THE STRUCTURE OF THE POLYTOPE ALGEBRA

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Abstract. We construct an isomorphism from McMullen's polytope algebra, onto the quotient of the algebra of continuous, piecewise polynomial functions with integral value at 0, by its ideal generated by coordinate functions. This explains the non-trivial grading of the polytope algebra, by the obvious grading of piecewise polynomial functions. In the process of the proof, we make explicit many connections between convex polytopes and piecewise polynomials.

Introduction

In the study of valuations (or finitely additive measures) on convex polytopes in a finite-dimensional real vector space, a fundamental role is played by the polytope algebra: the universal group for translation-invariant valuations. This group is endowed with a multiplication, via Minkowski sum of polytopes, and with many other structures, discovered by McMullen, Morelli, Khovanskii-Pukhlikov and others. In particular, the polytope algebra is almost a graded algebra over $\mathbf{R}$; its grading is defined by diagonalizing the action of the group of dilatations (see [Mc1]). The proof of existence of this grading uses the logarithm of a polytope P, defined by $\log(P) = \sum_{n=1}^{\infty} (-1)^{n-1}(P-1)^n/n$ (this makes sense in the polytope algebra, because $P-1$ is nilpotent there).

In this paper, we recover some of the most important properties of the polytope algebra, as corollaries of a structure theorem for this algebra. To state our main result, we need some notation.

Let V be a vector space over $\mathbf{R}$ of finite dimension $d \geq 2$, and let V^* be its dual. To any convex polytope P in V^* is associated its support function H_P on V; then H_P is continuous, and piecewise linear with respect to some subdivision of V into polyhedral cones having the origin as their common vertex. We denote by R the algebra of all continuous functions on V that are piecewise polynomial (in the same sense). Then R is a graded algebra over $\mathbf{R}$ for the operations of pointwise addition and multiplication; it turns out that R is generated by support functions of polytopes. We denote by $\overline{R}$ the quotient of R by its graded ideal generated by all (globally) linear functions on V.

Theorem. *(i) The graded algebra $\overline{R} = \oplus_{n=0}^{\infty} \overline{R}_n$ vanishes in all degrees $n > d$. Moreover, the vector space $\overline{R}_d$ is one-dimensional, and multiplication in $\overline{R}$ induces non-degenerate pairings $\overline{R}_j \times \overline{R}_{d-j} \to \overline{R}_d$ for $1 \leq j \leq d-1$.*
(ii) The map $P \to \exp(H_P) = \sum_{n=0}^{\infty} H_P^n/n!$ extends to an isomorphism of the polytope algebra, onto the subalgebra $\overline{R}^{\mathrm{int}} = \mathbf{Z} \oplus \overline{R}_1 \oplus \overline{R}_2 \oplus \cdots \oplus \overline{R}_d$ of $\overline{R}$.

This statement explains the grading of the polytope algebra, and the role of the logarithm as well: namely, $\log(P)$ is identified with the support function of P.

1991 *Mathematics Subject Classification*: Primary 52B45, Secondary 14M25.

In fact, our structure theorem is proved here when $\mathbf{R}$ is replaced by any subfield (it can be proved for arbitrary ordered fields). In the case of the field of rational numbers, a version of this theorem was obtained in [Br], motivated by previous work of Fulton and Sturmfels [Fu-St]; there the algebra R was studied in relation to cohomology of toric varieties, using (and adding to) the dictionary between convex polytopes over $\mathbf{Q}$ and projective toric varieties with an ample $\mathbf{Q}$-divisor class. The approach of the present paper is direct and essentially self-contained; connections to toric geometry are indicated at the end of each of the first three sections.

We now summarize the contents of this paper, and its relation to earlier work of Billera, Khovanskii-Pukhlikov, McMullen, Morelli and Oda. We rely on the classical correspondence between convex polytopes in V^* with prescribed directions of faces, and convex, piecewise linear functions on a fixed complete fan in V, that is, on a subdivision of V by polyhedral, convex cones having the origin as their common vertex.

In Section 1, we introduce and study the Hodge spaces of a fan, an analog in combinatorial geometry of Hodge spaces of an algebraic variety. Both notions are compatible in the case of a rational fan associated with a smooth, complete toric variety; a related, but somewhat more complicated definition appears in [Od2], as a combinatorial version of Ishida's complexes in toric geometry. For any d-dimensional fan Σ, we obtain finite-dimensional vector spaces $H^{i,j}(\Sigma)$ indexed by pairs of integers between 0 and d. If Σ is the normal fan of a convex polytope P, then each diagonal Hodge space $H^{j,j}(\Sigma)$ is identified with the space of Minkowski j-weights on P (see 1.5 below). If moreover P is simple, then all non-diagonal Hodge spaces vanish, and the dimension of $H^{j,j}(\Sigma)$ is the j-th component of the h-vector of P (1.2, 1.4). For an arbitrary complete fan Σ, all upper diagonal Hodge spaces vanish, whereas the lower diagonal spaces are rather mysterious combinatorial invariants of Σ; an interpretation of $H^{2,1}$ is proposed in 1.3.

In Section 2, we study the space R_Σ of continuous, piecewise polynomial functions on a complete, simplicial fan Σ; then R_Σ is a subalgebra of R, and it contains the algebra S of (globally) polynomial functions on V. As a special case of results of Billera [Bi1], [Bi2], the graded S-module R_Σ is free of finite rank: the number of maximal cones in Σ. We prove that each diagonal Hodge space $H^{j,j}(\Sigma)$ is identified with the space of generators of degree j of this module (2.1). We define a canonical homogeneous S-linear map $\pi : R \to S$ of degree $-d$, and we prove that the S-bilinear map $R_\Sigma \times R_\Sigma \to S : (f,g) \to \pi(fg)$ is a perfect pairing. This induces a duality between Hodge spaces $H^{j,j}(\Sigma)$ and $H^{d-j,d-j}(\Sigma)$. Remembering the connections between diagonal Hodge spaces and h-vectors, we may see this duality as an algebraic version of the Dehn-Sommerville equations (2.4).

In Section 3, we turn to the ring E_Σ of continuous, piecewise exponential functions on a complete, simplicial fan Σ. This ring appears under a different disguise in [Mo1], [Mo2] and [Kh-Pu], as the space of piecewise linear functions from V to $\mathbf{Z}[\mathbf{R}]$. Our approach to it is naive, but new; it leads in 3.3 to a short proof of a refinement of the main result in [Kh-Pu]. Then, in Section 4, we prove that both algebras E_Σ and R_Σ have the same completion: the algebra of compatible, formal power series on Σ. Moreover, we obtain our key technical result in 4.3: the quotient of E by its ideal generated by functions $e^x - 1$ (x a globally linear function) is isomorphic to $\overline{R}$. Here E is the ring of continuous, piecewise exponential functions (with respect to no specified fan).

In Section 5, we prove that the polytope algebra is isomorphic to the quotient of E defined above. This latter result was known in slightly different formulations; see [Kh-Pu] and [Mo1]. Then our main theorem follows by putting everything together. Moreover, our map $\pi : R \to S$ turns out to be related to volume by $\pi(H_P^d) = d!\,\mathrm{vol}(P)$ (more generally, π is related to Fourier transform, see 5.3) and this fact implies a separation result for the polytope algebra, originally due to McMullen (see 5.4).

1. The Hodge spaces of fans

1.1. Let K be a subfield of $\mathbf{R}$, and let V be a K-vector space of finite dimension d. Let $V_{\mathbf{R}} := V \otimes_K \mathbf{R}$ be the associated $\mathbf{R}$-vector space.

A (polyhedral, convex) *cone* σ in V is an intersection of finitely many closed half-spaces of V. We denote by $\sigma_{\mathbf{R}}$ the associated cone in $V_{\mathbf{R}}$, and by $L(\sigma)$ the linear span of σ in V. A *fan* in V is a finite set Σ of cones, such that:

(i) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.

(ii) If $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of σ.

(iii) If $\sigma \in \Sigma$ then σ contains no line.

For $0 \leq i \leq d$, the set of i-dimensional cones of Σ is denoted by $\Sigma(i)$. The *support* $|\Sigma|$ of Σ is the union of its cones; Σ is *complete* if $|\Sigma| = V$.

A *sheaf* $\mathcal{F}$ on a fan Σ is a collection of abelian groups $(\mathcal{F}_\sigma)_{\sigma \in \Sigma}$ and of maps $\rho_{\sigma\tau} : \mathcal{F}_\sigma \to \mathcal{F}_\tau$ ($\sigma \in \Sigma$, τ a face of σ) such that:

(i) $\rho_{\sigma\sigma} : \mathcal{F}_\sigma \to \mathcal{F}_\sigma$ is the identity map for any $\sigma \in \Sigma$.

(ii) $\rho_{\sigma_1\sigma_3} = \rho_{\sigma_2\sigma_3} \circ \rho_{\sigma_1\sigma_2}$ whenever $\sigma_3 \subset \sigma_2 \subset \sigma_1 \in \Sigma$.

There are obvious notions of morphisms of sheaves on Σ, and of exact sequences.

Any abelian group F defines a sheaf on Σ with value F at all cones of Σ, each map $\rho_{\sigma\tau}$ being the identity. We denote this constant sheaf by F.

To any sheaf $\mathcal{F}$ on Σ we associate cohomology groups $H^i(\mathcal{F})$ ($i \geq 0$) as follows. Choose an orientation on each $\sigma_{\mathbf{R}} \in \Sigma_{\mathbf{R}}$. For $\sigma \in \Sigma$ and a face $\tau \subset \sigma$ of codimension 1, set $\varepsilon_{\sigma\tau} = 1$ if the orientations of σ and τ agree, and $\varepsilon_{\sigma\tau} = -1$ otherwise. Set

$$C^i(\mathcal{F}) = \bigoplus_{\sigma \in \Sigma(d-i)} \mathcal{F}_\sigma$$

and let $\delta^i : C^i(\mathcal{F}) \to C^{i+1}(\mathcal{F})$ be the direct sum of the maps

$$\sum_{\tau \subset \sigma} \varepsilon_{\sigma\tau}\rho_{\sigma\tau} : \mathcal{F}_\sigma \to \bigoplus_{\tau \in \Sigma(d-i-1), \tau \subset \sigma} \mathcal{F}_\tau \ .$$

It is easily checked that $\delta^{i+1} \circ \delta^i = 0$, i.e., $(C^*(\mathcal{F}), \delta)$ is a complex; let $H^i(\mathcal{F})$ be the i-th cohomology group of this complex. If Σ is complete, then $H^0(\mathcal{F})$ consists of all elements in $\oplus_{\sigma \in \Sigma(d)} \mathcal{F}_\sigma$ that agree on $(d-1)$-dimensional cones.

In our study of the cohomology groups of certain sheaves, we will use the following observations.

Lemma. *(i) Any exact sequence of sheaves on* Σ*:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

induces a long exact sequence of cohomology groups

$$\cdots \to H^{i-1}(\mathcal{F}'') \to H^i(\mathcal{F}') \to H^i(\mathcal{F}) \to H^i(\mathcal{F}'') \to H^{i+1}(\mathcal{F}') \to \cdots$$

(ii) If F *is a constant sheaf, then* $H^d(F) = 0$*. Moreover, for* $0 \leq i \leq d$*, the group* $H^i(F)$ *is identified with the* $(d-i-1)$*-th homology group of* $|\Sigma|_{\mathbf{R}} \cap S^{d-1}$ *with coefficients in* F*, where* S^{d-1} *is a sphere centered at 0.*

Proof. (i) The exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

induces an exact sequence of complexes

$$0 \to C^*(\mathcal{F}') \to C^*(\mathcal{F}) \to C^*(\mathcal{F}'') \to 0$$

and hence a long exact sequence of cohomology groups.
(ii) The vanishing of $H^d(F)$ is immediate. Intersecting each cone in $\Sigma_{\mathbf{R}}$ with S^{d-1}, we obtain a polyhedral decomposition of $|\Sigma|_{\mathbf{R}} \cap S^{d-1}$. Moreover, the complex

$$(C^{d-i-1}(F))_{0 \leq i \leq d-1}$$

is identified with the usual chain complex associated to this polyhedral decomposition.

Finally, observe that the tensor product over K of any two sheaves of K-vector spaces is a sheaf. It follows that for any sheaf $\mathcal{F}$ of K-vector spaces, and for any integer $n \geq 0$, we have symmetric powers $S^n\mathcal{F}$ and exterior powers $\wedge^n\mathcal{F}$.

1.2. Let Σ be a fan in V, and let V^* be the dual space of V over K. For any $\sigma \in \Sigma$ we denote by $\sigma^\perp$ the set of all $f \in V^*$ that vanish identically on σ. The assignment $\sigma \to \sigma^\perp$ defines a sheaf $\mathcal{F}$ of K-vector spaces on Σ, the maps $\rho_{\sigma\tau} : \mathcal{F}_\sigma \to \mathcal{F}_\tau$ being the inclusions $\sigma^\perp \subset \tau^\perp$.

For any non-negative integer j, we have the j-th exterior power $\wedge^j\mathcal{F}$. We set:

$$H^{i,j}(\Sigma) := H^i(\wedge^j\mathcal{F}) \ .$$

The spaces $(H^{i,j}(\Sigma))_{i,j}$ will be called the *Hodge spaces* of Σ. A related construction can be found in [Od2] for complete, simplicial fans.

Proposition. *With the notation above, we have:*
(i) $H^{i,j}(\Sigma) = 0$ *for* $i < j$*.*
(ii) If $|\Sigma|$ *is not contained in any hyperplane, then* $H^{d,j}(\Sigma) = 0$ *for* $j < d$*, and* $H^{d,d}(\Sigma)$ *is isomorphic to* K*.*

(iii) If Σ is complete, and if e is a positive integer such that $\Sigma(e)$ consists of simplicial cones, then $H^{i,j}(\Sigma) = 0$ for $i - j > d - e$.

Proof. (i) Observe that the dimension of $\mathcal{F}_\sigma$ is the codimension of σ, and hence $\wedge^j \mathcal{F}_\sigma = 0$ for all $\sigma \in \Sigma(d-j)$. By the definition of cohomology groups, we have $H^i(\wedge^j \mathcal{F}) = 0$ for $i < j$.
(ii) The group $H^{d,j}(\Sigma)$ is the cokernel of the map

$$\delta : \oplus_{\sigma \in \Sigma(1)} \wedge^j \sigma^\perp \to \wedge^j V^*$$

the direct sum of the inclusion maps $\wedge^j \sigma^\perp \to \wedge^j V^*$. We check that δ is surjective for $j < d$. Because $|\Sigma|$ is not contained in any hyperplane, we can choose linearly independent vectors $e_1, \ldots, e_d$ in V such that each e_i generates an edge of σ; call this edge σ_i. Then a basis of $\wedge^j \sigma_i^\perp$ consists of the wedge products of any j vectors among the e_n $(n \neq i)$. It follows that the map

$$\oplus_{i=1}^d \wedge^j \sigma_i^\perp \to \wedge^j V^*$$

is surjective, and this proves our assertion.
(iii) The proof of this statement is somewhat technical, and hence we begin with the simplest case, where Σ is simplicial (that is, $e = d$). Then, for any $\sigma \in \Sigma$, we have an exact sequence of K-vector spaces

$$0 \to \sigma^\perp \to V^* \to \oplus_{\tau \in \sigma(1)} L(\tau)^* \to 0 \,,$$

where $L(\tau)$ denotes the line generated by the edge τ of σ; the map on the right is the direct sum of the restriction maps from V^* to the duals of the $L(\tau)$'s. For any $\tau \in \Sigma(1)$ and any $\sigma \in \Sigma$, we set

$$K(\tau)_\sigma = \begin{cases} K & \text{if } \tau \subset \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then $K(\tau)$ is a sheaf: the constant sheaf on the star of τ. We set

$$\mathcal{G} = \oplus_{\tau \in \Sigma(1)} K(\tau) \ .$$

Then we have an exact sequence of sheaves

$$0 \to \mathcal{F} \to V^* \to \mathcal{G} \to 0$$

and hence a long exact sequence (the Koszul complex)

$$0 \to \wedge^j \mathcal{F} \to \wedge^j V^* \to \wedge^{j-1} V^* \otimes \mathcal{G} \to \cdots \to \wedge^{j-n} V^* \otimes S^n \mathcal{G} \to \cdots \to S^j \mathcal{G} \to 0 \ .$$

We claim that each sheaf $\wedge^{j-n} V^* \otimes S^n \mathcal{G}$ is acyclic, that is, $H^i(\wedge^{j-n} V^* \otimes S^n \mathcal{G}) = 0$ for all $i \geq 1$. Cutting the Koszul complex into short exact sequences and repeatedly using Lemma 1.1 (i), we see that the claim implies the vanishing of $H^i(\wedge^j \mathcal{F})$ for $i > j$, as required.

To prove the claim, observe that $H^i(\wedge^{j-n} V^* \otimes S^n \mathcal{G}) \simeq \wedge^{j-n} V^* \otimes H^i(S^n \mathcal{G})$. If $n = 0$, then the vanishing of $H^i(\wedge^j V^*)$ follows from Lemma 1.1 (ii). If $n \geq 1$, we have

$$S^n \mathcal{G} = \oplus_{\tau_1, \ldots, \tau_n \in \Sigma(1)} \mathcal{G}(\tau_1, \ldots, \tau_n)$$

where the sheaf $\mathcal{G}(\tau_1, \dots, \tau_n)$ is defined by

$$\mathcal{G}(\tau_1, \dots, \tau_n)_\sigma = \begin{cases} K & \text{if } \sigma \text{ contains } \tau_1, \dots, \tau_n \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathrm{St}(\tau_1, \dots, \tau_n)$ the union of cones in $\Sigma_{\mathbf{R}}$ that contain $\tau_1, \dots, \tau_n$, and denote by $\overline{\mathrm{St}}(\tau_1, \dots, \tau_n)$ its closure. Then, as in the proof of Lemma 1.1 (ii), we obtain the vanishing of $H^d(\mathcal{G}(\tau_1, \dots, \tau_n))$ and isomorphisms

$$H^i(\mathcal{G}(\tau_1, \dots, \tau_n)) = H_{d-i-1}(\overline{\mathrm{St}}(\tau_1, \dots, \tau_n) \cap S^{d-1}, \partial\overline{\mathrm{St}}(\tau_1, \dots, \tau_n) \cap S^{d-1}, K)\,,$$

where the latter are homology groups of the pair consisting of $\overline{\mathrm{St}}(\tau_1, \dots, \tau_n) \cap S^{d-1}$ and in its boundary. But these groups vanish, because the space $\mathrm{St}(\tau_1, \dots, \tau_n) \cap S^{d-1}$ is contractible. This ends the proof of the claim.

Now we turn to the general case, where e is arbitrary. Then the sequence

$$0 \to \sigma^\perp \to V^* \to \oplus_{\tau \in \sigma(1)} L(\tau)^* \to 0$$

is left exact; this sequence is exact if and only if σ is simplicial. Defining $K(\tau)$ (for $\tau \in \Sigma(1)$) and $\mathcal{G}$ as before, we obtain a left exact sequence of sheaves

$$0 \to \mathcal{F} \to V^* \to \mathcal{G} \to 0\,.$$

We complete it to an exact sequence

$$0 \to \mathcal{F} \to V^* \to \mathcal{G} \to \mathcal{H} \to 0$$

for some sheaf $\mathcal{H}$ on Σ, such that $\mathcal{H}_\sigma = 0$ if and only if σ is simplicial. Now, using [Le] or [Ni], we obtain a long exact sequence of sheaves

$$0 \to \wedge^j \mathcal{F} \to \mathcal{F}_{j,0} \to \mathcal{F}_{j,1} \to \cdots \to \mathcal{F}_{j,n} \to \cdots \to,$$

where $\mathcal{F}_{j,n}$ denotes the sheaf

$$\bigoplus_{a+b+c=j,\, b+2c=n} \wedge^a V^* \otimes S^b \mathcal{G} \otimes \wedge^c \mathcal{H}\,.$$

In particular, $\mathcal{F}_{j,n}$ contains $\wedge^{j-n} V^* \otimes S^n \mathcal{G}$ as a direct factor, for $n \le j$. Moreover, for a simplicial cone σ, we have

$$(\mathcal{F}_{j,n})_\sigma = \begin{cases} \wedge^{j-n} V^* \otimes S^n \mathcal{G} & \text{if } n \le j \\ 0 & \text{if } n > j. \end{cases}$$

Therefore we have a long exact sequence of sheaves

$$\begin{aligned} 0 \to \wedge^j \mathcal{F} \to \wedge^j V^* \to \wedge^{j-1} V^* \otimes \mathcal{G} \to \cdots \to (\wedge^{j-n} V^* \otimes S^n \mathcal{G}) \oplus \mathcal{F}'_{j,n} \\ \to \cdots \to S^j \mathcal{G} \oplus \mathcal{F}'_{j,j} \to \mathcal{F}'_{j,j+1} \to 0 \end{aligned}$$

with sheaves $\mathcal{F}'_{j,n}$ $(2 \leq n \leq j+1)$ that vanish on any simplicial cone, in particular on any cone of dimension at most e. By the definition of cohomology groups, we have then $H^i(\mathcal{F}'_{j,n}) = 0$ for $i \geq d-e$, and for arbitrary n. On the other hand, the sheaves $\wedge^{j-n} V^* \otimes S^n \mathcal{G}$ are ayclic by the first part of the proof. It follows that $H^i(\wedge^j \mathcal{F}) = 0$ for $i - j > d - e$.

As a special case, we obtain the following result, a version of which appears in [Od2].

Corollary. *For any complete, simplicial fan Σ, we have $H^{i,j}(\Sigma) = 0$ if $i \neq j$, and $H^{d,d}(\Sigma) \simeq K$.*

1.3. For any cone σ, we denote by $\mathrm{rel}(\sigma)$ the kernel of the (surjective) summation map

$$\oplus_{\tau \in \sigma(1)} L(\tau) \to L(\sigma)$$

(recall that $L(\sigma)$ denotes the linear span of σ). Then $\mathrm{rel}(\sigma)$ is the space of linear relations among the edges of σ. If τ is any face of σ, then $\mathrm{rel}(\tau)$ is identified with a subspace of $\mathrm{rel}(\sigma)$.

Similarly, for any fan Σ, denote by $\mathrm{rel}(\Sigma)$ the kernel of the summation map

$$\oplus_{\tau \in \Sigma(1)} L(\tau) \to V \ .$$

Then the dimension of $\mathrm{rel}(\Sigma)$ is the number of edges of Σ, minus the dimension of the linear span of $|\Sigma|$.

Finally, denote by $\mathrm{Rel}(\Sigma)$ the cokernel of the map

$$\prod_{\tau \in \Sigma(d-1)} \mathrm{rel}(\tau) \to \prod_{\sigma \in \Sigma(d)} \mathrm{rel}(\sigma)$$

defined in a way dual to 1.1. Then $\mathrm{Rel}(\Sigma)$ is a "globalization" of the spaces of linear relations among the edges of d-dimensional cones in Σ. The compatible injective maps $\mathrm{rel}(\sigma) \to \mathrm{rel}(\Sigma)$ induce a linear map $u : \mathrm{Rel}(\Sigma) \to \mathrm{rel}(\Sigma)$. The following statement describes the first non-trivial Hodge spaces $H^{1,1}(\Sigma)$ and $H^{2,1}(\Sigma)$ in terms of the map u. Another interpretation of $H^{1,1}(\Sigma)$ will be given in 2.1 below.

Proposition. *With the notation above, the transpose map $u^* : \mathrm{rel}(\Sigma)^* \to \mathrm{Rel}(\Sigma)^*$ fits into an exact sequence*

$$0 \to H^{1,1}(\Sigma) \to \mathrm{rel}(\Sigma)^* \to \mathrm{Rel}(\Sigma)^* \to H^{2,1}(\Sigma) \to 0 \ .$$

Proof. We use the notation of the proof of Proposition 1.2: there is an exact sequence

$$0 \to \mathcal{F} \to V^* \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{G} = \oplus_{\tau \in \Sigma(1)} K(\tau)$. Moreover, each $\mathcal{H}_\sigma$ is the cokernel of the map

$$V^* \to \oplus_{\tau \in \Sigma(1)} L(\tau)^* \ .$$

Therefore, $\mathcal{H}_\sigma$ is identified with $\mathrm{rel}(\sigma)^*$, and $H^0(\mathcal{H})$ is identified with $\mathrm{Rel}(\Sigma)^*$. On the other hand, there is an exact sequence

$$0 \to V^* \to H^0(\mathcal{G}) \to \mathrm{rel}(\Sigma)^* \to 0 \ .$$

Denote by $\mathcal{F}_1$ the cokernel of the map $\mathcal{F} \to V^*$. Then, from the exact sequences

$$0 \to \mathcal{F} \to V^* \to \mathcal{F}_1 \to 0, \ 0 \to \mathcal{F}_1 \to \mathcal{G} \to \mathcal{H} \to 0$$

and from the vanishing of $H^0(\mathcal{H})$, $H^1(V^*)$, $H^2(V^*)$ and $H^1(\mathcal{G})$, we obtain exact sequences

$$0 \to V^* \to H^0(\mathcal{F}_1) \to \mathcal{H}^1(\mathcal{F}) \to 0,$$

$$0 \to H^0(\mathcal{F}_1) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}_1) \to 0$$

and an isomorphism $H^1(\mathcal{F}_1) \simeq H^2(\mathcal{F})$. Therefore, we have an exact sequence

$$0 \to H^1(\mathcal{F}) \to H^0(\mathcal{G})/V^* \to H^0(\mathcal{H}) \to H^2(\mathcal{F}) \to 0 \ .$$

Moreover, the map $H^0(\mathcal{G})/V^* \to H^0(\mathcal{H})$ is identified with $u^* : \mathrm{rel}(\Sigma)^* \to \mathrm{Rel}(\Sigma)^*$.

Corollary. *Let Σ be a complete fan such that any two non-simplicial cones in Σ intersect only at the origin. Then $H^{2,1}(\Sigma) = 0$.*

Proof. The assumption implies that any $(d-1)$-dimensional cone in Σ is simplicial. Then $\mathrm{Rel}(\Sigma)$ is the direct sum of the $\mathrm{rel}(\sigma)$ (σ a non-simplicial, d-dimensional cone in Σ). Any two such cones have no common edge, and hence the map $\mathrm{Rel}(\Sigma) \to \mathrm{rel}(\Sigma)$ is injective.

1.4. Consider a d-dimensional convex polytope P in V^*. To each face F of P, we associate the dual cone σ_F of the convex cone generated by the vectors $f - p$ with $f \in F$ and $p \in P$. Observe that the dimension of σ_F is the codimension of F. The set $(\sigma_F)_{F \subset P}$ is a complete fan: the *outer normal fan of P*. We denote this fan by Σ_P. The assignement $F \mapsto \sigma_F$ sets up a bijective, order-reversing correspondence between faces of P and cones in Σ_P.

For $0 \leq i \leq d$, denote by $f_i(P)$ the number of i-dimensional faces of P. Recall that the h-vector $(h_0(P), h_1(P), \ldots, h_d(P))$ is defined by

$$h_j(P) = \sum_{i=j}^{d} (-1)^{i-j} \binom{i}{j} f_i(P) \ .$$

Proposition. *For any convex d-polytope P, we have*

$$h_j(P) = \sum_{i=j}^{d} (-1)^{i-j} \dim(H^{i,j}(\Sigma_P)) \ .$$

Proof. The Euler-Poincaré characteristic of the complex $C^*(\wedge^j \mathcal{F})$ is equal to

$$\sum_i (-1)^i \sum_{\sigma\in\Sigma_P(d-i)} \dim(\wedge^j \sigma^\perp) = \sum_{i=j}^d (-1)^i \binom{i}{j} f_i(P)$$

on one hand, and to

$$\sum_i (-1)^i \dim(H^i(\wedge^j \mathcal{F})) = \sum_{i=j}^d (-1)^i \dim(H^{i,j}(\Sigma_P))$$

on the other hand.

Corollary. *For any d-dimensional convex polytope P such that each edge of P lies in exactly $d-1$ facets, we have*

$$h_j(P) \le \dim(H^{j,j}(\Sigma_P))$$

with equality if P is simple.

Proof. The assumption on P means that any $(d-1)$-dimensional cone in Σ_P is simplicial. Then, by Proposition 1.2 (iii), we have $H^{i,j}(\Sigma_P) = 0$ for $i > j+1$. It follows that

$$h_j(P) = \dim(H^{j,j}(\Sigma_P)) - \dim(H^{j+1,j}(\Sigma_P)) \ .$$

1.5. We maintain the notation of 1.4. In the case where $K = \mathbf{R}$, we have the notion of a *Minkowski weight* on P, defined as follows (see [Mc1, §5], [Mc2]). For any faces F and G of P such that F is a facet of G, denote by $n_{F,G}$ the outer unit normal vector to F in G (for some fixed Euclidean norm on V^*). Then a j-weight on P is the assignement to each j-dimensional face F, of a real number a_F such that $\sum_{F\subset G} a_F\, n_{F,G} = 0$ for each $(j+1)$-dimensional face G. The set $\Omega_j(P)$ of all j-weights on P is a real vector space; it turns out to be independant of the Euclidean norm. In fact, $\Omega_j(P)$ only depends on Σ_P, as shown by the following:

Proposition. *For $0 \le j \le d$, the space $\Omega_j(P)$ is isomorphic to $H^{j,j}(\Sigma_P)$.*

Proof. The complex $C^*(\wedge^j \mathcal{F})$ is zero in degree $< j$. Therefore, by definition, $H^{j,j}(\Sigma_P)$ is the kernel of the differential

$$\delta : \oplus_{\sigma\in\Sigma_P(d-j)} \wedge^j \sigma^\perp \to \oplus_{\tau\in\Sigma_P(d-j-1)} \wedge^j \tau^\perp \ .$$

We identify $(d-j)$-dimensional cones in Σ_P and j-dimensional faces of P. For such a face F, the space $\sigma_F^\perp$ is identified with $\mathrm{lin}(F)$ (the direction of the affine space generated by F). The Euclidean structure on V defines a volume form on $\mathrm{lin}(F)$ and hence an identification of $\wedge^j\mathrm{lin(F)}$ to $\mathbf{R}$. Therefore, the space $\oplus_{\sigma\in\Sigma_P(d-j)} \wedge^j \sigma^\perp$ is identified with the space of real-valued functions on the set of j-dimensional faces of P. On the other hand, for any $(j+1)$-dimensional face G, we have an isomorphism $\wedge^j\mathrm{lin(G)} \to \mathrm{lin}(G)$ that sends the canonical generator of $\wedge^j \mathrm{lin}(F)$ (where F is any facet of G) to $\varepsilon_{F,G} n_{F,G}$. Therefore, δ is identified with the map

$$(a_F)_F \to (\sum_{F \subset G} a_G \, n_{F,G})_G \ .$$

But the kernel of this map is $\Omega_j(P)$.

This isomorphism, combined with Corollary 1.4, implies the following refinement of Theorem 6.1 in [Mc1].

Corollary. *For any d-dimensional convex polytope P, such that each edge of P lies in exactly $d-1$ facets, we have*

$$\dim \Omega_j(P) \geq h_j(P) \ .$$

Moreover, equality holds if P is simple.

Remark. To a fan Σ in a vector space V over $\mathbf{Q}$, and to a lattice in V, is associated a complex toric variety $X = X_\Sigma$, see [Od1]. Denoting by Ω_X^j the sheaf of differential j-forms on X (in the sense of Zariski-Steenbrink), we have isomorphisms

$$H^i(X, \Omega_X^j) \simeq H^{i,j}(\Sigma) \otimes_{\mathbf{Q}} \mathbf{C},$$

see [Da, 12.4.1]. In this setting, the statements (i) and (ii) in proposition 1.2, and its corollary as well, are due to Danilov, see [Da, §10].

If moreover Σ is complete, then the group $H^{1,1}(\Sigma)$ is identified with the rational Picard group of X; the presentation

$$0 \to H^{1,1}(\Sigma) \to \mathrm{rel}(\Sigma)^* \to \mathrm{Rel}(\Sigma)^*$$

is equivalent to Eikelberg's determination of the rank of the Picard group, see [Ei1] and [Ei2].

Finally, the notion of a Minkowski weight can be adapted to a rational, complete fan; for such a fan Σ, the space of all Minkowski j-weights is isomorphic to the j-th Chow cohomology group of X_Σ, with rational coefficients, see [Fu-St, Theorem 1].

2. The algebra of continuous, piecewise polynomial functions

2.1. We denote by S the algebra of K-valued polynomial functions on V. Given a fan Σ and a cone $\sigma \in \Sigma$, we denote by R_σ the space of K-valued polynomial functions on the linear span of σ; then R_σ is the quotient of S by its ideal generated by $\sigma^\perp$. For $\tau \subset \sigma$, we have the restriction map $R_\sigma \to R_\tau, f \mapsto f|_\tau$, and this defines a sheaf of S-algebras $\mathcal{R} = (R_\sigma)_{\sigma \in \Sigma}$ on Σ. Moreover, S, R_σ and $\mathcal{R}$ carry a natural grading.

We set

$$R_\Sigma := \{(f_\sigma)_{\sigma \in \Sigma} \mid f_\sigma \in R_\sigma, f_\sigma|_\tau = f_\tau \, , \forall \, \tau \subset \sigma\} \ .$$

Then R_Σ is a graded algebra over S: the *algebra of continuous, piecewise polynomial functions on* Σ. If $|\Sigma|$ is purely d-dimensional, then R_Σ is the space of global sections of $\mathcal{R}$.

For any non-negative integer n, we denote by $R_{\Sigma,n}$ the homogeneous component of degree n in R_Σ. In particular, $R_{\Sigma,1}$ consists of all continuous, piecewise linear functions on Σ. If Σ is complete, then $R_{\Sigma,1}$ contains the space V^* of globally linear functions, and the quotient $R_{\Sigma,1}/V^*$ is identified with $H^{1,1}(\Sigma)$. Namely, the exact sequence

$$0 \to \mathcal{F} \to V^* \to \mathcal{R}_{\Sigma,1} \to 0$$

induces a long exact sequence of cohomology groups, beginning with

$$0 \to V^* \to R_{\Sigma,1} \to H^1(\mathcal{F}) \to 0\ .$$

The components of higher degree in R_Σ are related to higher Hodge spaces in a more complicated way, by the following statement.

Theorem. *Let Σ be a complete fan.*
(i) The graded S-module R_Σ has a canonical increasing filtration such that

$$\operatorname{gr} R_\Sigma \subset S \otimes \bigoplus_{i=0}^{d} H^{i,i}(\Sigma)$$

where $\operatorname{gr} R_\Sigma$ is the associated graded module, and where each space $H^{i,i}(\Sigma)$ occurs in degree i.
(ii) If moreover $H^{i+1,i}(\Sigma) = 0$ for $1 \le i \le d-2$, then equality holds in (i).
(iii) Finally, if $H^{i,j}(\Sigma) = 0$ for $i \neq j$, then $H^i(\mathcal{R}) = 0$ for all $i \ge 1$.

Proof. We have an exact sequence

$$0 \to \mathcal{I}_\sigma \to S \to \mathcal{R}_\sigma \to 0$$

where $\mathcal{I}_\sigma$ denotes the ideal of S generated by $\sigma^\perp$. Therefore, we have an exact sequence (the Koszul complex)

$$0 \to S \otimes \wedge^d \sigma^\perp \to S \otimes \wedge^{d-1}\sigma^\perp \to \cdots \to S \otimes \sigma^\perp \to S \to \mathcal{R}_\sigma \to 0$$

and this defines a resolution of the sheaf $\mathcal{R}$ by sheaves $S \otimes \wedge^j \mathcal{F}$; the differentials are homogeneous of degree -1. We cut this resolution into short exact sequences

$$\begin{gathered}
0 \to S \otimes \wedge^d \mathcal{F} \to S \otimes \wedge^{d-1}\mathcal{F} \to \mathcal{E}_{d-1} \to 0 \\
0 \to \mathcal{E}_{d-1} \to S \otimes \wedge^{d-2}\mathcal{F} \to \mathcal{E}_{d-2} \to 0 \\
\cdots \\
0 \to \mathcal{E}_2 \to S \otimes \mathcal{F} \to \mathcal{E}_1 \to 0 \\
0 \to \mathcal{E}_1 \to S \to \mathcal{R} \to 0\ .
\end{gathered}$$

Because $(\mathcal{E}_j)_\sigma$ is a quotient of $\wedge^j \sigma^\perp$, we have $(\mathcal{E}_j)_\sigma = 0$ for $\dim(\sigma) > d - j$, whence $H^i(\mathcal{E}_j) = 0$ for $i < j$. The same holds for $H^i(\wedge^j \mathcal{F})$ by 1.2. Therefore, we have long exact sequences

$$0 \to S \otimes H^{d-1}(\wedge^{d-1}\mathcal{F}) \to H^{d-1}(\mathcal{E}_{d-1}) \to S \otimes H^d(\wedge^d \mathcal{F}) \to S \otimes H^d(\wedge^{d-1}\mathcal{F})$$
$$0 \to S \otimes H^{d-2}(\wedge^{d-2}\mathcal{F}) \to H^{d-2}(\mathcal{E}_{d-2}) \to H^{d-1}(\mathcal{E}_{d-1}) \to S \otimes H^{d-1}(\wedge^{d-2}\mathcal{F})$$
$$\cdots$$
$$0 \to S \otimes H^2(\wedge^2\mathcal{F}) \to H^2(\mathcal{E}_2) \to H^3(\mathcal{E}_3) \to S \otimes H^3(\wedge^2\mathcal{F})$$
$$0 \to S \otimes H^1(\mathcal{F}) \to H^1(\mathcal{E}_1) \to H^2(\mathcal{E}_2) \to S \otimes H^2(\mathcal{F})$$
$$0 \to S \to H^0(\mathcal{R}) \to H^1(\mathcal{E}_1) \to S \otimes H^1(K) \ .$$

But $H^1(K)$ vanishes by Lemma 1.1 (ii). It follows that the quotient $R_\Sigma^{(1)} = R_\Sigma/S$ contains an S-submodule isomorphic to $S \otimes H^{1,1}(\Sigma)$. Moreover, the quotient

$$R_\Sigma^{(2)} = R_\Sigma/(S \oplus S \otimes H^{1,1}(\Sigma))$$

is an S-submodule of $H^2(\mathcal{E}_2)$, with equality if $H^{2,1}(\Sigma) = 0$. Further, the exact sequence

$$0 \to S \otimes H^{2,2}(\Sigma) \to H^2(\mathcal{E}_2) \to H^3(\mathcal{E}_3) \to S \otimes H^{3,2}(\Sigma)$$

presents $R_\Sigma^{(2)}$ as an extension of a submodule of $S \otimes H^{2,2}(\Sigma)$, by a submodule of $H^3(\mathcal{E}_3)$. Continuing this way, we construct the filtration of R_Σ, and this proves (i) and (ii).

If moreover $H^i(\wedge^j \mathcal{F}) = 0$ for $i \neq j$, then one obtains by descending induction over j: $H^i(\mathcal{E}_j) = 0$ for $i \neq j$. In particular, $H^i(\mathcal{E}_1) = 0$ for $i \geq 2$, and this implies the vanishing of $H^i(\mathcal{R})$ for $i \geq 1$.

Corollary. *Let Σ be a complete, simplicial fan. Then the graded S-module R_Σ is free of finite rank, with generators in degrees $0, 1, \ldots, d$. Moreover, the space of generators of degree i is isomorphic with $H^{i,i}(\Sigma)$; in particular, the space of generators of degree d is one-dimensional. Finally, the complex*

$$0 \to R_\Sigma \to \oplus_{\sigma \in \Sigma(d)} R_\sigma \to \oplus_{\sigma \in \Sigma(d-1)} R_\sigma \to \cdots$$

is exact.

The last statement answers a question of Bernstein and Lunts; [Be-Lu, p. 128]. Observe that the results of the corollary hold for certain non-simplicial fans too, for example for three-dimensional fans Σ such that any two non-simplicial cones in Σ intersect only at the origin. Then the spaces $H^{2,1}(\Sigma)$ and $H^{3,2}(\Sigma)$ vanish in this case, by 1.3 and 1.2 (ii).

The methods of this section can be used to study the algebra of piecewise polynomial functions which are continuously differentiable of a fixed order (such algebras are considered in [Bi1] and [Bi-Ro], as modules over the algebra of polynomial functions). This will be developed elsewhere.

2.2. For each simplicial, d-dimensional cone σ, we denote by Φ_σ the product of the equations of the facets of σ. Then $\Phi_\sigma \in S$ is uniquely defined up to scalar multiplication.

We normalize Φ_σ as follows: we choose a non-zero element in $\wedge^d V$, and we impose that the equations of facets of σ are non-negative on σ, and that the absolute value of their wedge product is 1. We denote by φ_σ the function on V such that

$$\varphi_\sigma(v) = \begin{cases} \Phi_\sigma(v) & \text{if } v \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then φ_σ is a continuous, piecewise polynomial function that vanishes outside the interior of σ.

Theorem. *Let Σ be a complete, simplicial fan. Then there exists a non-zero linear map $\pi_\Sigma : R_\Sigma \to S$ such that*
(i) π_Σ is S-linear,
(ii) π_Σ is homogeneous of degree $-d$.
Moreover, (i) and (ii) define π_Σ uniquely up to scalar multiplication, and a choice of π_Σ is given by

$$\pi_\Sigma(f) = \sum_{\sigma\in\Sigma(d)} \frac{f_\sigma}{\Phi_\sigma}$$

for any $f = (f_\sigma) \in R_\Sigma$. Then $\pi_\Sigma(\varphi_\sigma) = 1$ for any $\sigma \in \Sigma(d)$.

Proof. If π_Σ exists, then it vanishes on any element of R_Σ of degree $0, 1, \ldots, d-1$, by assumption (ii). Now (i) and Corollary 2.1 imply that π_Σ is unique up to scalar multiplication.

By Corollary 2.1 again, the quotient of R_Σ by its S-submodule generated by elements of degree at most $d-1$, is isomorphic to S. The resulting map $R_\Sigma \to S$ satisfies the conditions (i) and (ii), and hence it can be taken as π_Σ.

For $f = (f_\sigma)$ in R_Σ, set

$$g = \sum_{\sigma\in\Sigma(d)} \frac{f_\sigma}{\Phi_\sigma} \; .$$

Then g is a rational function on V, and the denominator for g is the product of the equations of $(d-1)$-dimensional cones of Σ. We claim that g is a polynomial function on V; for this, it is enough to check that no $\sigma \in \Sigma(d-1)$ is a pole set of g. Denote by σ' and σ'' the cones in $\Sigma(d)$ having σ as their common face. We can find generators $e_1, \ldots, e_{d-1}, e'_d$ (resp. e''_d) of edges of σ' (resp. σ'') such that $e_1 \wedge \cdots \wedge e_{d-1} \wedge e'_d = 1$ and that $e_1 \wedge \cdots \wedge e_{d-1} \wedge e''_d = -1$. Then there exist $a_1, \ldots, a_{d-1}$ in K such that $\sum_{i=1}^{d-1} a_i\, e_i = e'_d + e''_d$. Let $x_1, \ldots, x_d$ in V^* form the dual basis of $e_1, \ldots, e_{d-1}, e'_d$. Then we have

$$\Phi_{\sigma'} = x_1 \cdots x_d,\ \Phi_{\sigma''} = -x_d \prod_{i=1}^{d-1} (x_i + a_i\, x_d) \; .$$

It follows that

$$\frac{f_{\sigma'}}{\Phi_{\sigma'}} + \frac{f_{\sigma''}}{\Phi_{\sigma''}} = x_d^{-1} \prod_{i=1}^{d} x_i^{-1}(x_i + a_i x_d)^{-1} \, (f_{\sigma'} \prod_{i=1}^{d-1}(x_i + a_i x_d) - f_{\sigma''} \prod_{i=1}^{d-1} x_i)$$

has no pole along $x_d = 0$, because $f_{\sigma'} - f_{\sigma''}$ is divisible by x_d. Therefore, g has no pole along σ, and this proves our claim.

Now the map $p_\Sigma : f \mapsto g$ sends R_Σ to S, and p_Σ satisfies the conditions (i) and (ii). Moreover, we have $p_\Sigma(\varphi_\sigma) = 1$ for all $\sigma \in \Sigma(d)$, and hence p_Σ is non-zero. By the first step of the proof, p_Σ is proportional to π_Σ.

Example. Choose affinely independant points $x_0, x_1 \ldots, x_d$ in V^*. Let P be the simplex with vertices $x_0, x_1, \ldots, x_d$; let Σ be the normal fan to P. Then the function

$$\begin{array}{rcll} f: & V & \to & K \\ & v & \to & \max(\langle x_0, v\rangle, \ldots, \langle x_d, v\rangle) \end{array}$$

(the support function of P) is piecewise linear on Σ, and we can normalize π_Σ so that, for any integer $n \geq 1$:

$$\pi_\Sigma(f^n) = \sum_{i=0}^{d} \frac{x_i^n}{\prod_{j\neq i}(x_i - x_j)} \ .$$

Denote by h_n the complete symmetric function of degree n, i.e. the sum of all monomials of degree n. Then

$$\pi_\Sigma(f^n) = \begin{cases} h_{n-d}(x_0, x_1, \ldots, x_d) & \text{if } n \geq d \\ 0 & \text{otherwise.} \end{cases}$$

Namely, this follows from the identity

$$\prod_{i=0}^{d} \frac{1}{1 - tx_i} = \sum_{i=0}^{d} \left(\frac{1}{1 - tx_i} \prod_{j\neq i} \frac{1}{1 - x_i^{-1}x_j}\right)$$

by expanding both sides into power series in t.

2.3. We keep the notation of 2.1 and 2.2.

Theorem. *Let Σ be a complete, simplicial fan; let Σ' be a simplicial subdivision of Σ. Then there exists a unique map $\pi_{\Sigma',\Sigma} : R_{\Sigma'} \to R_\Sigma$ such that*

(i) $\pi_{\Sigma',\Sigma}(1) = 1$,

(ii) $\pi_{\Sigma',\Sigma}$ *is R_Σ-linear,*

(iii) $\pi_{\Sigma',\Sigma}$ *is homogeneous of degree zero.*

Moreover, we have for any $\sigma \in \Sigma$ and $f = (f_{\sigma'}) \in R_{\Sigma'}$:

$$\pi_{\Sigma',\Sigma}(f)_\sigma = \Phi_\sigma \sum_{\sigma'\subset\sigma, \sigma'\in\Sigma'(d)} \frac{f_{\sigma'}}{\Phi_{\sigma'}} \ .$$

Proof. Let $\pi : R_{\Sigma'} \to R_\Sigma$ satisfy (i), (ii) and (iii). Then the map $\pi_\Sigma \circ \pi : R_{\Sigma'} \to S$ is S-linear and homogeneous of degree $-d$. Moreover, by (i) and (ii), the restriction of this map to R_Σ is the (non-zero) map π_Σ. Therefore, by 2.2, we have $\pi_\Sigma \circ \pi = \pi_{\Sigma'}$.

Now choose $\sigma \in \Sigma(d)$, whence $\varphi_\sigma \in R_\Sigma$. Then we have for all $f \in R_{\Sigma'}$:

$$\pi_{\Sigma'}(f\varphi_\sigma) = \pi_\Sigma(\pi(f\varphi_\sigma)) = \pi_\Sigma(\varphi_\sigma \pi(f)) = \pi(f)_\sigma$$

(the last equality follows from the formula for π_Σ given in 2.2, because $\varphi_\sigma\pi(f)$ vanishes outside σ). Using 2.2 again, we obtain

$$\pi(f)_\sigma = \Phi_\sigma \sum_{\sigma' \subset \sigma, \sigma \in \Sigma'(d)} \frac{f_{\sigma'}}{\Phi_{\sigma'}} \ .$$

This proves the uniqueness of $\pi = \pi_{\Sigma',\Sigma}$. For the existence, we define π by the formula above. Then (ii) and (iii) are obvious, whereas (i) and the fact that π has values in R_Σ can be checked as in the proof of 2.2.

Corollary. *(i) For any $f \in R_\Sigma \subset R_{\Sigma'}$, we have $\pi_{\Sigma'}(f) = \pi_\Sigma(f)$.*
(ii) For any simplicial subdivision Σ'' of Σ', we have $\pi_{\Sigma',\Sigma} \circ \pi_{\Sigma'',\Sigma'} = \pi_{\Sigma'',\Sigma}$.

The first assertion follows from (i) and (ii), and the second one from the uniqueness of $\pi_{\Sigma',\Sigma}$.

Denoting by R the algebra of all continuous, piecewise polynomial functions on V (with respect to no specified fan), we conclude that there is a canonical map $\pi : R \to S$ that is S-linear and homogeneous of degree $-d$. Moreover, for any complete, simplicial fan Σ, there is a canonical, R_Σ-linear projection $R \to R_\Sigma$ that is compatible with π.

Remark. Let V' be a K-vector space, and let $u : V \to V'$ be a K-linear map. Then composition by u induces an algebra homomorphism $u^* : R' \to R$ where R' denotes the algebra of continuous, piecewise polynomial functions on V'. We claim that π vanishes on the image of u^*, whenever u is not an isomorphism. To check this claim, we may replace V' by the image of u, and hence assume that u is surjective. Now the composition $\pi \circ u^* : R' \to S$ is a homogeneous, S'-linear map of degree $-d$. But the S'-module R' is generated in degree at most $\dim(V') < d$, and this implies our assertion.

In other words, π vanishes on functions that do not depend on all variables.

2.4. We keep the notation of 2.1 and 2.2.

Theorem. *Let Σ be a complete, simplicial fan. Then the S-bilinear symmetric map*

$$\begin{array}{ccc} R_\Sigma \times R_\Sigma & \to & S \\ (f,g) & \to & \pi_\Sigma(fg) \end{array}$$

is a perfect pairing, i.e., it induces an isomorphism $R_\Sigma \to \mathrm{Hom}_S(R_\Sigma, S)$.

Proof. We first check that the map $R_\Sigma \to \mathrm{Hom}_S(R_\Sigma, S)$ is injective. Let $f \in R_\Sigma$ such that $\pi_\Sigma(fg) = 0$ for all $g \in R_\Sigma$. For any $\sigma \in \Sigma(d)$, choose $h_\sigma \in S$. Then the functions $\varphi_\sigma h_\sigma$ glue together into a continuous piecewise polynomial function g on Σ, because these functions vanish on every $(d-1)$-dimensional cone. Therefore, we have:

$$0 = \pi_\Sigma(fg) = \sum_{\sigma \in \Sigma(d)} f_\sigma h_\sigma \ .$$

This holds for an arbitrary family of h_σ's, whence $f = 0$.

Now we check that the map $R_\Sigma \to \mathrm{Hom}_S(R_\Sigma, S)$ is surjective. Let $u : R_\Sigma \to S$ be a S-linear map. Define a function g_σ on each $\sigma \in \Sigma(d)$ by $g_\sigma = u(\varphi_\sigma)$. We check that these function glue together into g in R_Σ. Namely, let $\sigma \in \Sigma(d-1)$ separate two maximal cones σ' and σ''. Then $\varphi_{\sigma'} - \varphi_{\sigma''} = f_\sigma h_{\sigma',\sigma''}$ where f_σ is an equation of σ, and where $h_{\sigma',\sigma''} \in R_\Sigma$. Therefore, $g_{\sigma'} - g_{\sigma''} = f_\sigma u(h_{\sigma',\sigma''})$, i.e. $g_{\sigma'}$ and $g_{\sigma''}$ agree on σ.

Denote by Φ_Σ the product of the equations of all $(d-1)$-dimensional cones of Σ. Then for any $f \in R_\Sigma$, we claim that

$$f\Phi_\Sigma = \sum_{\sigma\in\Sigma(d)} \varphi_\sigma f_\sigma \Phi_\Sigma/\Phi_\sigma \ .$$

Indeed, both sides agree on any given $\sigma \in \Sigma(d)$, because $\varphi_\sigma|\sigma = \Phi_\sigma$ and $\varphi_\tau|\sigma = 0$ for $\tau \neq \sigma$. Moreover, because $f_\sigma \Phi_\Sigma/\Phi_\sigma \in S$, we have

$$u(f\Phi_\Sigma) = \sum_{\sigma\in\Sigma(d)} u(\varphi_\sigma) f_\sigma \Phi_\sigma/\Phi_\Sigma = \sum_{\sigma\in\Sigma(d)} g_\sigma f_\sigma \Phi_\Sigma/\Phi_\sigma$$

and hence

$$u(f) = \sum_{\sigma\in\Sigma(d)} g_\sigma f_\sigma/\Phi_\sigma = \pi_\Sigma(fg) \ .$$

This concludes the proof.

Let $\overline{R_\Sigma}$ be the quotient of the algebra R_Σ by its ideal generated by homogeneous, globally polynomial functions (i.e. by V^*); for $f \in R_\Sigma$, let $\overline{f}$ be its image in $\overline{R_\Sigma}$. By Corollary 2.1, we have an isomorphism of graded vector spaces

$$\overline{R_\Sigma} \simeq \bigoplus_{j=0}^{d} H^{j,j}(\Sigma) \ .$$

Using Nakayama's lemma, we derive easily the following:

Corollary. *For any complete, simplicial fan Σ, the K-bilinear, symmetric map*

$$\begin{array}{ccc} \overline{R_\Sigma} \times \overline{R_\Sigma} & \to & K \\ (\overline{f}, \overline{g}) & \to & \pi_\Sigma(fg)(0) \end{array}$$

is well-defined, and it induces non-degenerate pairings

$$H^{j,j}(\Sigma) \times H^{d-j,d-j}(\Sigma) \to K \ .$$

In particular, the spaces $H^{j,j}(\Sigma)$ and $H^{d-j,d-j}(\Sigma)$ have the same dimension. This statement implies the Dehn-Sommerville equations, by Corollary 1.4: for any simple polytope P of dimension d, we have $h_j(P) = h_{d-j}(P)$.

Remark. Let Σ be a complete, simplicial fan in a vector space V over $\mathbf{Q}$; choose a lattice in V. These data define a toric variety X_Σ; the algebraic objects of this section have the following interpretations in terms of the geometry of X_Σ, see [Br] for details.

The algebra R_Σ is isomorphic to the equivariant cohomology ring of X_Σ with rational coefficients. Moreover, for any simplicial subdivision Σ' of Σ, the map $\pi_{\Sigma',\Sigma} : R_{\Sigma'} \to R_\Sigma$ is identified with the push-forward map defined by the morphism $X_{\Sigma'} \to X_\Sigma$. Finally, the map $\pi_\Sigma : R_\Sigma \to S$ is the push-forward defined by the constant morphism $X_\Sigma \to$ point. It follows that the (ordinary) cohomology ring of X_Σ with rational coefficients, is isomorphic to $\overline{R}_\Sigma$; recall that this ring coincides with the Chow ring with rational coefficients, see [Da, §10]. In this identification, the bilinear symmetric map in the corollary above, becomes the intersection product.

So the algebra $\overline{R}$ is the direct limit of rational Chow rings of smooth, complete toric varieties. In turn, by work of Fulton and Sturmfels, this direct limit is isomorphic with the rational polytope algebra, see [Fu-St, Theorem 4.2]. The latter result was one of the motivations for [Mc4], [Br] and the present paper.

3. The ring of continuous, piecewise exponential functions

3.1. We denote by $\mathbf{Z}[V^*]$ the group ring over $\mathbf{Z}$ of the abelian group V^*. Then $\mathbf{Z}[V^*]$ is a free abelian group over the symbols e^x, $x \in V^*$. The multiplication in $\mathbf{Z}[V^*]$ is defined by $e^x\, e^y = e^{x+y}$. The subgroup of $\mathbf{Z}[V^*]$ generated by the $e^x - 1$ ($x \in V^*$) is an ideal; we denote it by I.

We will need the following description of the quotients I^n/I^{n+1}, where I^n denotes the n-th power of the ideal I.

Proposition. *The map $V^* \to I/I^2$, $x \mapsto e^x - 1 \pmod{I^2}$ is a group isomorphism. Furthermore, this map induces a ring isomorphism*

$$S^\bullet_{\mathbf{Z}}(V^*) \to \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

where $S^\bullet_{\mathbf{Z}}(V^*)$ denotes the symmetric algebra over $\mathbf{Z}$ of the abelian group V^*.

Observe that, since the additive group V^* is divisible, the canonical map

$$S^\bullet_{\mathbf{Z}}(V^*) \to S^\bullet_{\mathbf{Q}}(V^*)$$

is an isomorphism.

Proof. Denote by (x) the image of $e^x - 1$ in I/I^2. Then $(x-y) = (x) - (y)$ by the following relation:

$$e^{x-y} - 1 = (e^x - 1) + (e^{-y} - 1) + (e^x - 1)(e^{-y} - 1)\ .$$

Therefore, the map $V^* \to I/I^2$, $x \mapsto (x)$ is a group homomorphism. Observe that any $u \in I/I^2$ can be representated in I by some $\sum_{i=1}^r a_i(e^{x_i} - 1)$ with $a_i \in \mathbf{Z}$ and $x_i \in V^*$. So $(u) = \sum_{i=1}^r a_i(x_i) = (\sum_{i=1}^r a_i x_i)$ and our map is surjective. On the other hand, if $(x) = 0$

then $e^x - 1$ is in I^2, and hence the Taylor expansion at the origin of $\exp(x) - 1$ has order at least two: then $x = 0$. So our map is injective, and the first assertion is proved.

For the second assertion, observe that the map $x \mapsto (x)$ extends uniquely to a homomorphism of graded rings

$$S^{\bullet}_{\mathbf{Z}}(V^*) \to \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

by the universal property of the symmetric algebra. This homomorphism is surjective, by the first assertion. To check the injectivity, we may replace V^* by a finitely generated subgroup G. Then G is a free abelian group, and hence its group ring $\mathbf{Z}[G]$ is identified with the ring of Laurent polynomials in r variables $x_1, \ldots, x_r$ with integral coefficients; here r is the rank of G. Moreover, the ideal I is generated by $x_1 - 1, \ldots, x_r - 1$, and these elements form a regular sequence in $\mathbf{Z}[G]$. But our statement is well-known in this case.

3.2. Let Σ be a fan. For any cone σ, let E_σ be the group ring over $\mathbf{Z}$ of the abelian group $L(\sigma)^*$ (the dual of the linear span of σ). For any $\sigma \in \Sigma$, and for any face τ of σ, the inclusion $L(\tau) \subset L(\sigma)$ induces a surjective map $L(\sigma)^* \to L(\tau)^*$ and hence a surjective homomorphism $E_\sigma \to E_\tau, f \mapsto f|_\tau$. We set

$$E_\Sigma := \{(f_\sigma)_{\sigma\in\Sigma} \mid f_\sigma \in E_\sigma,\ f_\sigma|_\tau = f_\tau\ \forall\, \tau \subset \sigma\}\ .$$

Then E_Σ is the *ring of continuous, piecewise exponential functions on* Σ; there is an obvious structure of $\mathbf{Z}[V^*]$-module on E_Σ.

In contrast to the algebra of continuous, piecewise polynomial functions as a module over the algebra of polynomial functions, the $\mathbf{Z}[V^*]$-module E_Σ is not finitely generated in general. Indeed, consider the case where $V = K$ is one-dimensional, and where Σ consists of the two half-lines K^+ and K^-, together with the origin. Then elements of E_Σ are pairs (f^+, f^-) in $\mathbf{Z}[K]$ such that $f^+(0) = f^-(0)$. Therefore, the map

$$E_\Sigma \to \mathbf{Z}[K] \times I,\ f \to (f^+, (f^+ - f^-)|K^+$$

is a ring isomorphism. Using 3.1, it follows that

$$E_\Sigma/IE_\Sigma \simeq \mathbf{Z} \times K\ .$$

In particular, the abelian group E_Σ/IE_Σ is not finitely generated, and hence the $\mathbf{Z}[K]$-module E_Σ is not finitely generated. Observe that this module is not free either.

So it would be difficult to study E_Σ by using the homological methods of the previous sections. We will use a different approach, by induction on the number of cones in Σ. This approach was used in [Br] for the algebra of continuous, piecewise polynomial functions.

For any maximal cone $\sigma \in \Sigma$, we denote by E_{σ^0} the set of all $f \in E_\Sigma$ such that f vanishes identically outside σ^0. Then E_{σ^0} is an ideal of E_Σ. In the case where Σ is simplicial, we construct elements of E_{σ^0} as follows. Let τ be an edge of σ. Then there exists a non-zero continuous, piecewise linear function φ_τ that vanishes at all edges of Σ, except for τ. Moreover, φ_τ is uniquely defined up to scalar multiplication; it is called a

Courant function in [Bi2]. Observe that $e^{\varphi_\tau}-1$ is in E_Σ and vanishes outside the star of τ. Therefore, $\prod_{\tau\in\sigma(1)}(e^{\varphi_\tau}-1)$ is in E_{σ^0} whenever σ is a maximal cone in Σ.

Proposition. *Let Σ be a simplicial fan; let σ be a maximal cone in Σ.*
(i) The sequence

$$0 \to E_{\sigma^0} \to E_\Sigma \to E_{\Sigma\setminus\{\sigma\}} \to 0$$

is exact.
(ii) The $\mathbf{Z}[V^*]$-module E_{σ^0} is generated by all $\prod_{\tau\in\sigma(1)}(e^{\varphi_\tau}-1)$ where φ_τ is a Courant function associated with τ.

Proof. (i) Clearly, the sequence is left exact. To prove the surjectivity of the restriction $E_\Sigma \to E_{\Sigma\setminus\{\sigma\}}$, it is enough to check that any continuous, piecewise exponential function on the boundary of σ extends to an exponential function on σ. Choose coordinates $x_1,\ldots,x_d$ on V such that

$$\sigma = \Big(\bigcap_{i=1}^{r}(x_i\geq 0)\Big)\cap\Big(\bigcap_{j=r+1}^{d}(x_j=0)\Big)\ .$$

For any subset J of $\{1,\ldots,r\}$, set

$$\sigma_J := \sigma\cap\Big(\bigcap_{j\notin J}(x_j=0)\Big)\ .$$

This sets up a bijection between subsets J of $\{1,\ldots,r\}$ and faces σ_J of σ. By assumption, for any proper subset $J\subset\{1,\ldots,r\}$, we have an exponential function $f_J(x_j)_{j\in J}$ on σ_J, and these functions are compatible on the boundary of σ. Now set

$$f(x_1,\ldots,x_d) := \sum_{J\subset\{1,\ldots,r\}}(-1)^{r-1-\mathrm{card}(J)}f_J(x_i)_{i\in J}$$

(sum over all proper subsets of $\{1,\ldots,r\}$). Then f is the desired extension.

(ii) For $J\subset\{1,\ldots,r\}$, denote by x_J the r-tuple whose j-th coordinate is x_j if $j\notin J$, and 0 otherwise. Define a map $p_\sigma : E_\sigma\to E_\sigma$ by

$$p_\sigma(f)(x) = \sum_{J\subset\{1,\ldots,r\}}(-1)^{r-1-\mathrm{card}(J)}f(x_J)$$

(sum over all subsets of $\{1,\ldots,r\}$). Then p_σ is a projection of E_σ onto E_{σ^0}. Moreover, we have

$$p_\sigma(e^{a_1x_1+\cdots+a_dx_d}) = e^{a_{r+1}x_{r+1}+\cdots+a_dx_d}\prod_{j=1}^{r}(e^{a_jx_j}-1)\ .$$

This implies our statement, because we have

$$\prod_{j=1}^{r}(e^{a_jx_j}-1) = \prod_{\tau\in\sigma(1)}(e^{\varphi_\tau}-1)$$

for a suitable normalization of the φ_τ's.

Corollary. *For any simplicial fan Σ, the abelian group E_Σ is generated by exponentials of piecewise linear functions on Σ.*

Proof. Choose a maximal cone $\sigma \in \Sigma$, and let $f \in E_\Sigma$. By induction on the number of cones in Σ, we may assume that

$$f|_{\Sigma \setminus \{\sigma\}} = \sum_{j=1}^{r} a_j \, e^{f_j}$$

with $a_j \in \mathbf{Z}$ and f_j continuous and piecewise linear on $\Sigma \setminus \{\sigma\}$. Then by the argument of the proof of 3.2, each f_j extends to a continuous, piecewise linear function on Σ. Therefore, we may assume that $f|_{\Sigma \setminus \{\sigma\}} = 0$, i.e. that $f \in E_{\sigma^0}$. Now we conclude the argument by statement (ii) above.

3.3. Let Σ be a fan, let E_Σ be the ring of piecewise exponential functions on Σ, and let $J_\Sigma \subset E_\Sigma$ be the kernel of the evaluation at 0. Clearly, we have $IE_\Sigma \subset J_\Sigma$ and hence $I^n E_\Sigma \subset J_\Sigma^n$ for all integers $n \geq 1$.

Theorem. *For any simplicial d-dimensional fan Σ, and for any integer $n \geq 1$, we have $J_\Sigma^{n+d} \subset I^n E_\Sigma$.*

Proof. We prove this theorem by induction over the number of cones in Σ. The first step of the induction is trivial. Choose a maximal cone $\sigma \in \Sigma$, and let $f \in J_\Sigma^{n+d}$. Then $f|_{\Sigma \setminus \{\sigma\}} \in J_{\Sigma \setminus \{\sigma\}}^{n+d}$. Using the induction hypothesis and the surjectivity of the restriction $E_\Sigma \to E_{\Sigma \setminus \{\sigma\}}$, we may assume that $f|_{\Sigma \setminus \{\sigma\}} = 0$. Then $f \in E_{\sigma^0} \cap J_\Sigma^{n+d}$, i.e. $f_\sigma \in E_{\sigma^0} \cap I^{n+d} E_\sigma$. It is enough to prove that $f_\sigma \in I^n E_{\sigma^0}$. For this, we use the notation of the proof of Proposition 3.2. Then $f_\sigma = p_\sigma(f_\sigma) \in p_\sigma(I^{n+d} E_\sigma)$. Therefore, it is enough to check that

$$p_\sigma(I^{n+d} E_\sigma) \subset I^n p_\sigma(E_\sigma) \ .$$

We observe that $p_\sigma = p_1 \cdots p_r$ where $p_i(f) = f - f|_{x_i=0}$. Moreover, the p_i's commute pairwise. For any f, g in E_σ, we have

$$(*) \qquad p_i(fg) = p_i(f)g + f|_{x_i=0} p_i(g) \ .$$

It follows that $p_i(I^2 E_\sigma) \subset I p_i(E_\sigma)$ (observe that $f \in I$ implies $f|_{x_i=0} \in I$) and, by induction on n, that

$$(**) \qquad p_i(I^{n+1} E_\sigma) \subset I^n p_i(E_\sigma) \ .$$

To end the proof, it suffices to check by induction on i that

$$p_i \cdots p_1(I^{n+i} E_\sigma) \subset I^n p_i \cdots p_1(E_\sigma) \ .$$

This statement holds for $i = 1$; if it holds for $i-1$, then we have (using $(*)$ for the second inclusion, and $(**)$ for the third one)

$$\begin{aligned} p_i \cdots p_1(I^{n+i}E_\sigma) &\subset p_i(I^{n+1}p_{i-1}\cdots p_1(E_\sigma)) \\ &\subset p_i(I^{n+1})p_{i-1}\cdots p_1(E_\sigma) + I^{n+1}p_i\cdots p_1(E_\sigma) \\ &\subset I^n p_i(E_\sigma)p_{i-1}\cdots p_1(E_\sigma) + I^n p_i \cdots p_1(E_\sigma) \\ &\subset I^n p_i \cdots p_1(E_\sigma)\ . \end{aligned}$$

Remark. Corollary 3.2 can also be deduced from [Mo1, §5], whereas Theorem 3.3 is a sharpening of the main result of [Kh-Pu]; see 5.1 below. In the case where $K = \mathbf{Q}$, there are close connections between the algebra of piecewise exponential functions on a fan, and the equivariant K-theory of the corresponding toric variety, see [Mo2].

4. Piecewise polynomials and piecewise exponentials

4.1. Recall that R_Σ denotes the graded algebra of continuous, piecewise polynomial functions on the fan Σ, and that S denotes the graded algebra of polynomial functions on V. We denote by $R_{\Sigma,\geq n}$ (resp. $S_{\geq n}$) the sum of the homogeneous components of degree at least n in R_Σ (resp. S). For any maximal cone $\sigma \in \Sigma$, let R_{σ^0} be the ideal of R_Σ consisting of functions that vanish identically outside the relative interior of σ.

Proposition. *Let Σ be a simplicial fan, and let σ be a maximal cone in Σ.*
(i) The sequence

$$0 \to R_{\sigma^0} \to R_\Sigma \to R_{\Sigma\setminus\{\sigma\}} \to 0$$

is exact.
(ii) The S-module R_{σ^0} is generated by the function $\prod_{\tau\in\sigma(1)} \varphi_\tau$ where φ_τ is a Courant function associated with τ (see 3.2).
(iii) For any $n \geq 0$, we have $R_{\Sigma,\geq n+d} \subset S_{\geq n}R_\Sigma \subset R_{\Sigma,\geq n}$.
(iv) For any $n \geq 0$, we have $R_{\sigma^0} \cap S_{\geq n}R_\Sigma = S_{\geq n}R_{\sigma^0}$.

Proof. The statements (i) and (ii) (resp. (iii)) are checked as in 3.2 (resp. 3.3). For (iv), it is enough to prove that $R_{\sigma^0} \cap S_{\geq n}R$ is contained in $S_{\geq n}R_{\sigma^0}$. For this, we may replace σ by its linear span, and hence we may assume that σ is d-dimensional. Then the S-module R_{σ^0} is generated by

$$\varphi_\sigma = \prod_{\tau\in\sigma(1)} \varphi_\tau$$

with the notation of 2.2 and 2.3. Moreover, there is an S-linear map $\pi : R \to S$ such that $\pi(\varphi_\sigma) = 1$ (see 2.2). Therefore, the S-module R_{σ^0} is a direct factor of R, and this implies our statement.

As in 3.2, we deduce the following:

Corollary. *The K-algebra R_Σ is generated by the continuous, piecewise linear functions on Σ.*

4.2. Let Σ be a simplicial fan. The algebra R_Σ is endowed with two filtrations, by powers of the ideals $R_{\Sigma,\geq 1}$ and $S_{\geq 1}R_\Sigma$. It follows from Corollary 4.1 that $(R_{\Sigma,\geq 1})^n = R_{\Sigma,\geq n}$, and from Proposition 4.1 (iii) that both filtrations define the same topology on R_Σ. We denote by $\hat{R}_\Sigma$ the completion of R_Σ with respect to this topology. Then $\hat{R}_\Sigma$ is an algebra over the ring $\hat{S}$ of formal power series on V.

We will need the following variant of R_Σ: Define R_Σ^{int} as the subset of R_Σ consisting of all functions f such that $f(0)$ is an integer. Then R_Σ^{int} is a graded subring of R_Σ, with $R_{\Sigma,0}^{\mathrm{int}} = \mathbf{Z}$ and $R_{\Sigma,n}^{\mathrm{int}} = R_{\Sigma,n}$ for all $n \geq 1$.

Proposition. *(i) The algebra $\hat{R}_\Sigma$ consists of all compatible piecewise formal power series on Σ; in other words,*

$$\hat{R}_\Sigma = \{(f_\sigma)_{\sigma\in\Sigma} \mid f_\sigma \in \hat{R}_\sigma, f_\sigma|_\tau = f_\tau \ \forall\, \tau \subset \sigma\} \ .$$

(ii) The map $E_\Sigma \to \hat{R}_\Sigma$ that sends any continuous, piecewise exponential function (f_σ) to the collection of the Taylor expansions of each f_σ, is injective. Moreover, the closure of its image consists of $\hat{R}_\Sigma^{\mathrm{int}}$.

Proof. (i) is checked by induction on the number of cones in Σ, the case of one cone being trivial. Choose a maximal cone σ in Σ. Observe that the S-module R_Σ is finitely generated (this follows, e.g., from 4.1 (i) and (ii)). Therefore, the sequence

$$0 \to \hat{R}_{\sigma^0} \to \hat{R}_\Sigma \to \hat{R}_{\Sigma\setminus\{\sigma\}} \to 0$$

is exact. Moreover, because $R_{\sigma^0} = \varphi_\sigma S$, we can identify $\hat{R}_{\sigma^0}$ with $\varphi_\sigma \hat{S}$.

On the other hand, denote by C_Σ the algebra of compatible piecewise power series on Σ. Then C_Σ is complete, as a closed subalgebra of the product of all $\hat{R}_\sigma$'s. Therefore, R_Σ maps to C_Σ, and this map induces a morphism from the exact sequence above, to the analogous exact sequence satisfied by C_Σ. By the induction hypothesis, this morphism is an isomorphism.

(ii) Observe that the map $\mathbf{Z}[V^*] \to \hat{S}$, $e^x \mapsto \sum_{n=0}^\infty x^n/n!$ is well-defined and injective. Therefore, the map $E_\Sigma \to \hat{R}_\Sigma$ is injective, too; clearly, its image is contained in $\hat{R}_\Sigma^{\mathrm{int}}$. We identify E_Σ with its image, and we check that E_Σ is dense in $\hat{R}_\Sigma^{\mathrm{int}}$. In more concrete terms, given $f = (f_\sigma) \in R_\Sigma^{\mathrm{int}}$ and $N \geq 0$, we must find $g = (g_\sigma) \in E_\Sigma$ such that $g_\sigma - f_\sigma \in R_{\sigma,\geq N}^{\mathrm{int}}$ for all $\sigma \in \Sigma$. We may assume that f is homogeneous. If the degree of f is zero, then $f = f(0) \in \mathbf{Z}$ and we simply take $g = f(0)$. Otherwise, we may assume that $f = cl^n$ for some $c \in K$, some continuous, piecewise linear function l on Σ, and some integer $n \geq 1$; namely, the abelian group $R_{\Sigma,n}^{\mathrm{int}} = R_{\Sigma,n}$ is generated by such functions (see Corollary 4.1). Now there exists a formal power series $u(t) = t + \sum_{i\geq 2} u_i t^i$ such that $u(\exp(t) - 1) = t$ as formal power series. Then

$$T_N(l) := (e^l - 1) + \sum_{i=2}^{N} u_i(e^l - 1)^i$$

is in E_Σ, and $T_N(l)$ is an approximation of l at the order N. Now the function $g := T_N(cl)T_N(l)^{n-1}$ is continuous and piecewise exponential, and g approximates $f = cl^n$ at the order N.

4.3. Denote by E the ring of continuous, piecewise exponential functions on V (with respect to no specified fan). Then E injects into $\hat{R}$ as a dense subring of $\hat{R}^{\mathrm{int}}$; we will identify E with its image.

Theorem. *We have $IE = E \cap S_{\geq 1}\hat{R}$.*

Proof. Clearly, IE is contained in $E \cap S_{\geq 1}\hat{R}$. Therefore, it is enough to prove that $E_\Sigma \cap S_{\geq 1}\hat{R}$ is contained in IE for any complete, simplicial fan Σ. But this statement makes sense for any (non-complete) fan Σ, if we replace IE by its restriction to $|\Sigma|$. Now we can use induction on the number of cones in Σ, because of Proposition 3.2; then we reduce to checking that $E_{\sigma^0} \cap S_{\geq 1}\hat{R}$ is contained in IE for any simplicial cone σ. We may assume further that σ is d-dimensional.

First we consider the case where $d = 1$. Choose the coordinate x on V such that $\sigma = (x \geq 0)$. By Proposition 3.2, the $\mathbf{Z}[V^*]$-module E_{σ^0} is generated by functions f_a such that

$$f_a(x) = \begin{cases} e^{ax} - 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in K$. The identity

$$(e^{ax} - 1) + (e^{bx} - 1) = (e^{(a+b)x} - 1) - (e^{ax} - 1)(e^{bx} - 1)$$

implies that $f_a + f_b = f_{a+b} - (e^{ax} - 1)f_b$, and hence that

$$(*) \qquad f_a + f_b - f_{a+b} \in IE\ .$$

Given $f \in E_{\sigma^0}$, we can write $f = \sum_{i=1}^r u_i f_{a_i}$ where $u_i \in \mathbf{Z}[V^*]$ and $a_i \in K$. Moreover, there exist integers n_i such that $f_i - n_i \in I$ for $1 \leq i \leq r$, and hence $f = \sum_{i=1}^r n_i f_{a_i} + g$ where $g \in IE$. Using $(*)$, we can even write $f = f_c + h$ where $c \in K$ and $h \in IE$. If moreover $f \in S_{\geq 1}\hat{R}$, then $f_c \in S_{\geq 1}\hat{R}$. But the Taylor expansion of f_c at the origin begins with cx, hence $c = 0$ and $f \in IE$ as required.

Now we consider the more involved case where $d = 2$. Choose coordinates x_1, x_2 on V such that $\sigma = (x_1 \geq 0, x_2 \geq 0)$. Then, as before, the $\mathbf{Z}[V^*]$-module E_{σ^0} is generated by functions f_{a_1,a_2} such that

$$f_{a_1,a_2}(x_1, x_2) := \begin{cases} (e^{a_1x_1} - 1)(e^{a_2x_2} - 1) & \text{if } (x_1, x_2) \in \sigma \\ 0 & \text{otherwise,} \end{cases}$$

where a_1, a_2 are in K. For any $a \in K$, define a function g_a on V by

$$g_a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0 \\ e^{-ax_1} & \text{if } x_1 \leq 0 \text{ and } x_1 + x_2 \leq 0 \\ e^{-ax_2} & \text{if } x_2 \leq 0 \text{ and } x_1 + x_2 \geq 0. \end{cases}$$

Then g_a is continuous and piecewise exponential. Moreover, it is easy to check that

$$f_{a_1,a_2}(x_1, x_2) = e^{a_1x_1 + a_2x_2} g_{a_1+a_2}(x_1, x_2) - e^{a_1x_1} g_{a_1}(x_1, x_2) - e^{a_2x_2} g_{a_2}(x_1, x_2) + 1\ .$$

It follows that $f_{a_1,a_2} - f_{a_2,a_1} \in IE$. Replacing (x_1, x_2) by (b_1x_1, b_2x_2) for arbitrary b_1 and b_2, we deduce that $f_{a_1b_1,a_2b_2} - f_{a_2b_1,a_1b_2} \in IE$. In particular, we have for arbitrary a, b in K:

$$(**) \qquad f_{a,b} - f_{1,ab} \in IE \ .$$

It follows that any $f \in E_{\sigma^0}$ can be written as

$$f = \sum_{i=1}^{r} u_i f_{1,a_i} + g \, ,$$

where $u_i \in \mathbf{Z}[V^*]$, $a_i \in K$ and $g \in IE$. We can further assume that $u_i \in \mathbf{Z}$. Now we have as in the first step of the proof:

$$f_{1,a} + f_{1,b} - f_{1,a+b} \in IE \ .$$

It follows that $f = f_{1,c} + h$ for some $c \in K$ and $h \in IE$. In particular, $h \in S_{\geq 1}\hat{R}$.

Now assume further that f is in $S_{\geq 1}\hat{R}$. Then $f_{1,c}$ is in $S_{\geq 1}\hat{R}$, too. We have to check that $c = 0$. Otherwise, expanding $f_{1,c}$ into a power series, we obtain $\varphi_\sigma \in S_{\geq 1}\hat{R}$ and hence $\varphi_\sigma \in S_1R_1 + S_2$ by homogeneity. But $\pi(\varphi_\sigma) = 1$ with the notation of 2.2 and 2.3, while $\pi(S_1R_1 + S_2) = 0$, a contradiction.

In the general case of a d-dimensional cone σ, we can write σ as $\cap_{i=1}^{d}(x_i \geq 0)$. Then the $\mathbf{Z}[V^*]$-module E_{σ^0} is generated by functions $f_{a_1,a_2,\ldots,a_d}$ such that

$$f_{a_1,a_2\ldots,a_d}(x_1, \ldots, x_d) = \begin{cases} \prod_{i=1}^{d} (e^{a_ix_i} - 1) & \text{if } (x_1, x_2, \ldots, x_d) \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

We observe that

$$f_{a_1,a_2,\ldots,a_d}(x_1, x_2, \ldots, x_d) = f_{a_1,a_2}(x_1, x_2)\, f_{a_3,\ldots,a_d}(x_3, \ldots, x_d)$$

with the obvious notation. Now repeatedly use equation $(**)$ to obtain

$$f_{a_1,a_2,\ldots,a_d} - f_{1,1,\ldots,1,a_1\cdots a_d} \in IE$$

and conclude as before.

Corollary. *The maps $R^{\mathrm{int}} \to \hat{R}^{\mathrm{int}} \leftarrow E$ induce isomorphisms*

$$R^{\mathrm{int}}/S_{\geq 1}R^{\mathrm{int}} \to \hat{R}^{\mathrm{int}}/S_{\geq 1}\hat{R}^{\mathrm{int}} \leftarrow E/IE \ .$$

Proof. The map $R^{\mathrm{int}}/S_{\geq 1}R^{\mathrm{int}} \to \hat{R}^{\mathrm{int}}/S_{\geq 1}\hat{R}^{\mathrm{int}}$ is an isomorphism, because R^{int} is graded. On the other hand, the map $E/IE \to \hat{R}^{\mathrm{int}}/S_{\geq 1}\hat{R}^{\mathrm{int}}$ is injective by 4.3, and surjective by 4.2.

4.4. We keep the notation and conventions of 4.3.

Proposition. *(i) If $K \neq \mathbf{Q}$, then $I^n E$ is strictly contained in $E \cap S_{\geq n}\hat{R}$ for any integer $n \geq 2$.*
(ii) If $K = \mathbf{Q}$, then $I^n E_\Sigma = E_\Sigma \cap S_{\geq n}\hat{R}_\Sigma$ for any integer $n \geq 1$, and for any simplicial fan Σ.

Proof. (i) We argue by contradiction, and we first handle the case where $n \geq 3$. If $I^n E = E \cap S_{\geq n}\hat{R}$, then, considering globally exponential functions, we deduce that $I^n = \mathbf{Z}[V^*] \cap \hat{S}_{\geq n}$. Therefore, the map

$$I^{n-1}/I^n \to \hat{S}_{\geq n-1}/\hat{S}_{\geq n}$$

(induced by the inclusions $I^m \subset \hat{S}_{\geq m}$) is injective. But this map is identified with the canonical map between $(n-1)$-st symmetric powers

$$S^{n-1}_{\mathbf{Z}}(V^*) \to S^{n-1}_K(V^*)$$

by using 3.1 and the isomorphisms

$$\hat{S}_{\geq n-1}/\hat{S}_{\geq n} \simeq S_{\geq n-1}/S_{\geq n} \simeq S^{n-1}_K(V^*) \ .$$

Now choose $t \in K \setminus \mathbf{Q}$, and choose two linearly independant vectors x, y in V^*. Then $x(ty) - (tx)y$ is non-zero in $S^2_{\mathbf{Z}}(V^*)$ and therefore, $x^{n-2}(ty) - x^{n-3}(tx)y$ is non-zero in $S^{n-1}_{\mathbf{Z}}(V^*)$. But the image of this element in $S^{n-1}_K(V^*)$ vanishes, a contradiction.

Now we consider the case where $n = 2$. Choose $t \in K \setminus \mathbf{Q}$ as before, and set $u = 1/t$. Choose coordinates $x = x_1, x_2, \ldots, x_d$ on V. We claim that the function f such that

$$f(x, x_2, \ldots, x_d) = \begin{cases} (e^{tx} - 1)(e^{ux} - 1) - (e^x - 1)^2 & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is in $E \cap S_{\geq 2}\hat{R}$ but not in $I^2 E$.

Clearly, f is in E and moreover $x^{-2} f(x, x_2, \ldots, x_d)$ vanishes along $x = 0$, whence $f \in E \cap x^2\hat{R}$. Assume that $f \in I^2 E$; then we can find a complete fan Σ such that $f \in I^2 E_\Sigma$ and that the hyperplane $(x = 0)$ is a common wall to at least two cones σ^+, σ^- in $\Sigma(d)$. Writing explicitly that $f \in I^2 E_\Sigma$ and making $x_2 = \cdots x_d = 0$, we obtain the existence of two families of functions $f^+_{a,b}$ and $f^-_{a,b}$ in $\mathbf{Z}[Kx]$ (indexed by a, b in K) such that $f^+_{a,b}(0) = f^-_{a,b}(0)$ and

$$(e^{tx} - 1)(e^{ux} - 1) - (e^x - 1)^2 = \sum_{a,b} (e^{ax} - 1)(e^{bx} - 1) f^+_{a,b},$$

$$0 = \sum_{a,b} (e^{ax} - 1)(e^{bx} - 1) f^-_{a,b} \ .$$

Subtracting, we have

$$(e^{tx}-1)(e^{ux}-1)-(e^{x}-1)^2=\sum_{a,b}(e^{ax}-1)(e^{bx}-1)f_{a,b}$$

with $f_{a,b}$ in $\mathbf{Z}[Kx]$ such that $f_{a,b}(0)=0$. Then each $f_{a,b}$ is in I_x (the augmentation ideal of $\mathbf{Z}[Kx]$) and hence

$$(e^{tx}-1)(e^{ux}-1)-(e^{x}-1)^2\in I_x^3\ .$$

But this contradicts Proposition 3.1 applied to Kx, because $tu-1$ is non-zero in $S^2_{\mathbf{Z}}(K)$.

(ii) Using 3.2, we are reduced to checking that $E_{\sigma^0}\cap S_{\geq n}\hat{R}_\Sigma$ is contained in I^nE_Σ, where σ is a maximal cone in Σ. We may assume that σ is d-dimensional; we use the notation of the proof of 4.3. Let f be in $E_{\sigma^0}\cap S_{\geq n}\hat{R}_\Sigma$. Write

$$f=\sum_{a_1,\ldots,a_d} f_{a_1,\ldots,a_d}g_{a_1,\ldots,a_d}$$

where the $g_{a_1,\ldots,a_d}$ are in $\mathbf{Z}[V^*]$. Let q be a denominator common to all the rational numbers $a_1,\ldots,a_d$ such that $g_{a_1,\ldots,a_d}\neq 0$. Replacing the coordinates $x_1,x_2,\ldots,x_d$ by $q^{-1}x_1,\ldots,q^{-1}x_d$, we may assume that $a_1,\ldots,a_d$ are integers. Then each $f_{a_1,\ldots,a_d}$ is the product of $f_{1,1,\ldots,1}$ by some element of $\mathbf{Z}[V^*]$, and hence we can write $f=f_{1,\ldots,1}g$ for some $g\in\mathbf{Z}[V^*]$. Because $f\in\hat{R}_{\sigma^0}\cap S_{\geq n}\hat{R}$, we have $g\in\hat{S}_{\geq n}$ (this follows from 4.1 (iv) and from the fact that $f_{1,\ldots,1}$ is the product of φ_σ by a power series with constant term 1). So g is in $\mathbf{Z}[V^*]\cap\hat{S}_{\geq n}$, but this space coincides with I^n by reversing the argument of the proof of (i). Therefore, $g\in I^n$ as required.

5. The polytope algebra

5.1. Let $\mathcal{P}$ be the set of all convex polytopes in V^*. Let $\tilde{\Pi}$ be the abelian group generated by $\mathcal{P}$, subject to the relations

$$[P\cup Q]+[P\cap Q]-[P]-[Q]\,,$$

whenever P, Q and $P\cup Q$ are in $\mathcal{P}$. The group $\tilde{\Pi}$ is endowed with a ring structure, the multiplication being defined by $[P][Q]=[P+Q]$ (see [Mc1, p. 86]). Moreover, the group V^* of translations acts on $\tilde{\Pi}$ by ring automorphisms. In other words, $\tilde{\Pi}$ is an algebra over $\mathbf{Z}[V^*]$. Furthermore, $\tilde{\Pi}$ is equipped with a ring homomorphism $\deg:\tilde{\Pi}\to\mathbf{Z}$ which sends the class of any convex polytope to 1.

To any convex polytope P in V^*, we associate its support function $H_P:V\to K$ defined by

$$H_P(v)=\max\nolimits_{x\in P}\langle x,v\rangle\ .$$

Then H_P is a continuous, piecewise linear function on the outer normal fan of P; moreover, H_P is strictly convex with respect to this fan.

For any complete fan Σ in V, we denote by $\mathcal{P}_\Sigma$ the set of all $P\in\mathcal{P}$ whose support function H_P is linear on each cone of Σ. Then $\mathcal{P}_\Sigma$ is closed under Minkowski sum; if moreover Σ is the outer normal fan to some convex polytope P, then $\mathcal{P}_\Sigma$ consists of all Minkowski summands of a multiple of P. We denote by $\tilde{\Pi}_\Sigma$ the subgroup of $\tilde{\Pi}$ generated by the classes of polytopes in $\mathcal{P}_\Sigma$; then $\tilde{\Pi}_\Sigma$ is a $\mathbf{Z}[V^*]$-subalgebra of $\tilde{\Pi}$.

Proposition. *For any complete fan Σ, the map*

$$\gamma : \mathcal{P}_\Sigma \to E_\Sigma,\ P \mapsto e^{H_P}$$

induces an injective homomorphism of $\mathbf{Z}[V^*]$-algebras $\gamma : \tilde{\Pi}_\Sigma \to E_\Sigma$. If moreover Σ is the normal fan of a simple convex polytope, then this homomorphism is surjective.

Proof. For P and Q in $\mathcal{P}$, we have $H_{P\cup Q} = \max(H_P, H_Q)$. If moreover $P \cup Q$ is in $\mathcal{P}$, then $(P \cup Q) + (P \cap Q) = P + Q$ and hence $H_{P\cup Q} + H_{P\cap Q} = H_P + H_Q$. It follows that $H_{P\cap Q} = \min(H_P, H_Q)$, and therefore that

$$e^{H_{P\cup Q}} + e^{H_{P\cap Q}} = e^{H_P} + e^{H_Q},\ \text{i.e.}\ \gamma(P \cup Q) + \gamma(P \cap Q) = \gamma(P) + \gamma(Q)\ .$$

Therefore, γ extends uniquely to a group homomorphism $\gamma : \tilde{\Pi} \to E$. Clearly, γ maps $\tilde{\Pi}_\Sigma$ to E_Σ. Moreover, we have

$$\gamma(P + Q) = e^{H_{P+Q}} = e^{H_P + H_Q} = e^{H_P} e^{H_Q} = \gamma(P)\gamma(Q)\,,$$

i.e. γ is a ring homomorphism. Finally, we have $\gamma(x) = e^x$ for any point-polytope x, and hence $\gamma(uv) = u\gamma(v)$ for any $u \in \mathbf{Z}[V^*]$ and $v \in \tilde{\Pi}$.

We check that γ is injective. Let $u \in \tilde{\Pi}$ be such that $\gamma(u) = 0$. We can write

$$u = \sum_{i=1}^{m} a_i\,[P_i] - \sum_{j=1}^{n} b_j\,[Q_j]$$

with P_i and Q_j in $\mathcal{P}$ and with positive integral coefficients a_i, b_j. Then

$$\sum_{i=1}^{m} a_i\, e^{H_{P_i}} = \sum_{j=1}^{n} b_j\, e^{H_{Q_j}}\ .$$

This equality holds in $\mathbf{Z}[V^*]$ and hence we have for all $v \in V$:

$$(*) \qquad \sum_{i=1}^{m} a_i\, \exp\, H_{P_i}(v) = \sum_{j=1}^{n} b_j\, \exp\, H_{Q_j}(v)\,,$$

where $\exp : \mathbf{R} \to \mathbf{R}$ is the usual exponential function. Now observe that, for $x \in V^*$:

$$\inf_{v\in V}(H_P(v) - \langle x, v\rangle) = \begin{cases} 0 & \text{if } x \in P \\ -\infty & \text{otherwise} \end{cases}$$

In other words, the function

$$x \to \inf_{v\in V}\ \exp(H_P(v) - \langle x, v\rangle)$$

is the characteristic function of P; denote it by $\mathbf{1}_P$. Therefore, $(*)$ implies

$$\sum_{i=1}^{m} a_i\,\mathbf{1}_{P_i} = \sum_{j=1}^{n} b_j\,\mathbf{1}_{Q_j}$$

and this in turn implies

$$\sum_{i=1}^{m} a_i\,[P_i] = \sum_{j=1}^{n} b_j\,[Q_j]$$

in $\tilde{\Pi}$, see e.g. [Mo1, p. 11].

To check the surjectivity of $\gamma : \tilde{\Pi}_\Sigma \to E_\Sigma$ in the case where Σ is the normal fan of a simple polytope Δ, recall that the abelian group E_Σ is generated by the e^f where f is a piecewise linear function on Σ. For such a function f and for large $t \in K$, the function $f + tH_\Delta = f + H_{t\Delta}$ is strictly convex on Σ, and hence $f + H_{t\Delta} = H_P$ for some $P \in \mathcal{P}_\Delta$. Now $e^f = e^{H_P}\,e^{-H_{t\Delta}}$ is in the image of γ; indeed, $[t\Delta]$ is invertible in $\tilde{\Pi}_\Sigma$ (see, e.g., [Mo1, §5]) and moreover $\gamma([t\Delta]^{-1}) = e^{-H_{t\Delta}}$.

It follows that γ induces an isomorphism of $\mathbf{Z}[V^*]$-algebras, from $\tilde{\Pi}$ onto E. Through this isomorphism, the degree on $\tilde{\Pi}$ is identified with the evaluation at the origin on E. So Proposition 3.2 implies the following statement, which is the main result of [Kh-Pu].

Corollary. *Let J be the kernel of* $\deg : \tilde{\Pi} \to \mathbf{Z}$. *Then J^{n+d} is contained in $I^n\tilde{\Pi}$ for any integer $n \geq 1$.*

5.2. Recall that I denotes the ideal of $\mathbf{Z}[V^*]$ generated by all $e^x - 1$ ($x \in V^*$). The polytope algebra Π is the quotient of $\tilde{\Pi}$ by its ideal $I\tilde{\Pi}$. More concretely, Π is generated by classes of convex polytopes $[P]$, with relations $[P \cup Q] + [P \cap Q] - [P] - [Q]$ whenever P, Q and $P \cup Q$ are convex polytopes, and $[x + P] - [P]$ whenever $x \in V^*$ and P is a convex polytope.

Theorem. *The polytope algebra is isomorphic to the quotient of the algebra of continuous, piecewise polynomial functions with integral value at 0, by its ideal generated by (globally) linear functions. The isomorphism sends the class of any polytope [P] to the image in the quotient of $\sum_{n=0}^{d} H_P^n/n!$, where H_P is the support function of P.*

Proof. By 5.1, the map $\gamma : P \mapsto e^{H_P}$ induces an isomorphism $\Pi \to E/IE$. Then we conclude the argument by Corollary 4.3 and Corollary 2.1, the latter impliying that any element in $\hat{R}^{\mathrm{int}}/S_{\geq 1}\hat{R}^{\mathrm{int}}$ has a representative in R^{int} of degree at most d.

Using Corollary 2.1 and 2.4, we derive the following statement, one of the main results in [Mc1].

Corollary. *There exists a unique abelian group decomposition*

$$\Pi = \Pi_0 \oplus \Pi_1 \oplus \cdots \oplus \Pi_d\,,$$

where $\Pi_0 \simeq \mathbf{Z}$ via deg, *and $\Pi_1, \ldots \Pi_d$ are K-vector spaces, such that $[tP] = \sum_{j=0}^{d} t^j [P]_j$ for any $[P] = \sum_{j=0}^{d} [P]_j$ with $[P]_j \in \Pi_j$, and for any $t \in K_{>0}$. Moreover, we have*

(i) $\Pi_i\Pi_j \subset \Pi_{i+j}$ *for all* i *and* j.
(ii) The K*-vector space* Π_d *is one-dimensional, and multiplication induces non-degenerate pairings* $\Pi_j \times \Pi_{d-j} \to \Pi_d$ *for* $1 \leq j \leq d-1$.

Remark. The polytope algebra is the universal group for translation-invariant valuations on convex polytopes. More generally, the universal group for valuations which are polynomial of degree at most n (with respect to translations) is the quotient $\tilde{\Pi}/I^{n+1}$. It follows from 4.2 that $\tilde{\Pi}/I^{n+1}$ maps onto $R/S_{\geq n+1}R$. But this map is not injective for $n \geq 1$, except when $K = \mathbf{Q}$; see 4.4.

In other words, our structure theorem for Π has a natural extension to its higher versions $\tilde{\Pi}/I^{n+1}$ in the only case of rational polytopes. This explains the complications of the theory of polynomial valuations, and the role played there by continuity assumptions.

5.3. In this section, we assume that K is the field of real numbers. Recall the S-linear map $\pi : R \to S$ introduced in 2.2; because π is homogeneous (of degree $-d$), it extends uniquely to $\pi : \hat{R} \to \hat{S}$ with the notation of 4.2. In particular, $\pi(\exp(H_P))$ makes sense for any convex polytope P. In fact, π is defined up to a multiplicative constant, and a normalization of π depends on the choice of a non-zero element in $\wedge^d V$. Such a choice normalizes the volume element on V^*.

Proposition. *For any convex polytope* P *in* V^**, and for any* $v \in V$*, the formal power series* $\pi(\exp(H_P))$ *represents an entire function, and we have for all* $v \in V$*:*

$$\pi(\exp(H_P))(v) = \int_P \exp\langle x, v\rangle dx \ .$$

Proof. Recall that

$$\exp(H_{P\cup Q}) + \exp(H_{P\cap Q}) = \exp(H_P) + \exp(H_Q)\,,$$

whenever P, Q and $P \cup Q$ are in $\mathcal{P}$. It follows that the map

$$\mathcal{P} \to \hat{S},\ P \mapsto \pi(\exp(H_P))$$

extends to a map $\tilde{\Pi} \to \hat{S}$. The same holds for the map

$$\mathcal{P} \to \hat{S},\ P \mapsto (v \mapsto \sum_{n=0}^{\infty} \int_P \frac{\langle x, v\rangle^n}{n!} dx) \ .$$

But the abelian group $\tilde{\Pi}$ is generated by the classes of simplices (see [Mc1, p.85]) and therefore, it is enough to check our statement when P is a simplex. Further, we may assume that the volume of P is $1/d!$.

Observe that the support function of P factors through the quotient map $u : V \to V'$ where V' is the vector space dual to the direction of the affine span of P. Using the remark at the end of 2.3, it follows that $\pi(\exp(H_P)) = 0$ whenever P is not d-dimensional. On the other hand, for a d-dimensional simplex P with vertices $x_0, x_1, \ldots, x_d$, we have by the example at the end of 2.2:

$$\pi(H_P^n)(v) = \begin{cases} h_{n-d}(\langle x_0, v\rangle, \ldots, \langle x_d, v\rangle) & \text{if } n \geq d \\ 0 & \text{otherwise,} \end{cases}$$

where h_n denotes the complete symmetric function of degree n. But it is easily checked that

$$\int_P \langle x, v\rangle^n \, dx = \frac{h_n(\langle x_0, v\rangle, \ldots, \langle x_d, v\rangle)}{(n+1)(n+2)\cdots(n+d)}$$

and this implies our formula.

Corollary. *For any d convex polytopes $P_1, P_2, \ldots, P_d$ in V^*, we have*

$$\pi(H_{P_1} H_{P_2} \cdots H_{P_d}) = d! V(P_1, P_2, \ldots, P_d),$$

where V denotes the mixed volume.

Proof. The statement makes sense, because the left-hand side is a constant, π being homogeneous of degree $-d$. To prove it, we consider the constant term in the identity of the proposition above:

$$\pi(\frac{H_P^d}{d!}) = \mathrm{vol}(P)$$

for any convex polytope P. Then we take $P = t_1 P_1 + t_2 P_2 + \cdots + t_d P_d$ where $t_1, t_2, \ldots, t_d$ are arbitrary positive numbers, and we consider the coefficient of $t_1 t_2 \cdots t_d$ in the resulting polynomial expansion of the left-hand side.

Remark. In fact, the existence of mixed volumes for convex polytopes follows from the proof above (of course, it can be checked in a more straightforward way!).

5.4. We still assume that K is the field of real numbers. Using 5.3, we obtain the following separation theorem, first proved in [Mc3].

Theorem. *For any convex polytopes $P_1, P_2, \ldots, P_r$ and for any integers a_1, a_2, ..., a_r, the following conditions are equivalent:*
(i) $a_1[P_1] + a_2[P_2] + \cdots + a_r[P_r]$ is zero in Π.
(ii) $a_1 \mathrm{vol}(P_1 + Q) + a_2 \mathrm{vol}(P_2 + Q) + \cdots + a_r \mathrm{vol}(P_r + Q) = 0$ for any convex polytope Q.

Proof. The map $(P, Q) \to \mathrm{vol}(P + Q)$ extends to a bilinear map $\rho : \Pi \times \Pi \to \mathbf{R}$. Moreover, the identity

$$\mathrm{vol}(P + Q) = \pi(\exp(H_P)\exp(H_Q))(0)$$

(a consequence of 5.3) means that ρ is identified with the bilinear form $(\overline{f}, \overline{g}) \to \pi(fg)(0)$ through the identification of Π with $R^{\mathrm{int}}/S_{\geq 1} R^{\mathrm{int}}$. Now our statement follows from 2.4.

Remark. For $P_1, P_2, \ldots, P_r$ as above, let Σ be a complete, simplicial fan such that $H_{P_1}, H_{P_2}, \ldots, H_{P_r}$ are piecewise linear on Σ, and that Σ is the normal fan to some convex polytope Δ. Then in the statement of the theorem above, it is enough to consider polytopes Q that are Minkowski summands of Δ.

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