto, M., and N. Tokushige
[1989] The exact bound in the Erdős–Ko–Rado Theorem for cross-intersecting families, *J. Combin. Theory A* **22**, 90–97.

Meyer, J.C.
[1974] Quelques problèmes concernant les cliques des hypergraphes h-complets et q-parti h-complets, in: *Hypergraph Seminar, Columbus, Ohio, 1972, Lecture Notes in Mathematics*, Vol. 411, eds. C. Berge et al. (Springer, Berlin) pp. 127–139.

Miklos, D.
[1984] Great intersecting families of edges in hereditary hypergraphs, *Discrete Math.* **48**, 95–99.

Mörs, M.
[1985] A generalization of a theorem of Kruskal, *Graphs and Combin.* **1**, 167–183.

Quinn, F.
[1987] Ph.D. Thesis (MIT, Cambridge, MA).

Rödl, V.
[1985] On a packing and covering problem, *European J. Combin.* **6**, 69–78.

Ruzsa, I.Z., and E. Szemerédi
[1978] Triple systems with no six points carrying three triangles, *Colloq. Math. Soc. János Bolyai* **1** 939–945.

Sidorenko, F.
[1987] On a problem of Bollobás on 4-graphs (in Russian), *Mat. Zametki* **41**, 433–455.

Sperner, E.
[1928] Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27**, 544–548.

Tuza, Z.
[1984] Helly-type hypergraphs and Sperner families, *European J. Combin.* **5**, 185–187.
[1985] Critical hypergraphs and intersecting set-pair systems, *J. Combin. Theory B* **39**, 134–145.
[1988] *An Inequality for Minimal Covering Sets in Set Systems of Given Rank*. Preprint (MTA SzTAK Budapest).

Wilson, R.M.
[1984] The exact bound in the Erdős–Ko–Rado theorem, *Combinatorica* **4**, 247–257.

Yamamoto, K.
[1954] Logarithmic order of free distributive lattices, *J. Math. Soc. Jpn.* **6**, 343–353.