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A NONPOLYHEDRAL TRIANGULATED MÖBIUS STRIP

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ABSTRACT. We construct a triangulated Möbius strip with 9 vertices which is not embeddable into $\mathbf{R}^3$ such that all edges are straight line segments. It even cannot be immersed polyhedrally into $\mathbf{R}^3$.

For every integer $g \geqslant 1$ it is still an open problem whether there exists a triangulated closed orientable two-dimensional manifold of genus g which is not embeddable into $\mathbf{R}^3$ such that all triangles are planar triangles with straight edges (cf. [**1**, **2**]). It is natural to ask the corresponding question for nonorientable two-dimensional manifolds with boundary. Theorems 1 and 2 (and the remark) answer that question completely also for immersions.

THEOREM 1. *There exists a triangulated Möbius strip with 9 vertices which is not embeddable into* $\mathbf{R}^3$ *such that all edges are straight line segments.*

PROOF. (1) Let I and J be disjoint closed polygonal curves in $\mathbf{R}^3$ with three vertices. Then $|\operatorname{lk}(I, J)| \leqslant 1$ where $\operatorname{lk}(I, J)$ denotes the linking number of I and J (for the definition of linking number cf. [**3**]).

(2) The homotopy group of the Möbius strip is isomorphic to $\mathbf{Z}$. Let $\omega(J)$ denote the integer corresponding to the homotopy class of a closed curve J on the Möbius strip. Let I be the boundary curve of an embedded polyhedral Möbius strip in $\mathbf{R}^3$ and J, J' be simple closed polygonal curves in the relative interior of the Möbius strip, such that $|\omega(J)| = 1$ and $|w(J')| = 2$ (cf. Figure 1). Then $\operatorname{lk}(I, J)$ is an odd number (it is equal to the number of "twists" in the Möbius strip) and $|\operatorname{lk}(I, J')| = 2|\operatorname{lk}(I, J)|$, thus $|\operatorname{lk}(I, J')| \geqslant 2$.

(3) The triangulated Möbius strip given in Figure 2a cannot be embedded into $\mathbf{R}^3$ with straight edges, since for the triangular curves $I = 1231$ and $J' = 4564$ we have $|\operatorname{lk}(I, J')| \geqslant 2$ by (2), (choose $J = 7897$), which is impossible with straight edges by (1). In Figure 2b we give another representation of the same triangulation (1 2 3 is not a triangle of the triangulation).

Figure 2a presents the Möbius strip as a rectangle with a pair of opposite edges identified with a twist and Figure 2b presents it as a real projective plane with a triangle removed.

DEFINITION. Let us define a polyhedral immersion of a two-dimensional simplicial complex to be an immersion such that all triangles are planar and all edges are straight line segments.

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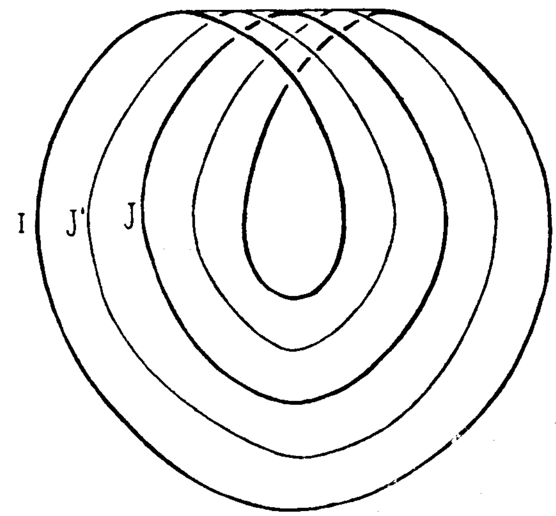

FIGURE 1

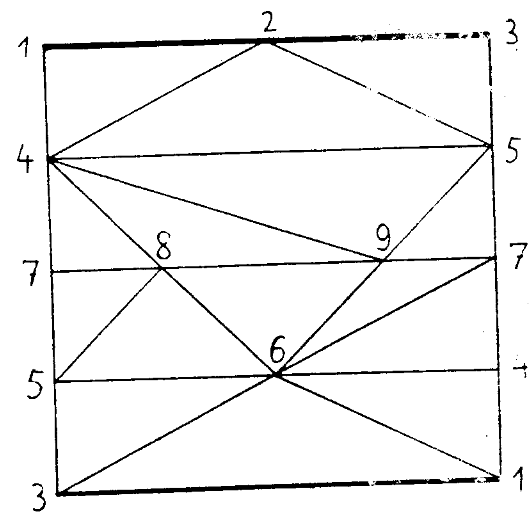

FIGURE 2a

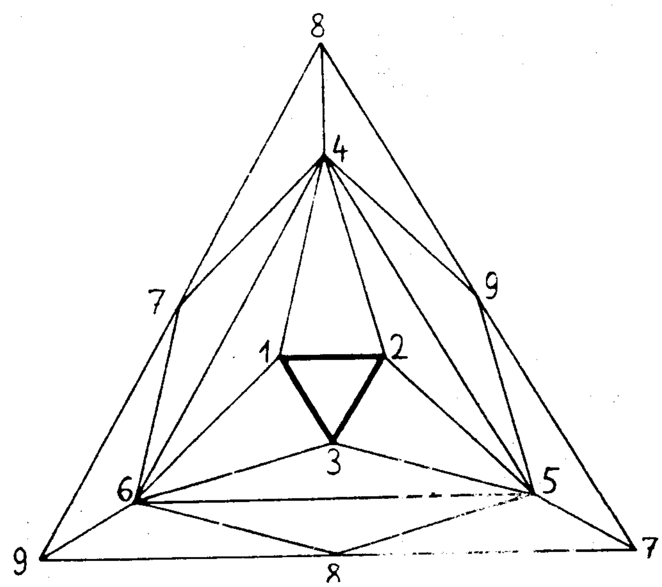

FIGURE 2b

THEOREM 2. *The triangulated Möbius strip given in Figure 2a cannot be polyhedrally immersed into* $\mathbf{R}^3$.

PROOF. (4) If a two-dimensional simplicial complex is polyhedrally immersed into $\mathbf{R}^3$ and if an edge a_1a_2 meets a triangle $b_1b_2b_3$ of the complex and if $a_i \neq b_j$ ($i = 1, 2$; $j = 1, 2, 3$) then none of the triangles $a_1a_2b_j$ belongs to the complex.

(5) Let us assume that the Möbius strip given in Figure 2a is polyhedrally immersed into $\mathbf{R}^3$. Let us define M_1 to be the compact Möbius strip which is contained in the given one and has boundary curve $J' = 4564$. (4) implies that an edge and a triangle of M_1 can only meet if they have a vertex in common. Consequently the immersion restricted to M_1 is already an embedding.

Now there exists also a neighborhood of M_1 in the given Möbius strip such that the immersion restricted to that neighborhood is an embedding. There exists a hexagonal curve I_1 (cf. Figure 3a or Figure 3b) in that neighborhood such that I_1 is disjoint to M_1 and homotopic to 4564. So the immersion restricted to the compact Möbius strip which is contained in the given one and has I_1 as boundary curve is also an embedding.

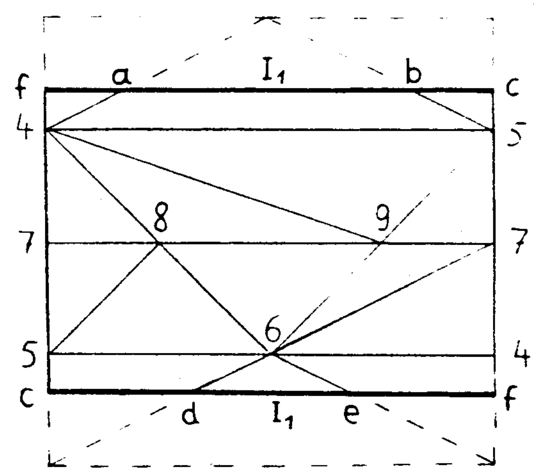

FIGURE 3a

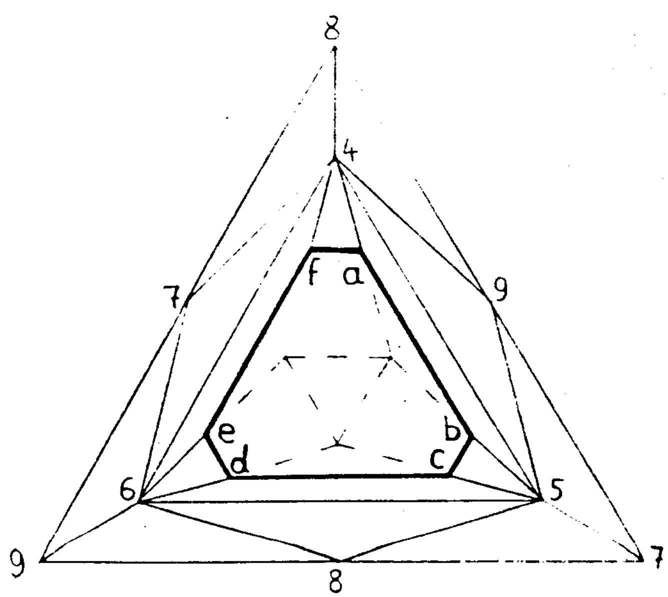

FIGURE 3b

The edges ab, cd and ef of I_1 (cf. Figure 3a or Figure 3b) cannot pierce the plane spanned by 4, 5 and 6. This implies $\mathrm{lk}(I_1, J') \leq 1$ in contradiction to (2).

REMARK. Since every two-dimensional nonorientable manifold M with boundary contains a Möbius strip, we can triangulate M such that it cannot be immersed (and consequently not be embedded) polyhedrally into $\mathbf{R}^3$.

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