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Hilbert Polynomials in Combinatorics

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Abstract. We prove that several polynomials naturally arising in combinatorics are Hilbert polynomials of standard graded commutative k-algebras.

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1. Introduction

The purpose of this paper is to investigate which polynomials naturally arising in combinatorics are Hilbert polynomials of standard graded (commutative) k-algebras. Our motivation comes from the fact (first proved by R. Stanley [34]) that the order polynomial of a partially ordered set is a Hilbert polynomial. Since Stanley informally told me of this result I have been wondering whether it was an isolated one or an instance of a more general phenomenon. Several works of Stanley (see, e.g., [31, 32], and the references cited there) show that many sequences arising in combinatorics are Hilbert functions, but Stanley never explicitly considered Hilbert polynomials.

In this paper we begin such a systematic investigation. Our results show that several polynomials arising in combinatorics are Hilbert polynomials, and in many (but not all) cases we find general reasons for this. The techniques that we use are based on combinatorial characterizations of Hilbert functions and polynomials obtained by Macaulay in 1927 [24]. Though the characterization of Hilbert functions is very well-known and has been extensively used since then, the one for Hilbert polynomials is not, and is our main tool. Most of our results are non-constructive. More precisely, we often prove that a given combinatorial polynomial is Hilbert but we are unable to construct (in a natural way) a standard graded k-algebra having the given Hilbert polynomial.

The organization of the paper is as follows. In the next section we collect several definitions, notation, and results that will be used in the rest of this work. In Section 3 we develop a general theory of Hilbert polynomials. More precisely, using Macaulay's result, and other techniques, we present several operations on polynomials that preserve the Hilbert property, as well as results that give sufficient conditions on the coefficients of a polynomial (when expanded in terms of several different bases) that insure that the polynomial is

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ring of symmetric functions. Also, given $\lambda \in \mathcal{P}$, we denote by λ' its conjugate, and by s_λ (respectively e_λ, h_λ, p_λ, m_λ) the *Schur* (respectively *elementary*, *complete homogeneous*, *power sum*, *monomial*) symmetric function associated to λ. We will usually identify a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with its diagram $\{(i, j) \in \mathbf{P} \times \mathbf{P} : 1 \leq i \leq r,\ 1 \leq j \leq \lambda_i\}$.

We follow [31] for notation and terminology concerning graded algebras and Hilbert functions. In particular, by a *graded k-algebra* (k being a field, fixed once and for all) we mean a commutative, associative ring R, with identity, containing a copy of the field k (so that R is a vector space over k) together with a collection of k-subspaces $\{R_i\}_{i \in \mathbf{N}}$ such that:

(i) $R = \bigoplus_{i \geq 0} R_i$ (as a k-vector space);
(ii) $R_0 = k$;
(iii) $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbf{N}$;
(iv) R is finitely generated as a k-algebra.

Note that this implies that each R_i is a finite dimensional vector space over k. The *Hilbert series* of R is the formal power series

$$P(R; x) \stackrel{\text{def}}{=} \sum_{i \geq 0} \dim_k(R_i) x^i.$$

The following fundamental result is well-known, and a proof of it can be found, e.g., in [3], Theorem 11.1, or in [31], Theorem 8.

Theorem 2.1 *Let R be a graded k-algebra as above. Then*

$$P(R; x) = \frac{h(R; x)}{\prod_{i=1}^{r} (1 - x^{k_i})},$$

in $\mathbf{Z}[[x]]$, where $h(R; x) \in \mathbf{Z}[x]$ and $k_1, \ldots, k_r$ are the degrees of a homogeneous generating set of R (as a k-algebra).

We call

$$H(R; i) \stackrel{\text{def}}{=} \dim_k(R_i)$$

the *Hilbert function* of R. We say that a k-algebra R as above is *standard* if it can be finitely generated (as a k-algebra) by elements of R_1. From now on we will always assume that all our graded k-algebras are standard. If R is a standard graded k-algebra then we can take $k_1 = \cdots = k_r = 1$ in Theorem 2.1 and this, by well known results from the theory of rational generating functions (see, e.g., [33], Proposition 4.2.2(iii)), implies the following fundamental result which was first proved by Hilbert (in a more general setting, see, e.g., [32], Corollary 9, [3], Corollary 11.2, or [12], Theorem 4.1.3).

Theorem 2.2 *Let R be a standard graded k-algebra. Then there exists a polynomial $P_R(x) \in \mathbf{Q}[x]$ and $N \in \mathbf{P}$ such that $H(R; i) = P_R(i)$ for all $i \geq N$.*

Proposition 2.4 *Let $\sum_{i\geq 0} a_i x^i$ and $\sum_{i\geq 0} b_i x^i$ be two O-series, and $j \in \mathbf{P}$. Then the following are also O-series*:

(i) $(\sum_{i\geq 0} a_i x^i)(\sum_{i\geq 0} b_i x^i)$;
(ii) $\sum_{i\geq 0} a_i x^i + \sum_{i\geq 0} b_i x^i - 1$;
(iii) $\sum_{i=0}^{j} a_i x^i$;
(iv) $\sum_{i\geq 0} a_i b_i x^i$;
(v) $\sum_{i\geq 0} a_{ji} x^i$.

Proof: (iii) is immediate from the definition of an O-series. The other statements all follow from corresponding constructions in the theory of graded algebras and Theorem 2.3. More precisely, let $R = \bigoplus_{i\geq 0} R_i$ and $S = \bigoplus_{i\geq 0} S_i$ be two standard graded k-algebras such that $P(R; x) = \sum_{i\geq 0} a_i x^i$ and $P(S; x) = \sum_{i\geq 0} b_i x^i$. Then $R \oplus S$, $R \otimes_k S$, $R * S$ (where $*$ denotes the Segre product, i.e., $R * S \overset{\text{def}}{=} \bigoplus_{i\geq 0}(R_i \otimes_k S_i)$) and $R^{(j)}$ (where $R^{(j)}$ denotes the jth Veronese subalgebra of R, i.e., $R^{(j)} \overset{\text{def}}{=} \oplus_{i\geq 0} R_{ij}$) are again standard graded k-algebras and $P(R \oplus S; x) = P(R; x) + P(S; x) - 1$, $P(R \otimes_k S; x) = P(R; x)\, P(S; x)$, $P(R * S; x) = \sum_{i\geq 0} a_i b_i x^i$ and $P(R^{(j)}; x) = \sum_{i\geq 0} a_{ji} x^i$ which, by Theorem 2.3, proves (i), (ii), (iv), and (v). □

Note that it is also possible to prove the preceding result by using the equivalence of parts (i) and (iii) in Theorem 2.3, thus avoiding commutative algebra.

Throughout this work, we say that a sequence $\{h_i\}_{i\in\mathbf{N}}$ (respectively, a polynomial $H(x)$) is a *Hilbert function* (respectively, a *Hilbert polynomial*) if there exists a standard graded k-algebra R such that $h_i = H(R; i)$ for all $i \in \mathbf{N}$ (respectively, $H(x) = P_R(x)$). We say that a finite sequence $\{h_0, h_1, \ldots, h_d\}$ is a Hilbert function if the sequence $\{h_0, h_1, \ldots, h_d, 0, 0, \ldots\}$ is a Hilbert function.

Just as Theorem 2.3 provides a numerical characterization of Hilbert functions, there is a numerical characterization of Hilbert polynomials, also due to Macaulay.

Theorem 2.5 *Let $P(x) \in \mathbf{Q}[x]$ be such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$, and let $m_0, \ldots, m_d$ be the unique integers such that*

$$P(x) = \sum_{i=0}^{d} \left[\left(\left({x \atop i+1} \right) \right) - \left(\left({x - m_i \atop i+1} \right) \right) \right], \tag{2}$$

(where $d = \deg(P(x))$). Then $P(x)$ is a Hilbert polynomial if and only if $m_0 \geq m_1 \geq \cdots \geq m_d \geq 0$.

The existence and uniqueness of the integers $m_0, \ldots, m_d$ is an elementary statement, and the "if" part of the above theorem is easy to show. A proof of the "only if" part of Theorem 2.5 is given, e.g., in [24], p. 536, [20], Corollary 5.7, p. 47, and [28], Theorem 2.1, (see also [12], Exercise 4.2.15, p. 165).

Because of the previous result, given a polynomial $P(x) \in \mathbf{Q}[x]$ such that $P(\mathbf{Z}) \subseteq \mathbf{Z}$, we call the integers $m_0, \ldots, m_d$ uniquely determined by (2) the *Macaulay parameters* of $P(x)$, and we write $M(P) = (m_0, \ldots, m_d)$.

where $d = \deg(P(x))$. *Then*

$$c_i = \sum_{j=0}^{d-i} (-1)^j \binom{m_{i+j}}{j+1}, \tag{5}$$

for $i = 0, \ldots, d$.

Proof: It is easy to see that

$$\left(\!\binom{x-m}{i+1}\!\right) = \sum_{j=0}^{i+1} (-1)^{i+1-j} \binom{m}{i+1-j} \left(\!\binom{x}{j}\!\right) \tag{6}$$

for all $m, i \in \mathbf{N}$. Therefore

$$\left(\!\binom{x}{i+1}\!\right) - \left(\!\binom{x-m}{i+1}\!\right) = \sum_{j=0}^{i} (-1)^{i-j} \binom{m}{i+1-j} \left(\!\binom{x}{j}\!\right). \tag{7}$$

Summing (7) (with $m = m_i$) for $i = 0, \ldots, d$ and comparing with (4) yields (5), as desired. □

Note that the previous result makes it easy to compute the coefficients of a polynomial with respect to the basis of twisted binomial coefficients from its Macaulay parameters (as implicitly noted also in [24], p. 537), but not conversely (even though the relations (5) are, of course, invertible). Hence, even a reasonably detailed knowledge of the coefficients $\{c_0, \ldots, c_d\}$ in (4) will not make it easy to decide if the polynomial is Hilbert. However, the relations (5) do have the following interesting consequence.

Theorem 3.2 *For* $i \in \mathbf{N}$ *there exist* $\Phi_i \in \mathbf{Q}[x_0, \ldots, x_i]$ *such that*:

(i) $\deg(\Phi_i) = 2^i$;

(ii) *if* $P(x) = \sum_{i=0}^{d} c_i (\!\binom{x}{i}\!) \in \mathbf{Q}[x]$ *is such that* $P(\mathbf{Z}) \subseteq \mathbf{Z}$ *and* $M(P(x)) = (m_0, \ldots, m_d)$, *then* $m_{d-i} = \Phi_i(c_d, \ldots, c_{d-i})$ *for* $i = 0, \ldots, d$;

(iii) *the leading monomial of* Φ_i *is* $2\,(\frac{x_0}{2})^{2^i}$.

Proof: We define $\Phi_i \in \mathbf{Q}[x_0, \ldots, x_i]$ inductively as follows,

$$\Phi_0 \stackrel{\text{def}}{=} x_0, \tag{8}$$

$$\Phi_i \stackrel{\text{def}}{=} x_i - \sum_{j=1}^{i} (-1)^j \binom{\Phi_{i-j}}{j+1}, \tag{9}$$

if $i \geq 1$. Then (i), (ii), and (iii) follow easily by induction on $i \in \mathbf{P}$. In fact, by our induction hypotheses, $\deg(\binom{\Phi_{i-j}}{j+1}) = (j+1)2^{i-j}$ for $j = 1, \ldots, i$ and hence, by (9), $\deg(\Phi_i) =$

(i) $A(x) + B(x)$;
(ii) $A(x)B(x)$;
(iii) $A(kx + m)$;
(iv) $A(x) - A(x - 1)$;
(v) $k\,A(x) + m$;
(vi) $\sum_{i=0}^{r} h_i A(x - i)$.

Proof: By hypothesis there exist $H_1, H_2 : \mathbf{N} \to \mathbf{N}$ such that $\{H_1(n)\}_{n\in\mathbf{N}}$ and $\{H_2(n)\}_{n\in\mathbf{N}}$ are O-sequences, and $H_1(n) = A(n)$, $H_2(n) = B(n)$ if $n \geq n_0$ (for some $n_0 \in \mathbf{N}$). Hence, by (ii) of Proposition 2.4, $\{1, H_1(1) + H_2(1), H_1(2) + H_2(2), \ldots\}$ is an O-sequence and $H_1(n) + H_2(n) = A(n) + B(n)$ for $n \geq n_0$ and this shows that $A(x) + B(x)$ is a Hilbert polynomial. In an exactly analogous way (using (iv) and (v) of Proposition 2.4) one proves (ii), and (iii) for $m = 0$.

To prove (iv) note that by Theorem 2.5 and our hypotheses we have that

$$A(x) = \sum_{i=0}^{d} \left[\left(\binom{x}{i+1}\right) - \left(\binom{x - m_i}{i+1}\right) \right] \tag{10}$$

where $m_0 \geq m_1 \geq \cdots \geq m_d \geq 0$, and $d \stackrel{\text{def}}{=} \deg(A(x))$. Therefore

$$\begin{aligned} A(x) - A(x-1) &= \sum_{i=0}^{d} \left[\left(\binom{x}{i+1}\right) - \left(\binom{x-1}{i+1}\right) - \left(\binom{x - m_i}{i+1}\right) \right. \\ &\qquad \left. + \left(\binom{x - 1 - m_i}{i+1}\right) \right] \\ &= \sum_{i=0}^{d-1} \left[\left(\binom{x}{i+1}\right) - \left(\binom{x - m_{i+1}}{i+1}\right) \right] \end{aligned}$$

and (iv) follows from Theorem 2.5. Also, (10) implies that

$$A(x+1) = \sum_{i=0}^{d} \left[\left(\binom{x+1}{i+1}\right) - \left(\binom{x + 1 - m_i}{i+1}\right) \right]. \tag{11}$$

Now note that,

$$\left(\binom{x+1}{i+1}\right) - \left(\binom{x+1-m}{i+1}\right) = \sum_{j=0}^{i} \left[\left(\binom{x}{j+1}\right) - \left(\binom{x-m}{j+1}\right) \right]$$

and hence

$$M\left(\left(\binom{x+1}{i+1}\right) - \left(\binom{x+1-m}{i+1}\right) \right) = \underbrace{(m, m, \ldots, m)}_{i+1} \tag{12}$$

Proof: Let $B(x) \stackrel{\text{def}}{=} A(x) - A(x-1)$, for brevity. Then we have from our hypothesis and the definition of $(\nabla B)(x)$ that, for all $n \in \mathbf{N}$,

$$\begin{aligned}
(\nabla B)(n) &= \sum_{j=0}^{n} B(j) \\
&= \sum_{i=0}^{d}\sum_{j=0}^{n}\left[\left(\!\!\binom{j}{i+1}\!\!\right) - \left(\!\!\binom{j-m_i}{i+1}\!\!\right)\right] \\
&= \sum_{i=0}^{d}\left[\left(\!\!\binom{n}{i+2}\!\!\right) - \left(\!\!\binom{n-m_i}{i+2}\!\!\right) + \left(\!\!\binom{-m_i-1}{i+2}\!\!\right)\right] \\
&= \sum_{i=0}^{d+1}\left[\left(\!\!\binom{n}{i+1}\!\!\right) - \left(\!\!\binom{n-m_i'}{i+1}\!\!\right)\right]
\end{aligned}$$

where $m_0' = \sum_{i=0}^{d}((\binom{-m_i-1}{i+2}))$, and $m_i' = m_{i-1}$, for $i = 1, \ldots, d+1$. Therefore $M(\nabla B) = (m_0', m_0, \ldots, m_d)$ and hence $M((\nabla B)(x) + A(-1)) = (A(-1) + m_0', m_0, \ldots, m_d)$, as desired. □

Corollary 3.7 *Let $A(x) \in \mathbf{Q}[x]$ be a Hilbert polynomial of degree d with Macaulay parameters $(m_0, \ldots, m_d)$, and $B(x) \in \mathbf{Q}[x]$ be such that $B(x) - B(x-1) = A(x)$. Then $B(x)$ is a Hilbert polynomial if and only if*

$$m_0 \leq \sum_{i=0}^{d}(-1)^i\binom{m_i+1}{i+2} + B(-1).$$

We now wish to study which polynomials of the bases defined in the previous section are Hilbert polynomials.

Theorem 3.8 *Let $d \in \mathbf{P}$, $a_0, \ldots, a_d \in \mathbf{P}$, and $i \in [0, d]$. Then:*
(i) *x^d is a Hilbert polynomial if and only if $d \geq 3$;*
(ii) *$\langle x\rangle_d$ is a Hilbert polynomial if and only if $d \geq 3$;*
(iii) *$a_0(x + a_1)\cdots(x + a_d)$ is a Hilbert polynomial;*
(iv) *$\binom{x}{d}$ is not a Hilbert polynomial;*
(v) *$((\binom{x}{d}))$ is not a Hilbert polynomial;*
(vi) *$\binom{x+d-i}{d}$ is a Hilbert polynomial if and only if $i = 0$.*

Proof: A straightforward computation using Theorem 2.5 shows that x and x^2 are not Hilbert polynomials, while x^3, x^4, and x^5 are. So (i) follows from part (ii) of Theorem 3.5. Also, it is easily verified, using Theorem 2.5, that $\langle x\rangle_2$ is not a Hilbert polynomial, while $\langle x\rangle_3$ is. But, by Proposition 3.4, r and $x + r$ are Hilbert polynomials whenever $r \geq 1$, so

W → (n)

(i) $c_0 \geq c_1 \geq \cdots \geq c_d \geq 0$;
(ii) $w_0 \geq w_1 \geq \cdots \geq w_d \geq 0$;
(iii) $a_0, \ldots, a_d \in \mathbf{N}$, $d \geq 3$, *and* $a_0 \leq a_1 \leq a_2 \leq a_3$.
Then $P(x)$ *is a Hilbert polynomial.*

Proof: Since $P(\mathbf{Z}) \subseteq \mathbf{Z}$ we conclude easily (considering $P(0), P(-1), \ldots, P(-d+1)$) that $c_0, \ldots, c_d \in \mathbf{Z}$, and $w_0, \ldots, w_d \in \mathbf{Z}$.

Assume now that (i) holds. Then there exist $\beta_0, \ldots, \beta_d \in \mathbf{N}$ such that $c_i = \beta_i + \beta_{i+1} + \cdots + \beta_d$ for $i = 0, \ldots, d$. Hence

$$P(x) = \sum_{i=0}^{d} c_i \left(\!\binom{x}{i}\!\right) = \sum_{i=0}^{d}\sum_{j=i}^{d} \beta_j \left(\!\binom{x}{i}\!\right) = \sum_{j=0}^{d} \beta_j \left(\sum_{i=0}^{j} \left(\!\binom{x}{i}\!\right)\right)$$
$$= \sum_{j=0}^{d} \beta_j \left(\!\binom{x+1}{j}\!\right),$$

and the thesis follows from Theorems 3.5 and 3.8.

Similarly, if (ii) holds then there exist $b_0, \ldots, b_d \in \mathbf{N}$ such that $w_i = b_i + b_{i+1} + \cdots + b_d$ for $i = 0, \ldots, d$. Hence

$$P(x) = \sum_{i=0}^{d} w_i \binom{x+d-i}{d} = \sum_{i=0}^{d}\sum_{j=i}^{d} b_j \binom{x+d-i}{d}$$
$$= \sum_{j=0}^{d} b_j \left(\sum_{i=0}^{j} \binom{x+d-i}{d}\right). \tag{13}$$

Now note that

$$\sum_{i=0}^{j} \binom{x+d-i}{d} = \sum_{i=0}^{d} \left[\left(\!\binom{x+1-i}{d+1}\!\right) - \left(\!\binom{x-i}{d+1}\!\right)\right]$$
$$= \left(\!\binom{x+1}{d+1}\!\right) - \left(\!\binom{x-j}{d+1}\!\right).$$

Hence $\sum_{i=0}^{j} \binom{x+d-i}{d}$ is a Hilbert polynomial by (12) and the thesis follows from (13), and Theorem 3.5.

Finally, assume that (iii) holds. It is easily verified (using Theorem 2.5) that $x^2 + x^3$, $x + x^2 + x^3$, and $1 + x + x^2 + x^3$ are Hilbert polynomials. But

$$P(x) = a_0(1 + x + x^2 + x^3) + (a_1 - a_0)(x + x^2 + x^3) + (a_2 - a_1)(x^2 + x^3)$$
$$+ (a_3 - a_2)x^3 + \sum_{i=4}^{d} a_i x^i,$$

so the thesis follows from our hypotheses and Theorems 3.5 and 3.8. □

[12], Theorem 5.1.7, p. 204), and also easy to see, that the Hilbert function of R_Δ is given by

$$H(R_\Delta; n) = \begin{cases} 1, & \text{if } n = 0, \\ \sum_{i=0}^{d-1} f_i \binom{n-1}{i}, & \text{if } n \in \mathbf{P}, \end{cases} \tag{14}$$

and (i) follows from part (iii) of Theorem 3.5.

To prove (ii) and (iii) note that if $f_2 \geq 3$ then necessarily $f_1 \geq 3$ and (ii) and (iii) follow from Corollary 3.10. If $f_2 \leq 2$ then $\dim(\Delta) \leq 2$ and it is easy to check, using Proposition 3.4, that $\sum_{i=1}^{2} f_i x^i$ and $\sum_{i=0}^{2} f_i \langle x \rangle_i$ are always Hilbert polynomials in this case. □

Note that there exists a complete numerical characterization, similar to Theorem 2.3, of the sequences that are the f-vector of some simplicial complex (see, e.g., [12], Section 5.1, p. 201, or [31], Theorem 2.1, p. 64). Therefore, one could state Theorem 3.13 without any reference to simplicial complexes.

We conclude our general discussion on Hilbert polynomials by introducing a concept which measures "how far" a polynomial is from being Hilbert. The crucial result for this definition is the following.

Theorem 3.14 *Let $P(x) \in \mathbf{Z}[x]$ be a polynomial with positive leading term. Then there exists $M \in \mathbf{N}$ such that $P(x + i)$ is a Hilbert polynomial for any $i \geq M$.*

Proof: Let $P(x) = \sum_{j=0}^{d} a_j x^j$ where $a_j \in \mathbf{Z}$ and $a_d \in \mathbf{P}$. Then

$$\begin{aligned} P(x+i) &= \sum_{j=0}^{d} a_j (x+i)^j \\ &= \sum_{j=0}^{d} a_j \sum_{k=0}^{j} \binom{j}{k} x^k i^{j-k} \\ &= \sum_{k=0}^{d} \left(\sum_{j=k}^{d} a_j \binom{j}{k} i^{j-k} \right) x^k. \end{aligned}$$

Hence the coefficient of x^k in $P(x + i)$ is a polynomial in i of degree $d - k$ and positive leading term, for $k = 0, \ldots, d$. Therefore there exists $N \in \mathbf{N}$ such that $P(x + i) \in \mathbf{N}[x]$ if $i > N$. The thesis follows from Corollary 3.9. □

The preceding theorem suggests, and allows us to make, the following definition. Given a polynomial $P(x) \in \mathbf{Z}[x]$ with positive leading term we let

$$H\{P\} \stackrel{\text{def}}{=} \max\{i \in \mathbf{N} : P(x+i) \text{ is not a Hilbert polynomial}\} + 1 \tag{15}$$

We begin by considering several polynomials associated to graph colorings. Let $G = (V, E)$ be a graph (without loops and multiple edges). A map $\varphi\colon V \to \mathbf{P}$ is said to be a *coloring* of G if $\varphi(x) \neq \varphi(y)$ for all $x, y \in V$ such that $(x, y) \in E$. Given $n \in \mathbf{P}$ we denote by $P_G(n)$ the number of colorings $\varphi\colon V \to \mathbf{P}$ such that $\varphi(V) \subseteq [n]$. It is then well known (see, e.g., [26], or [14], Section 4.1, p. 179) that there exists a polynomial $\chi(G; x) \in \mathbf{Z}[x]$, of degree $|V|$, such that $\chi(G; n) = P_G(n)$ for all $n \in \mathbf{P}$. This polynomial is called the *chromatic polynomial* of G and has been extensively studied (see, e.g., [27], for a survey). Since $\chi(G; x)$ is a polynomial one may write

$$\chi(G; x) = \sum_{i=0}^{|V|} a_i (x)_i = \sum_{i=0}^{|V|} (-1)^{|V|-i} c_i \langle x \rangle_i .$$

Then the polynomials $\sigma(G; x) \stackrel{\text{def}}{=} \sum_{i=0}^{|V|} a_i x^i$ and $\tau(G; x) \stackrel{\text{def}}{=} \sum_{i=0}^{|V|} c_i x^i$ are called the *σ-polynomial* and the *τ-polynomial* of G, respectively. Despite the fact that knowledge of one of these three polynomials implies knowledge of the other two it is often the case that $\sigma(G; x)$ and $\tau(G; x)$ are more convenient to handle then $\chi(G; x)$ itself. For this reason $\sigma(G; x)$ and $\tau(G; x)$ have also been studied, and we refer the reader to [9, 10], and the references cited therein, for more information on these two polynomials.

Theorem 4.1 *Let $G = (V, E)$ be a graph on p vertices, with $p \geq 3$. Then the following are Hilbert polynomials*:
(i) $\sigma(G; x)$;
(ii) $\tau(G; x)$;
(iii) $(-1)^p \chi(G; -(x + 1))$.

Proof: It is easy to verify directly (using Theorem 2.5 and some patience) that the theorem holds if $p = 3$.

We first prove (i) by induction on $p \geq 3$. Assume that $p \geq 4$. If $G = K_p$ (the complete graph on p vertices) then $\sigma(G; x) = x^p$ and (i) holds by Theorem 3.8. If $G \neq K_p$ then it follows from Theorem 1 of [26] that

$$\chi(G; x) = \chi(K_p; x) + \sum_{j=1}^{\binom{p}{2}-|E|} \chi(G_j; x),$$

and (therefore) that

$$\sigma(G; x) = \sigma(K_p; x) + \sum_{j=1}^{\binom{p}{2}-|E|} \sigma(G_j; x),$$

where each G_j has $p - 1$ vertices, and (i) follows from our induction hypothesis and Theorem 3.5.

enumerative invariants of the labeled poset (P, ω) and has been studied extensively (see, e.g., [29], and [8]). In particular, it is known (see, e.g., [29], Section 1.2, Definition 3.2, p. 8, and Proposition 8.3, p. 24) that if ω is a linear extension of P then the numbers $w_i(P; \omega)$ do not depend on ω. In this case we write $w_i(P)$ instead of $w_i(P; \omega)$.

Theorem 4.3 *Let P be a finite poset of size p. Then:*
(i) *$Z(P; x+1)$ is a Hilbert polynomial;*
(ii) *$\Omega(P; x+1)$ is a Hilbert polynomial;*
(iii) *$(w_1(P), \ldots, w_p(P))$ is a Hilbert function.*

Proof: It is well known (see, e.g., [33], Proposition 3.11.1, p. 129) that

$$Z(P; x+1) = \sum_{i=0}^{l} b_i \binom{x-1}{i} \tag{17}$$

where b_i is the number of chains of P of length i (i.e., totally ordered subsets of P of cardinality $i+1$), and l is the length of the longest chain of P. But the collection of all chains of P is clearly a simplicial complex (usually denoted $\Delta(P)$ and called the *order complex* of P, see, e.g., [33], p. 120) and its f-vector is $(b_0, b_1, \ldots, b_l)$. Hence (i) follows from (17) and (14). Also, it is well known (see, e.g., [33], Section 3.11, p. 130), and easy to see, that

$$\Omega(P; x) = Z(J(P); x) \tag{18}$$

(where $J(P)$ denotes the lattice of order ideals of P, see, e.g., [33], Section 3.4) and so (ii) follows from (i). To prove (iii) note that using (17) and (18) we conclude that

$$\begin{aligned} 1 + \sum_{n\geq 1}\sum_{i=0}^{p} f_i \binom{n-1}{i} x^n &= \sum_{n\geq 0} Z(J(P); n+1)x^n \\ &= \sum_{n\geq 0} \Omega(P; n+1)x^n \\ &= \frac{\sum_{i=1}^{p} w_i(P)x^{i-1}}{(1-x)^{p+1}} \end{aligned} \tag{19}$$

by a well-known result from the theory of P-partitions (see, e.g., [33], Theorem 4.5.14, p. 219), where f_i is the number of chains of $J(P)$ of length i (i.e., the number of i-dimensional faces of $\Delta(J(P))$). This implies, by (3) and the binomial theorem (see, e.g., [33], p. 16), that $(w_1(P), \ldots, w_p(P))$ is the h-vector of $\Delta(J(P))$. But it is well-known (see, e.g., [33], Section 3.4) that $J(P)$ is always a distributive lattice. This, in turn, implies that $\Delta(J(P))$ is shellable (see, e.g., [12], Theorem 5.1.12, p. 208, and [33], Section 3.3) and (iii) follows from the fact that h-vectors of shellable complexes are O-sequences (see, e.g., Theorem 5.1.15 of [12]). □

one can verify directly (using Theorem 2.5) that $(x+1)^2$, $x(x+1)^2$, and $x^2(x+1)^2$ are all Hilbert polynomials and the result follows. □

Theorem 4.6 *Let (W, S) be a Coxeter system and $u, v \in W$, $u \preceq v$, be such that $l(v) - l(u) \geq 3$. Then $R_{u,v}(x+1)$ is a Hilbert polynomial.*

Proof: If $l(v) - l(u) = 3$ then it is easy to see (see, e.g., [21], Section 7.5) that $R_{u,v}(x)$ equals either $(x-1)^3$ or $(x-1)^3 + (x-1)x$ and one can check that the result holds in this case. So assume that $l(v) - l(u) \geq 4$. It is then well known (see, e.g., [21], Section 7.5, p. 154, or [15], Theorem 1.3) that

$$R_{u,v}(x+1) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} a_i (x+1)^i x^{d-2i} \tag{21}$$

where $d \stackrel{\text{def}}{=} l(v) - l(u)$ and $a_i \in \mathbf{N}$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$. If $i = 1$ then $d - 2i \geq 2$ and hence $(x+1)x^{d-2i}$ is a Hilbert polynomial by Theorems 3.5 and 3.8 and the fact that $(x+1)x^2$ is a Hilbert polynomial. If $i = 0$ then $d - 2i \geq 4$ and hence x^{d-2i} is a Hilbert polynomial by Theorem 3.8. If $i \geq 2$ then $(x+1)^i x^{d-2i}$ is a Hilbert polynomial by Lemma 4.5. Hence the result follows from (21) and Theorem 3.5. □

Given a finite Coxeter system (W, S) we let

$$d_i(W) \stackrel{\text{def}}{=} |\{v \in W : d(v) = i\}| \tag{22}$$

for $i \in \mathbf{N}$.

Theorem 4.7 *Let (W, S) be a finite Coxeter system and $u, v \in W$, $u \preceq v$. Then:*
(i) *$\sum_{w \in W} x^{d(w)} = \sum_{i \geq 0} d_i(W) x^i$ is a Hilbert polynomial;*
(ii) *$P_{u,v}(x+1)$ is a Hilbert polynomial;*
(iii) *$\{d_0(W), d_1(W), \ldots, d_{|S|}(W)\}$ is a Hilbert function.*

Proof: Let, for brevity, $P(W; x) \stackrel{\text{def}}{=} \sum_{w \in W} x^{d(w)}$. If $|S| \leq 2$ then either $P(W; x) = 1 + x$ (if $|S| = 1$) or W is a finite dihedral group, in which case $P(W; x) = 1 + (2p-2)x + x^2$ for some $p \geq 2$. In both cases (i) holds by Proposition 3.4. So assume that $|S| \geq 3$. Note that it follows immediately from the definition (22) that

$$d_0(W) = 1, \quad d_1(W) \geq |S|$$

for any (W, S) (since $d(s) = 1$ for all $s \in S$). Furthermore, it is known (see, e.g., [11], Theorem 2.4) that $P(W; x)$ is always a symmetric unimodal polynomial. Hence

$$d_2(W) \geq |S|$$

Proof: It is clear from the definitions that $1 + ix$ is an O-series for all $i \geq 1$. But it is well-known (see, e.g., [33], Proposition 1.3.4, p. 19) that

$$\sum_{k=0}^{n-1} c(n, n-k)x^k = \prod_{i=1}^{n-1}(1+ix)$$

so (i) follows from part (i) of Proposition 2.4.

To prove (ii) let $V = \{S \subseteq [n-1] : n-r \geq |S| \geq r\}$ and $\Delta \stackrel{\text{def}}{=} \{F \subseteq V : S \cap T = \emptyset$ for all $S, T \in F$ such that $S \neq T$, and $\sum_{S \in F} |S| \leq n-r\}$. It is then clear that Δ is a simplicial complex on vertex set V. Also,

$$f_{k-1}(\Delta) = S_r(n, k+1)$$

for all $k = 0, \ldots, \lfloor \frac{n}{r} \rfloor - 1$ (since if $\{S_1, \ldots, S_k\} \in \Delta$ then $\{S_1, \ldots, S_k, [n]\backslash(\bigcup_{i=1}^k S_i)\}$ is a partition of $[n]$ into $k+1$ blocks, each of size $\geq r$, and this is a bijection). Hence $\{S_r(n,k)\}_{k=2,\ldots,\lfloor \frac{n}{r} \rfloor}$ is the f-vector of a simplicial complex and therefore $\{S_r(n,k)\}_{k=1,\ldots,\lfloor \frac{n}{r} \rfloor}$, by Theorem 2.3, is an O-sequence. □

We now prove that a rather general class of polynomials arising from the enumeration of Stirling permutations are always Hilbert polynomials. Fix $k \in \mathbf{P}$ and $m_1, m_2, \ldots \in \mathbf{P}$. Recall (see, e.g., [8], Section 6.6) that a permutation $a_1 a_2 \cdots a_{m_1+\cdots+m_k}$ of the multiset $M_k \stackrel{\text{def}}{=} \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}$ is called a *Stirling permutation* if $1 \leq u < v < w \leq m_1 + \cdots + m_k$ and $a_u = a_w$ imply $a_v \geq a_u$. Stirling permutations have been first introduced and studied in [19] in the case $m_1 = \cdots = m_k = 2$, and later in [18] (in the case $m_1 = \cdots = m_k$) and [8] (in the general case). We denote by Q_k the set of all Stirling permutations of M_k. So, for example, $Q_k = S_k$ if $m_1 = \cdots = m_k = 1$. Given a permutation $\pi = a_1 a_2 \cdots a_{m_1+\cdots+m_k}$ of M_k a *descent* of π is an index $j \in [m_1 + \cdots + m_k - 1]$ such that $a_j > a_{j+1}$. For $0 \leq i \leq |M_k| - 1$ we let $B_{k,i}$ be the number of Stirling permutations of M_k with exactly i descents. So, if $m_1 = \cdots = m_k = 1$, then $B_{k,i}$ is just the Eulerian number $A(k, i+1)$. There are (at least) two important generating functions associated with the numbers $B_{k,i}$, namely

$$B_k(x) \stackrel{\text{def}}{=} \sum_{i=0}^{|M_k|-1} B_{k,i} x^i, \tag{23}$$

and

$$f_k(x) \stackrel{\text{def}}{=} \sum_{i=1}^{|M_k|} B_{k,i-1} \binom{x + |M_k| - i}{|M_k|} \tag{24}$$

(see, [8], Section 6.6, for further information on these two polynomials). As noted in [8], p. 78, (24) is usually the better behaved of these two generating functions. This turns out to be true also from our present point of view. In fact, we will prove that (23) is always a Hilbert polynomial while $\{f_k(n+1)\}_{n \in \mathbf{N}}$ is always a Hilbert function.

does not provide such a combinatorial interpretation, it does provide an algebraic interpretation. Furthermore, if the rings R_k referred to above can be constructed explicitly, then a combinatorial interpretation of their Hilbert function would probably follow.

It is well-known (see, e.g., [19]), and also easy to see, that $S(n+k,n)$ is a polynomial function of n, for each $k \in \mathbf{N}$. An interesting consequence of Theorem 4.10 is the following.

Corollary 4.11 *Let $k \in \mathbf{N}$. Then $\{S(n+1+k, n+1)\}_{n\in\mathbf{N}}$ is a Hilbert function. In particular, $S(x+1+k, x+1)$ is a Hilbert polynomial.*

Proof: Taking $m_i = 2$ for all $i \in \mathbf{P}$ yields, by part (ii) of Proposition 6.6.4 of [8], that $f_k(n+1) = S(n+1+k, n+1)$ for all $n \in \mathbf{N}$, and the result follows from part (i) of Theorem 4.10. □

The polynomials $S(x+k, x)$ are usually called *Stirling polynomials* (see, e.g., [19], or [8], Section 6.6, p. 80).

Corollary 4.11 can, in turn, be generalized in another direction using the theory of symmetric functions. We need first the following simple observation.

Proposition 4.12 *Let $f \in \Lambda$. Then there exists a (necessarily unique) polynomial $\bar{f}(x) \in \mathbf{Q}[x]$ such that*

$$\bar{f}(n) = f(1, 2, \ldots, n, 0, 0, \ldots) \tag{27}$$

for all $n \in \mathbf{P}$.

Proof: It is well-known (see, e.g., [25], Chapter I, Section 2, Ex. 11, p. 23), and easy to see, that

$$S(n+k, n) = h_k(1, 2, \ldots, n, 0, 0, \ldots) \tag{28}$$

for all $n \in \mathbf{P}$ and $k \in \mathbf{N}$, and that, as noted before Corollary 4.11, $S(n+k, n)$ is a polynomial function of n for all $k \in \mathbf{N}$. By definition (see, e.g., [25], Chapter I, Section 2) we have that

$$h_\lambda(x_1, x_2, \ldots) = \prod_{i=1}^{l} h_{\lambda_i}(x_1, x_2, \ldots) \tag{29}$$

if $\lambda = (\lambda_1, \ldots, \lambda_l)$, hence the result holds for the complete homogeneous symmetric functions h_λ, $\lambda \in \mathcal{P}$. But every $f \in \Lambda$ can be expressed as a finite linear combination of h_λs, and the result follows. □

Thus Corollary 4.11 is asserting (by (28) and (27)) that $\{h_k(1, 2, \ldots, n+1)\}_{n\in\mathbf{N}}$ is a Hilbert function and $\bar{h}_k(x+1)$ is a Hilbert polynomial. This naturally suggests the problem of determining those symmetric functions $f \in \Lambda$ for which these properties hold.

Theorem 3.5), there are many sequences and polynomials naturally arising in enumerative and algebraic combinatorics for which we have been unable to decide whether they are Hilbert. In this section we survey the most striking such cases, and we present some conjectures together with the evidence we have in their favor.

Our first conjecture is naturally suggested by Theorem 4.9.

Conjecture 5.1 *Let $n \in \mathbf{P}$. Then $\{S(n, n-k)\}_{k=0,\ldots,n-1}$ is a Hilbert function.*

We have verified this conjecture for $n \leq 24$. In addition to the numerical evidence, there is a heuristic reasoning that suggests the validity of Conjecture 5.1. A sequence of positive integers is a Hilbert function if it "does not grow too fast". Now, it is well-known (see, e.g., [14], Section 7.1, Theorem D, p. 271) that the sequence $\{S(n,k)\}_{k=1,\ldots,n}$ is log-concave and unimodal, hence the real content of Conjecture 5.1 is for the values of k that precede the mode of the sequence. But it is known (see, e.g., [33], Chapter 1, Exercise 18, p. 47) that the mode of $\{S(n,k)\}_{k=1,\ldots,n}$ is less than $\lfloor \frac{n}{2} \rfloor$. Hence one expects the sequence $\{S(n, n-k)\}_{k=0,\ldots,n-1}$ to grow "less rapidly" than $\{S(n,k)\}_{k=1,\ldots,n}$ and therefore we expect Conjecture 5.1 to be true since Theorem 4.9 holds.

Theorem 3.8 allows one to settle the question of whether a given polynomial is Hilbert pretty easily if its coefficients with respect to the basis $\{x^i\}_{i \in \mathbf{N}}$ are nonnegative and have a combinatorial interpretation. However, there are many polynomials for which this is not the case (especially polynomials that "count something" when evaluated at nonnegative integers) but that seem to be Hilbert. In this respect, we feel that the following is the most interesting and outstanding open problem arising from the present work.

Conjecture 5.2 *Let G be a graph on at least 4 vertices, and $\chi(G; x)$ be its chromatic polynomial. Then $\chi(G; x)$ is a Hilbert polynomial.*

We have verified the above conjecture for all graphs with at most 15 vertices. Two related conjectures are the following:

Conjecture 5.3 *Let $d \in \mathbf{P}$, $d \geq 4$. Then $(x)_d$ is a Hilbert polynomial.*

Conjecture 5.4 *Let $d \in \mathbf{P}$. Then $3d\binom{x}{d}$ is a Hilbert polynomial.*

We have verified these conjectures for $d \leq 15$. Note that, by Proposition 4.2, Conjectures 5.2 and 5.3 are equivalent, while by Theorem 3.5 Conjecture 5.4 implies Conjecture 5.3.

For what concerns the symmetric functions $f \in \Lambda$ such that $\bar{f}(x+1)$ is a Hilbert polynomial we have the following conjectures.

Conjecture 5.5 *Let $\lambda \in \mathcal{P}$. Then $\bar{s}_\lambda(x+1)$ is a Hilbert polynomial if and only if $|\lambda| \geq 3$.*

Conjecture 5.6 *Let $\lambda \in \mathcal{P}$. Then $\bar{m}_\lambda(x+1)$ is a Hilbert polynomial if and only if $|\lambda| \geq 3$.*

We have verified the above conjectures for $|\lambda| \leq 7$. Note that since any Schur symmetric function is m-positive (see, e.g., [25], Chapter I, Section 6), Conjecture 5.6 implies Conjecture 5.5 as well as Corollary 4.14.

Finally, there is a general "open problem" that arises naturally with almost any result presented in this work. Namely, whenever we prove that a certain polynomial (or sequence) is Hilbert it is natural to ask whether one can construct, in a natural way, a standard graded k-algebra having the given Hilbert polynomial or function. Besides giving a more illuminating proof of the original result, such a graded algebra would probably have interesting properties in its own right. We have not investigated this problem. However, we do believe that natural constructions of graded algebras exist that "explain" all parts of Theorems 3.5, 3.12, 3.13, and 4.7.

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