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Degree Bounds in Monomial Subrings

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1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K and let $F = \{f_1, \ldots, f_q\}$ be a finite set of monomials. The *monomial subring* generated by F is the K-subalgebra $K[F] \subset R$. In general it is difficult to estimate several of the fundamental invariants of $K[F]$—degrees of the generators of its integral closure $\overline{K[F]}$, fine details of its Hilbert function (e.g. multiplicity and a–invariant)—or to carry out basic algebraic manipulations such as those required in Noether normalization.

In this work we study monomial subrings of R generated by monomials (including Rees algebras). One of the goals is to find bounds for the generators of the normalizations of those subrings benefitting from the fact that $\overline{K[F]}$ is a rational singularity.

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Our more specific aim here is to consider some of these issues when F is a set of squarefree monomials of the same degree k. In this case there are embeddings

$$K[F] \subset S \subset R^{(k)},$$

where

$$S = K[\{x_{i_1} \cdots x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}],$$

and $R^{(k)}$ is the kth Veronese subring of R. All the constructions of $K[F]$ take place inside of one of these algebras. This will be very useful despite the fact that these embeddings are not rational.

Especially we want to present an explicit generating set for the canonical module ω_S of S. In particular we fully describe the algebras S with the Gorenstein property and compute the a-invariant of S. In this setting, our results complement and refine those of De Negri and Hibi [6] for the class of algebras considered here. Our analysis will determine the Cohen–Macaulay type of S but also leads to the control of degrees in $\overline{K[F]}$.

2 A description of the canonical module

Let us fix some of the notation that will be used throughout this work; for unexplained terminology and notation see [1] and [4, Chapter 6]. Our main references for polyhedral geometry are [1, 14].

Let $n \geq 2k \geq 4$ be two integers (this is not an essential restriction; see Remark 2.13). We set

$$\mathcal{A} = \{e_{i_1} + \cdots + e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\},$$

where $e_1, \ldots, e_n$ are the canonical vectors in $\mathbb{R}^n$. The affine subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A}$ will be denoted by C, that is, we have $C = \sum_{\alpha \in \mathcal{A}} \mathbb{N}\alpha$. The *cone generated* by C will be denoted by $\mathbb{R}_+C$, it is defined as

$$\mathbb{R}_+C = \{\sum_{i=1}^{r} a_i\alpha_i \mid a_i \in \mathbb{R}_+,\ \alpha_i \in \mathcal{A},\ r \in \mathbb{N}\}.$$

If $a \in \mathbb{R}^n$, $a \neq 0$, then the set H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with *normal vector* a. Thus $H_a = \{x \in \mathbb{R}^n \mid \langle x, a \rangle = 0\}$, and H_a determines two *closed half-spaces*

$$H_a^+ = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0\}.$$

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, the K-subring of R spanned by the set $\{x^a \mid a \in \mathcal{A}\}$ is equal to the *affine semigroup ring* $K[C]$.

Remark 2.1 Let φ be the matrix whose columns are the vectors of $\mathcal{A}$. By [9, Proposition 7.4], we obtain that $\mathrm{rank}(\varphi) = \dim K[C] = n$; in particular the vector subspace generated by $\mathcal{A}$ is equal to $\mathbb{R}^n$ and $\dim \mathbb{R}_+ C = n$.

The equations of the cone

Lemma 2.2 *Set* $N_1 = \{-e_1, \ldots, -e_n\}$, $N = N_1 \cup N_2$ *and*

$$N_2 = \{-e_1 - \cdots - e_{i-1} + (k-1)e_i - e_{i+1} - \cdots - e_n |\, 1 \leq i \leq n\}.$$

If H *is a supporting hyperplane of the cone* $\mathbb{R}_+ C$ *containing a set* $\alpha_1, \ldots, \alpha_{n-1}$ *of linearly independent vectors in* $\mathcal{A}$, *then* $H = H_a$ *for some* $a \in N$.

Proof. Let $a = (a_1, \ldots, a_n)$, $a \neq 0$, so that $H = H_a$. Let M be the $(n-1) \times n$ matrix whose rows are the vectors $\alpha_1, \ldots, \alpha_{n-1}$. First note that if the j-column of M is equal to zero, then $\langle \alpha_i, e_j \rangle = 0$ for all i and $H = H_{e_j}$; on the other hand, if all the entries of the j-column of M are equal to 1, then $H = H_a$ for some $a \in N_2$. Therefore we may assume that all the columns of M have some zero entries and also some entries equal to 1. Set $A = \{i \,|\, a_i > 0\}$, $B = \{i \,|\, a_i < 0\}$ and $\ell = |A|$; without loss of generality we may assume $A = \{a_1, \ldots, a_\ell\}$. By symmetry the proof can be reduced to the following four cases.

(a) Assume $A = \emptyset$. Using that all the columns of M have at least one entry equal to 1 we obtain that $a = 0$; hence this case cannot occur.

(b) Assume $2 \leq \ell \leq k-1$. We claim that there exist α_i so that in the set of the first ℓ entries of α_i there is at least one entry equal to zero and at least one entry equal to 1. Otherwise, using that $\mathrm{rank}(M) = n-1$ and row-reducing M to its normal form we obtain $\ell = 2$, and hence $a = a_1 e_1 - a_1 e_2$. Setting $\beta = e_1 + e_3 + \cdots + e_{k+1}$ and $\gamma = e_2 + e_3 + \cdots + e_{k+1}$ we get $\langle \beta, a \rangle = a_1 > 0$ and $\langle \gamma, a \rangle = -a_1 < 0$, which is impossible because H_a is a supporting hyperplane of $\mathbb{R}_+ C$ and the proof of the claim is complete. For simplicity we assume

$$\begin{array}{l} \alpha_1 = e_{i_1} + e_{i_2} + \cdots + e_{i_m} + e_{j_1} + \cdots + e_{j_r}, \text{ where } e_{i_1} = e_1,\ m + r = k, \\ 3 \leq i_2 < \cdots < i_m \leq \ell < j_1 < \cdots < j_r \leq n \text{ and } r, m \geq 1. \end{array}$$

Let us show $\langle \beta, a \rangle > 0$, where $\beta = e_1 + \cdots + e_\ell + e_{j_1} + \cdots + e_{j_{k-\ell}}$. Note that

$$\begin{aligned} \langle \beta, a \rangle &> a_1 + a_3 + \cdots + a_\ell + a_{j_1} + \cdots + a_{j_{k-\ell}} \\ &\geq a_{i_1} + a_{i_2} + \cdots + a_{i_m} + a_{j_1} + \cdots + a_{j_{k-\ell}} \geq 0, \end{aligned}$$

because $\langle \alpha_1, a \rangle = 0$. Hence $\beta \in \mathcal{A}$ and $\langle \beta, a \rangle > 0$, as required. If $2 \leq |B| \leq k-1$, we may apply similar arguments to prove that there is a $\gamma \in \mathcal{A}$ so that $\langle \gamma, a \rangle < 0$, which contradicts that H is a supporting hyperplane. Therefore we may further assume that $|B|$ is either equal to 1 or $|B| \geq k$; in this situation we can rapidly find $\gamma \in \mathcal{A}$ so that $\langle \gamma, a \rangle < 0$, which again yields a contradiction.

(c) Assume $|A| = 1$ and $|B| \geq k$. Set $r = |B|$ and say $B = \{a_2, \ldots, a_{r+1}\}$. If $r = n-1$, then using $\langle \alpha_i, a \rangle = 0$ for all i we derive that all the entries of the first column of M are equal to 1, hence $H = H_a$ for some $a \in N_2$. Next we assume $r \leq n-2$. Let α_i be a vector with its first entry equal to zero. Since $\langle \alpha_i, a \rangle = 0$ it follows that the first $r+1$ entries of α_i are equal to zero and $n > r+k$. Setting $\beta = e_1 + e_{r+2} + \cdots + e_{r+k}$ and $\gamma = \beta - e_1 + e_2$ we obtain $\langle \beta, a \rangle > 0$ and $\langle \gamma, a \rangle < 0$, which is impossible. The case $|A| = 1$ and $|B| = 1$ can be treated similarly.

(d) Assume $|A| \geq k$ and $|B| \geq k$. This case cannot occur because one can rapidly find vectors β, γ in $\mathcal{A}$ so that $\langle \beta, a \rangle > 0$ and $\langle \gamma, a \rangle < 0$. $\square$

Remark 2.3 The converse of Lemma 2.2 is also true because we are assuming $n \geq 2k \geq 4$, and this implies $n \geq k+2$. Note that if $n = 3$ and $k = 2$, then the cone $\mathbb{R}_+ C$ has only three facets.

Proposition 2.4 *A point $x \in \mathbb{R}^n$ is in $\mathbb{R}_+ C$ if and only if $x = (x_1, \ldots, x_n)$ is a feasible solution of the system of linear inequalities*

$$\begin{aligned} -x_i &\leq 0, \quad i = 1, \ldots, n \\ (k-1)x_i - \textstyle\sum_{j \neq i} x_j &\leq 0, \quad i = 1, \ldots, n. \end{aligned}$$

Proof. Let $\mathbb{R}_+ C = H_{b_1}^- \cap \cdots \cap H_{b_m}^-$ be the irreducible representation of $\mathbb{R}_+ C$ as an intersection of closed half spaces. By [1, Theorem 8.2] the set $H_{b_i} \cap \mathbb{R}_+ C$ is a facet of $\mathbb{R}_+ C$; note that this readily implies that H_{b_i} is generated by a set of vectors in $\mathcal{A}$. Therefore by Lemma 2.2 one obtains $H_{b_i} = H_a$, for some $a \in N$, here N denotes the set defined in Lemma 2.2. $\square$

A generating set for the canonical module

Let X be an arbitrary subset of $\mathbb{R}^n$. The *relative interior* of X, denoted relint X, is the interior of X relative to aff X, the affine hull of X.

Lemma 2.5 *Let a be a vector in $C \cap \mathrm{relint}(\mathbb{R}_+ C)$ and set $A = \{i \,|\, a_i \geq 2\}$. If $|A| \geq k$ and $i_1, \ldots, i_k$ are distinct integers in A, then $a' = a - e_{i_1} - \cdots - e_{i_k}$ also belongs to $C \cap \mathrm{relint}(\mathbb{R}_+ C)$.*

Proof. Without loss of generality one may assume $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_k \geq 2$ and $a' = a - e_1 - \cdots - e_k$. We claim that $a' \in \mathrm{relint}\, \mathbb{R}_+ C$. First observe that

$\mathbb{R}_+C$ has dimension n; thus one has

$$\text{relint } \mathbb{R}_+C = \text{int } \mathbb{R}_+C = \text{int} \bigcap_{c\in N} H_c^- = \bigcap_{c\in N} \text{int } H_c^- = \bigcap_{c\in N} H_c^- \setminus H_c \tag{1}$$

by Proposition 2.4. Hence, using that $a \in \text{relint}\, \mathbb{R}_+C$ we readily obtain

$$(k-1)(a_i - 1) \leq -1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} (a_j - 1) + \sum_{k+1 \leq j} a_j, \quad \text{for } \; 1 \leq i \leq k. \tag{2}$$

On the other hand, if $k < i \leq n$ we have

$$\sum_{j=1}^{n} a_j \;\; \geq \;\; a_1 + (k-1)a_i + \sum_{k+1 \leq j} a_j = a_1 + ka_i + \sum_{\substack{k+1 \leq j \\ j \neq i}} a_j \geq ka_i + n - k + 1.$$

As $n \geq 2k \geq 4$ we obtain $\sum_{j=1}^{n} a_j \geq ka_i + k + 1$, or equivalently, one has

$$(k-1)a_i \leq -1 + \sum_{j=1}^{k}(a_j - 1) + \sum_{\substack{k+1 \leq j \\ j \neq i}} a_j, \quad \text{for } \; k+1 \leq i \leq n. \tag{3}$$

Altogether by (1), (2) and (3) we get $a' \in \text{relint}\, \mathbb{R}_+C$. By [13] we obtain that $K[C]$ is a normal domain and therefore $C = \mathbb{Z}C \cap \mathbb{R}_+C$; since $a' \in \mathbb{Z}C$ we conclude $a' \in C$. $\square$

Let $R = \oplus_{i=0}^{\infty} R_i$ be the standard grading of the polynomial ring R and let $R^{(k)} = \oplus_{i=0}^{\infty} R_{ki}$ be the kth *Veronese subring* of R graded by $(R^{(k)})_i = R_{ki}$. Notice that $K[C]$ is a graded subring of $R^{(k)}$ with the *normalized grading*:

$$K[C] = \bigoplus_{i=0}^{\infty} (K[C])_i, \text{ where } (K[C])_i = K[C] \cap (R^{(k)})_i.$$

We shall always assume that $K[C]$ has the normalized grading.

Theorem 2.6 *Set $S = K[C]$. Let ω_S be the canonical module of S and let $\mathfrak{B}$ be the set of monomials $M = x_1^{a_1} \cdots x_n^{a_n}$ satisfying the following conditions:*

(a) $a_i \geq 1$ *and* $(k-1)a_i \leq -1 + \sum_{j \neq i} a_j$, *for all* i.

(b) $\sum_{i=1}^{n} a_i \equiv 0 \bmod (k)$.

(c) $|\{\, i \mid a_i \geq 2\}| \leq k - 1$.

If $n \geq 2k \geq 4$, then $\mathfrak{B}$ is a generating set for ω_S.

Proof. According to [5, 10] we have

$$\omega_S = (\{x^a \mid a \in C \cap \mathrm{relint}(\mathbb{R}_+ C)\}).$$

Taking into account the arguments of the proof of Lemma 2.5 and by a repeated use of Lemma 2.5 it is enough to prove that $\mathfrak{B} \subset \omega_S$. Let $M \in \mathfrak{B}$; without loss of generality we may assume $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$, where $a_1 \geq \cdots \geq a_{k-1} \geq 1$. The monomial $N = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}}$ can be factored as

$$N = \prod_{i=1}^{k-1} N_i, \text{ where } N_i = \begin{cases} (x_1 \cdots x_i)^{a_i - a_{i+1}}, & \text{if } 1 \leq i \leq k-2 \\ (x_1 \cdots x_{k-1})^{a_{k-1}}, & \text{if } i = k-1. \end{cases}$$

On the other hand, by the properties (a) and (b) it follows that we can write

$$\prod_{i=k}^{n} x_i = N' \prod_{i=1}^{k-1} N_i', \text{ where } \deg(N_i') = \begin{cases} (k-i)(a_i - a_{i+1}), & \text{if } 1 \leq i \leq k-2 \\ a_{k-1}, & \text{if } i = k-1, \end{cases}$$

and $\deg(N') \equiv 0 \bmod (k)$. Hence $M = N' \prod_{i=1}^{k-1}(N_i N_i')$ is in $K[C]$, which readily implies $M \in \omega_S$. □

Consider a polynomial ring over a field K

$$B = K[\{T_{i_1 \cdots i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}],$$

with one variable $T_{i_1 \cdots i_k}$ for each monomial $x_{i_1} \cdots x_{i_k}$. There is an homomorphism

$$\psi : B \longrightarrow K[C], \text{ induced by } T_{i_1 \cdots i_k} \stackrel{\psi}{\longrightarrow} x_{i_1} \cdots x_{i_k}.$$

The ideal $P = \mathrm{Ker}(\psi)$ is called the *presentation ideal* or *toric ideal* of $K[C]$.

Definition 2.7 Let B/P be the presentation of $K[C]$. The *Cohen-Macaulay type* of the ring $K[C]$ is the last Betti number in the minimal free resolution of B/P as a B-module; it will be denoted by $\mathrm{type}(K[C])$.

Remark 2.8 Set $S = K[C]$. We recall that the *type* of S is also equal to the minimal number of generators of the canonical module ω_S of S. To compute the type of S notice that a monomial $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$ is in $\mathfrak{B}$ if and only if for all $1 \leq i \leq k-1$ one has

$$\sum_{j=1}^{k-1} a_j = mk - n + k - 1 \text{ and } 1 \leq a_i \leq m-1, \text{ for some } m \geq 2.$$

These two conditions imply $n/k \leq m \leq n-2k+2$. Therefore, by Theorem 2.6, the computation of the type of S reduces to counting partitions of positive integers.

Corollary 2.9 *Let ω_S be the canonical module of $S = K[C]$. Assume $k = 2$ and $n \geq 2k$. If n is odd, then*

$$\omega_S = (\{x_1 \cdots x_{j-1} x_j^{2i} x_{j+1} \cdots x_n \,|\, 1 \leq j \leq n,\ 1 \leq i \leq (n-3)/2\}),$$

and $\operatorname{type}(S) = n(n-3)/2$. *If n is even, then* $\operatorname{type}(S) = (n^2 - 4n + 2)/2$.

Corollary 2.10 *If $n = 2k + 1 \geq 5$, then the type of $K[C]$ is equal to $\binom{n}{k-1}$.*

Definition 2.11 Let S be a Cohen–Macaulay positively graded K-algebra over a field K, and let ω_S be the canonical module of S. Then

$$a(S) = -\min\{\, i \,|\, (\omega_S)_i \neq 0\}$$

is the *a-invariant* of S.

Corollary 2.12 *If $n \geq 2k \geq 4$, then the a-invariant of $K[C]$ is given by*

$$a(K[C]) = -\left\lceil \frac{n}{k} \right\rceil,$$

where $\lceil x \rceil$ is the least integer greater or equal than x.

Proof. Set $S = K[C]$ and $m = \lceil \frac{n}{k} \rceil$. It follows from Remark 2.8 that the degree of the generators in least degree of ω_S is at least m. To complete the proof we exhibit some generators of ω_S living in degree m. Write $n = qk + r$, $0 \leq r < k$; note $q \geq 2$. If $r \geq 1$, observe that the monomials

$$x_1^2 \cdots x_{k-r}^2 x_{k-r+1} \cdots x_{k-1} x_k \cdots x_n \text{ and } x_1 \cdots x_{n-k+r} x_{n-k+r+1}^2 \cdots x_{n-1}^2 x_n^2$$

belong to $(\omega_S)_m$. In particular, S cannot be a Gorenstein ring in this case. If $r = 0$, then the monomial $M = x_1 \cdots x_n$ satisfies $M \in (\omega_S)_m$. □

Remark 2.13 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K, and k an integer such that $1 \leq k \leq n - 1$. Consider

$$S_{n,k} = K[\{x_{i_1} \cdots x_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}],$$

the K-subring of R spanned by the $x_{i_1} \cdots x_{i_k}$'s. Observe that there is a graded isomorphism of K-algebras of degree zero:

$$\rho : S_{n,k} \longrightarrow S_{n,n-k}, \text{ induced by } \rho(x_{i_1} \cdots x_{i_k}) = x_{j_1} \cdots x_{j_{n-k}},$$

where $\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. In particular if $n \leq 2k$, then

$$a(S_{n,k}) = a(S_{n,n-k}) = -\left\lceil \frac{n}{n-k} \right\rceil.$$

Because of this duality one may always assume that $n \geq 2k$.

The next corollary was shown independently by De Negri and Hibi [6] using different methods.

Corollary 2.14 *$S_{n,k}$ is a Gorenstein ring if and only if $k \in \{1, n-1\}$ or $n = 2k$.*

Proof. By duality one may assume $n \geq 2k \geq 4$. Set $S = S_{n,k}$. If S is Gorenstein, then by the proof of Corollary 2.12 we may assume $n = qk$. If $q \geq 3$, then $x_1 \cdots x_n$ and $x_1^3 x_2^2 \cdots x_{k-1}^2 x_k \cdots x_n$ belong to $(\omega_S)_q$ and $(\omega_S)_{q+1}$ respectively, which is impossible. Therefore $q = 2$, as required.

Conversely assume $n = 2k$. Let $\mathfrak{B}$ be as in Theorem 2.6. Take a monomial M in $\mathfrak{B}$; it suffices to verify that M is equal to $x_1 \cdots x_n$. One may assume that $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$, where $a_i \geq a_{i+1} \geq 1$. By hypothesis one has

$$\sum_{j=1}^{k-1} a_j = k(m-1) - 1, \tag{4}$$

for some $m \geq 2$. On the other hand one has

$$ka_i \leq k + \sum_{j=1}^{k-1} a_j, \text{ for all } 1 \leq i \leq k-1. \tag{5}$$

Next we combine (4) and (5) to obtain $a_i \leq m-1$ for all i; therefore using (4) again we rapidly derive $k(m-1) - 1 \leq (k-1)(m-1)$, which yields $m \leq 2$. As a consequence $a_i = 1$ for all i, as required. □

3 Degrees of the generators of normalizations

Let A be an affine domain over the field K and denote by $\overline{A}$ its normalization. When A is graded, it is rarely possible to have estimates of the degrees of the generators of $\overline{A}$ unless there are very circumscribed situations. Here we discuss one of these when A is generated by monomials permitting a sharp regrading.

We begin with a general statement on degrees valid when A and $\overline{A}$ are both Cohen–Macaulay.

Proposition 3.1 *Let A be a standard graded algebra and let $M \subset N$ be finitely generated graded A-modules of the same dimension d and multiplicity e. If M and N are Cohen-Macaulay and*

$$H_M(t) = \frac{f(t)}{(1-t)^d}, \qquad H_N(t) = \frac{g(t)}{(1-t)^d}$$

are their Hilbert series, then $\deg f(t) \geq \deg g(t)$.

Proof. Consider the exact sequence

$$0 \to M \longrightarrow N \longrightarrow P \to 0$$

of graded modules. If $M \neq N$, P is a module of dimension $< d$ since M and N have the same multiplicity. Since M and N are Cohen–Macaulay, standard depth chasing implies that P is Cohen–Macaulay of dimension $d-1$.

We have the equality of Hilbert series,

$$\frac{g(t)}{(1-t)^d} = \frac{f(t)}{(1-t)^d} + \frac{h(t)}{(1-t)^{d-1}},$$

and therefore

$$g(t) - f(t) = (1-t)h(t).$$

The assertion follows since the h–vectors of these two modules are positive (see [4, Corollary 4.1.10]). □

Corollary 3.2 *If $K[F]$ is a Cohen-Macaulay monomial subring generated by a finite set F of monomials over a field K, then $a(\overline{K[F]}) \leq a(K[F])$.*

An interesting class of monomial subrings is that of Rees algebras of monomial ideals. Next we present some bounds for the generators of the normalizations of those algebras.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over an arbitrary field K, let F be a finite set of monomials in R, and F_0 the subset of those elements of F that have lowest total degree. For any monomial f we denote the exponent vector of f by $\log f$, and $\log M$ is the set of exponent vectors of the monomials contained in a subset M of R.

Then the integral closure of $I = FR$ is generated by all those monomials g such that $\log g$ belongs to the convex hull $\mathrm{conv}(\log I)$ of $\log I$ in $\mathbb{R}^n$. Especially, if I is integrally closed, then the following condition is satisfied:

$$(\mathcal{P}) \qquad \mathrm{conv}(\log F_0) \cap \mathbb{Z}^n = \log F_0.$$

Condition $(\mathcal{P})$ says that $\log F_0$ is the set of all lattice points in the convex polytope spanned by itself. We may interpret it as asserting that the integral closedness of I is not violated in the lowest possible degree. If F_0 consists of squarefree monomials, then $\mathcal{P}$ is certainly satisfied.

For use below we denote by F_1 the set of monomials f such that $\log f \in \mathrm{conv}(\log F_0)$. Then $\mathcal{P}$ is equivalent to $F_0 = F_1$.

Theorem 3.3 *Let I and I_0 be the ideals generated by F and F_0, respectively, in $R = K[x_1, \dots, x_n]$. Suppose that I is integral over I_0. Let $\mathcal{R} = \bigoplus I^i T \subset R[T]$ and $\mathcal{R}_0 = \bigoplus I_0^i T \subset R[T]$ denote the Rees algebras of I and I_0.*

(a) *Then the normalization $\overline{\mathcal{R}}$ of $\mathcal{R}$ is generated as an $\mathcal{R}_0$-module, and thus as an $\mathcal{R}$-algebra, by elements $g \in R[T]$ of T-degree at most n.*

(b) *If I_0 is integrally closed or, more generally, if the condition $(\mathcal{P})$ holds, then n can be replaced by $n-1$ in* (a).

Proof. Since $\mathcal{R}$ and $\mathcal{R}_0$ have the same field of fractions and since $\mathcal{R}$ is integral over $\mathcal{R}_0$ by hypothesis, we can replace F by F_0 throughout. Let $\mathcal{R}_1$ be the Rees algebra of $I_1 = F_1 R$. Then $\mathcal{R}_1$ is also integral over $\mathcal{R}_0$ and has the same normalization.

The monomials $f_1, \dots, f_m \in F_1$ have constant total degree k. We introduce a new grading δ on $R[T]$ by setting $\delta(x_i) = 1$ and $\delta(T) = -(k-1)$. Then we map $R[T]$ into $R[T, U, U^{-1}]$ by the K-linear extension of the assignment $f \mapsto fU^{\delta(f)}$ for each monomial f. Under this K-algebra homomorphism $\mathcal{R}_1$ is isomorphic to the K-algebra

$$S_1 = K[x_1U, \dots, x_nU, f_1TU, \dots, f_mTU] \subset R[T, U],$$

and $\mathcal{R}_0$ is isomorphic to a subalgebra S_0 of S_1 in a natural way; furthermore $\overline{\mathcal{R}}_0$ is mapped onto a monomial subalgebra of $R[T, U]$. We grade $R[T, U]$ by setting $\deg U = 1$, $\deg x_i = 0$, $i = 1, \dots, n$, and $\deg T = 0$. Under this grading both S_1 and S_0 are generated by their elements of degree 1, and S_0 contains a Noether normalization S_{-1} of S_1 generated by degree 1 elements.

Let P be the convex polytope spanned by the unit vectors $e_1, \dots, e_n \in \mathbb{R}^{n+1}$ and the vectors $v_j = (\log f_j, 1) \in \mathbb{R}^{n+1}$. Suppose $z \in \mathbb{Z}^{n+1}$ is an integral point of P. Then, writing P as a convex linear combination of the e_i and the v_j we see that z is either one of the e_i or belongs to the convex hull P' of the set $\{v_1, \dots, v_m\}$. However, by construction of F_1, $P' \cap \mathbb{Z}^{n+1} = \{v_1, \dots, v_m\}$. It follows that S_1 is generated by all the monomials corresponding to the integral points of P. In the notation of Bruns, Gubeladze, and Trung [3], $S_1 \cong K[S_P]$ is the semigroup ring associated with the polytope P. By virtue of [3, Corollary 1.3.4], the S_1-module $\overline{\mathcal{R}}_0$ is generated by elements of degree at most $n-1$ as an S_1-module. (Note that P is in fact an n-dimensional lattice polytope embedded into $\mathbb{R}^{n+1}$; its elements lie in the hyperplane $\sum_{i=1}^n z_i + (1-k)z_{n+1} = 1$, because the f_i have all the same total degree.)

This covers part (b) because condition $\mathcal{P}$ just says that $F_1 = F_0$. Furthermore note that the T-degree of each element of $\overline{\mathcal{R}}_0$ is at most its U-degree. (This shows that the proof gives a slightly stronger result than asserted in the theorem.) For part (a) we use the argument in the proof of [3, Theorem 1.3.3], demonstrating that $\overline{\mathcal{R}}_0$ is generated as an S_{-1}-module by elements of degree at most n. □

We give a variant of the previous theorem for the subalgebra generated by monomials of the same degree k. A homogeneous polynomial of degree ik is said to have *normalized degree* i.

Theorem 3.4 *Suppose that F consists of monomials of the same degree k, and let $A = K[F] \subset R = K[x_1, \ldots, x_n]$ be the monomial subring generated by F.*

(a) *Then the normalization $\overline{A}$ of A is generated as an A-module, and thus as an A-algebra, by elements $g \in R$ of normalized degree at most* $\dim A - 1$.

(b) *If condition $(\mathcal{P})$ holds, then* $\dim A - 1$ *can be replaced by* $\dim A - 2$ *in* (a).

Proof. Both A and $\overline{A}$ are contained in the kth Veronese subalgebra $R^{(k)}$ of the polynomial ring R. We embed $R^{(k)}$ into $R[U]$ by sending a homogeneous element f of normalized degree i to fU^i. Then we are essentially in the same situation as in the previous proof, except that the dimension of the polytope is now $\dim A - 1$.
□

Proposition 3.5 *Let F be a finite set of squarefree monomials of degree k in R, and let $A = K[F]$ be the K-subring of R spanned by F. If* $\dim A = n$, *then*

$$a(\overline{A}) \leq a(K[C]),$$

where $\overline{A}$ is the normalization of A and C is the subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A} = \{e_{i_1} + \cdots + e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$.

Proof. Let C and C_F be the subsemigroups of $\mathbb{N}^n$ generated by $\mathcal{A}$ and $\log F$ respectively. Set $S = K[C]$. Since S is normal [13], we obtain $\overline{A} \subset S$. Let

$$M_1 = \{x^a \mid a \in \overline{C_F} \cap \operatorname{relint}(\mathbb{R}_+\overline{C_F})\} \text{ and } M_2 = \{x^a \mid a \in C \cap \operatorname{relint}(\mathbb{R}_+C)\},$$

where $\overline{C_F} = \mathbb{Z}C_F \cap \mathbb{R}_+C_F$. Notice $\mathbb{R}_+\overline{C_F} = \mathbb{R}_+C_F$ and $\operatorname{aff}(\mathbb{R}_+C_F) = \mathbb{R}^n$. Therefore the relative interior of $\mathbb{R}_+\overline{C_F}$ equals its interior in $\mathbb{R}^n$. For similar reasons we have $\operatorname{relint}(\mathbb{R}_+C) = (\mathbb{R}_+C)^{\circ}$. Hence $\operatorname{relint}(\mathbb{R}_+\overline{C_F}) \subset \operatorname{relint}(\mathbb{R}_+C)$. Altogether we obtain $M_1 \subset M_2$. Let x^b be an element of minimal degree in M_1 so that $\deg(x^b) = -a(\overline{A})$. Set $r = -a(\overline{A})$. Since x^b is in M_2 and $x^b \in \overline{A}_r \subset S_r$, we conclude

$$-a(S) = \min\{\deg(x^a) | a \in M_2\} \leq r = -a(\overline{A}).$$

Hence $a(\overline{A}) \leq a(S)$, as required. □

Corollary 3.6 *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and F a finite set of squarefree monomials of degree k. If* $\dim K[F] = n$, *then*

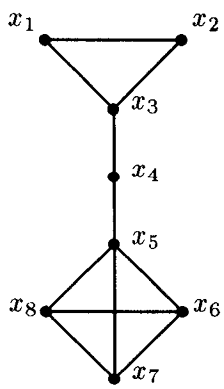
$$a(\overline{K[F]}) \leq \begin{cases} -\left\lceil \dfrac{n}{k} \right\rceil, & \text{if } n \geq 2k, \\ -\left\lceil \dfrac{n}{n-k} \right\rceil, & \text{if } n \leq 2k,\ n \neq k. \end{cases}$$

Remark 3.7 It is not hard to see that $a(R^{(k)}) = -\lceil \frac{n}{k} \rceil$, where $R^{(k)}$ is the kth Veronese subring of R. Let $K[C]$ be the subring generated by the squarefree monomials of degree k. Notice that $a(K[C]) \leq a(R^{(k)})$; for $n = 5$ and $k = 3$ we have $a(K[C]) = -3 < -2 = a(R^{(k)})$. Because of this, to keep better control of the a-invariant, it is preferable to embed $K[F]$ into $K[C]$, instead of $R^{(k)}$.

Corollary 3.8 *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and F a finite set of squarefree monomials of degree k in R. If* $\dim K[F] = n$ *and $n \geq 2k \geq 4$, then $\overline{K[F]}$ is generated as a K-algebra by elements of normalized degree less or equal than $n - \lceil \frac{n}{k} \rceil$.*

Proof. Use the argument in the proof of [3, Theorem 1.3.3], together with Corollary 3.6. □

Example 3.9 Let $K[F]$ be the subring of $R = K[x_1, \ldots, x_8]$ spanned by the monomials of R defining the edges of the graph shown below.



$$\begin{aligned} f_1 &= x_1x_2, \\ f_2 &= x_2x_3, \\ f_3 &= x_1x_3, \\ f_4 &= x_3x_4, \\ f_5 &= x_4x_5, \\ f_6 &= x_5x_6, \\ f_7 &= x_6x_7, \\ f_8 &= x_5x_7, \\ f_9 &= x_7x_8, \\ f_{10} &= x_5x_8, \\ f_{11} &= x_6x_8, \end{aligned}$$

$$\begin{aligned} f_{12} &= x_1x_2x_3x_5x_6x_8, \\ f_{13} &= x_1x_2x_3x_6x_7x_8, \\ f_{14} &= x_1x_2x_3x_5x_6x_7, \\ f_{15} &= x_1x_2x_3x_5x_7x_8. \end{aligned}$$

$$F := \{f_1, \ldots, f_{11}\},$$
$$\overline{K[F]} = K[F][f_{12}, f_{13}, f_{14}, f_{15}].$$

The generators of $\overline{K[F]}$ can be computed using [2], see also [12, Section 7.3]. A Noether normalization for $K[F]$ is given by

$$A_0 = K[h_1, \ldots, h_8] \hookrightarrow K[F] \hookrightarrow \overline{K[F]},$$

where

$$h_1 = f_1, \quad h_3 = f_8 - f_{11}, \quad h_5 = f_2 - f_3, \quad h_7 = f_5 - f_7 - f_9 - f_{11},$$
$$h_2 = f_6, \quad h_4 = f_9 - f_{10}, \quad h_6 = f_3 - f_5, \quad h_8 = f_1 - \sum_{i=2}^{11} f_i.$$

Since $\overline{K[F]}$ is Cohen–Macaulay we obtain a decomposition

$$\begin{aligned}\overline{K[F]} \;=\;& A_0 1 \oplus A_0 f_7 \oplus A_0 f_{10} \oplus a_0 f_{11} \oplus \\ & A_0 f_7^2 \oplus A_0 f_{10}^2 \oplus A_0 f_{10} f_{11} \oplus A_0 f_7 f_{11} \oplus \\ & A_0 f_7^3 \oplus A_0 f_7^2 f_{11} \oplus A_0 f_{10}^3 \oplus A_0 f_{10}^2 f_{11} \oplus \\ & A_0 f_{12} \oplus A_0 f_{13} \oplus A_0 f_{14} \oplus A_0 f_{15} \oplus A_0 f_{10}^3 f_{11}.\end{aligned}$$

Therefore the Hilbert series of $\overline{K[F]}$ is equal to

$$H(\overline{K[F]}, z) = \frac{1 + 3z + 4z^2 + 8z^3 + z^4}{(1-z)^8} \quad \text{and} \quad a(\overline{K[F]}) = -4.$$

Altogether $\overline{K[F]}$ is generated as an A_0-module by monomials of normalized degree less or equal than $\dim R + a(\overline{K[F]}) = 4$. However, as a K-algebra it is generated by squarefree monomials of normalized degree at most 3.

Question 3.10 Let F be a finite set of squarefree monomials of degree k in R. Is $\overline{K[F]}$ generated (as a K-algebra) by squarefree monomials ?

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