How to Build Minimal Polyhedral Models of the Boy Surface

Ulrich Brehm

Introduction and History

In the middle of the last century A. Möbius gave a combinatorial description of a closed, one-sided, polyhedral surface, which was soon recognized as a topological model of the real projective plane. It also turned out to be the surface that J. Steiner had defined geometrically. Soon, algebraic definitions followed which were used to construct plaster models of the cross-cap and the Roman surfaces. Until the Klein bottle, none of these non-orientable closed surfaces, whether smooth or polyhedral, was known to be "immersible" in $\mathbf{R}^3$. A topological *immersion* $i : M \rightarrow \mathbf{R}^3$ is a locally injective continuous mapping. An immersion $i : M \rightarrow \mathbf{R}^3$ of a compact 2-manifold (without boundary) is called *polyhedral* if the image of i is contained in the union of finitely many planes.

In 1903 D. Hilbert's student W. Boy proved in [3] that the real projective plane $\mathbf{R}P^2$ allows an immersion in $\mathbf{R}^3$ (with an axis of symmetry of order 3). Several efforts have been made to give an explicit description of such an immersion. A survey of explicit combinatorial, analytic, and algebraic descriptions of such immersions is included in F. Apéry's recent book on the subject [1]. In [5] polyhedral immersions of $\mathbf{R}P^2$ with eighteen vertices were described. The polyhedral immersions given in [1] have even more vertices. In this paper the existence of symmetric polyhedral versions of the Boy surface with only nine vertices and ten facets is shown and an easy recipe for building cardboard models of these objects is given. Formal definitions of the terms "vertex," "facet," and "edge" will be given near the end of the introduction.

T. Banchoff showed in [2] that an immersion of $\mathbf{R}P^2$ in $\mathbf{R}^3$ in general position must have a triple point. A polyhedral immersion can always be perturbated so that the vertices are in "very general" position (for example, with the coordinates of the vertices being algebraically independent). Thus we get the generic case with at least one triple point in the relative interior of three triangular facets. The intersection of any two of these triangles is a line segment which cannot contain a common vertex because a polyhedral immersion is injective in some neighborhood of any vertex. Thus the three triangles containing the triple point have together nine different vertices, so nine is a lower bound for the number of vertices of a polyhedral immersion of $\mathbf{R}P^2$. We will show that this lower bound can indeed be attained.

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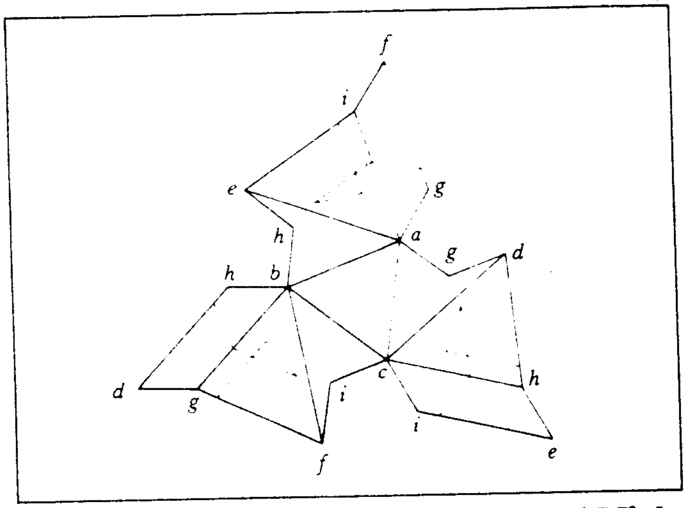

Figure 1. A net of a polyhedral immersion $P1$ of $\mathbf{R}P^2$. In Figure 2 and Figure 3 we show orthogonal projections of $P1$ in the direction of the axis of symmetry from "above" and from "below." We have indicated the self-intersection lines by dotted lines. Visible lines, dotted or solid, are drawn much thicker than invisible lines. For any two edges with intersecting projections we indicate which of the two edges is above the other one.

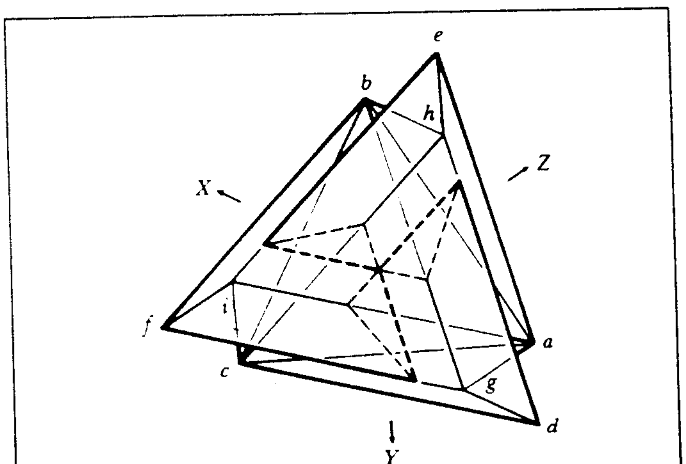

Figure 2. The orthogonal projection in the direction of the axis of symmetry from above.

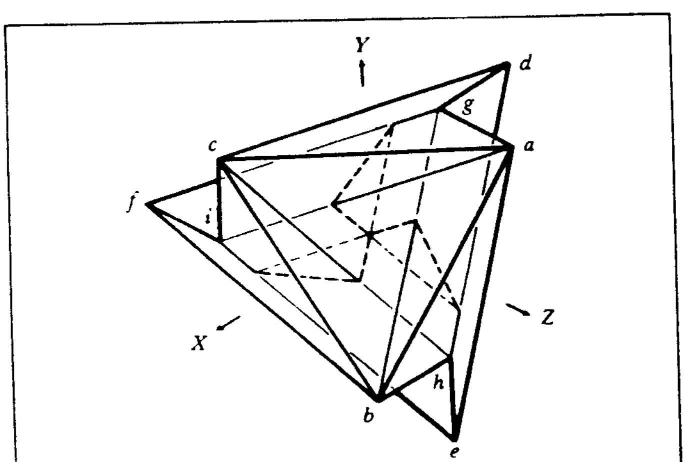

Figure 3. The orthogonal projection in the direction of the axis of symmetry from below.

The nicest way to prove this is to construct a 3-dimensional model of the object wanted. This paper contains an easy recipe for building your own cardboard models of minimal polyhedral versions of the Boy surface. We also give the coordinates of the vertices together with the combinatorial structure. Using these data you can also construct a computer model.

Definitions. Let M be a compact 2-manifold (without boundary) and $i : M \to \mathbf{R}^3$ a polyhedral immersion; (a) a *facet* is a connected component of a non-empty set of the form $\mathrm{int}(i^{-1}[H])$ where $H \subset \mathbf{R}^3$ is a plane and int denotes the interior of a set; (b) a *vertex* is a point of M that is in the intersection of the closures of (at least) three facets; (c) the connected components of the set of points of M that are neither vertices nor contained in some facet are called *edges*.

Thus if two vertices happen to be mapped onto the same point in $\mathbf{R}^3$, they are still counted as different vertices. On the other hand, the intersection of (the relative interior of) the images of a facet and an edge is not regarded as an additional vertex. If each facet is a topological open disc, then Euler's formula $f_2 - f_1 + f_0 = \chi(M)$ holds, where f_2, f_1, f_0 denote the numbers of facets, edges, vertices, respectively, and $\chi(M)$ denotes the Euler characteristic of M.

If no misunderstandings can occur we call the image of a vertex, edge, or facet also a vertex, edge, or facet, respectively. In particular, by coordinates of a vertex we mean always the coordinates of the image point in $\mathbf{R}^3$.

If M is triangulated such that i is piecewise linear, then the local injectivity of i has to be checked only in a neighborhood of each vertex of (the simplicial complex) M.

Polyhedral Immersions of $\mathbf{R}P^2$ with Nine Vertices and Ten Facets

We describe three combinatorially different symmetric polyhedral immersions $P1$, $P2$, $P3$ of $\mathbf{R}P^2$ with nine vertices and ten facets. $P1$ has six quadrangular and four triangular facets, whereas each of $P2$ and $P3$ has three pentagonal and seven triangular facets. The coordinates of the vertices of $P1$ are

a	$(-2,0,0)$	b	$(0,-2,0)$	c	$(0,0,-2)$
d	$(-1,2,1)$	e	$(1,-1,2)$	f	$(2,1,-1)$
g	$(-1,1,0)$	h	$(0,-1,1)$	i	$(1,0,-1)$

In Figure 1 we give a net of our immersed polyhedron $P1$. The dotted lines indicate the self-intersection lines. The list of coordinates of the vertices of $P1$ shows that the mapping $(x,y,z) \to (z,x,y)$ is a rotation by $2\pi/3$ with axis $\mathbf{R}(1,1,1)$ inducing the permutation (a,b,c) (d,e,f) (g,h,i) of the vertices. Because this permutation induces an automorphism of the net (see Figure 1), $P1$

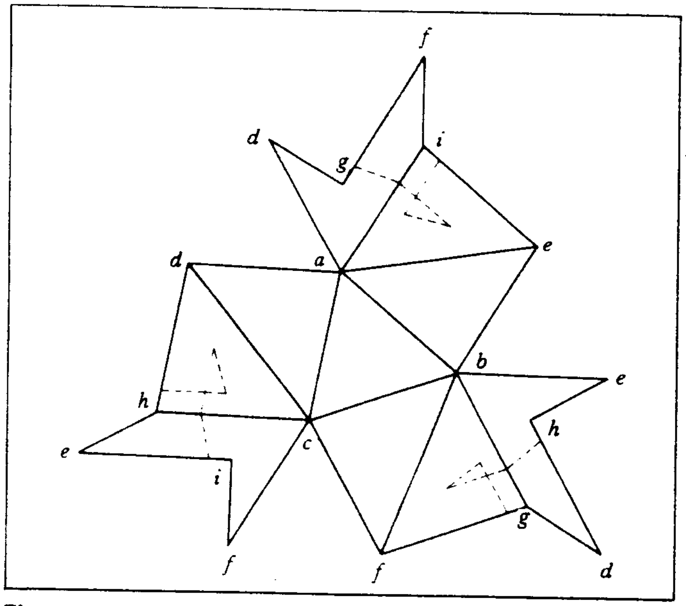

Figure 4. A net of a polyhedral immersion $P2$ of $\mathbf{R}P^2$.

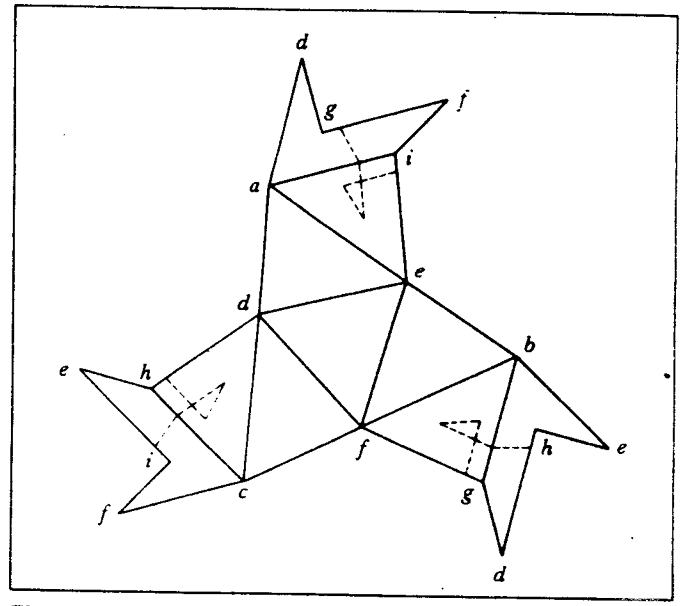

Figure 5. A net of a polyhedral immersion $P3$ of $\mathbf{R}P^2$.

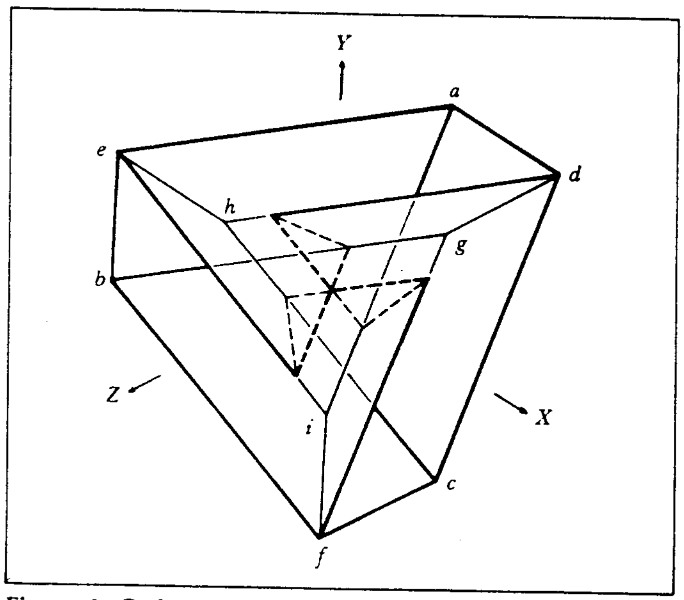

Figure 6. Orthogonal projection of the Möbius strip from above.

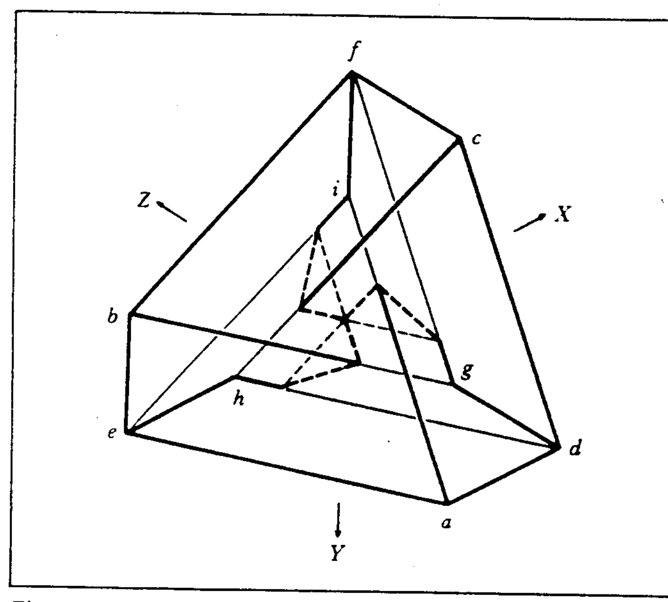

Figure 7. Orthogonal projection of the Möbius strip from below.

has an axis of symmetry of order 3. The cell-complex defined by Figure 1 (vertices and edges being identified in the obvious way) clearly is an $\mathbf{R}P^2$ with $f_0 = 9$, $f_1 = 18$, $f_2 = 10$.

In Figure 10 you can see some pictures of a cardboard model of $P1$. It is easy to check that Figure 2 is correct and that a,c,d,g are affinely dependent and $g - a = f - i$. With the symmetry this implies that $P1$ is indeed an immersed polyhedron with the combinatorial structure given in Figure 1. So we get the following result.

THEOREM 1: *$P1$ is a symmetric polyhedral immersion of $\mathbf{R}P^2$ into $\mathbf{R}^3$ with nine vertices, eighteen edges, and ten facets, six of which are quadrangles.*

Next we describe two minimal polyhedral versions of the Boy surface containing three pentagonal facets. The coordinates of the vertices of $P2$ and of $P3$ are

a	$(0,1,-1)$	b	$(-1,0,1)$	c	$(1,-1,0)$
d	$(2,2,0)$	e	$(0,2,2)$	f	$(2,0,2)$
g	$(1,1,0)$	h	$(0,1,1)$	i	$(1,0,1)$

In Figure 4 and Figure 5 we give nets of our immersed polyhedra $P2$ and $P3$. The dotted lines indicate the self-intersection lines. The list of coordinates of the vertices shows that the mapping $(x,y,z) \rightarrow (z,x,y)$ is a rotation by $2\pi/3$ with axis $\mathbf{R}(1,1,1)$ inducing the permutation (a,b,c) (d,e,f) (g,h,i) of the vertices. Because this permutation induces an automorphism of the nets

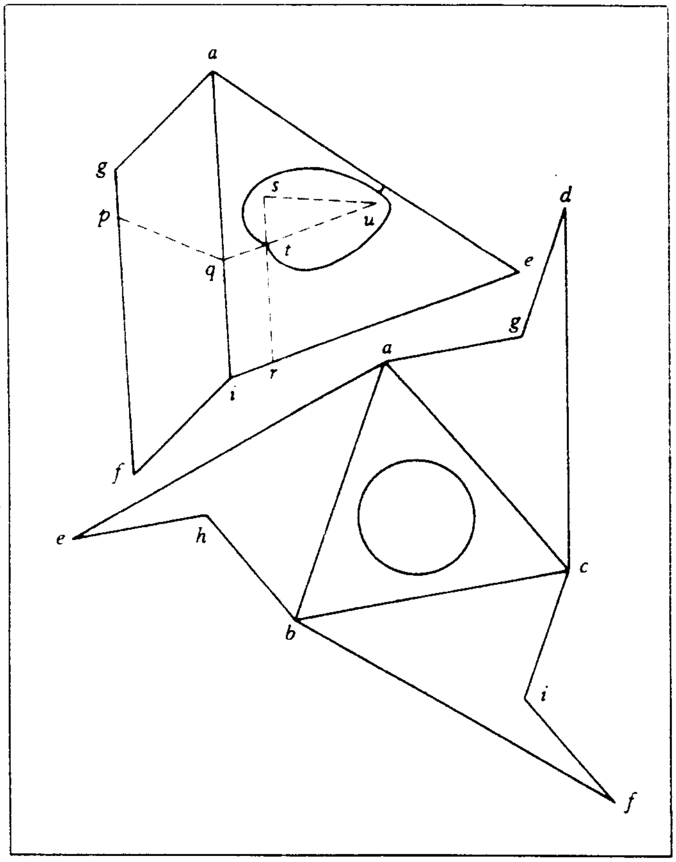

Figure 8. Part of $P1$.

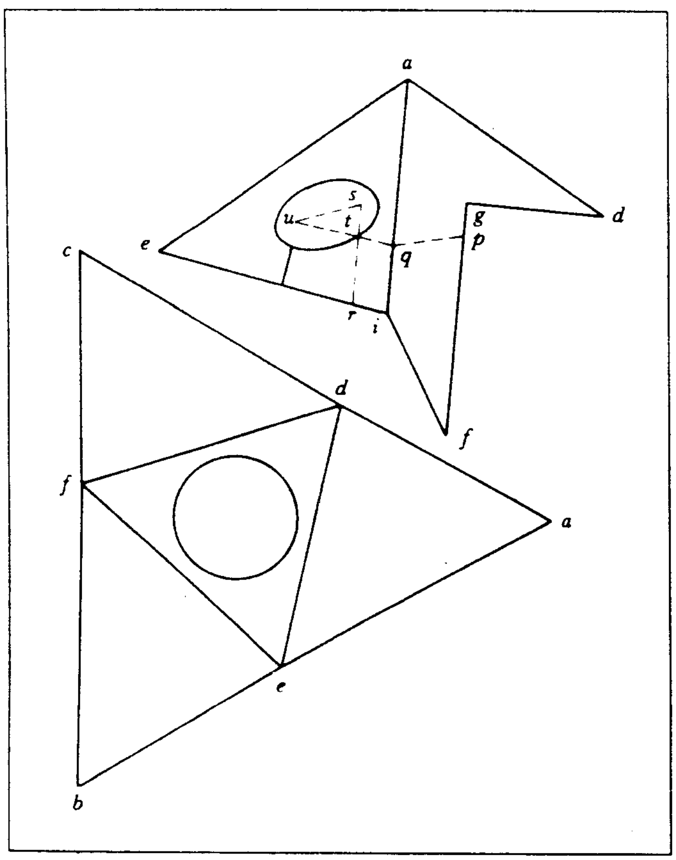

Figure 9. Part of $P3$.

How To Build Your Own Models of the Boy Surface

We call the symmetric parts of Figure 8 and Figure 9 consisting of four triangles the large parts and the other parts of Figure 8 and Figure 9 the small parts of the figure.

1. Make a sufficiently large copy of Figure 8 (for $P1$) or/and of Figure 9 (for $P3$), for example by running the figure (several times) through an enlarging copier; the lengths of the edges *ab, de,* respectively, should be at least 12 cm.
2. Make a copy of the large part and three copies of the small part of the enlarged Figure 8 (resp., Figure 9) on cardboard by piercing the vertices with a pin.
3. Draw the self-intersection lines on both sides of each of the three copies of the small part.
4. Scratch the edge *ai* on each of the three copies on one side and the three edges of the regular triangle on the reverse side for $P1$ (resp., the same side for $P3$) of the cardboard.
5. Cut out the four parts of the net along the contours. Also cut out the windows. In order to link the three windows, make a short cut from the edge *ae* to the edge of the window near *u* as indicated in Figure 8 (resp., Figure 9).
6. Fold the three copies of the small part along the edge *ai* and fold the large part in the opposite direction for $P1$ (resp., the same direction for $P3$).
7. Put the three congruent pieces together, creating the triple point; Figures 2, 3, 6, 7, 10, 11, 12 may be helpful.
8. Glue the three pieces together along the corresponding edges using a self-adhesive (transparent) strip and similarly close the cuts in the triangles. In the case of $P3$ you get the Möbius strip which $P2$ and $P3$ have in common. Note that some of the dihedral angles are quite sharp (between 23° and 30°), namely the angles at the edges *dh,ei,fg* (for $P1$, $P2$ and $P3$) and *ag, bh, cf* (for $P1$ and $P2$), whereas all other dihedral angles of $P1$, $P2$, and $P3$ are between 58° and 110°.
9. Add the large part to finish $P1$ (resp., $P3$) and glue the two pieces together (along all pairs of corresponding edges).

To build $P2$ you have to construct the central part of the net of $P2$ (cf. Figure 4) with a circular window in the equilateral triangle and glue this part to the Möbius strip you got in step 8 from the construction of $P3$.

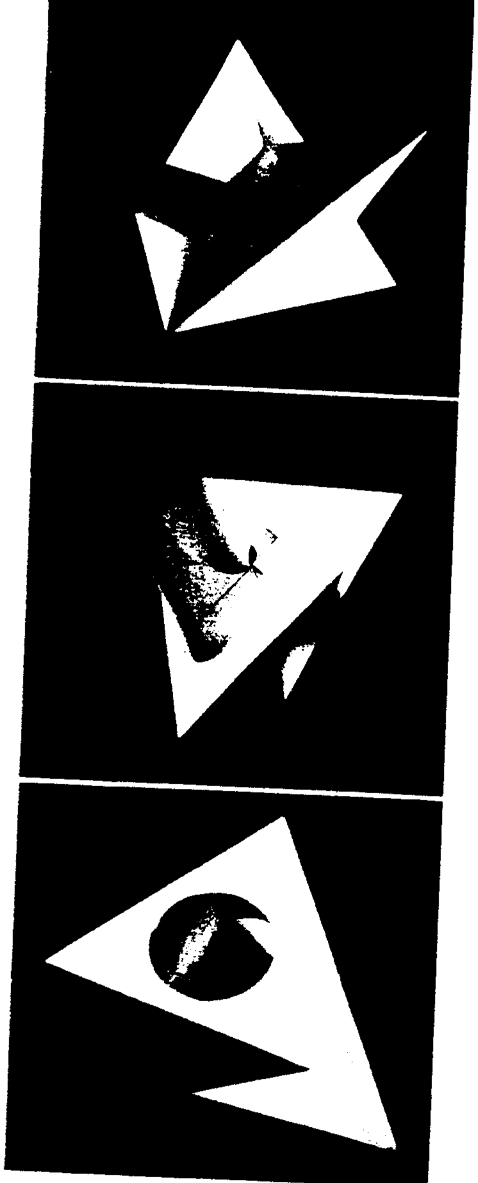

Figure 10. A model of *P*1.

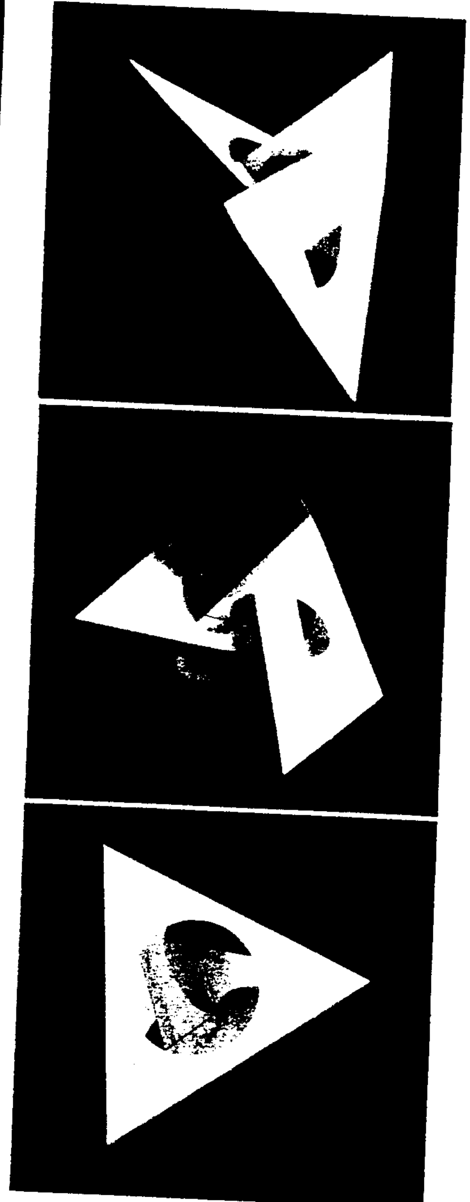

Figure 11. A model of *P*3.

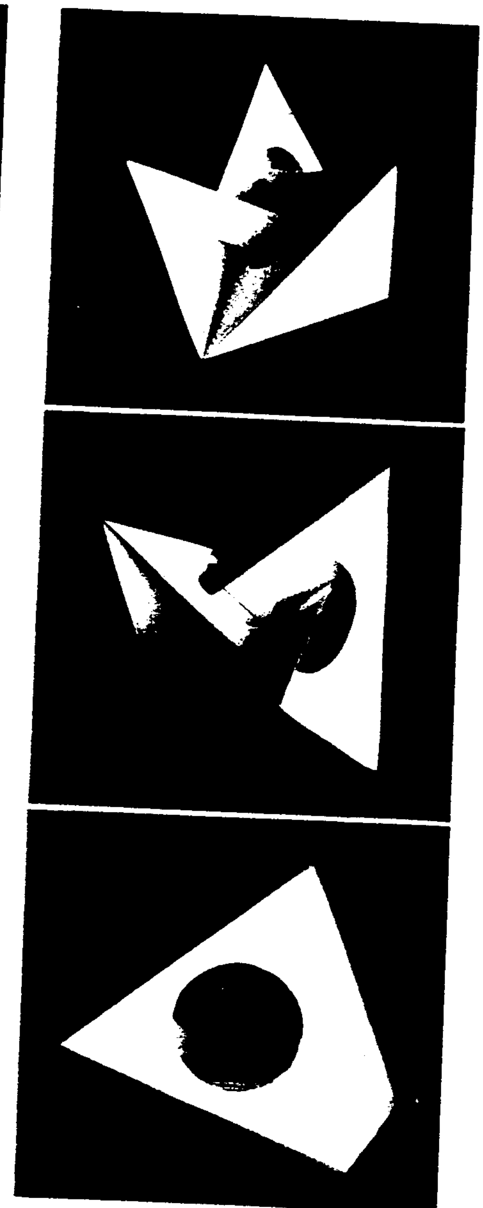

Figure 12. A model of *P*2.

(see Figure 4 and Figure 5), *P*2 and *P*3 have an axis of symmetry of order 3. The cell-complexes defined by Figure 4 and Figure 5 (vertices and edges being identified in the obvious way) clearly are $\mathbf{R}P^2$'s with $f_0 = 9$, $f_1 = 18$, $f_2 = 10$.

*P*2 and *P*3 are combinatorially different because *P*3 contains 6-valent vertices, whereas *P*2 does not contain such vertices.

Note that the Möbius strips arising from *P*2 and *P*3 when omitting the four triangles that do not contain the triple point are identical. In Figure 6 and Figure 7 we show orthogonal projections of this Möbius strip in the direction of the axis of symmetry from "above" and from "below." We have indicated the self-intersection lines by dotted lines. Visible lines, dotted or solid, are drawn much thicker than invisible lines. Note that the three pentagons form a symmetric polyhedral Möbius strip (without self-intersections).

In Figure 11 and Figure 12 you can see some pictures of *P*3 and *P*2, respectively. It is easy to check that Figure 6 is correct, that *a,d,g,f,i* are affinely dependent, and that *g* lies in the interior of the convex hull of the vertices. With the symmetry this implies that *P*2 and *P*3 are indeed immersed polyhedra with the combinatorial structure given in Figure 4 and Figure 5, respectively. So we get the following result.

THEOREM 2: *P*2 *and P*3 *are combinatorially different polyhedral immersions of* $\mathbf{R}P^2$ *into* $\mathbf{R}^3$ *with nine vertices, eighteen edges, and ten facets, three of which are pentagons.*

Now let us modify *P*1 by adding a new vertex $j = (-2, -2, -2)$ and replacing the triangle *abc* by the triangles *abj*, *acj*, *bcj*. Because *abj* and *abh* are coplanar, we can omit the edge *ab* and get a non-convex pentagon

$aehbj$. Symmetrically we omit the edges ac and bc. Thus we get a polyhedral immersion $P1'$ with $f_0 = 10$, $f_1 = 18$, $f_2 = 9$.

Similarly, we can modify $P2$ and $P3$, getting combinatorially different polyhedral immersions $P2'$ and $P3'$ with $f_0 = 10$, $f_1 = 18$, $f_2 = 9$. Thus we have shown:

THEOREM 3: *There exist symmetric polyhedral immersions of* $\mathbf{R}P^2$ *into* $\mathbf{R}^3$ *with ten vertices, eighteen edges, and nine facets.*

REMARKS: 1) One gets a symmetric modification of $P1$ such that the quadrangle $a\,c\,d\,g$ is convex if a,b,c,d,e,f are chosen as the vertices of a regular octahedron with diagonals af, bg, ch and g, h, e are chosen as the midpoints of the edges ad, be, cf, respectively, and the quadrangle $a\,g\,f\,i$ is split into the triangles $a\,g\,i$ and $f\,g\,i$ (splitting $c\,i\,e\,h$, $d\,g\,b\,h$ similarly).

2) A triangulation of the Möbius strip with 9 vertices, whose boundary forms a triangle and whose automorphism group has order 6, but which cannot be immersed in $\mathbf{R}^3$ was first described by the author in [4].

3) In order to build nice models I suggest cutting a circular "window" into the regular triangle and cutting curved "windows" into the triangles containing the triple point, such that no material self-intersections occur, but such that the full self-intersection figure, and in particular the triple point, are still visible (and marked by lines on the model on both sides).

In Figure 8 we show the central part of the net of $P1$ and a third of the net of the self-intersecting part of $P1$ (cf. Figure 1) indicating the self-intersection lines and the suggested windows. The lengths of the line segments are $ab = ac = bc = 2\sqrt{2}$, $ag = dg = bh = eh = ci = fi = \sqrt{2}$, $ad = be = cf = gi = \sqrt{6}$, $fg = ai = ei = \sqrt{10}$, $ae = bf = cd = \sqrt{14}$, $gp = ir = qt = st = \frac{2}{13}\sqrt{10}$, $tr = tu = iq = \frac{5}{13}\sqrt{10}$.

In Figure 9 we show the central part of the net of $P3$ and a third of the net of the Möbius strip which $P2$ and $P3$ have in common (cf. Figure 5) indicating the self-intersection lines and the suggested windows. The lengths of the line segments are $be = cf = ad = ai = di = ei = fg = \sqrt{6}$, $ag = dg = gi = fi = \sqrt{2}$, $ae = bf = cd = \sqrt{10}$, $de = df = ef = 2\sqrt{2}$, $gp = ir = qt = st = \frac{1}{7}\sqrt{6}$, $tr = tu = iq = \frac{2}{7}\sqrt{6}$, ($ab = ac = bc = \sqrt{6}$ for $P2$).

Acknowledgment: I wish to thank D. Ferus for taking the photos of the models (Figures 10, 11, 12) and B. Morin, U. Pinkall, D. Ferus, and E. Tjaden for helpful discussions.

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