On Flips in Triangulations

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Definition of a Flip

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In the literature, flips have been studied for many different types of graph classes including:

Trees, Planar Graphs, Triangulations, Graphs embedded on different surfaces (torus, higher genus, projective plane, Klein bottle, etc..), Geometric triangulations, Pseudo-triangulations, triangulated polygons, convex polygons, maximal outer-planar graphs . . .

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Flips have been studied in the context of various applications such as:

Meshes, Data Structures, Enumerating combinatorial and geometric structures, Rigidity . . .

Definition of a Flip

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For this talk, I will concentrate on results specifically related to flips in triangulations (in the combinatorial setting and the geometric setting)

Definition of a Flip

A flip operation in a graph is the deletion of an edge followed by the insertion of a distinct edge such that the resulting graph remains in the same graph class.

The main question that has driven the research in this area is essentially variants related to the following:

Given two n-vertex triangulations T_1 and T_2, is there a finite sequence of edge flips that transforms T_1 into a triangulation isomorphic to T_2?

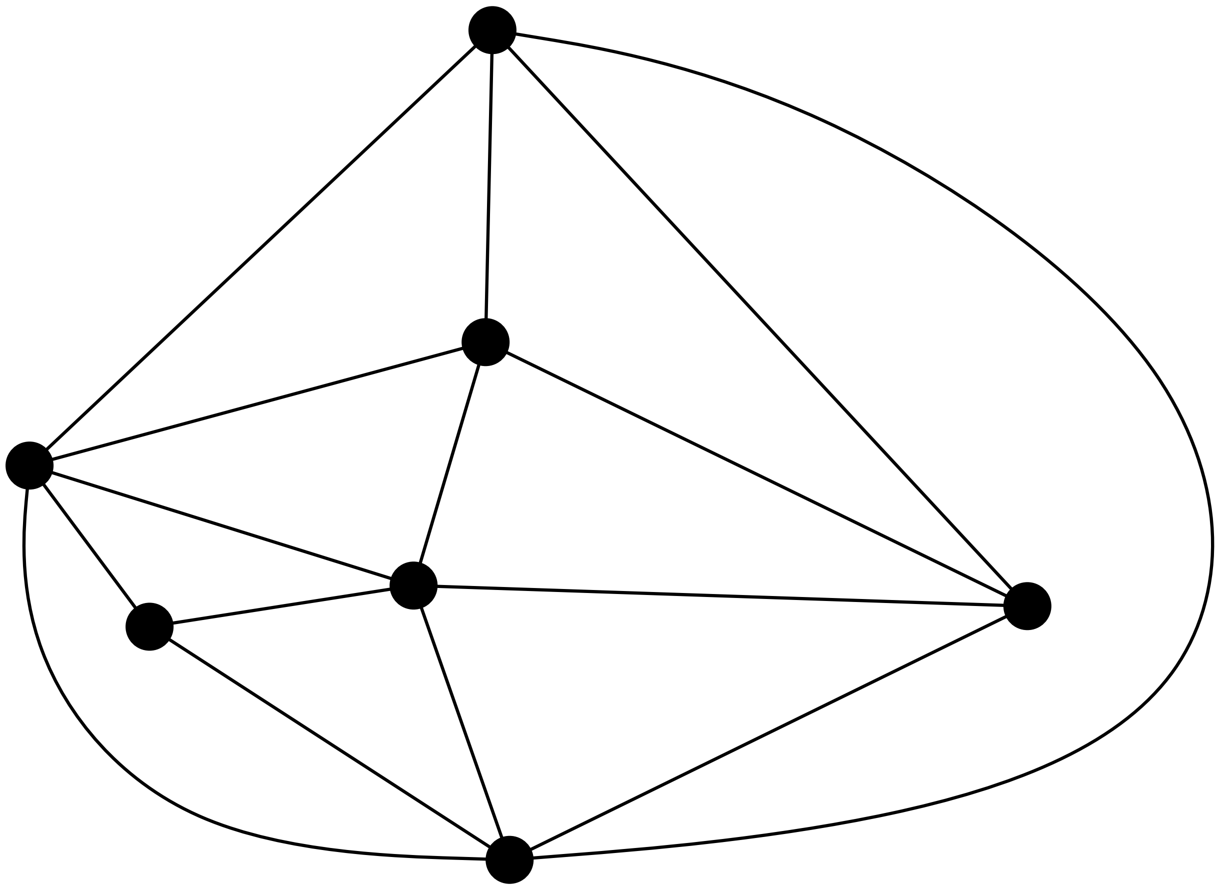
Combinatorial Setting

- In the combinatorial setting, the graphs are triangulations where every face including the outerface is a triangle.

- The triangulations are embedded combinatorially. That is: the cyclic order of the edges around each vertex is defined. This uniquely defines an embedding (Whitney, 1932).

Combinatorial Setting

Flips in combinatorial setting:



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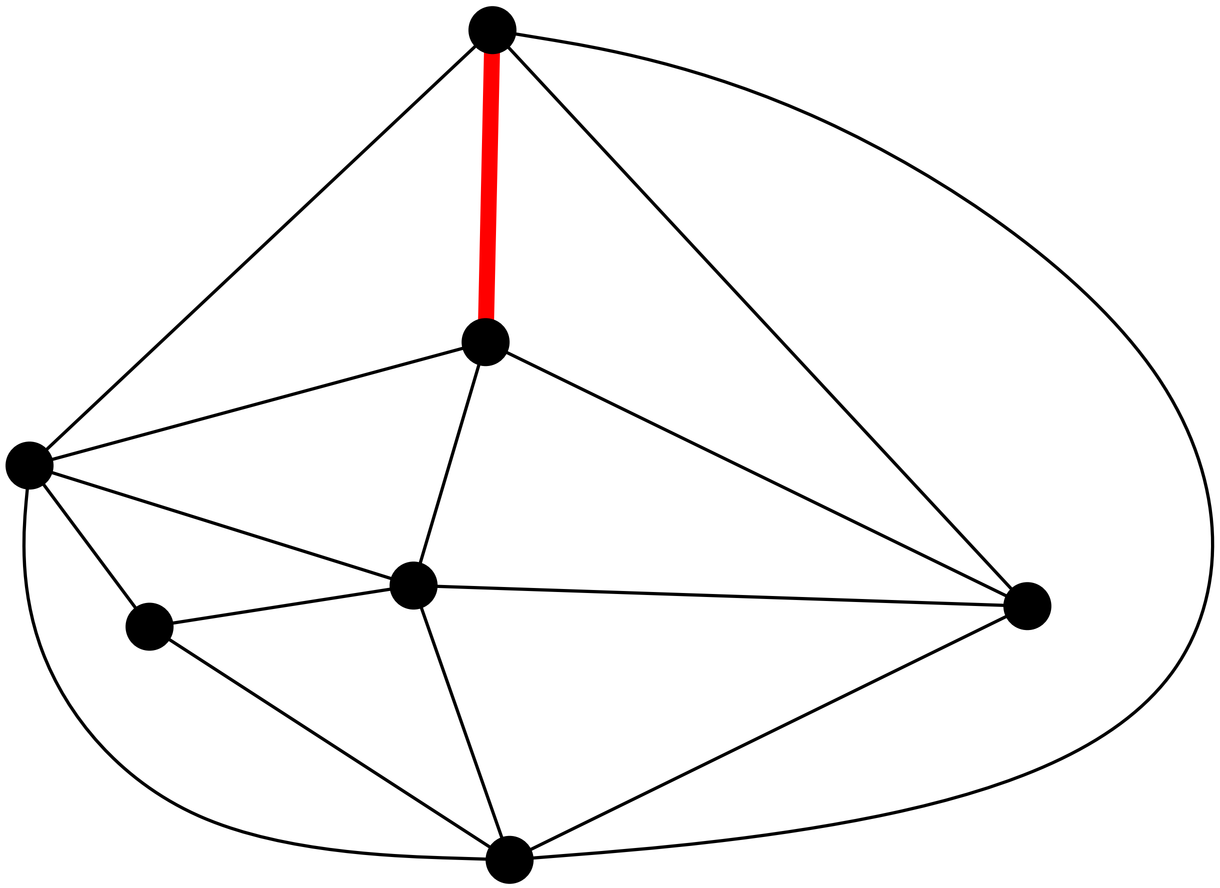

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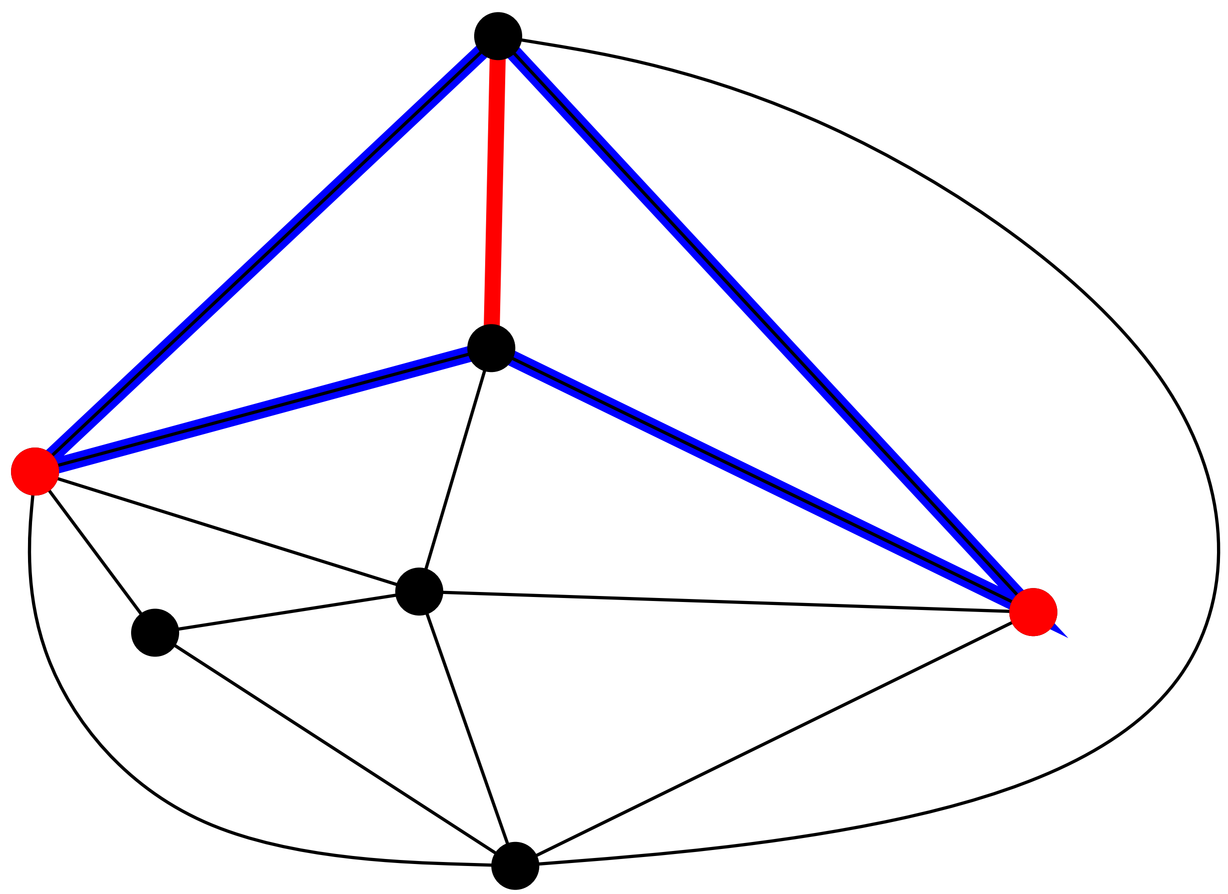

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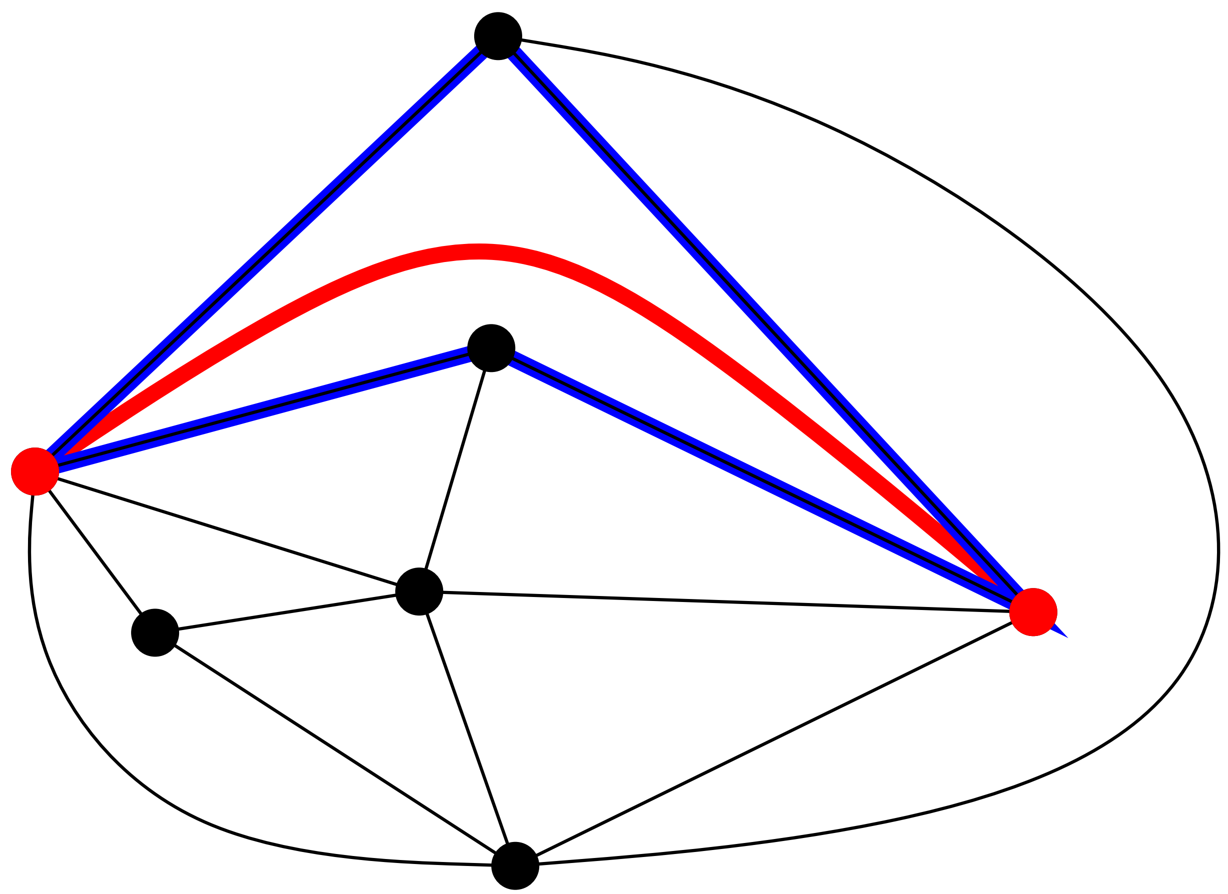

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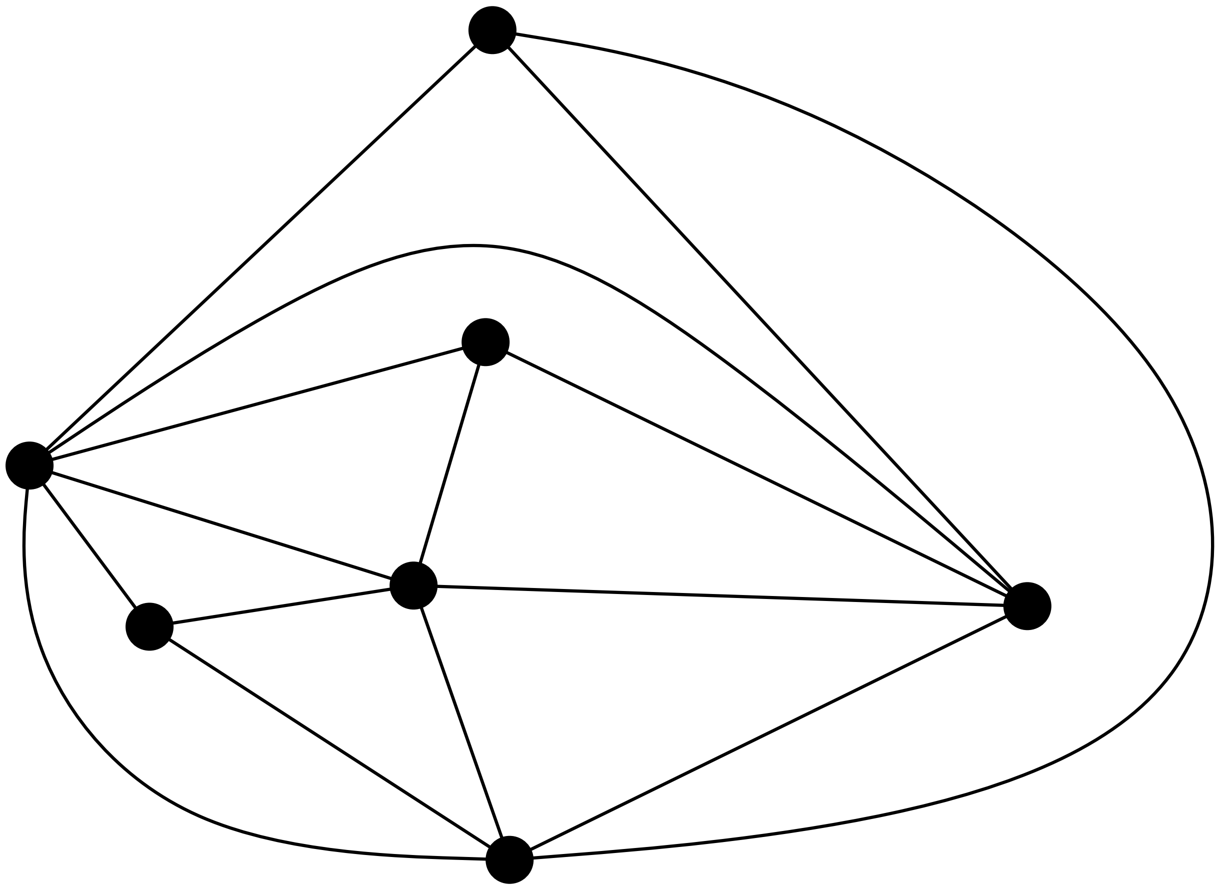

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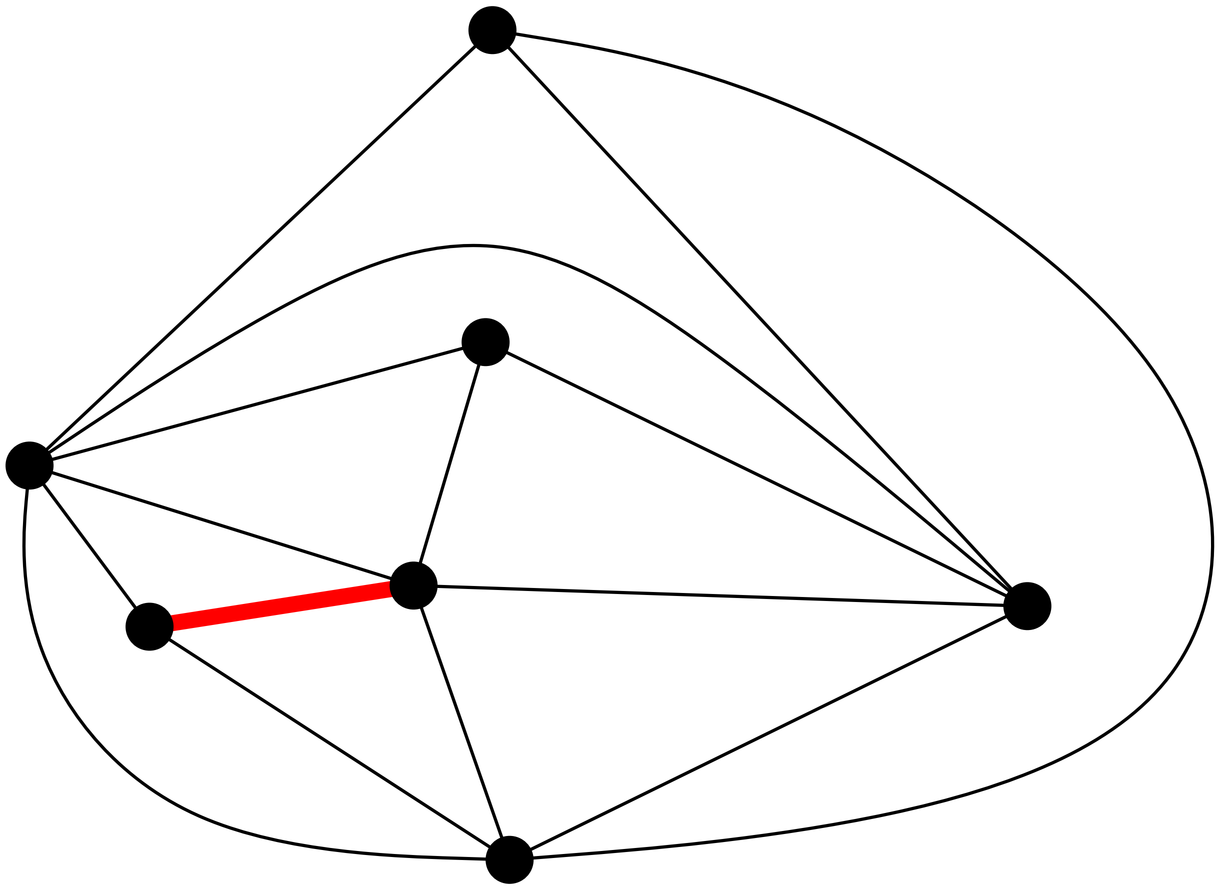

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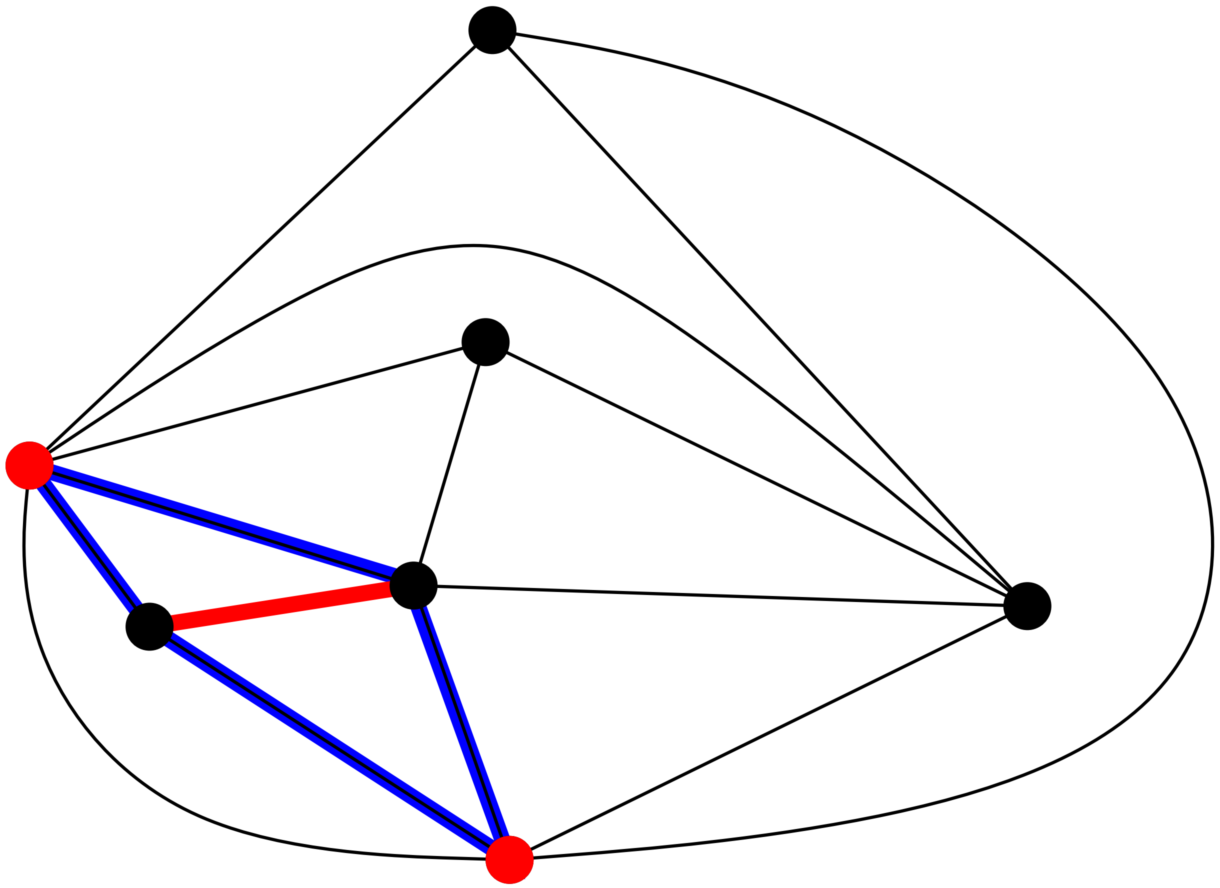

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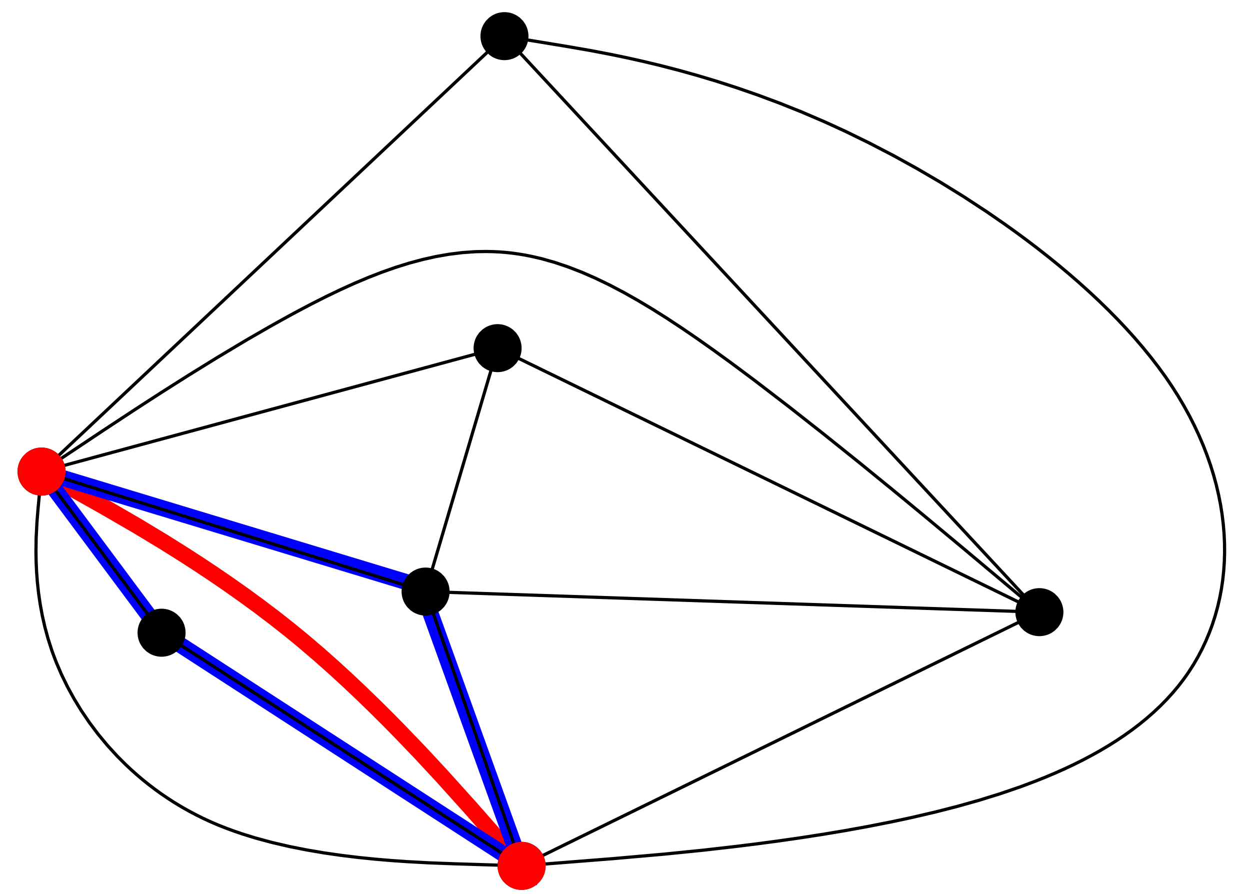

Illegal flip because it creates a parallel edge
so the graph is no longer simple

Combinatorial Setting

Wagner (1936) proved the seminal result in this area:

Given two n-vertex triangulations T_1 and T_2, the triangulation T_1 can be transformed into a triangulation isomorphic to T_2 with $O(n^2)$ flips.

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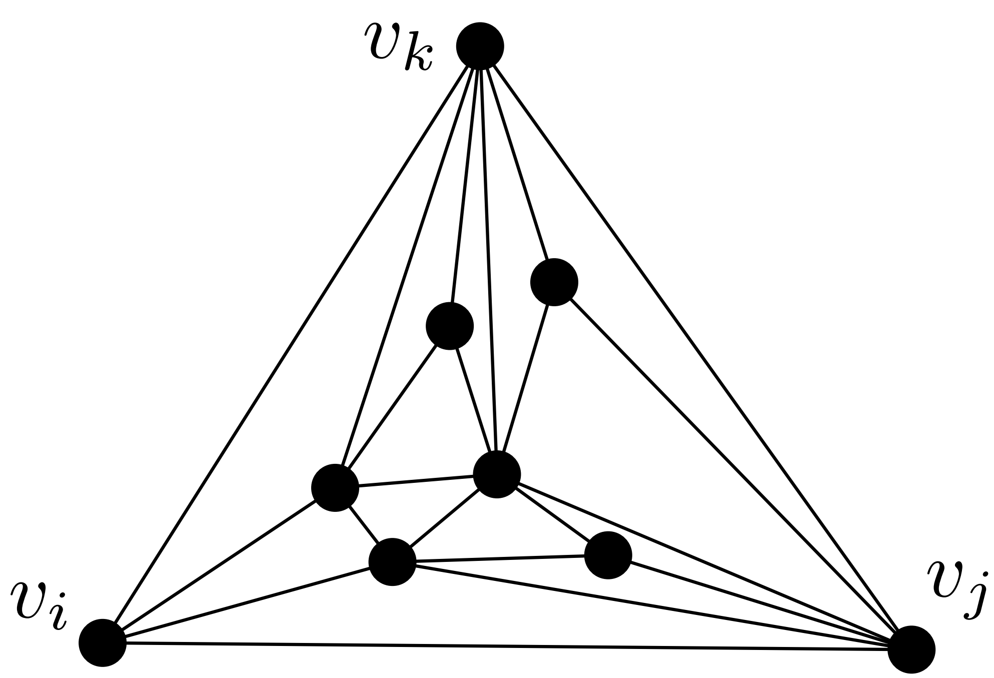

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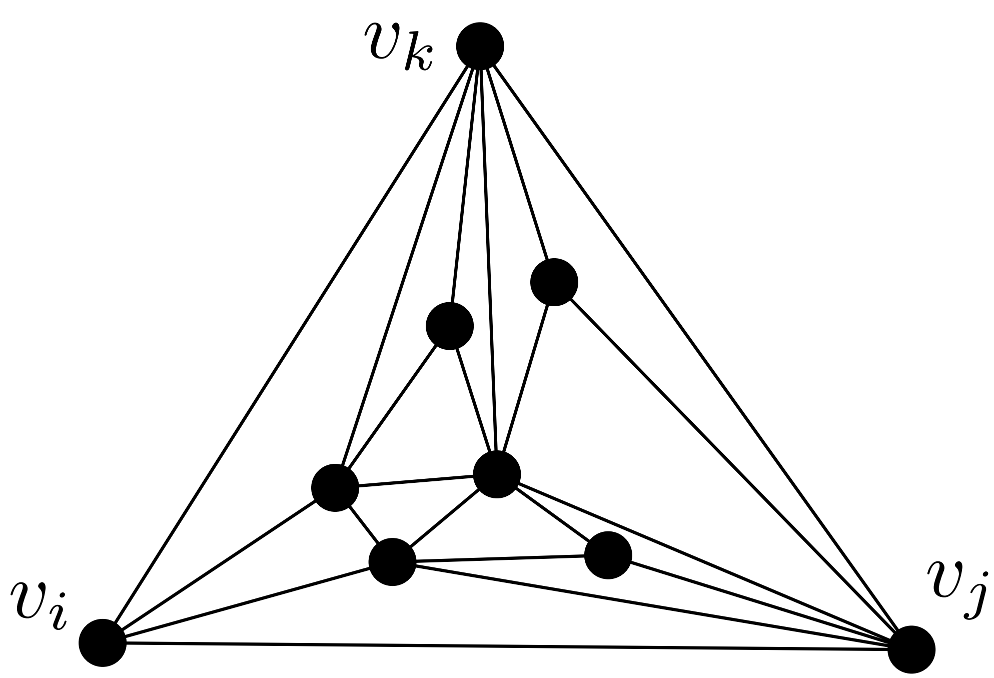

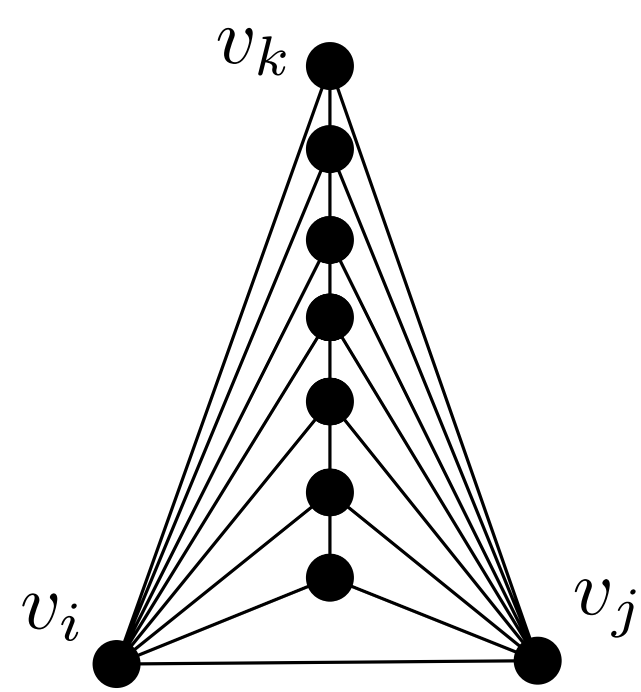

Wagner's Canonical Triangulation

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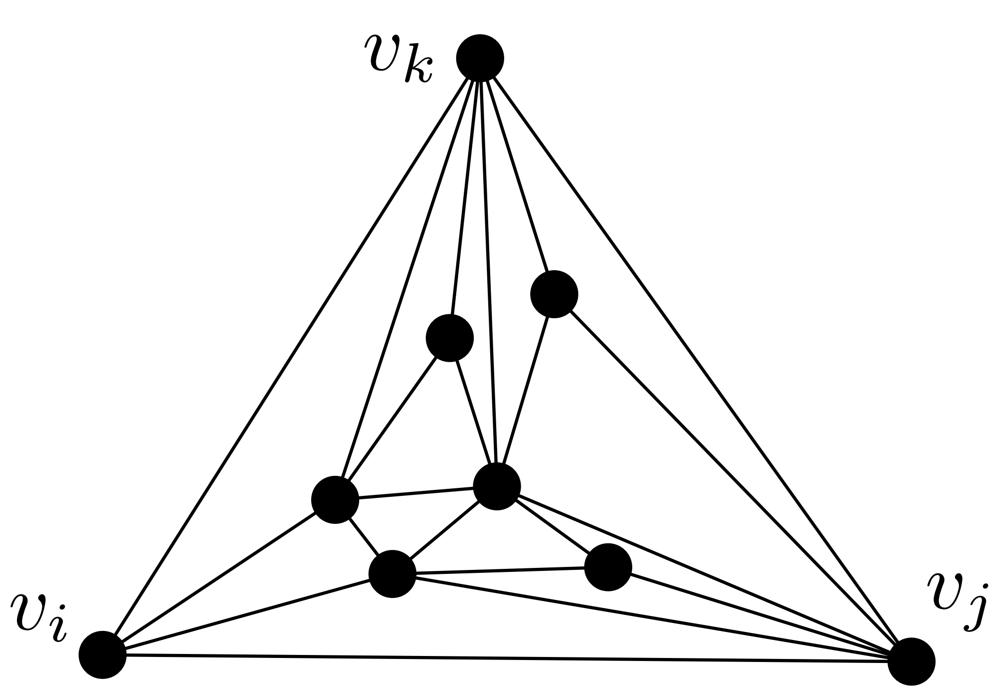
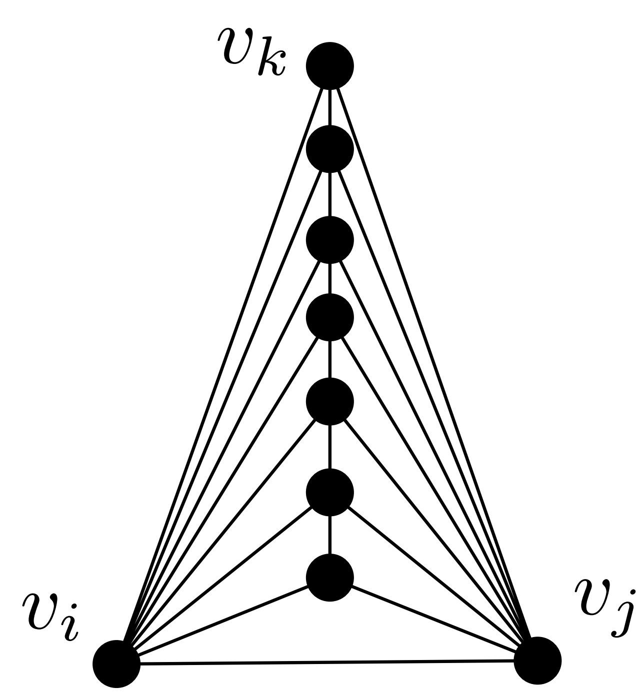
Given two n-vertex triangulations T_1 and T_2, the triangulation T_1 can be transformed into a triangulation isomorphic to T_2 with $O(n^2)$ flips.

$\Omega(n)$ is a lower bound on the number of flips required to convert one triangulation into another.





Wagner's Canonical Triangulation

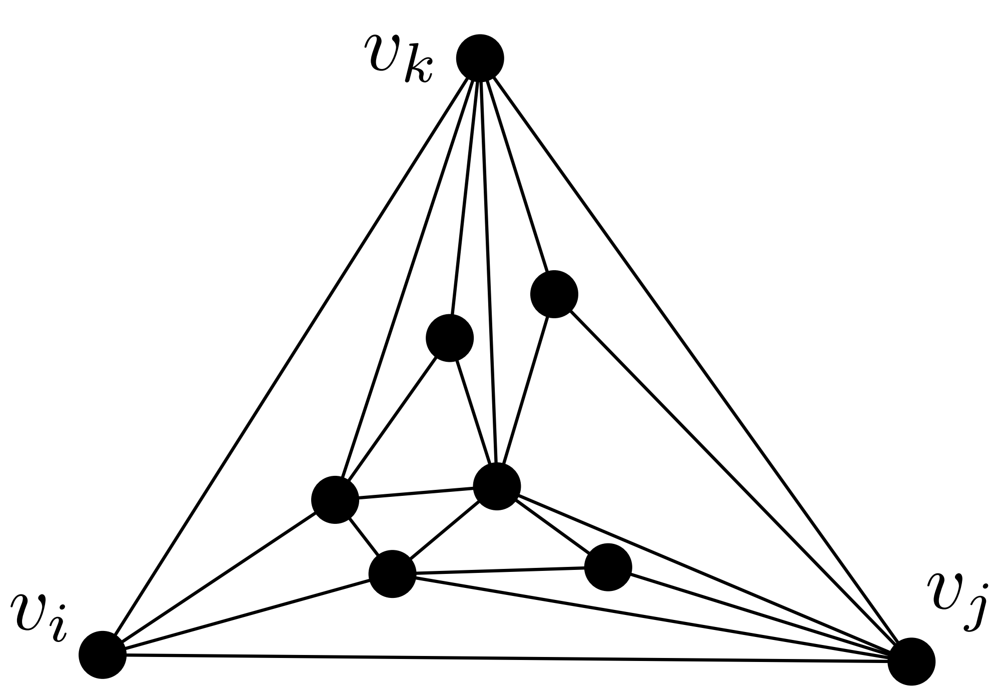
Combinatorial Setting

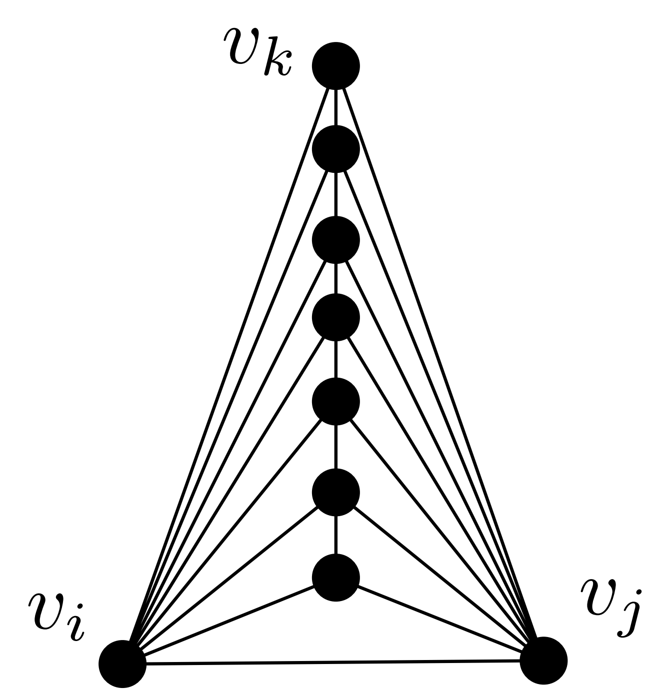
Komuro (1997) showed that $O(n)$ flips suffice.

The key observation is that if the triangulation is not in canonical form, two flips suffice to increase function by ≥ 1:

$P(v_i, v_j) = 3\deg(v_i) + \deg(v_j)$

Since $P(v_i, v_j) \leq 4(n-1)$ at most $8(n-1)$ flips suffice.

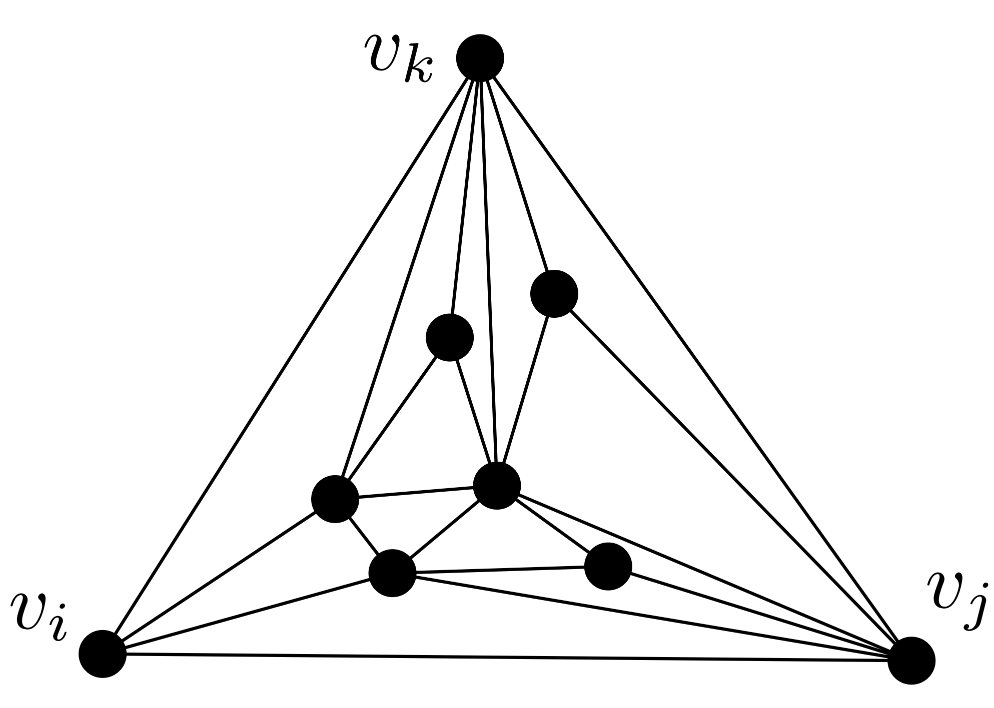


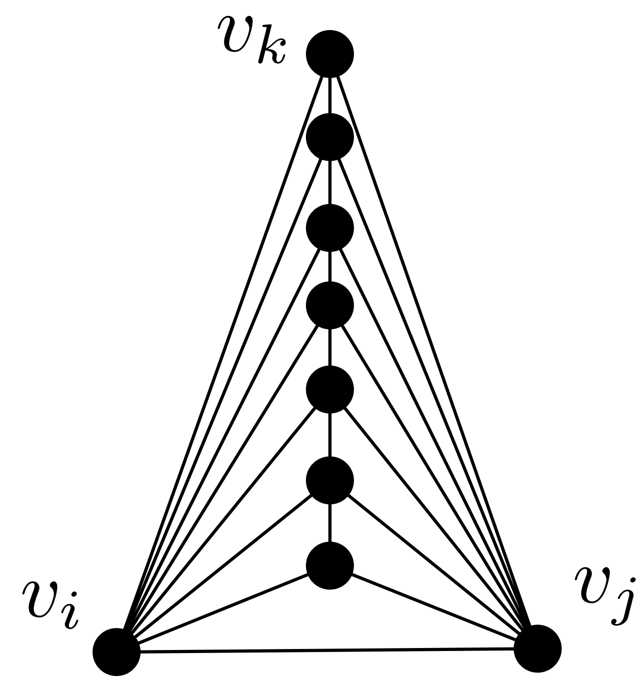


Wagner's Canonical Triangulation

Combinatorial Setting

Mori, Nakamoto and Ota (2003) improved this to $6n - 30$.





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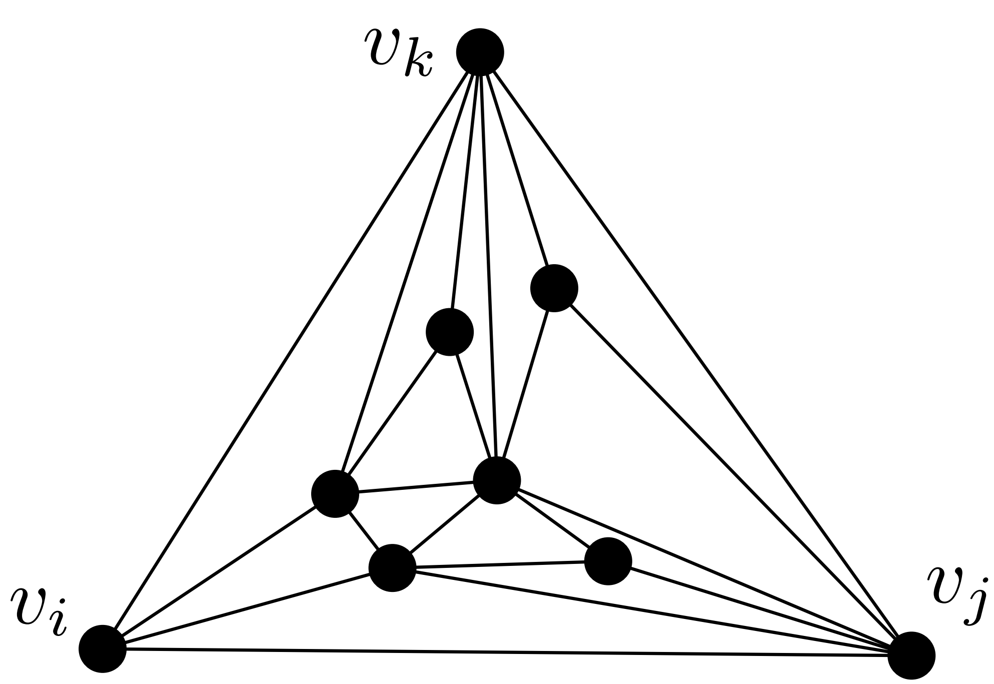
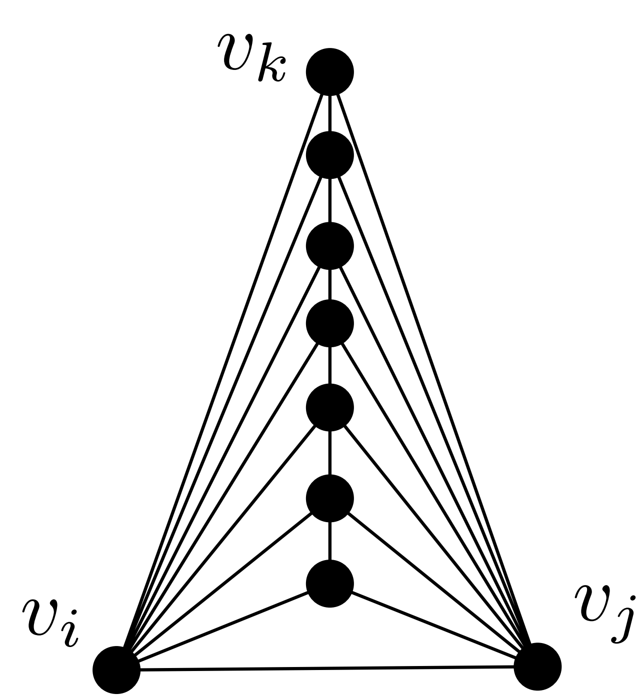
The key step behind their proof is that $n - 4$ flips suffice to convert any triangulation into a hamiltonian triangulation.





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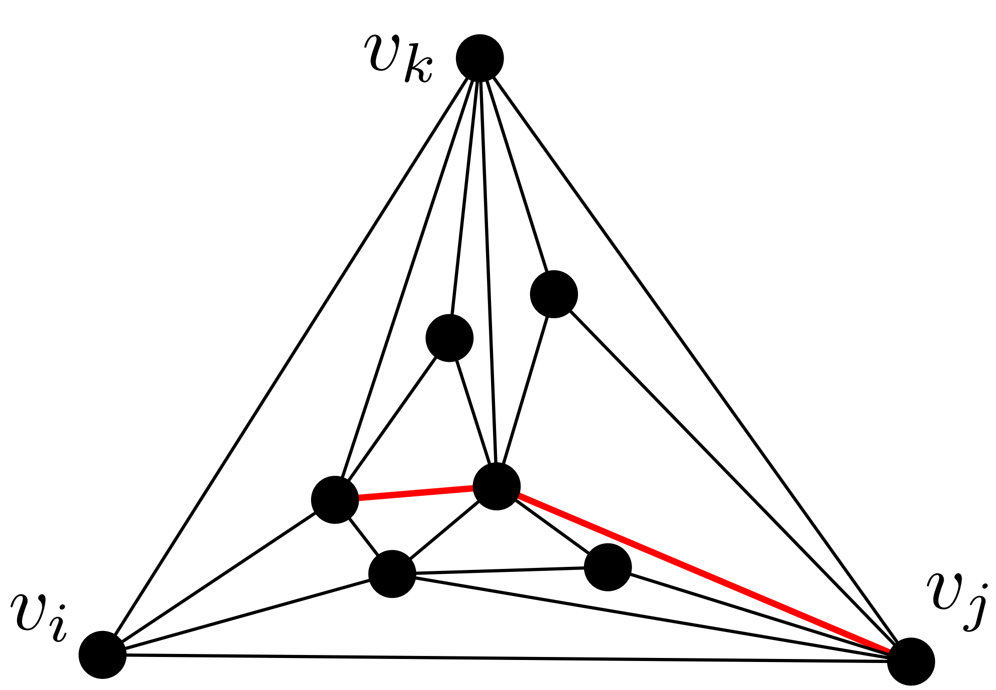

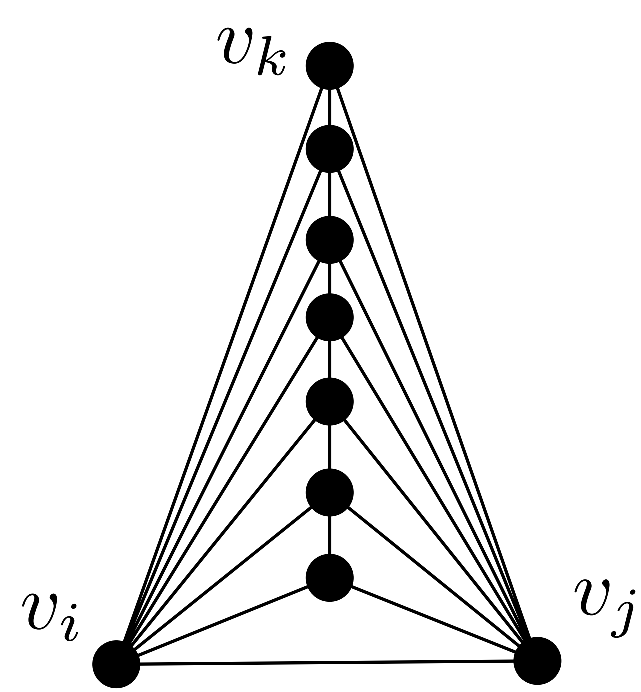

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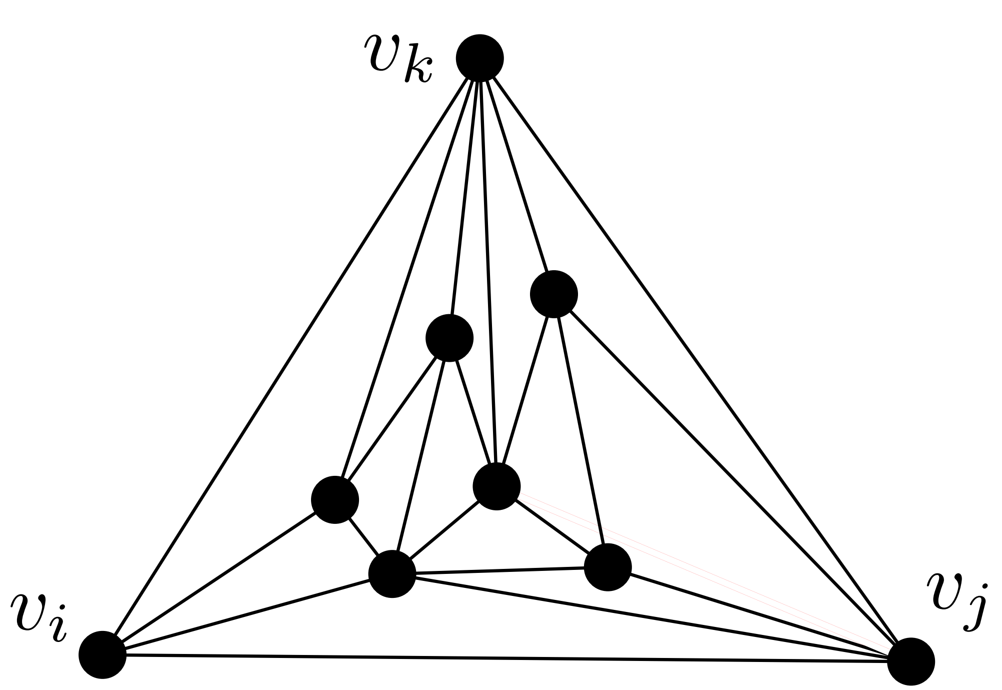

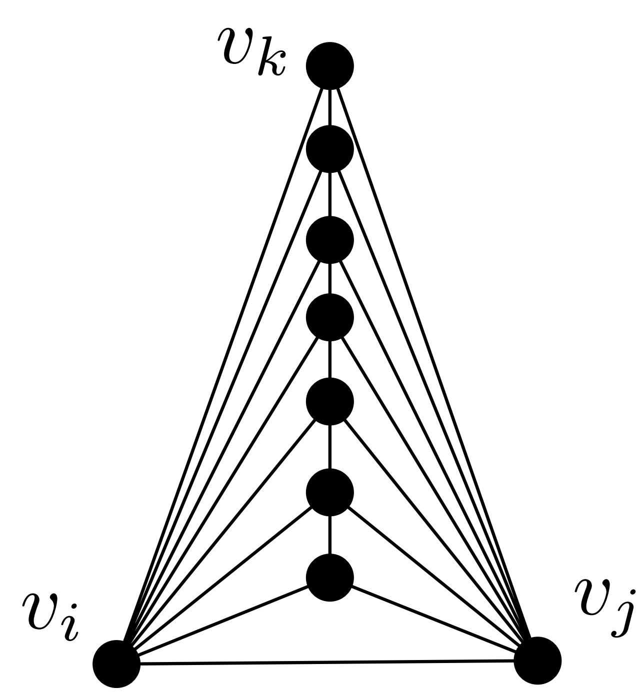

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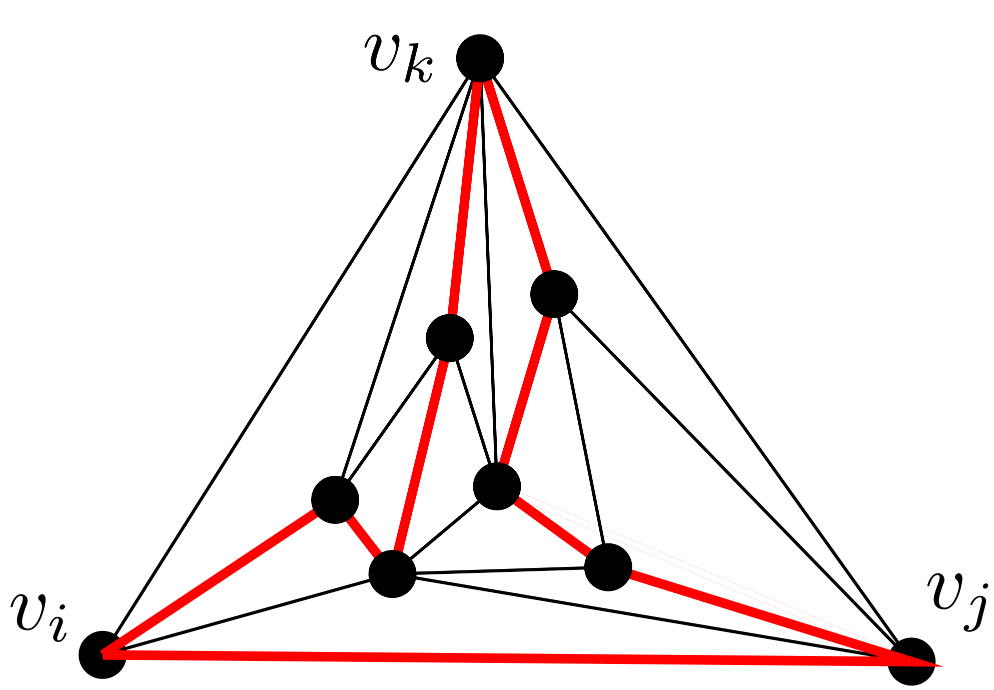

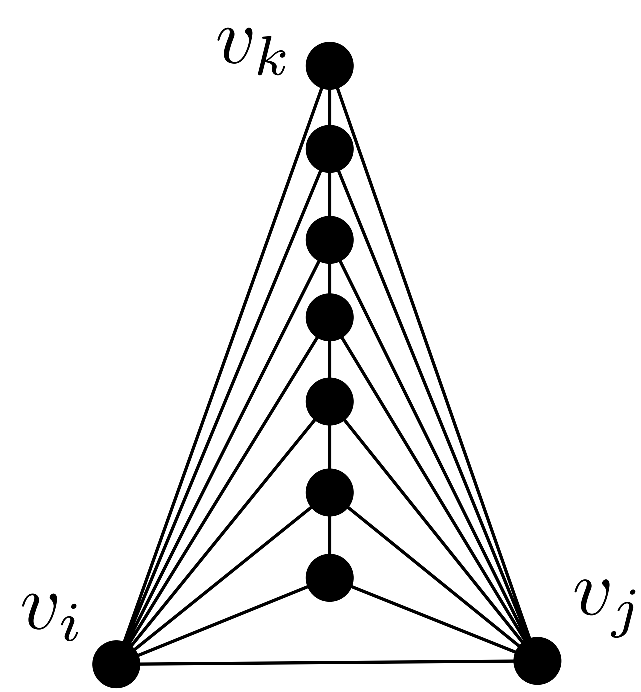

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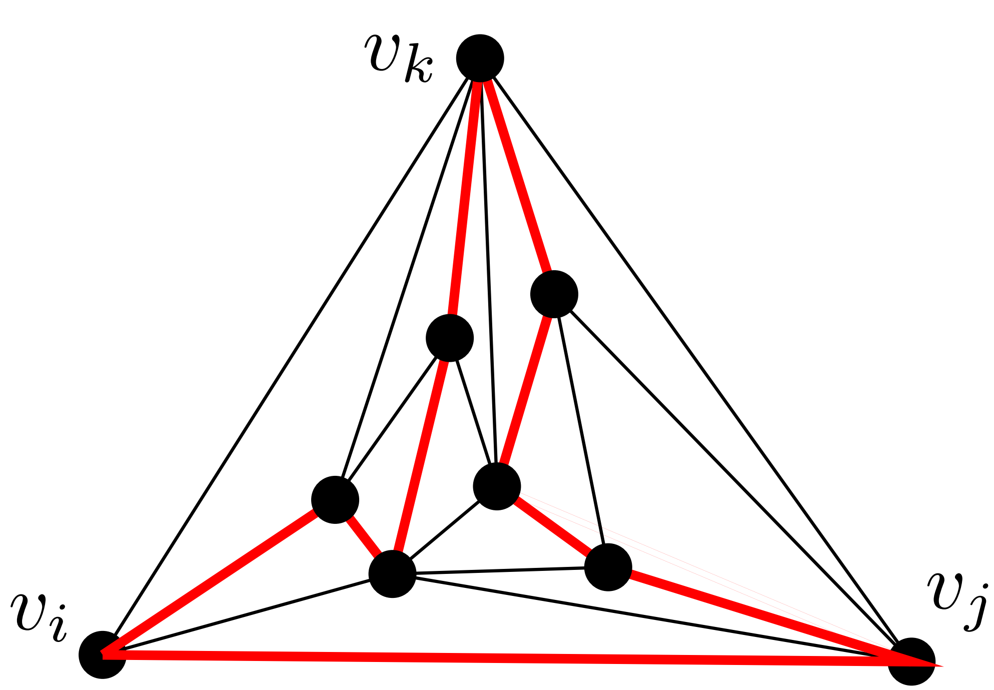
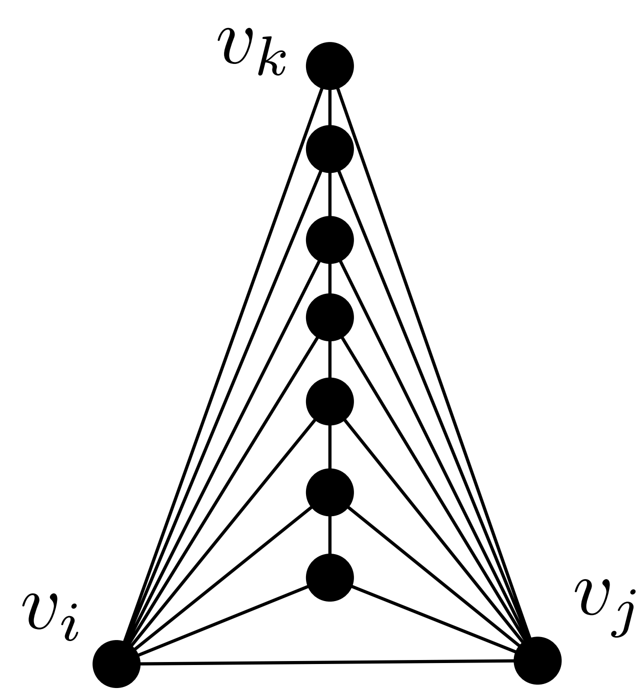
The key step behind their proof is that $n-4$ flips suffice to convert any triangulation into a hamiltonian triangulation.

$4n-22$ flips suffice to convert any hamiltonian triangulation to any other. Therefore, $6n-30$ flips suffice to convert any triangulation to any other.





Wagner's Canonical Triangulation

Combinatorial Setting

Flip Graph: Every combinatorially distinct n-vertex triangulation is a vertex. Two vertices are adjacent if the two triangulations differ by exactly one flip.

Combinatorial Setting

Open Problems

- Are there triangulations that require $n - 4$ edge flips to be converted to hamiltonian?

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Open Problems

Main Problem in this area:

Given two n-vertex triangulations, can you convert one into the other using the minimum number of flips?

Combinatorial Setting

What is the maximum number of flippable edges in an n-vertex triangulations?

Combinatorial Setting

Gao, Urrutia and Wang (2001) showed that in every n-vertex triangulation, there are always at least $(n-2)$ flippable edges and there exist triangulations where this bound is reached

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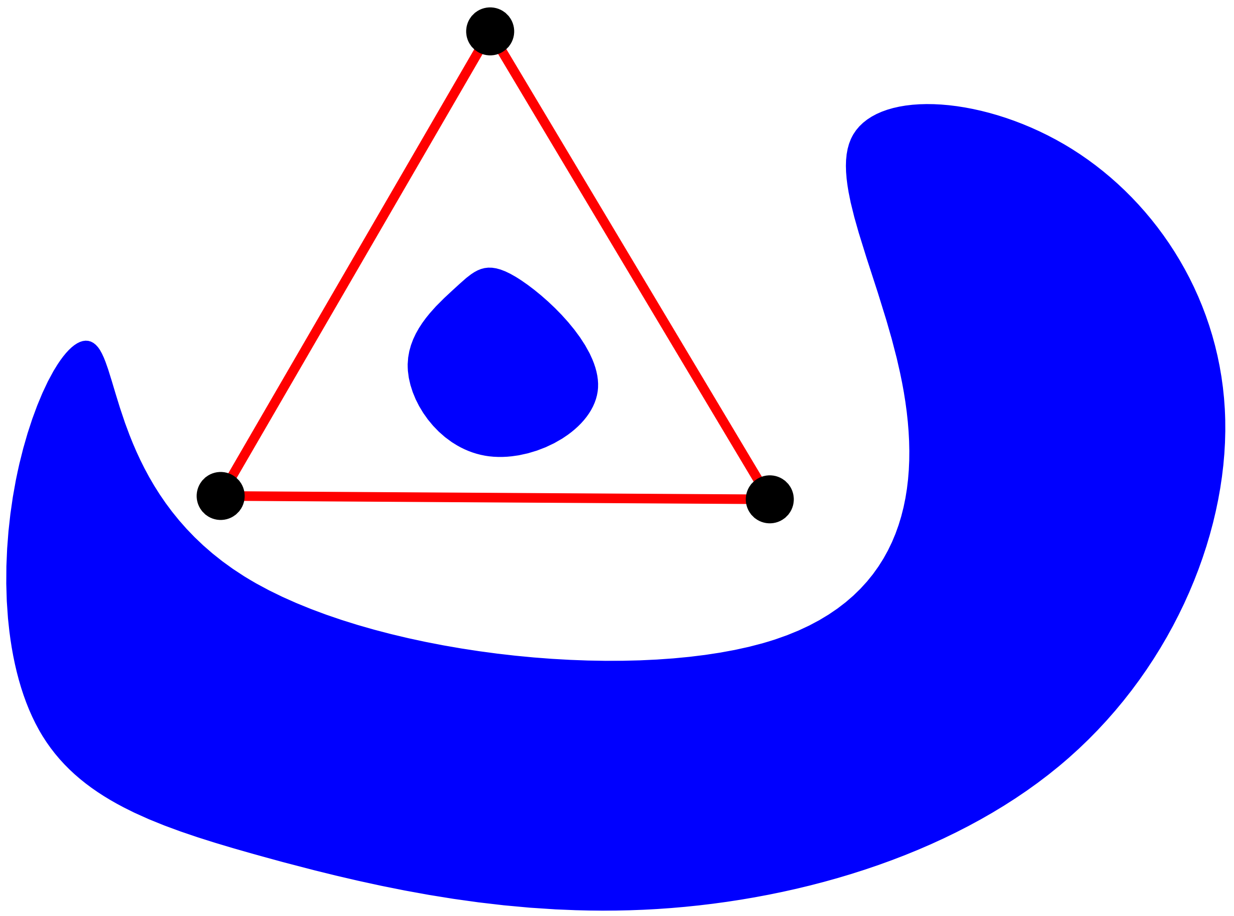

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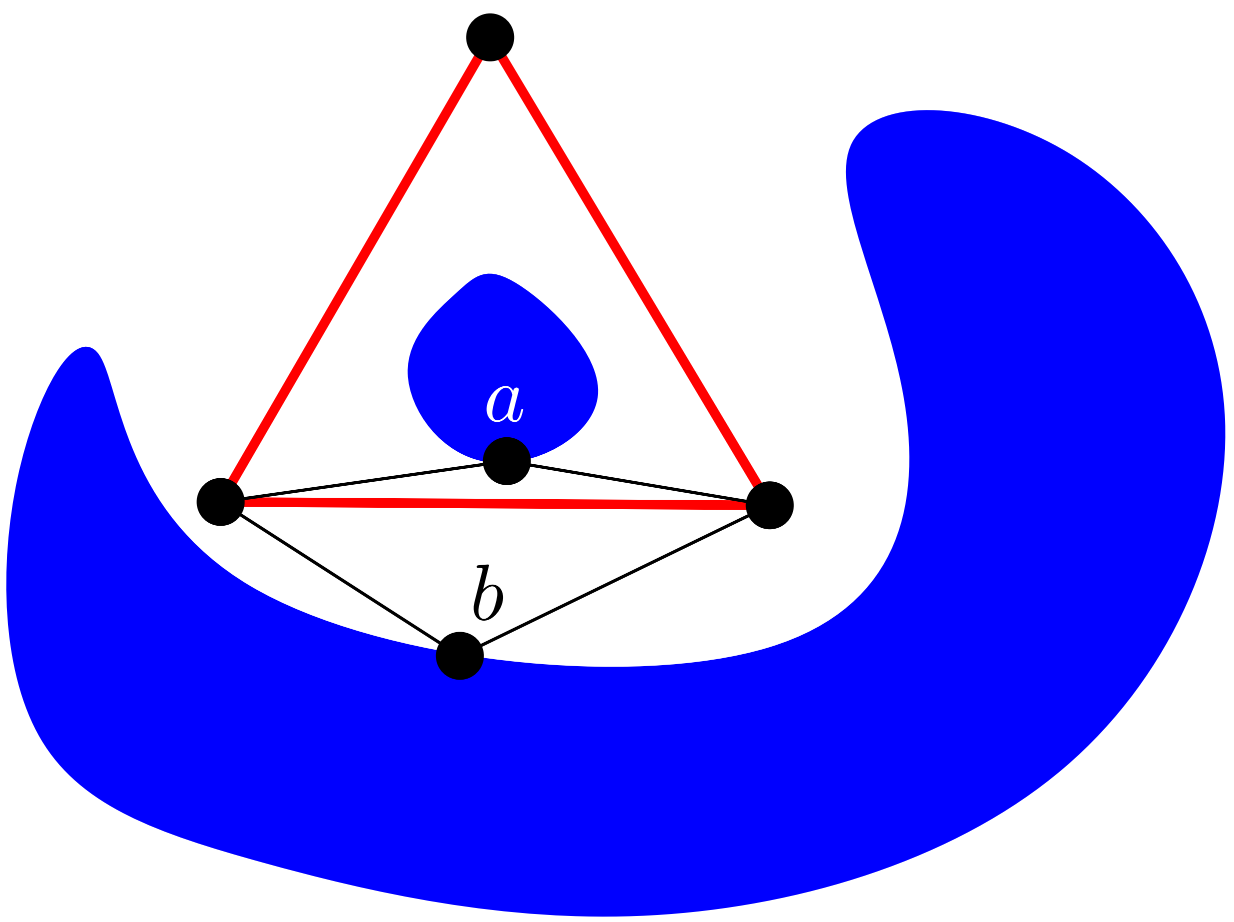

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Upper Bound: Every triangle in a triangulation has at least one flippable edge. Therefore, there are at least $n-2$ flippable edges.

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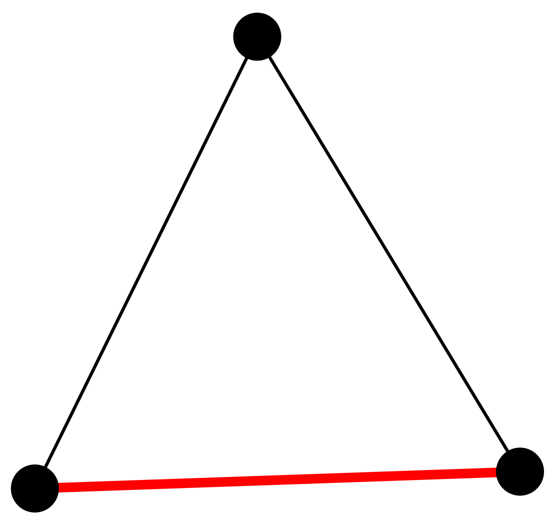

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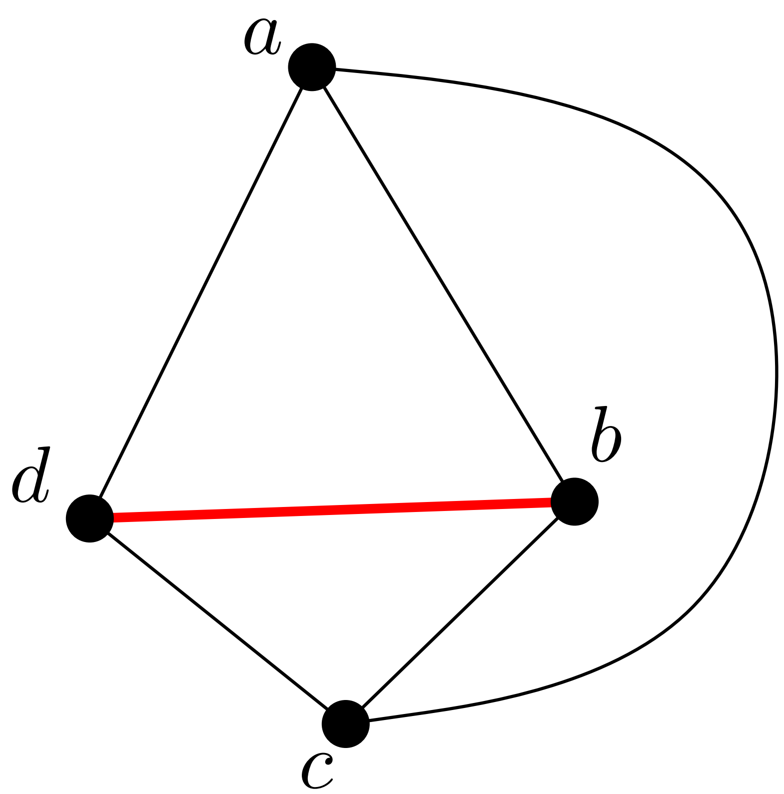

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Lower Bound: Start with any triangulation and add one vertex inside each face. This gives at most $n-2$ flippable edges.

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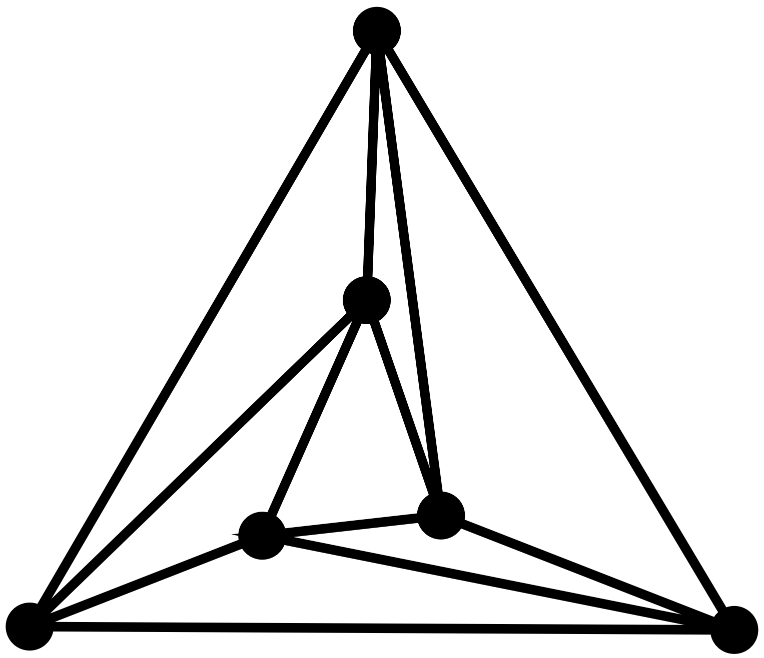

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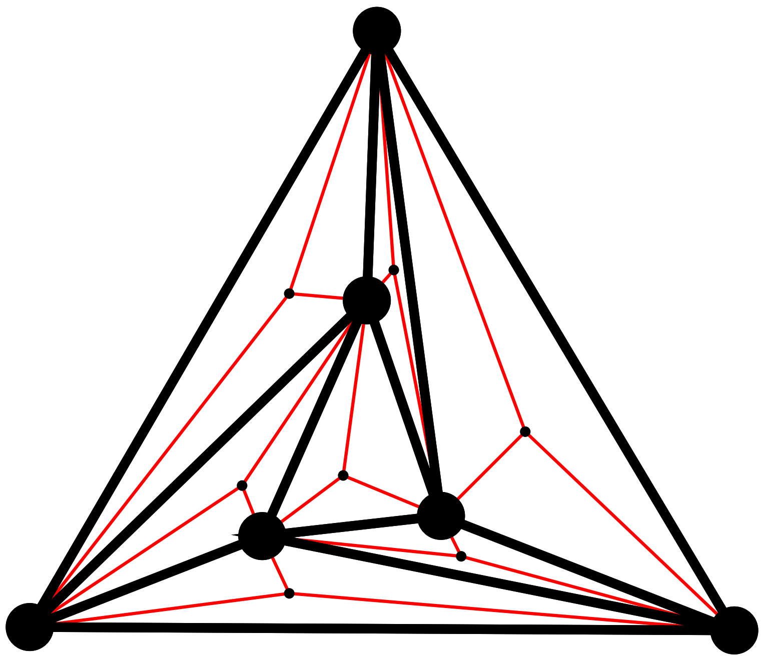

Combinatorial Setting

Open Problem

What is the minimum, maximum and average degree of a vertex in the flip graph?

Combinatorial Setting

Galtier, Hurtado, Noy, Pérennes and Urrutia (2003) introduced the notion of *Simultaneous Flips*

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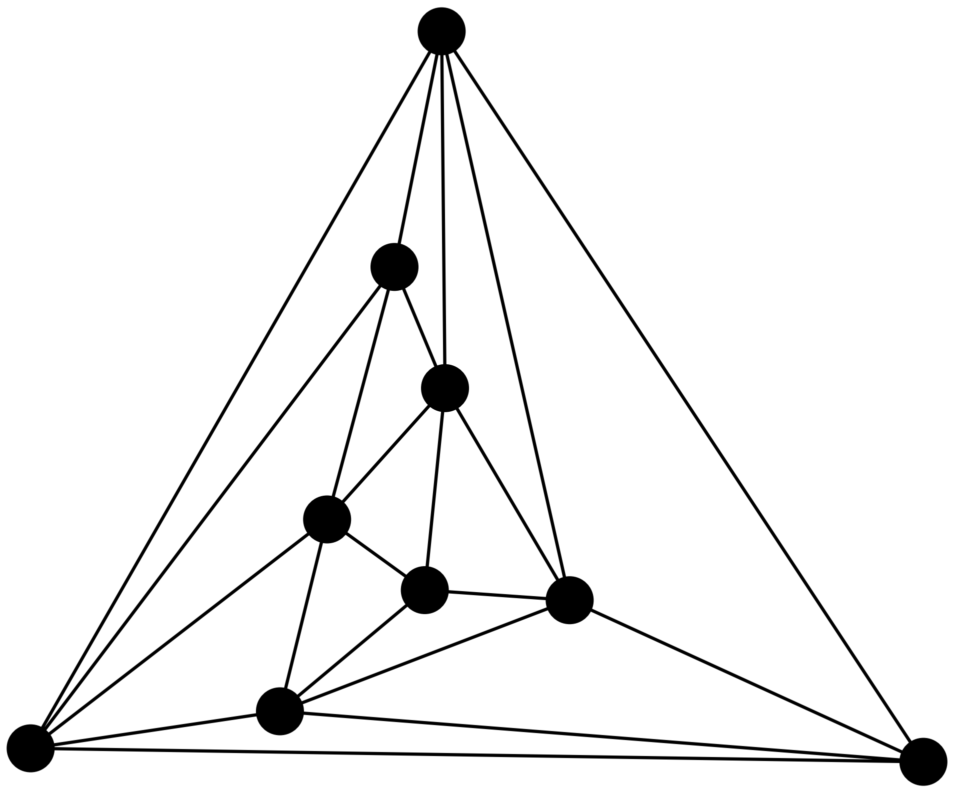

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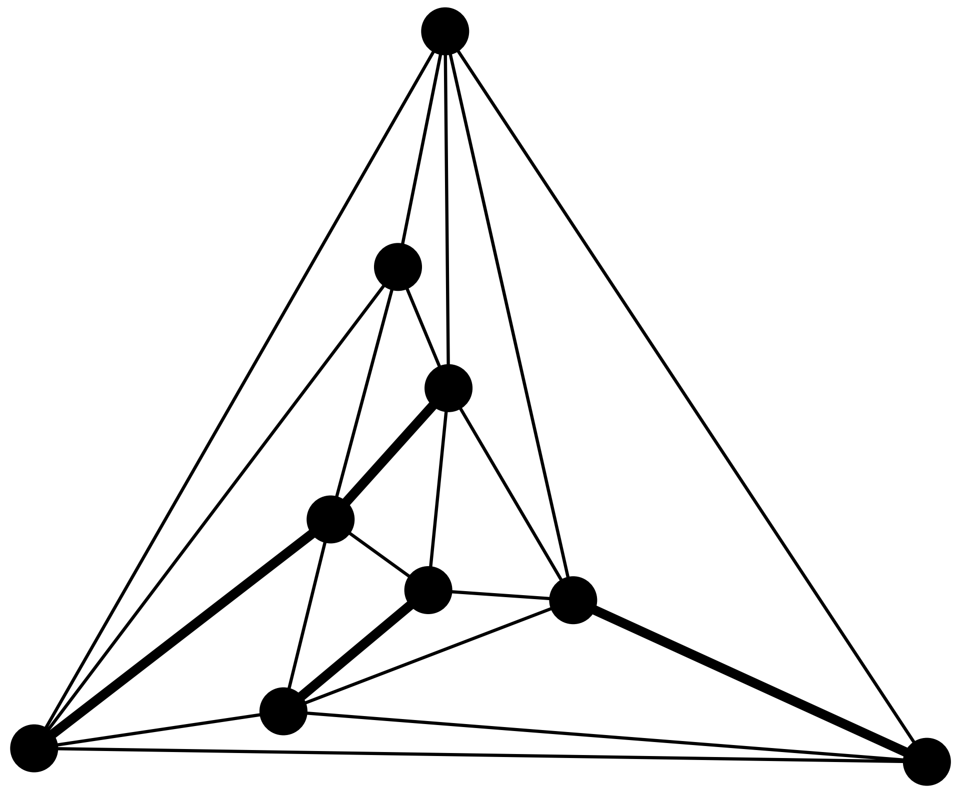

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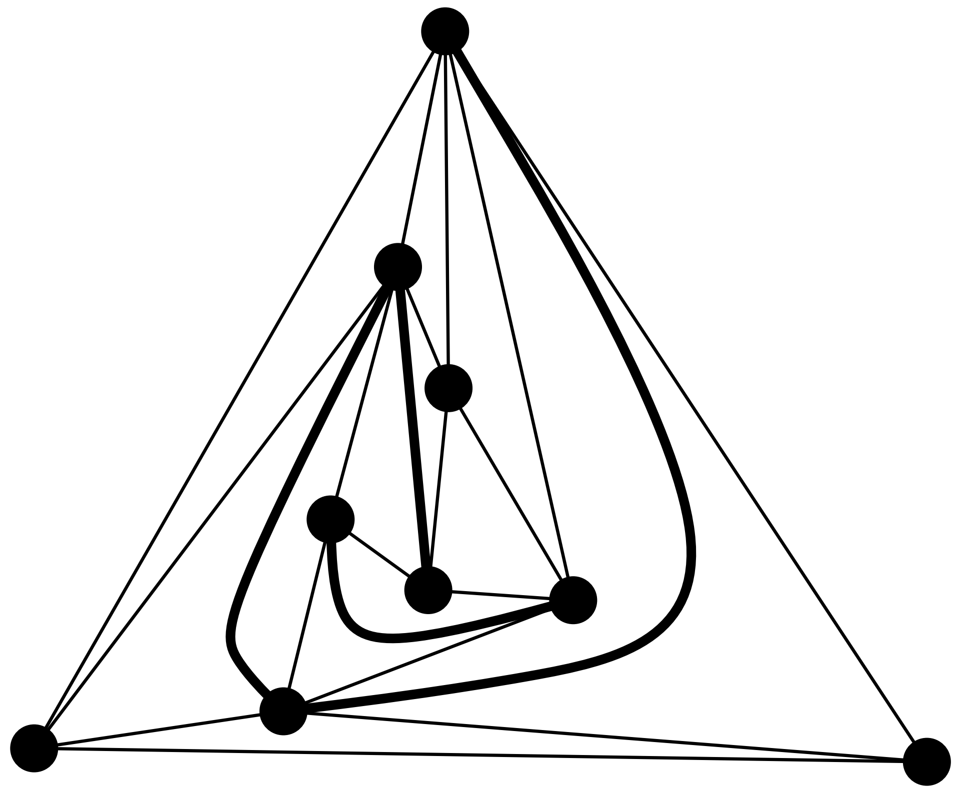

Overview of Results in Combinatorial Simultaneous Setting

Bose, Czyzowicz, Gao, Morin, and Wood (2005)

1. With one simultaneous flip, any triangulation can be converted to a hamiltonian triangulation.

2. $O(\log n)$ simultaneous flips suffice to convert any triangulation into another.

3. There exists a simple lower bound of $\Omega(\log n)$.

4. There exist triangulations with at most $6n/7$ edges that can be flipped simultaneously.

5. Every triangulation has at least $(n-2)/3$ edges that can be flipped simultaneously.

Converting one triangulation to another

Outline of the idea to convert G_1 to G_2

1. Convert G_1 to a hamiltonian triangulation with 1 parallel flip.
2. Convert interior of the hamiltonian cycle to a star with $O(\log n)$ flips.
3. Convert exterior of the hamiltonian cycle to a star with $O(\log n)$ flips.
4. Run this backwards to get G_2 with $O(\log n)$ flips.

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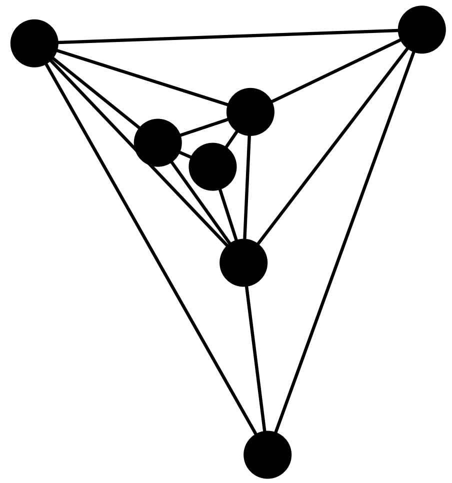

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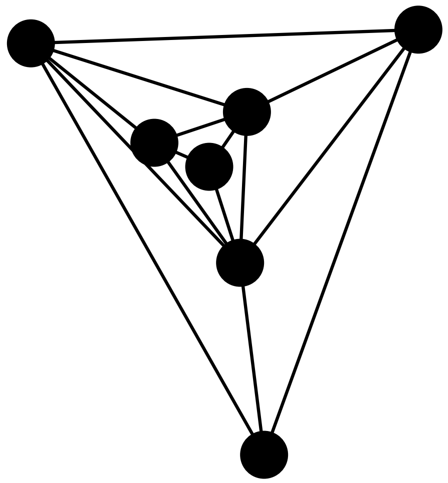

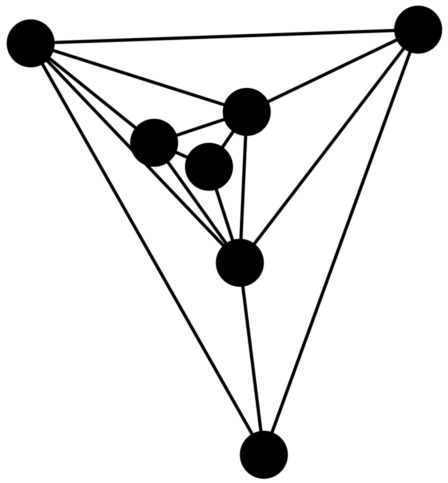

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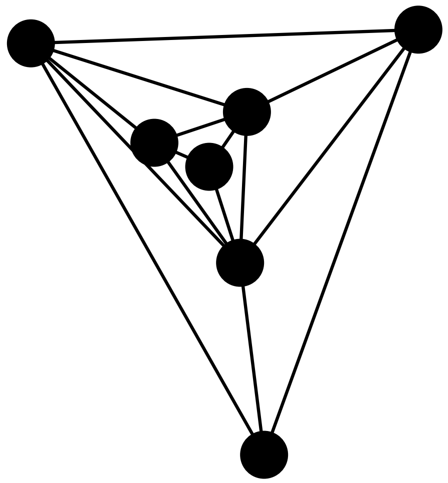

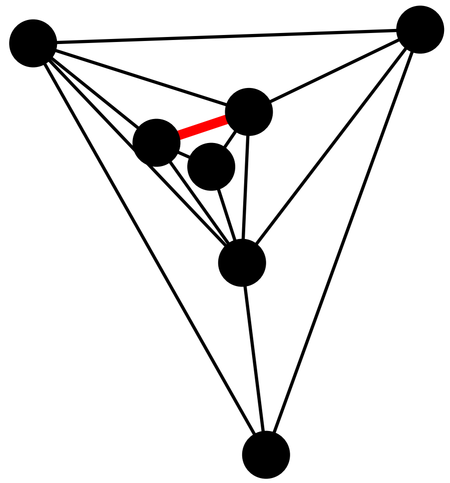

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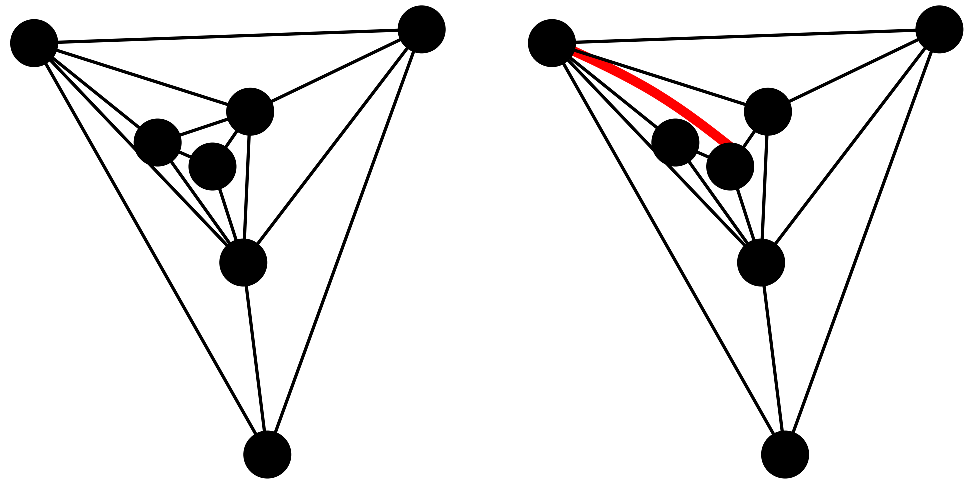

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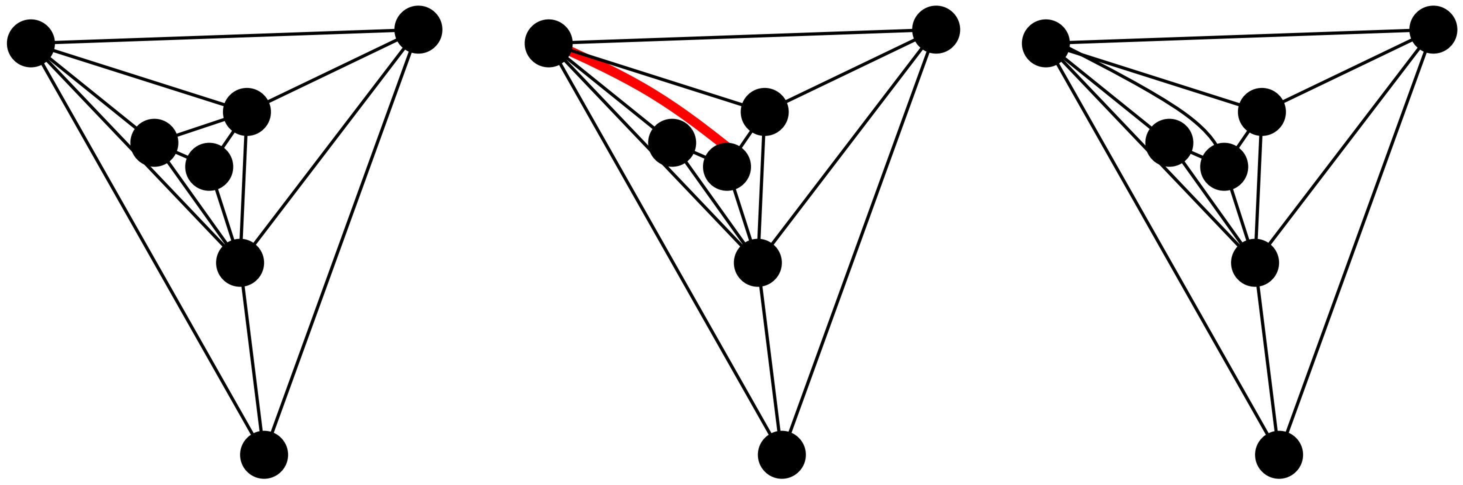

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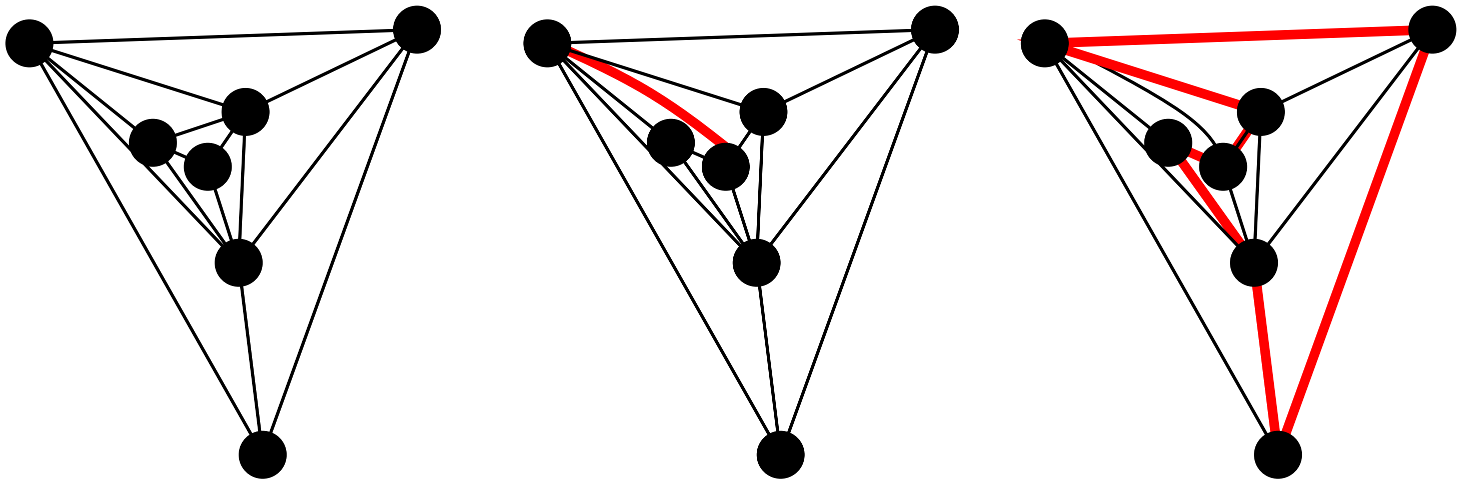
2. Convert interior of the hamiltonian cycle to a star with $O(\log n)$ flips.
3. Convert exterior of the hamiltonian cycle to a star with $O(\log n)$ flips.
4. Run this backwards to get G_2 with $O(\log n)$ flips.



Converting one triangulation to another

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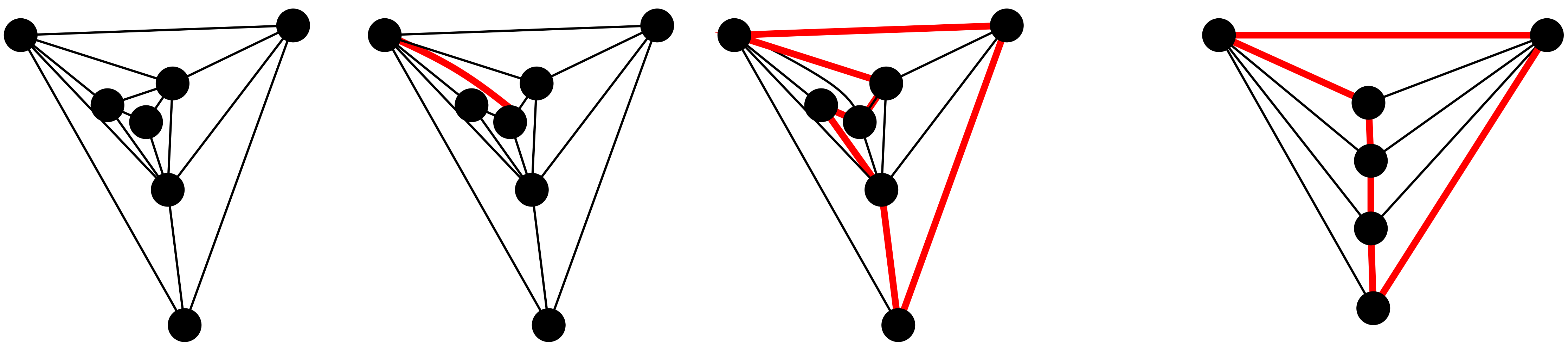
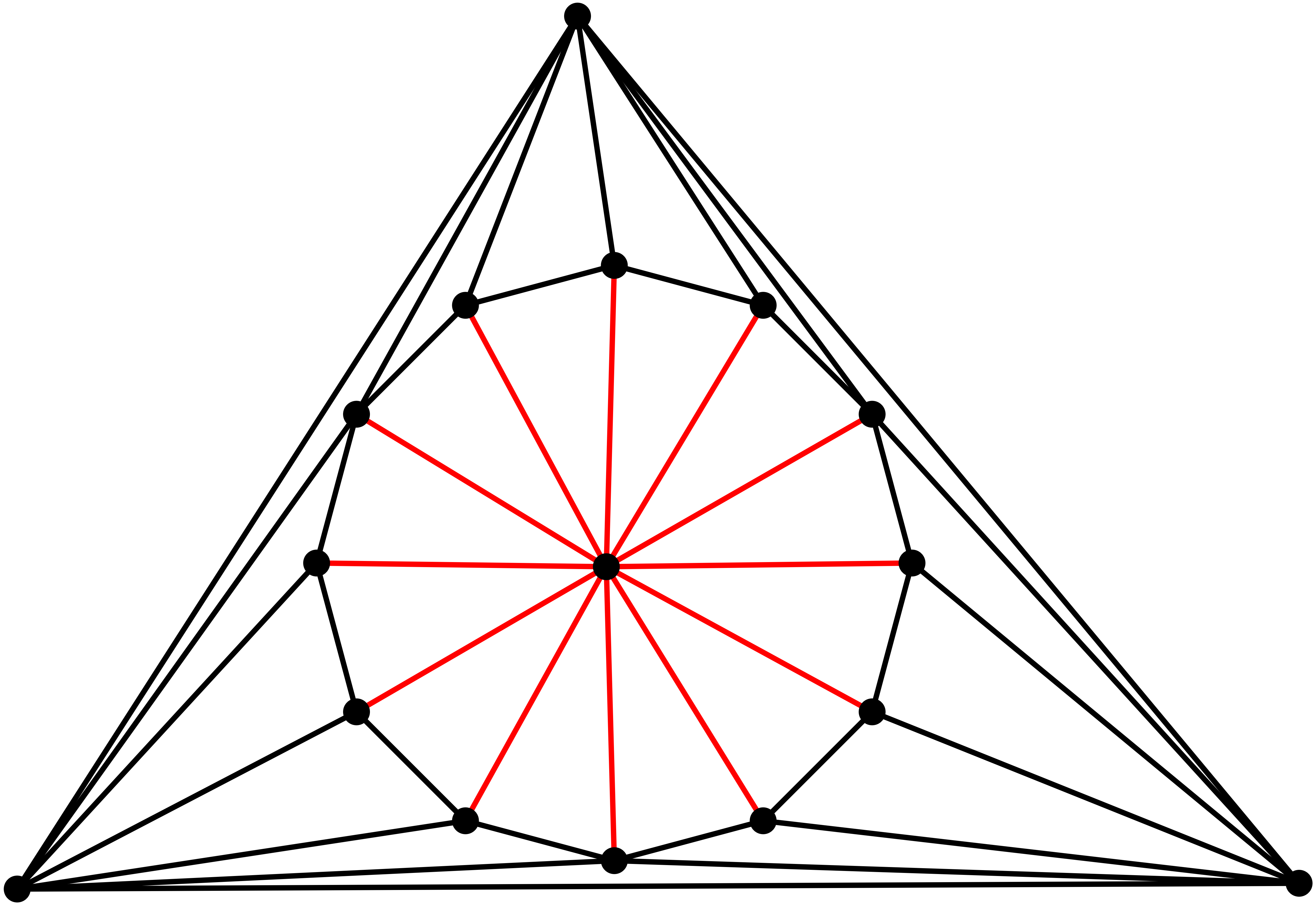


Lower Bound



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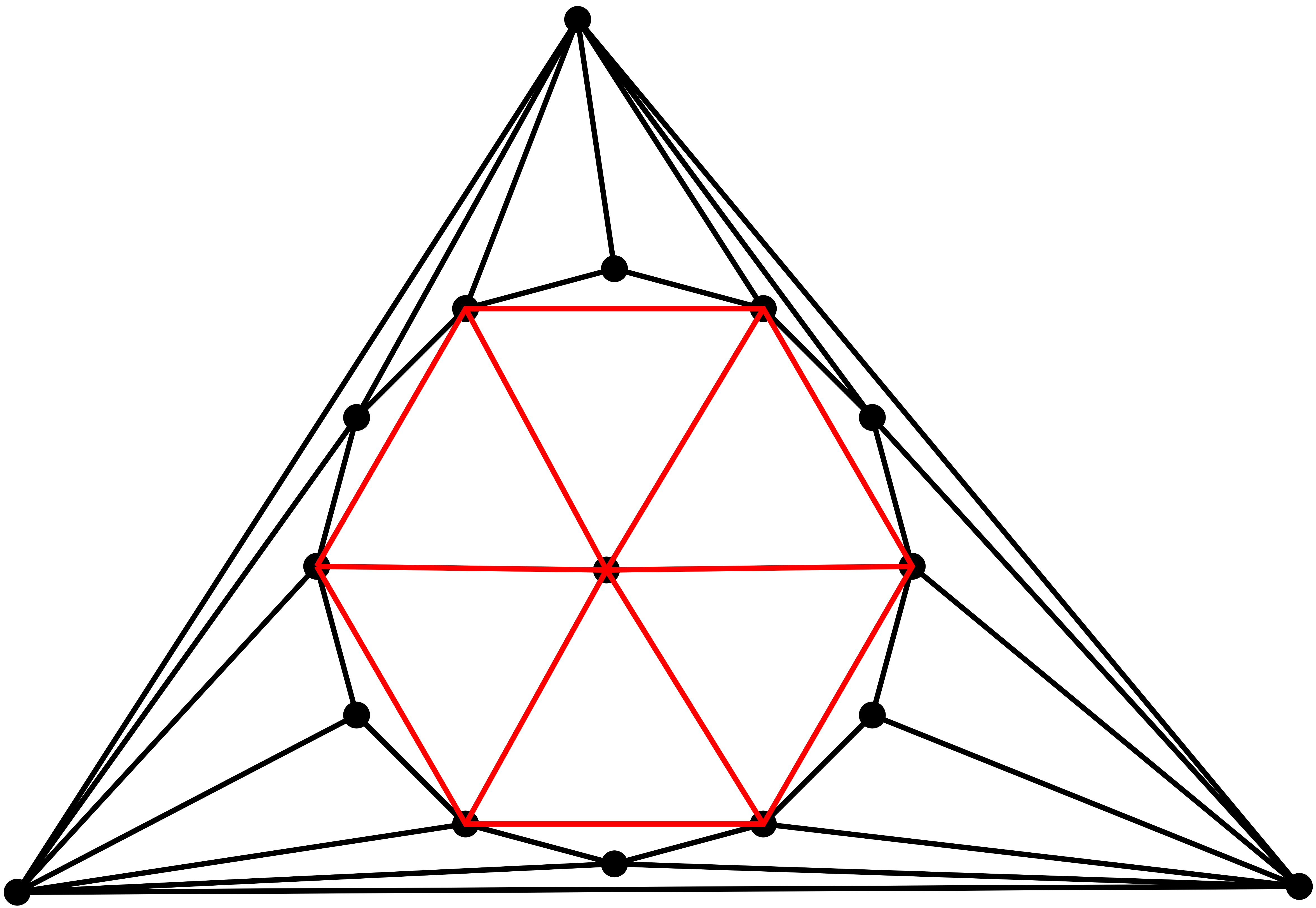

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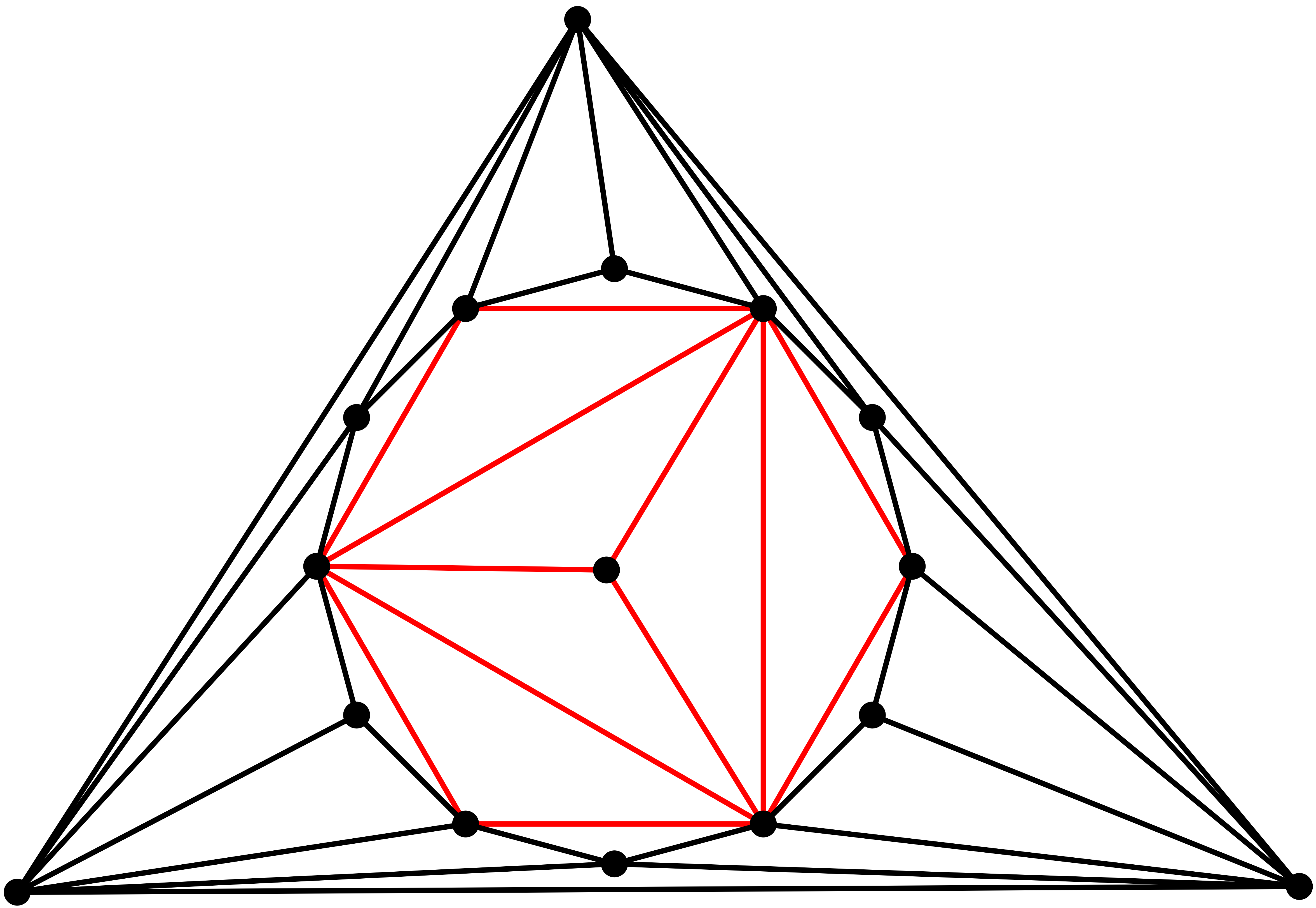
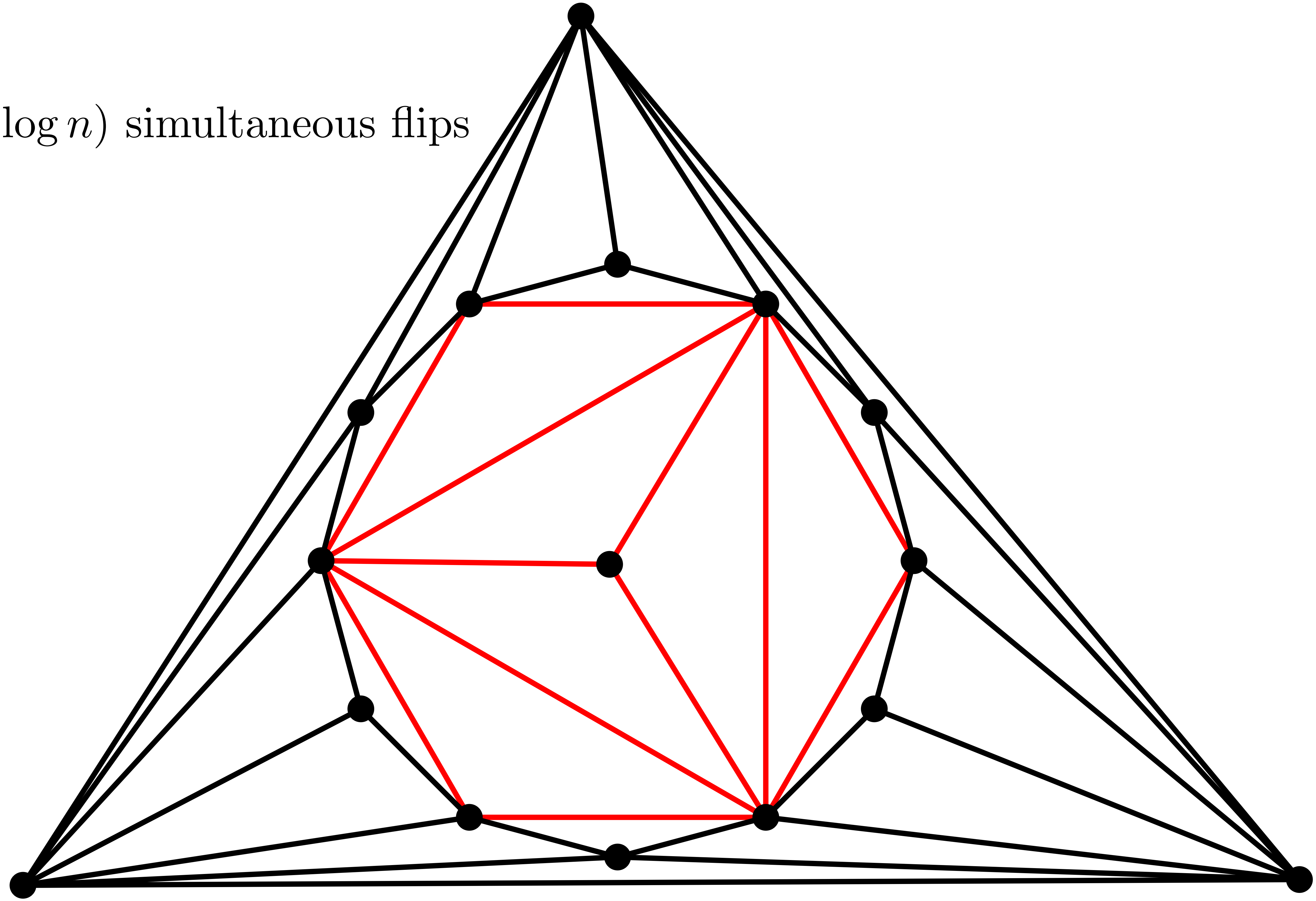


Lower Bound

$\Omega(\log n)$ simultaneous flips



Simultaneous Flips - Upper Bound

Theorem 1 *There exists an infinite family of triangulations that admits exactly* $6(n-2)/7$ *simultaneous flips.*

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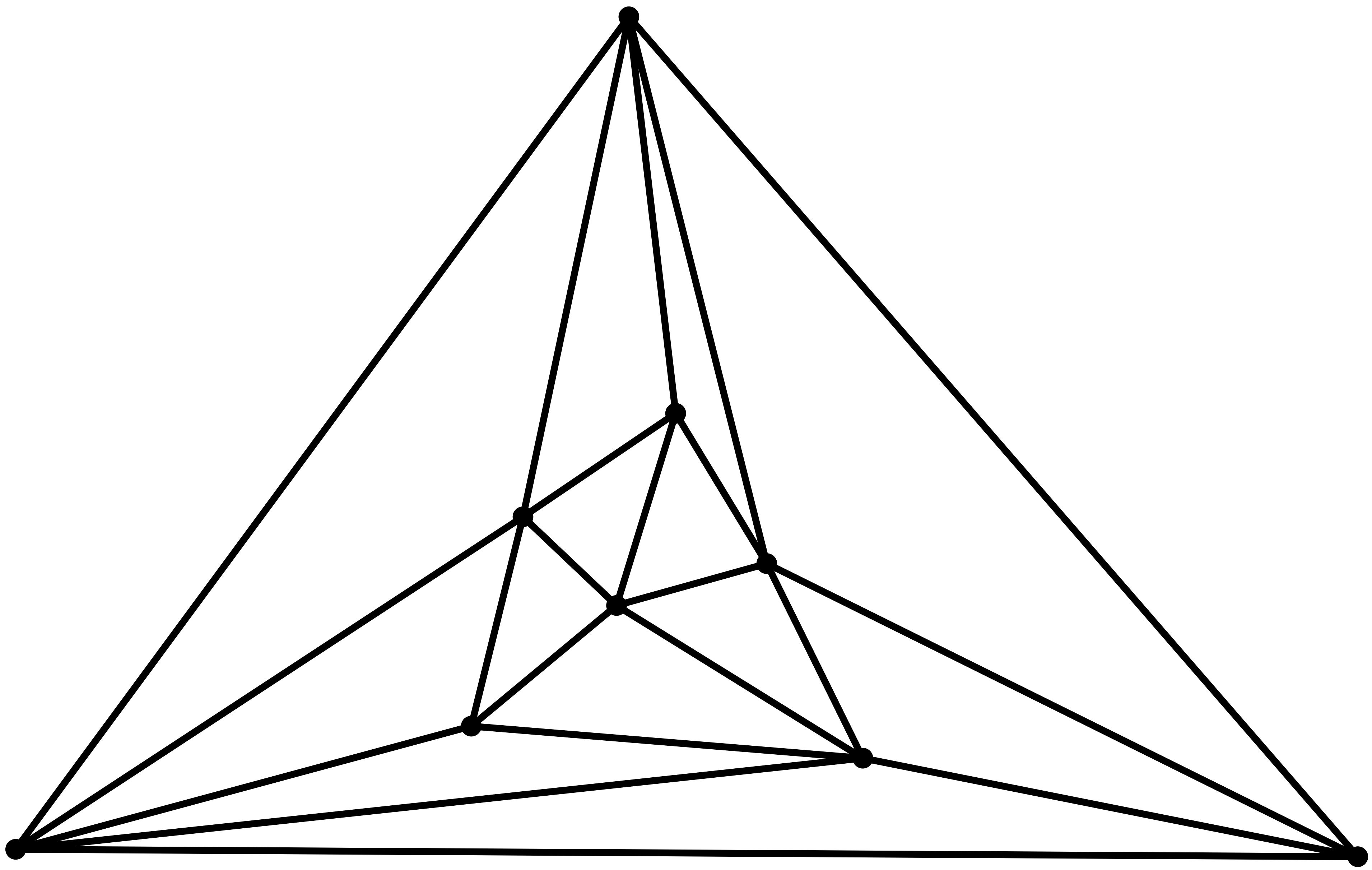

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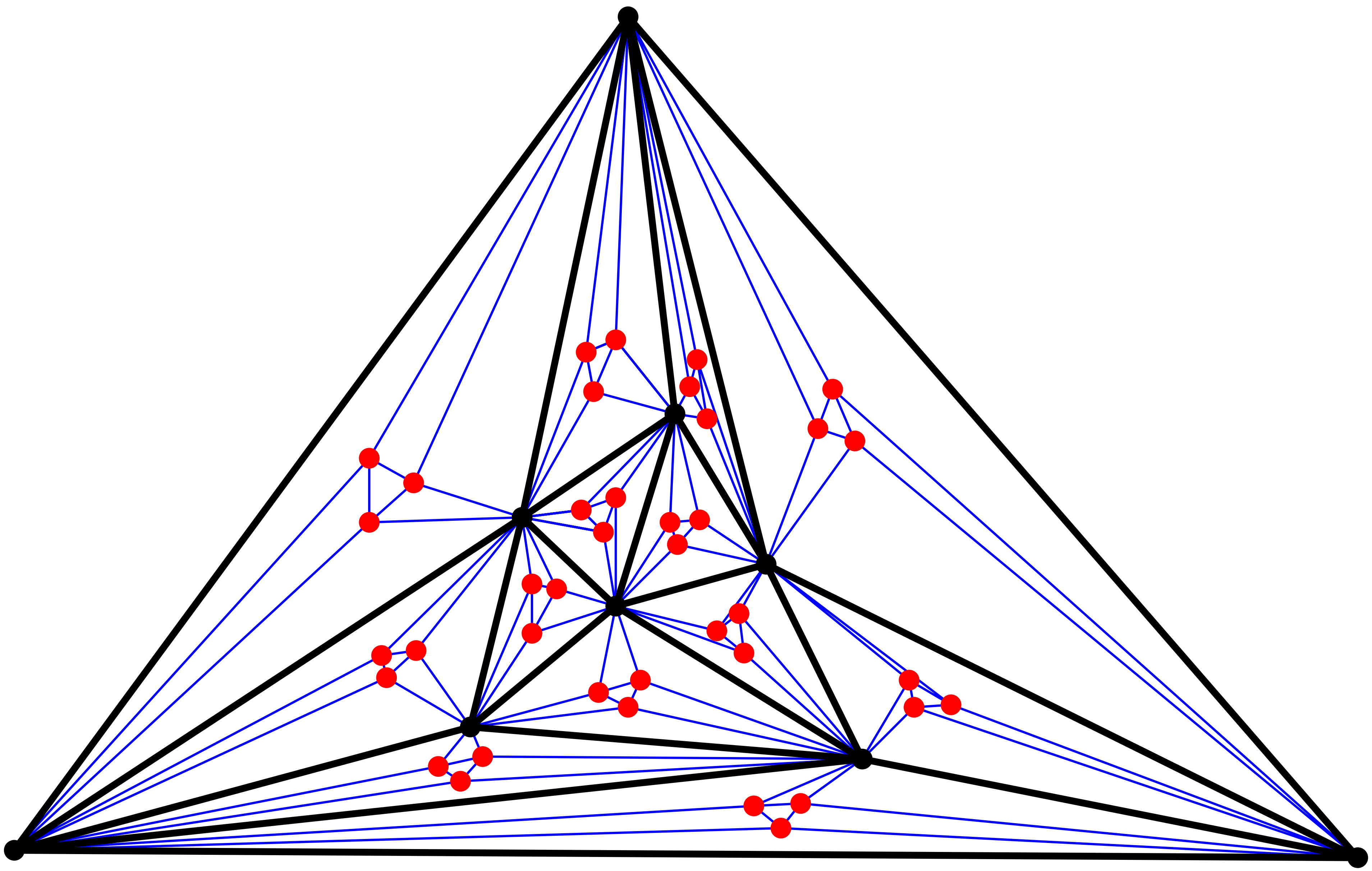

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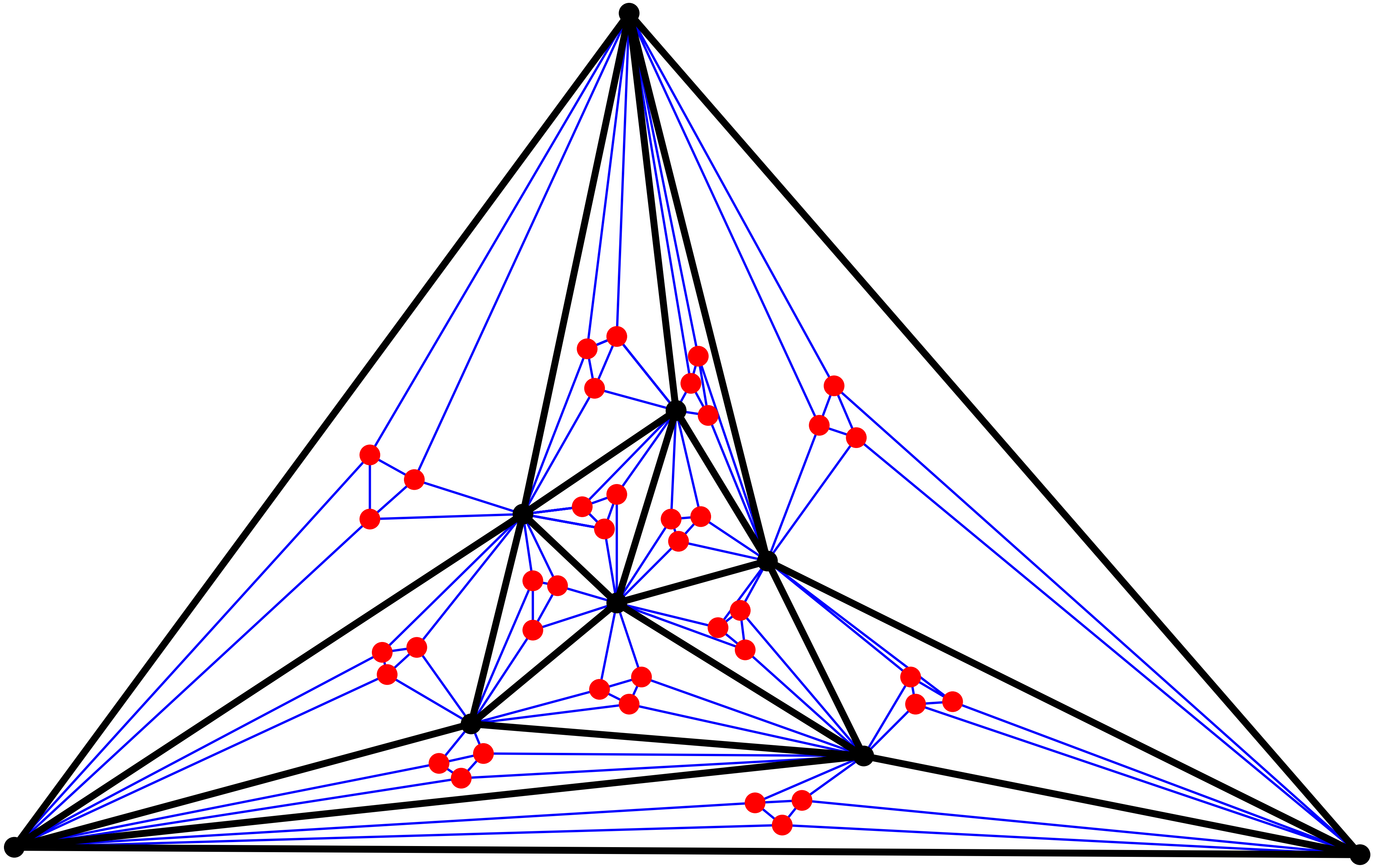
At least 1 of the 7 triangles does not have an edge in the optimal flip set.



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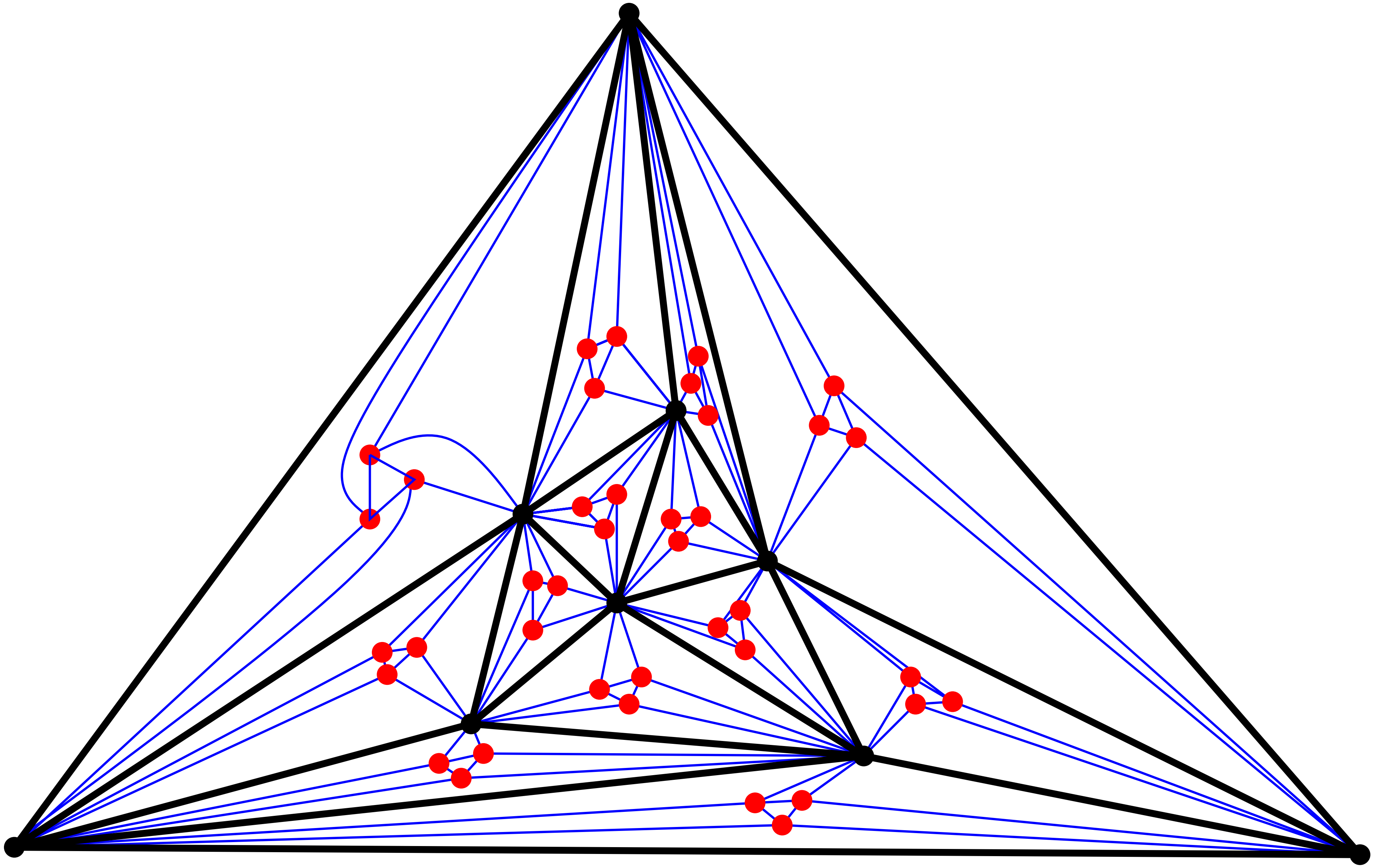


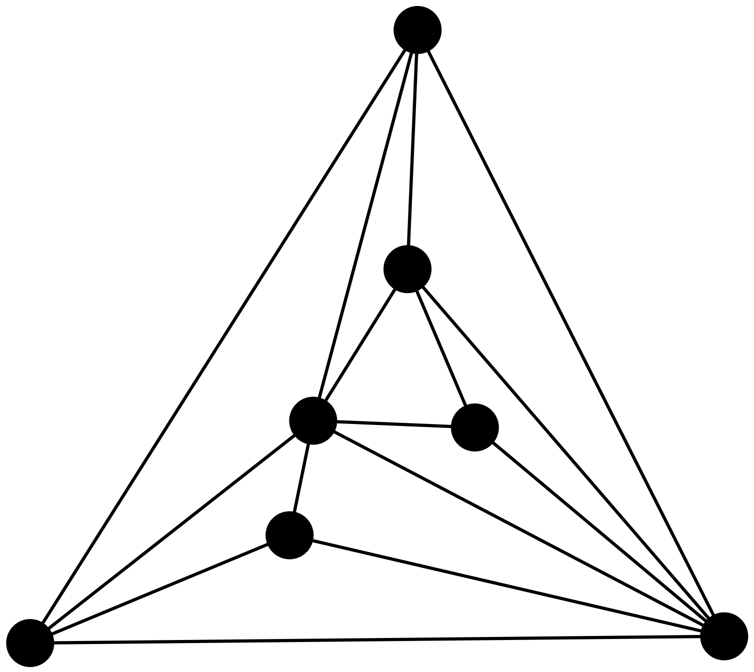
Open Problems

- Can one close the gap in the constant between the upper and lower bound on the number of simultaneous flips needed to convert one triangulation into another?

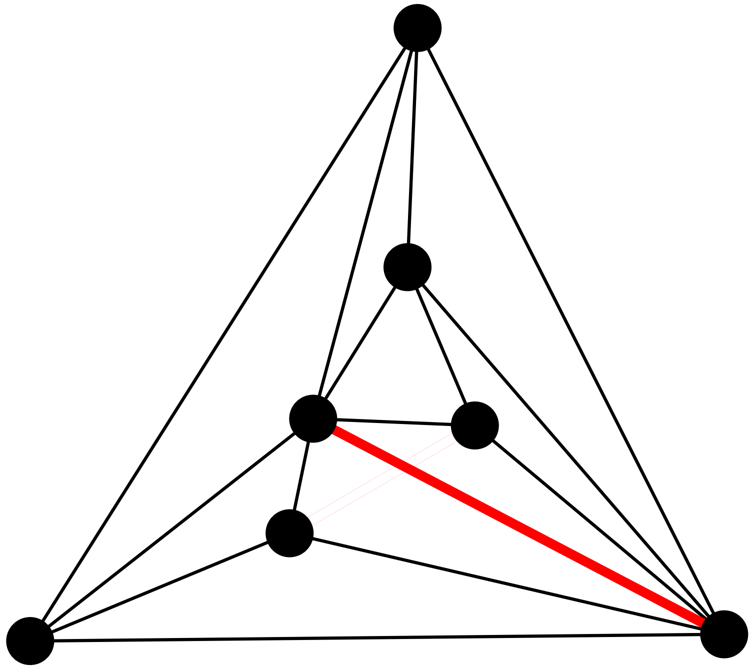
- Can one compute a set of simultaneous flips that converts one triangulation into another that is sensitive to the minimum number of simultaneous flips required?

- Can one close the gap between the upper bound of $6(n-2)/7$ and lower bound of $(n-2)/3$ on the number of simultaneously flippable edges?

Geometric Setting



Geometric Setting



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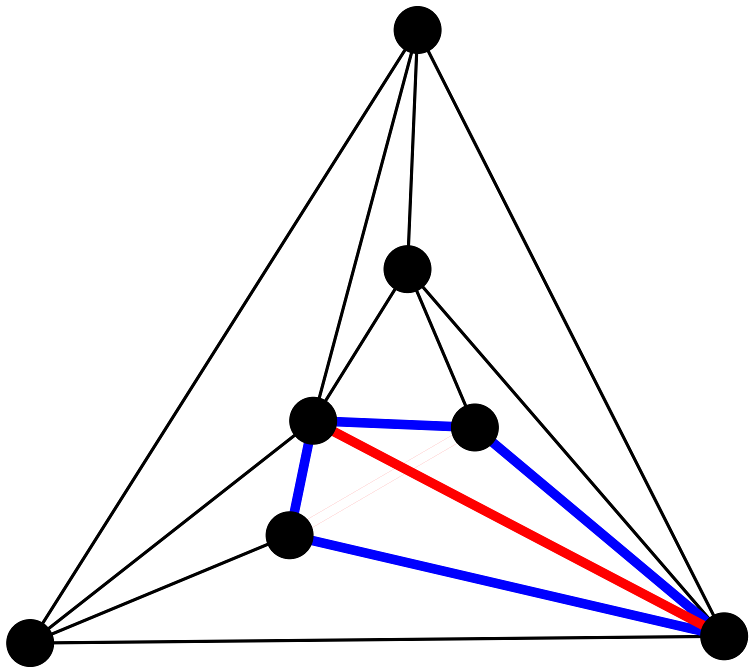

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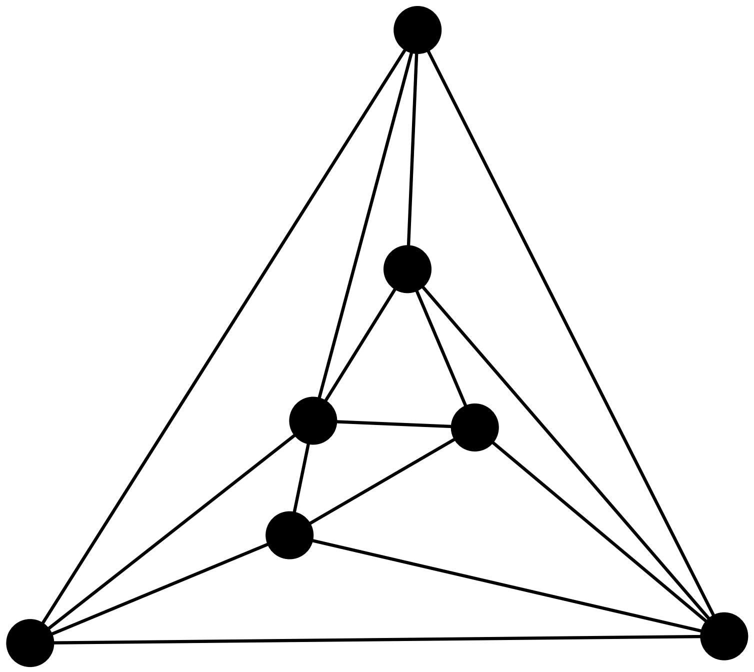

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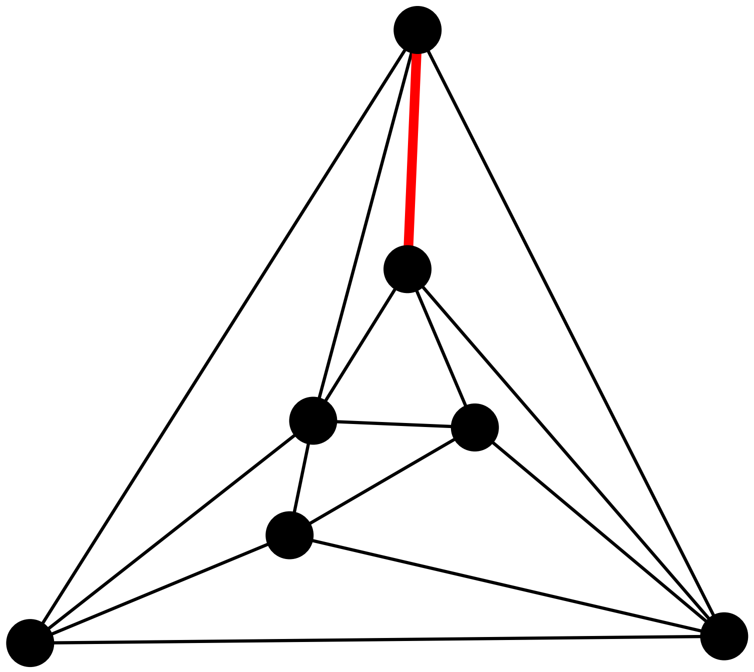
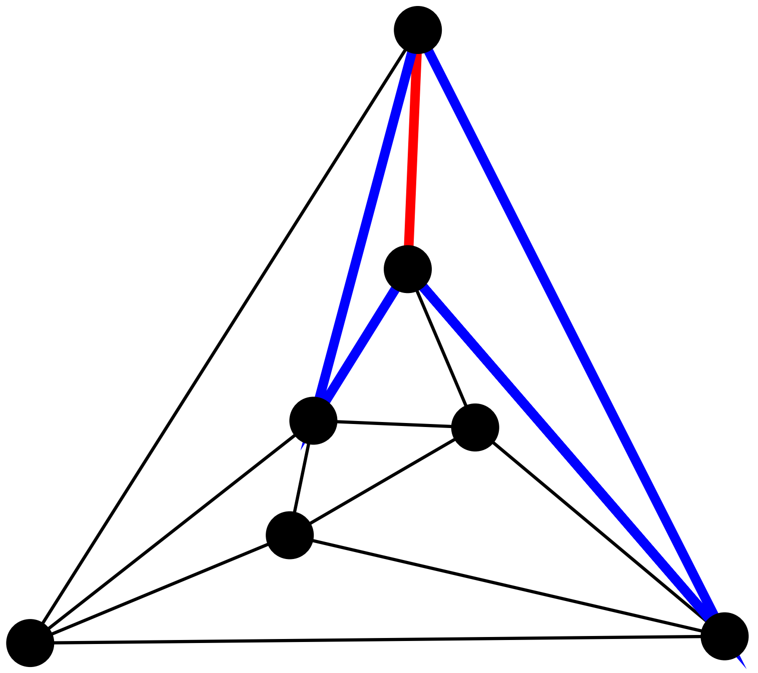


Geometric Setting



Geometric Setting

Lawson (1977) proved the seminal result in this setting. He showed that $O(n^2)$ flips are sufficient to convert any triangulation of n points to any other triangulation of the same n points in the plane.

The canonical triangulation that Lawson used is the Delaunay triangulation.

Key idea: Once an edge is flipped out, it is never flipped back in.

Geometric Setting

Hurtado, Noy and Urrutia (1999) showed that there exists a pair of triangulations that require $(n-1)^2$ flips to transform one into the other.

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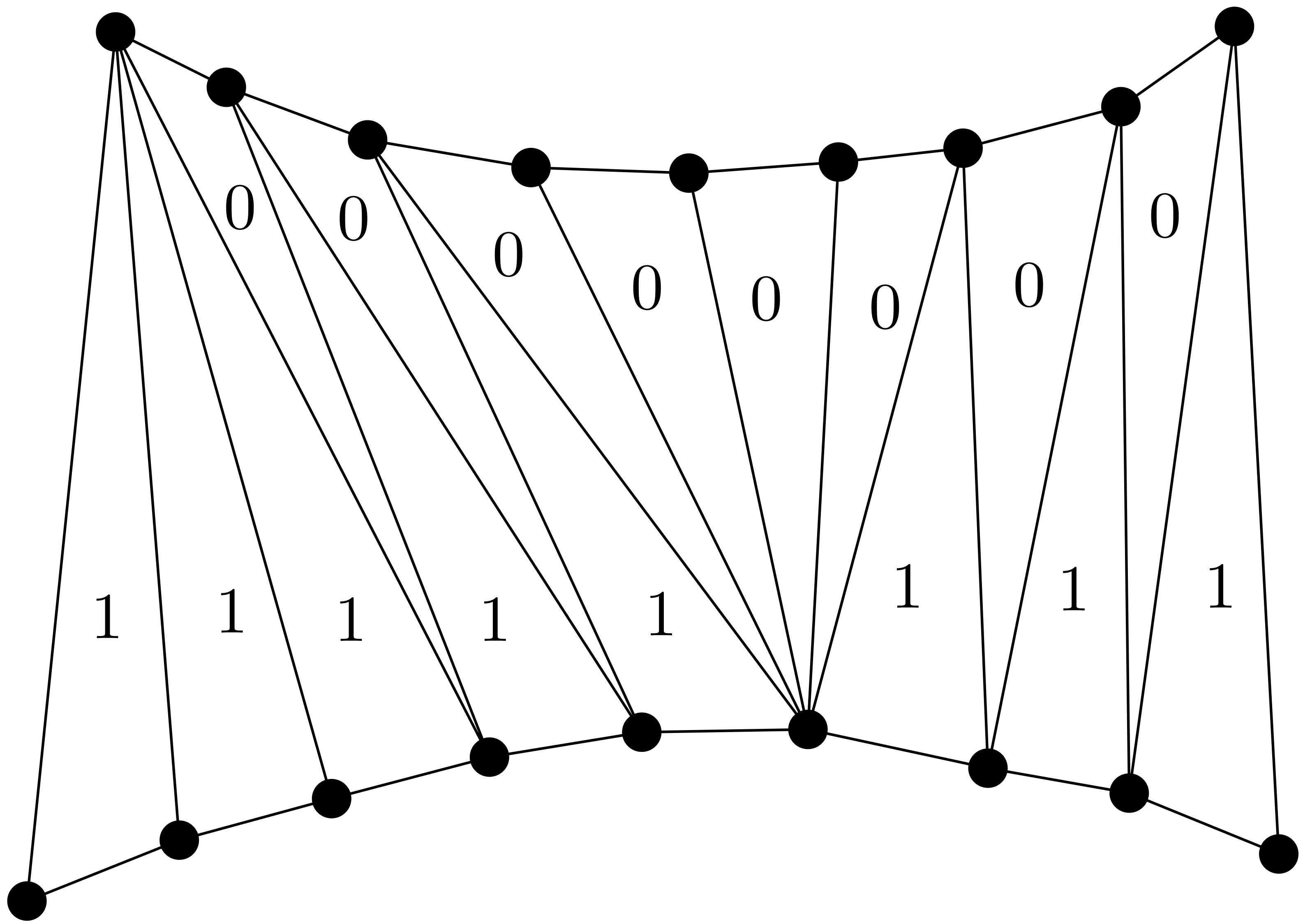

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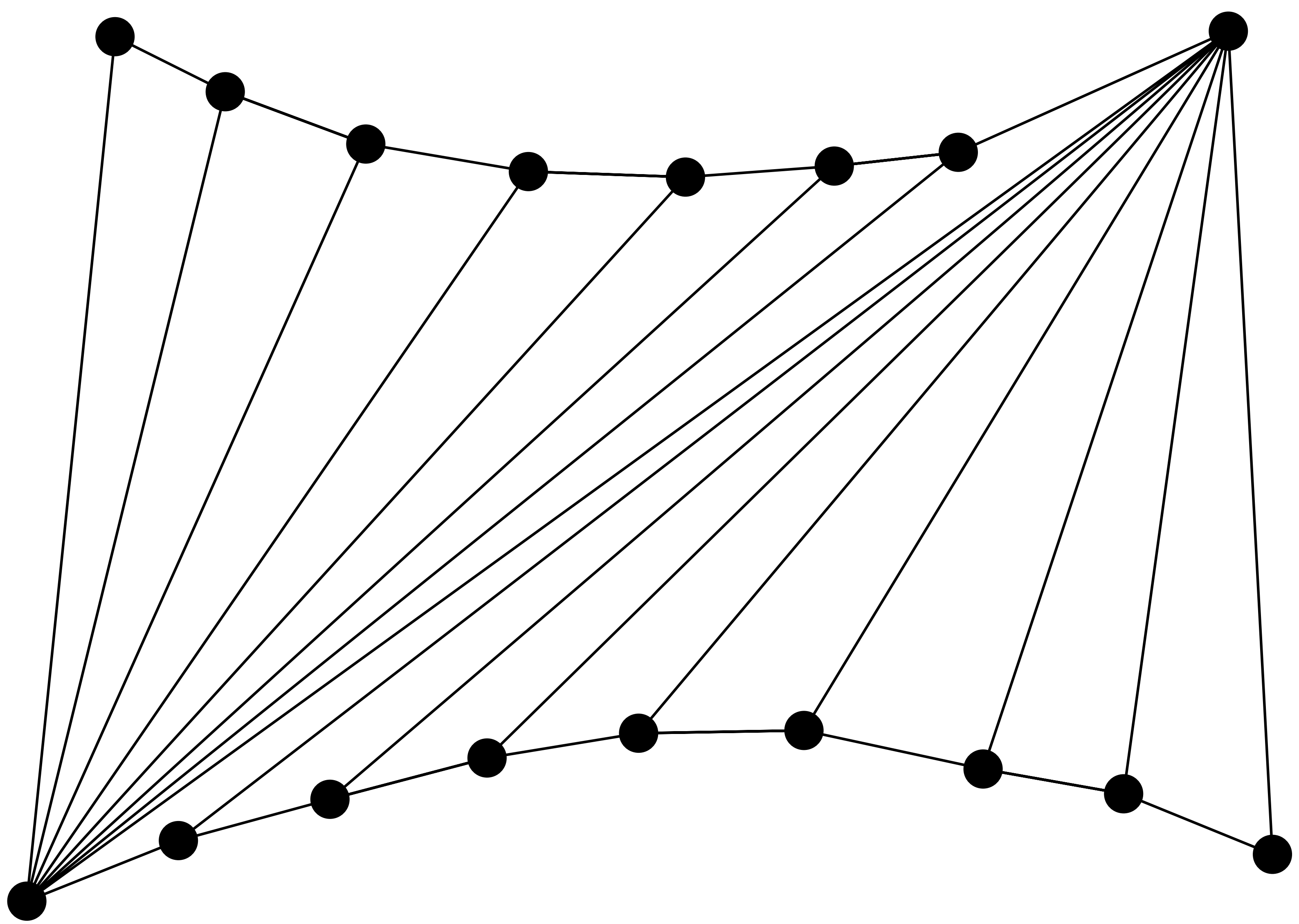

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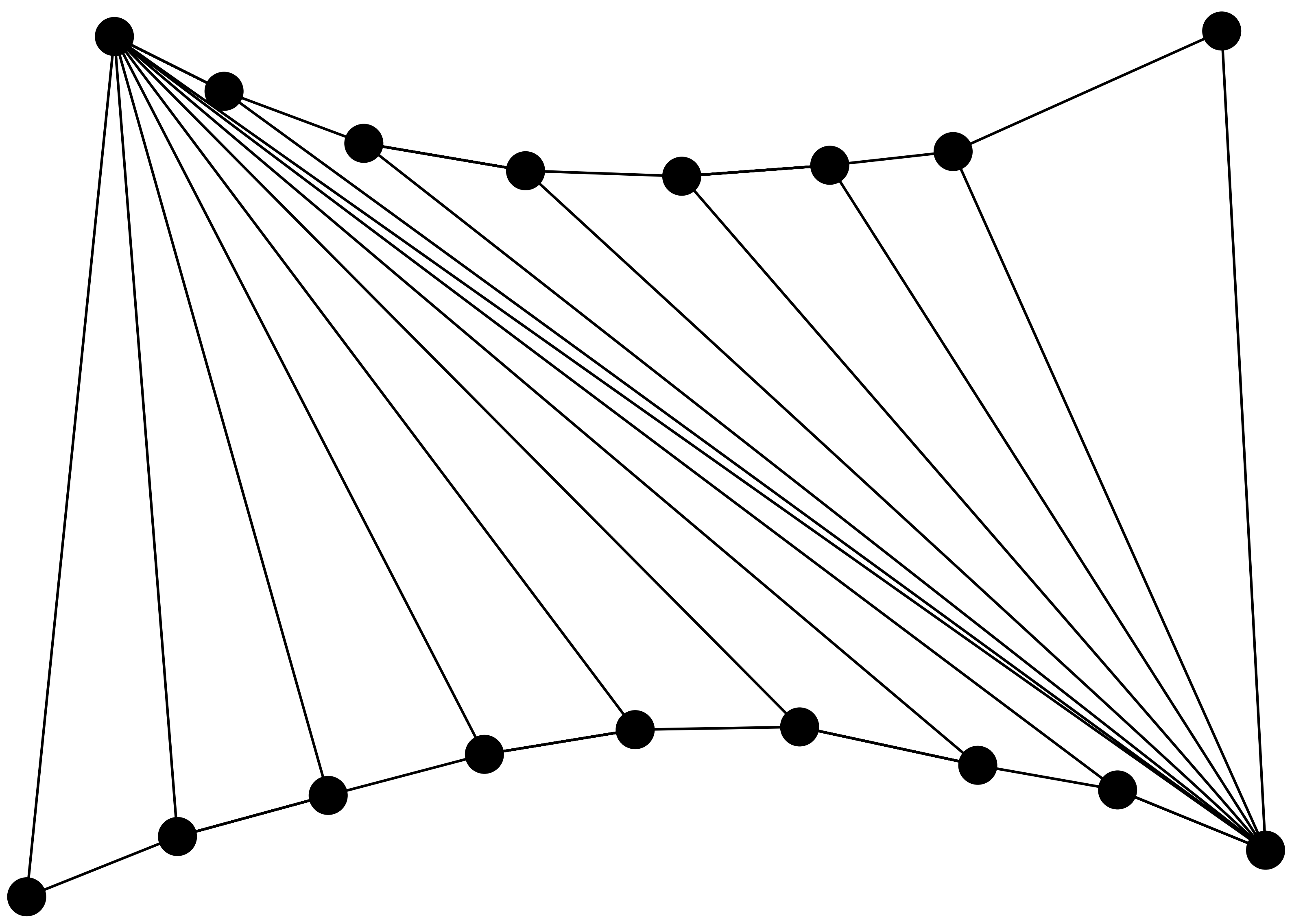

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Geometric Setting

Open Problem

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2. Can one compute a triangulation *simply* that allows you to flip to a (Greedy, Delaunay, etc) triangulation in $o(n^2)$ flips?

3. Given two triangulations, can one find the minimum number of flips to convert one triangulation into the other?

Geometric Setting

Hanke, Ottmann and Schuierer (1996) proved the following: Let T_1 and T_2 be two triangulations of the same n points in the plane. Let M be the number of intersections between the edges of T_1 and T_2. At most M flips are sufficient to convert T_1 into T_2.

Key Idea: There always exists an edge that can be flipped that reduces the number of intersections.

Simultaneous Flips in Geometric Setting

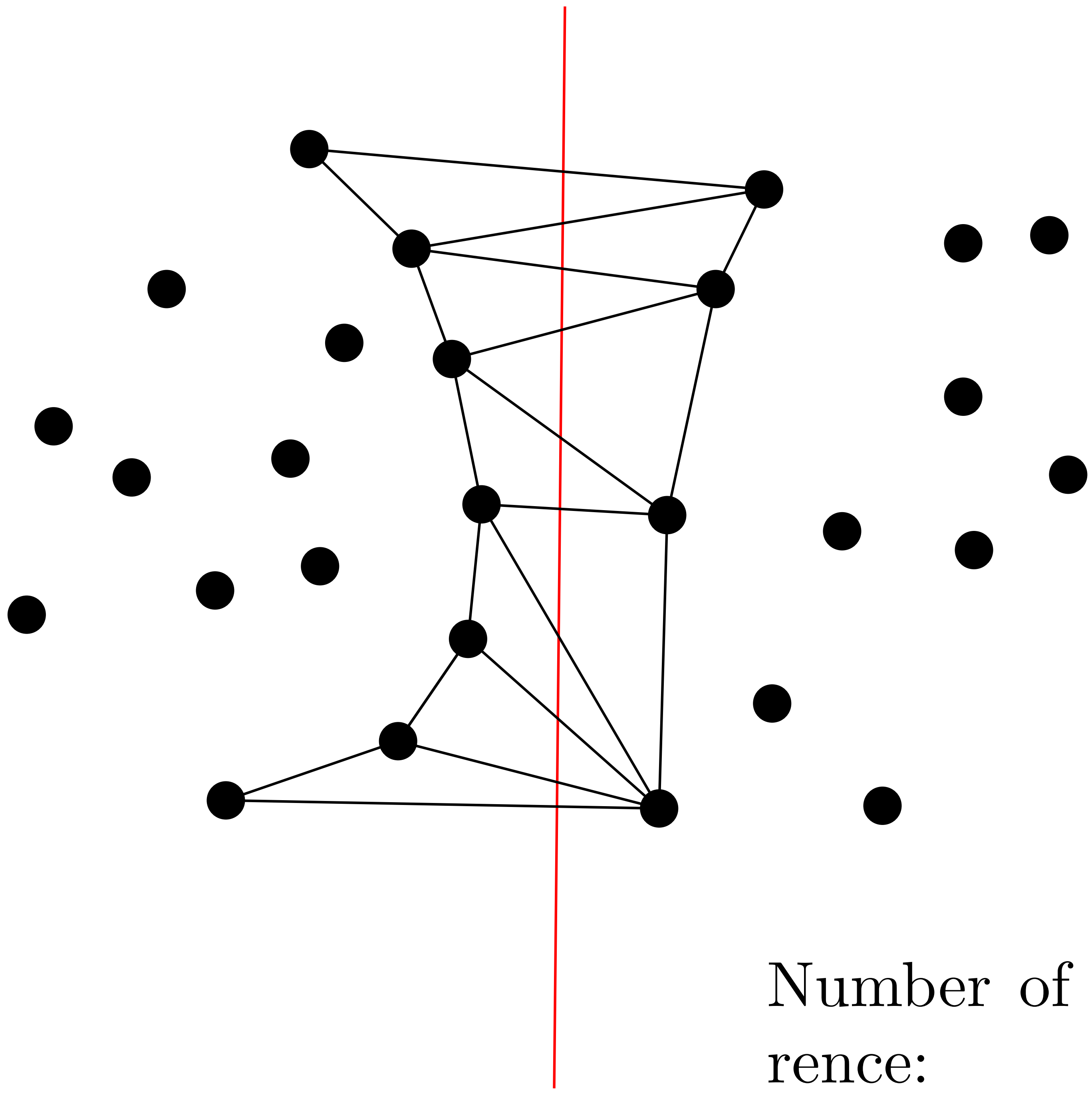
Gaultier, Hurtado, Noy, Pérennes and Urrutia (2003) introduced the notion of simultaneous flips. They proved the following:

1. $\Omega(n)$ simultaneous geometric flips are sometimes necessary to convert one triangulation to another.

2. $O(n)$ simultaneous geometric flips are sufficient to convert one triangulation to another.

3. There always exist $(n-6)/4$ edges that can be simultaneously flipped and there are triangulations where at most $(n-4)/5$ edges can be flipped simultaneously.

Simultaneous Flips in Geometric Setting



Number of flips satisfies the recurrence:
$F(n) = F(n/2) + O(n)$ which resolves to $O(n)$.

Simultaneous Flips in Geometric Setting

Key idea for lower bound: $(n-4)/2$ edges are individually flippable. At least a $1/3$ of them can be flipped simultaneously which give $(n-4)/6$.

Simultaneous Flips in Geometric Setting

Open Problems

1. Can the gap between 1/6 and 1/5 be closed between the upper and lower bound on number of edges that can be flipped simultaneously?

2. Can one flip simultaneously to a hamiltonian triangulation in $o(n)$ simultaneous flips?

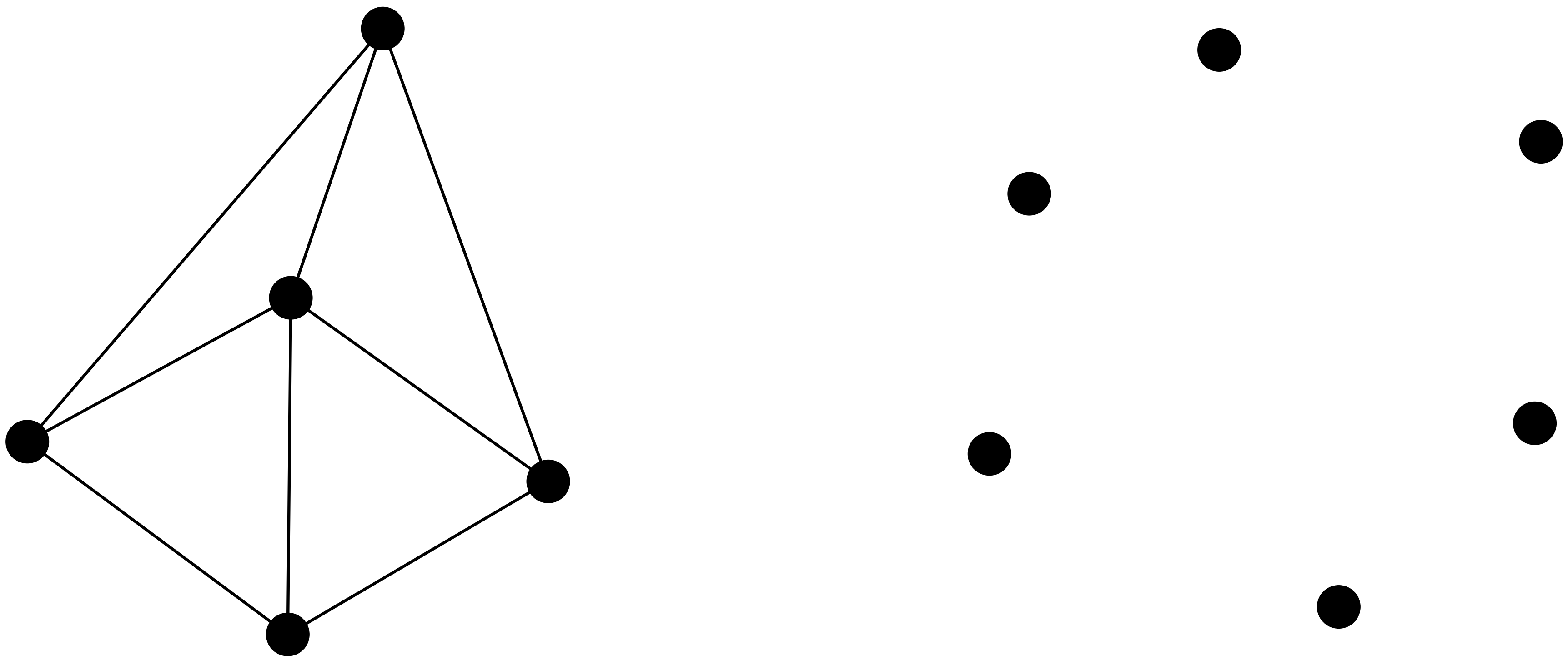
3. Can one flip to the (Greedy, Delaunay, etc) triangulation in $o(n^2)$ simultaneous flips?

4. Can one compute simultaneous flips in parallel?

5. Can one compute simultaneous flips that is sensitive to the minimum number of simultaneous flips required to convert a triangulation to another?

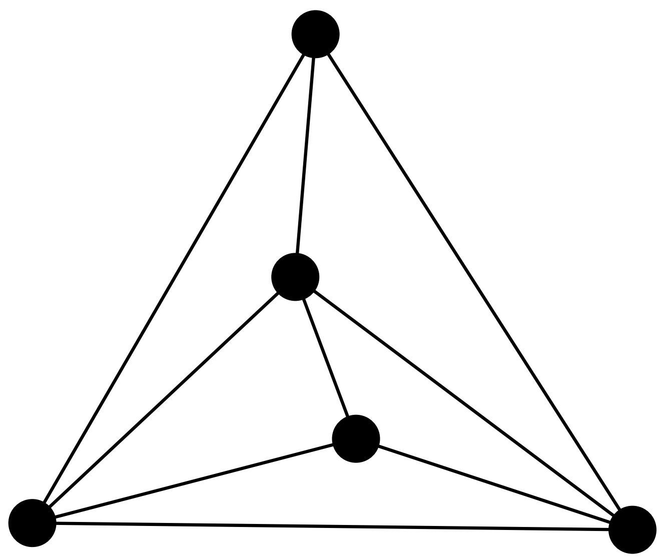
Extensions of the Geometric Setting

There exist discrepancies between the combinatorial setting and geometric setting.



Extensions of the Geometric Setting

These discrepancies are reduced if you allow the additional operation of a point move.



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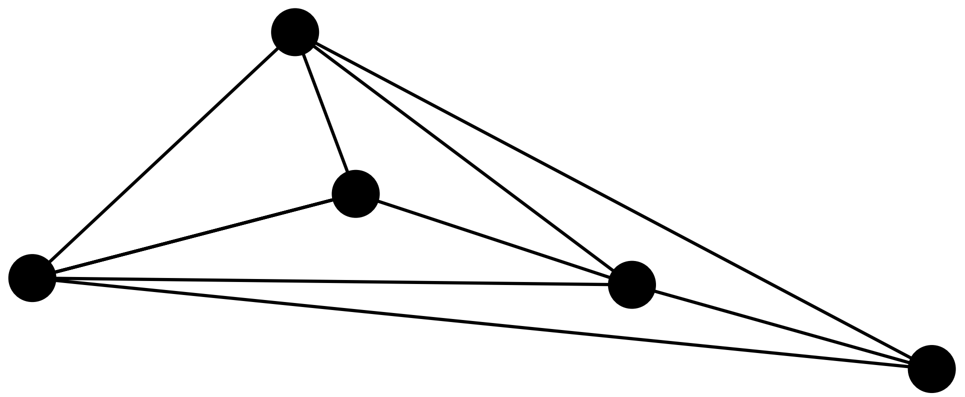

Extensions of the Geometric Setting

Abellanas, Bose, Olaverri, Hurtado, Ramos, Rivera-Campo and Tejel (2004) showed that $O(n)$ point moves and $O(n^2)$ edge flips are sufficient to convert any triangulation to any other triangulation.

Aloupis, Bose, and Morin (2004) showed that $O(n \log n)$ flips and moves are sufficient to convert any triangulation to any other.

Thank you! Question?