NORMALIZ – a program for computing normalizations of affine semigroups

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1.3.3 `doc/`

This contains the LaTeX 2_ε documentation file **`normaliz.tex`** and the compiled versions **`normaliz.dvi`** and `normaliz.ps`.

1.3.4 `example/`

Here are the input and output files of 8 examples, called **`rproj2`**, **`rafa1409`**, **`squaref0`**, **`squaref1`**, **`rafa2310`**, **`rafa2416`**, **`polytop`** and **`rees`**. We thank Rafael Villarreal, who sent us some of these examples.

1.4 Compilation

Under UNIX, the GNU C++ compiler is called by the following command line:

```
g++ -O3 normaliz.cc -o normaliz
(g++ -O3 enormalz.cc -o enormalz -lg++)
```

If working under DOS/WINDOWS, the DJGPP port of the GNU C++ compiler can be called by

```
gcc -O3 normaliz.cc -o normaliz.exe
(gcc -O3 enormalz.cc -o enormalz.exe -lgpp)
```

The executables **`*.exe`** need a DPMI server. The DOS boxes of WINDOWS and OS/2 supply this service automatically. But a DPMI server must be provided if the program is to be run under pure DOS.

The executables created by the above compiler calls are contained in the directory **`bin`** of the file `normaliz.zip`.

1.5 Running the program

The program **`normaliz`** is started by the following command:

```
normaliz [-fh] <filename>
```

It expects its input in the file **`<filename>.in`** and writes its output into **`<filename>.out`**. Therefore the argument **`<filename>`** in your command line must *not* contain the suffix **`.in`**.

With the option **`-f`**, not only the standard output file **`<filename>.out`** will be written, but also the files **`<filename>.gen`**, **`<filename>.sup`** and **`<filename>.val`** which separately contain the generators, the support hyperplanes and the values of the generators with respect to the linear forms representing the support hyperplanes, respectively. (See Section 1.5.2 for details.)

If you include the option **`-h`** in your command line, the h-vector and the (coefficients of the) Hilbert polynomial will be written into **`<filename>.out`**. But note that this will only work if the semigroup is homogeneous. (See Section 1.5.2 for a definition.)

1.5.1 The input file

The input file **`<filename>.in`** is structured as follows.

The first line contains the number of generators of the semigroup S (or the number of lattice points spanning the polytope, or the number of generators of the ideal I defining the Rees algebra).

The second line contains the dimension of the ambient lattice.

The next lines contain the generators of S (or the spanning lattice points, or the monomials generating the ideal I, respectively), as shown in the examples below.

```
16
7
1 0 0 0 0 0 0
0 1 0 0 0 0 0
0 0 1 0 0 0 0
0 0 0 1 0 0 0
0 0 0 0 1 0 0
0 0 0 0 0 1 0
1 1 1 0 0 0 1
1 1 0 1 0 0 1
1 0 1 0 1 0 1
1 0 0 1 0 1 1
1 0 0 0 1 1 1
0 1 1 0 0 1 1
0 1 0 1 1 0 1
0 1 0 0 1 1 1
0 0 1 1 1 0 1
0 0 1 1 0 1 1
0
```

This means that we wish to compute the integral closure of the semigroup generated by the 16 vectors

$$[1,0,0,0,0,0,0], \quad [0,1,0,0,0,0,0], \quad \ldots, \quad [0,0,1,1,0,1,1]$$

in dimension 7. We compute it in the ambient lattice $\mathbb{Z}^7$, which is indicated by the final digit 0. Calling NORMALIZ by the command

```
normaliz rproj2
```

produces the file `rproj2.out` which has the following content:

```
17 generators of integral closure:
  1  0  0  0  0  0  0
  0  1  0  0  0  0  0
  0  0  1  0  0  0  0
  0  0  0  1  0  0  0
  0  0  0  0  1  0  0
  0  0  0  0  0  1  0
  1  1  1  0  0  0  1
  1  1  0  1  0  0  1
  1  0  1  0  1  0  1
  1  0  0  1  0  1  1
  1  0  0  0  1  1  1
  0  1  1  0  0  1  1
  0  1  0  1  1  0  1
  0  1  0  0  1  1  1
  0  0  1  1  1  0  1
  0  0  1  1  0  1  1
  1  1  1  1  1  1  2

(original) semigroup has rank 7 (maximal)
(original) semigroup is of index 1

24 support hyperplanes:
   1   1   1   1   1   0  -2
   1   1   0   1   1   1  -2
   1   1   1   0   1   1  -2
   0   1   1   0   0   1  -1
   0   0   0   0   0   1   0
   1   0   0   1   0   1  -1
   1   0   1   1   1   1  -2
   1   1   1   0   0   0  -1
   1   1   0   1   0   0  -1
   1   0   1   0   1   0  -1
   0   0   0   0   1   0   0
   0   1   0   1   1   0  -1
   1   0   0   0   0   0   0
   1   1   1   1   1   1  -3
   1   0   0   0   1   1  -1
   0   1   0   0   1   1  -1
   0   1   0   0   0   0   0
   1   1   1   1   0   1  -2
   0   0   1   1   1   0  -1
   0   0   0   1   0   0   0
   0   0   1   1   0   1  -1
   0   0   1   0   0   0   0
   0   1   1   1   1   1  -2
   0   0   0   0   0   0   1

(original) semigroup is homogeneous

multiplicity = 72
```

From this, we see that there are 17 generators of the integral closure of the semigroup in $\mathbb{Z}^7$, that the semigroup has index 1 in $\mathbb{Z}^7$, and that the corresponding support hyperplanes are given by the

The desired lattice points are the 18 ones listed above. To complete the picture, we also provide all the generators of the Ehrhart ring of the polytope. (There are 19 of them in this example.) Furthermore, the original polytope is the solution of the system of the 4 inequalities

$$x_3 \geq 0\,, \quad x_2 \geq 0\,, \quad x_1 \geq 0 \quad \text{and} \quad 15x_1 + 10x_2 + 6x_3 \leq 30\,,$$

and has normalized volume 30.

Again, calling NORMALIZ by **normaliz -h polytop** writes additional output into **polytop.out**, namely

```
h-vector =  1  14  15  0

Ehrhart poly :  1  4  8  5
```

This provides the information that the h-vector of the Ehrhart ring is

$$(h_0, h_1, h_2, h_3) = (1, 14, 15, 0)\,,$$

and its Ehrhart polynomial is

$$P(t) = 1 + 4t + 8t^2 + 5t^3\,.$$

To complete the picture, let us discuss the example in **rees.in**:

```
10
6
1 1 1 0 0 0
1 1 0 1 0 0
1 0 1 0 1 0
1 0 0 1 0 1
1 0 0 0 1 1
0 1 1 0 0 1
0 1 0 1 1 0
0 1 0 0 1 1
0 0 1 1 1 0
0 0 1 1 0 1
3
```

Comparing with the data in **rproj2.in** shows that **rees** is the origin of **rproj2**. (For details see the comments on the reduction of item (3) on page 9.)

Here we want to compute the integral closure of the Rees algebra of the ideal generated by the monomials corresponding to the above 10 exponential vectors. (Note again the last digit, 3 in this case.) The output in **rees.out** coincides with that in **rproj2.out**, up to notions and the supplementary information on the integral closure of the ideal:

```
10 generators of integral closure of the ideal:
  1  1  1  0  0  0
  1  1  0  1  0  0
  1  0  1  0  1  0
  1  0  0  1  0  1
  1  0  0  0  1  1
  0  1  1  0  0  1
  0  1  0  1  1  0
  0  1  0  0  1  1
  0  0  1  1  1  0
  0  0  1  1  0  1
```

Remark 1: (i) *$\bar{S}_L = C \cap L$ for $L = \mathbb{Z}^n, \mathbb{Z}E$.*
(ii) *$\bar{S}_L$ is a finitely generated semigroup.*

General assumption: In the following, we always assume that C is a strictly convex cone, i.e. C does not contain any nontrivial linear subspace, i.e. for all $x \in C$ we have:

$$-x \in C \quad \Longrightarrow \quad x = 0\,.$$

Under this assumption, we call an element $v \in \bar{S}_L$ **irreducible** if a decomposition

$$v = v_1 + v_2 \quad \text{with } v_i \in \bar{S}_L$$

implies $v_1 = 0$ or $v_2 = 0$.

As you may recall, NORMALIZ is able to compute (compare with the four "modes")

(0) the integral closure of an affine semigroup in $\mathbb{Z}^n$;
(1) the normalization of an affine semigroup;
(2) the lattice points of an integral polytope and its Ehrhart ring;
(3) the integral closure of a monomial ideal $I \subseteq K[X_1, \dots, X_n]$ and the integral closure of its Rees algebra;
(H) the Hilbert series and Hilbert polynomial.

Obviously, (1) can be reduced to (0) by performing a suitable change of coordinates. As for (2), let $v_1, \dots, v_m \in \mathbb{Z}^n$ be the vertices of the polytope. The lattice points in the polytope are exactly the vectors $v \in \mathbb{Z}^n$ such that $[v, 1] \in \bar{S}_{\mathbb{Z}^{n+1}}$, where S is generated by $[v_1, 1], \dots, [v_m, 1] \in \mathbb{Z}^{n+1}$. This is how to reduce (2) to (0).

In order to solve (3), one starts as in (2). In fact this realizes the multiplication of the generators of I by an additional indeterminate. Furthermore, one adds the generators

$$[1, 0, \dots, 0, 0]\,, \dots,\ [0, \dots, 0, 1, 0] \in \mathbb{Z}^{n+1}$$

representing the indeterminates $X_1, \dots, X_n$, and again arrives at (0).

Therefore only the following two problems have to be solved.

(G) Find an irreducible system of generators of $\bar{S}_L$.
(H) Calculate the Hilbert series (and the Hilbert polynomial) of $\bar{S}_L$.

In order to solve problem (G), one proceeds according to the following steps:

(G1) Perform a change of coordinates (if necessary), such that $L = \mathbb{Z}^n$ and $\dim(C) = n$.
(G2) Decompose

$$C = \Sigma_1 \cup \dots \cup \Sigma_t \tag{1}$$

into **simplicial cones** $\Sigma_j \subseteq \mathbb{R}^n$ (i.e. Σ_j is spanned by exactly n linearly independent vectors). This process is called **triangulation**.
(G3) Solve problem (G) for each Σ_j.
(G4) Collect and reduce the generators found in step (G3).

Steps (G2)–(G4) will be discussed in the following three sections, and problem (H) is dealt with in Section 2.5. (You can immediately proceed to Section 2.5 if you are especially interested in problem (H).)

2.2 Constructing the triangulation

The triangulation is constructed inductively. The inductive step can be carried out as follows.

Assume that $v_1, \dots, v_s \in \mathbb{Z}^n$ span a cone C_0 with decomposition

$$C_0 = \Sigma_1 \cup \dots \cup \Sigma_r\,,$$

2.4 Collecting generators

For a decomposition (1) of the original cone C, step (G3) yields a set U_j of generators of $\Sigma_j \cap \mathbb{Z}^n$ for every Σ_j. If we put

$$U = \bigcup_{j=1}^{t} U_j ,$$

then the elements of U obviously generate $C \cap \mathbb{Z}^n$. Now it remains to construct a subset $V \subseteq U$ whose elements are irreducible *and* still generate $C \cap \mathbb{Z}^n$.

Let $U = \{u_1, \ldots, u_N\}$. Set V is constructed inductively. In the 0-th step, we put $V = \emptyset$. In the k-th step, we check if $u_k - u \in C$ for some $u \in V$. If so, we forget u_k and increase k by 1. If not, we remove all those u from V which satisfy $u - u_k \in C$, and add u_k to V before increasing k by 1.

To test whether a vector $v \in \mathbb{Z}^n$ lies in C, one evaluates the support hyperplanes in v.

2.5 Computing the Hilbert series

Let us recall and extend the notation from Section 2.1, where we start with a finite set $E = \{w_1, \ldots, w_m\} \subseteq \mathbb{Z}^n$. The integral closure of the affine semigroup $S = S(E) \subseteq \mathbb{Z}^n$ is denoted by $\bar{S} \subseteq \mathbb{Z}^n$. We may assume that the cone $C = \text{cone}(E)$ satisfies $\dim(C) = n$.

Next we define the corresponding semigroup rings

$$R := K[X^v \mid v \in S] \quad \text{and} \quad \bar{R} := K[X^v \mid v \in \bar{S}] .$$

(Here, of course, K is a field, and X is the n-tuple $(X_1, \ldots, X_n)$ of indeterminates.) By Remark 1 (ii), $\bar{R}$ is a finite R-module.

Of special interest is the **homogeneous** situation, i.e. there is $\varphi \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that $\varphi(w_i) = 1$ for all i. Then there is a natural grading of $\bar{R}$, given by

$$\deg X^v := \varphi(v) ,$$

and so R is generated in degree 1.

Throughout this section, we will assume that R is homogeneous. Then, according to the grading, write

$$\bar{R} = \bigoplus_{k=0}^{\infty} \bar{R}_k .$$

(Once more we refer the reader to [BH].) As is generally known, the **Hilbert function** of $\bar{R}$ is defined by

$$H(\bar{R}, k) = \dim_K(\bar{R}_k) \quad \text{for } k \geq 0$$

and coincides with the **Hilbert polynomial** $P_{\bar{R}}$ for large values of k:

$$H(\bar{R}, k) = P_{\bar{R}}(k) \quad \text{for } k \gg 0 .$$

The **Hilbert series** of $\bar{R}$ is

$$H_{\bar{R}}(t) = \sum_{k=0}^{\infty} H(\bar{R}, k) t^k .$$

and can be written as

$$H_{\bar{R}}(t) = \frac{h_0 + h_1 t + \cdots + h_{n-1} t^{n-1}}{(1-t)^n} ,$$

where $(h_0, h_1, \ldots, h_{n-1})$ is the ***h*-vector** of $\bar{R}$. In particular,

$$H(\bar{R}, k) = P_{\bar{R}}(k) \quad \text{for all } k \geq 0 .$$

Before discussing the general case, one should investigate the simplicial case.

For $T = \emptyset$, we set $H_\emptyset(t) = H(\emptyset, 0) = 1$.

Next we claim that this definition makes sense, i.e. $H(T,k) < \infty$ for all $k \geq 0$ and $T \subseteq C$. The proof is quite simple: It suffices to consider $H(C,k)$. Now if $w \in C$ satisfies $\varphi(w) = k$, then there is a representation

$$w = \sum_{v \in E} \alpha_v \cdot v$$

with $\alpha_v \geq 0$ and $k = \varphi(w) = \sum_{v \in E} \alpha_v$. Hence every α_v is bounded, and so is every coordinate of w. Altogether we have shown that there is only a finite number of possibilities for the choice of a vector $w \in C$ satisfying $\varphi(w) = k$ and $w \in \mathbb{Z}^n$.

The following remark shows that the Hilbert series H_T of a subset $T \subseteq C$ generalizes the Hilbert series $H_{\bar{R}}$ of $\bar{R}$.

Remark 4: *We have* $H(C,k) = H(\bar{R},k)$ *for all* $k \geq 0$.

Proof: Simply write

$$\bar{R}_k = \bigoplus_{\substack{v \in \bar{S} \\ \varphi(v) = k}} K \cdot X^v \, .$$

Therefore

$$H(\bar{R},k) = \#\{v \in \bar{S} \mid \varphi(v) = k\} = H(C,k)$$

by Remark 1 (i). □

In particular, the Hilbert series of any convex (in particular: simplicial) subcone $C' \subseteq C$ coincides, of course, with that of the corresponding semigroup ring $K[X^v \mid v \in C' \cap \mathbb{Z}^n]$.

Now there is an especially interesting connection between the Hilbert series of subsets of C. We only need the simplicial version.

Lemma 5: *Let* $\Sigma \subseteq C$ *be the simplicial cone spanned by the vectors* $v_1, \ldots, v_n \in \mathbb{Z}^n$. *Then*

$$H_\Sigma = \sum_{\sigma \subseteq \{v_1, \ldots, v_n\}} H_{\mathrm{int}(\mathrm{cone}(\sigma))} \, ,$$

where $\mathrm{cone}(\sigma) = \mathbb{R}_{\geq 0} \cdot \sigma$ *is the (possibly lower-dimensional) cone spanned by the vectors from* σ, *and* $\mathrm{int}(\mathrm{cone}(\sigma))$ *is its interior (with respect to the standard topology of* $\mathbb{R}^n$*).*

Proof: If $v \in \Sigma \cap \mathbb{Z}^n$ has a representation

$$v = \sum_{i \in I} \alpha_i v_i$$

with $\alpha_i > 0$ for all $i \in I$, then

$$v \in \mathrm{int}(\mathrm{cone}\{v_i \mid i \in I\}) \, ,$$

and vice versa. The zero vector is also counted correctly due to the convention $H_\emptyset = 1$. □

Finally, we are now able to discuss the general case. For this, one uses the decomposition

$$C = \Sigma_1 \cup \cdots \cup \Sigma_t \tag{1}$$

of C into simplicial subcones Σ_j found in step (G2). One then proceeds by induction. The case $t = 1$ is clear from Section 2.5.1 and Lemma 5. The inductive step can be derived from Lemma 5 (and its

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This program computes

(1) the normalization (or integral closure) of an affine semigroup or, in other terms, the Hilbert basis of a rational cone;
(2) the support hyperplanes of the cone;
(3) the lattice points and
(4) the support hyperplanes of an integral polytope;
(5) the generators of the integral closure of the Rees algebra of a monomial ideal I;
(6) the generators of the integral closure of I;
(7) the Hilbert series and Hilbert polynomial of the semigroup in the homogeneous case.

Download the file "normaliz.zip" to a directory of your choice and unzip it by InfoZip's unzip (or PKUNZIP or WinZip (take care of the option "create subdirectories")). The names of the subdirectories created are self-explanatory.

Detailed information can be found in the LaTeX2e documentation file "normaliz.tex" and its compiled versions "normaliz.dvi" and "normaliz.ps", which are all contained in the directory "doc/" of "normaliz.zip".