

# Computing Roadmaps of General Semi-Algebraic Sets

J.F. Canny<sup>1</sup>

University of California  
Berkeley, CA 94720

## Abstract

In this paper we study the problem of determining whether two points lie in the same connected component of a semi-algebraic set  $S$ . Although we are mostly concerned with sets  $S \subseteq \mathbb{R}^n$ , our algorithm can also decide if points in an arbitrary set  $S \subseteq R^n$  can be joined by a semi-algebraic path, for any real closed field  $R$ . Our algorithm computes a one-dimensional semi-algebraic subset  $\mathfrak{R}(S)$  of  $S$  (actually of an embedding of  $S$  in a space  $\hat{R}^n$  for a certain real extension field  $\hat{R}$  of the given field  $R$ ).  $\mathfrak{R}(S)$  is called the roadmap of  $S$ . Our construction uses the original roadmap algorithm described in [Can88a], [Can88b] which worked only for compact, regularly stratified sets.

We measure the complexity of the formula describing the set  $S$  by the number of polynomials  $k$ , their maximum degree  $d$ , the maximum length of their coefficients in bits  $c$ , and the number of variables  $n$ . With respect to the above measures, the complexity of our new algorithm is  $(k^n \log^2 k) d^{O(n^2)} c^2$  randomized, or  $(k^n \log k) d^{O(n^4)} c^2$  deterministic. Note that the combinatorial complexity (complexity in terms of  $k$ ) in both cases is within a polylog factor of the worst-case lower bound for the number of connected components  $\Omega(k^n)$ .

## 1 Introduction

Good sequential and parallel algorithms have been available for some time for deciding the theory of the reals [Gri88], [GV88], [Can88a], and for real quantifier elimination [BOKR86], [HRS90a], [Ren89]. Geometrically, these problems amount to deciding the emptiness or non-emptiness of semi-algebraic sets, and taking projections of these sets. Recently, attention has turned to computing geometric properties of semi-algebraic sets, such as counting the number of connected components. The first algorithm for connectivity was described by Schwartz and Sharir [SS83] who observed that a cylindrical algebraic cell decomposition provides a convenient substrate from which to extract connectivity information. This idea was pursued by Kozen and Yap [KY85], who gave a simple formula for deciding adjacency between cells, and made use of a parallel algorithm for constructing the cell decomposition [BOKR86] (this paper contains errors that were fixed in [FGM87], see also [Can91]). Their algorithm was fully parallel, and had single-exponential parallel running time. From Kozen and Yap's results it follows that all geometric properties of semialgebraic sets are at hand, because a semi-algebraic set can be represented as a regular cell complex.

---

<sup>1</sup>Supported by a David and Lucile Packard Foundation Fellowship and by NSF Presidential Young Investigator Grant IRI-8958577

However, all known methods for constructing algebraic cell decompositions require double exponential sequential time and improving this bound remains a major open problem. So the two previous algorithms run in sequential time double exponential in the dimension, with the latter taking single exponential time in parallel. Neither looks practical in the near future even in low dimensions.

A different approach was taken in [Can88a] and [Can88b], based on the construction of a one-dimensional skeleton of the set. This construction, called a roadmap, gave much lower (single exponential) complexity. The original roadmap papers [Can88a] and [Can88b], see also [Tro89], made use of regular stratification [GWDL76], rather than a cell decomposition, which allowed a very coarse (and efficient) partition of the set. Let us measure the complexity of the formula describing the set  $S$  by the number of polynomials  $k$ , their maximum degree  $d$ , the maximum length of their coefficients in bits  $c$ , and the number of variables  $n$ . The complexity of finding paths in [Can88a] was  $(k^n \log k) d^{O(n^2)} c^2$ , and since the lower bound on the number of components is  $\Omega((kd)^n)$ , this algorithm is nearly optimal in terms of  $k$ . This is important for applications in geometric modelling and robotics where  $d$  and  $n$  are small and fixed, but  $k$ , representing the number of surfaces, may be large. The disadvantage of the algorithm in [Can88a], which we will henceforth refer to as the roadmap algorithm, is that it required the semi-algebraic set to be compact and in general position.

Later Grigor'ev and Vorobjov [GV89] gave a  $(dk)^{O(n^{19})}$  algorithm for finding paths in arbitrary semi-algebraic sets, and the Franco-Argentine school in [HRS90b] and [HRS90c] gave a solution for the general case with a running time of  $(dk)^{n^{O(1)}}$ . However, the double exponents of both algorithms appears to make them impractical for robotics or geometric modelling applications. The problem of actually finding the connected components in exponential time, i.e. giving quantifier-free formulae for them, was solved in [CGV91]. See also [Gou91] for an explicit construction of roadmaps for compact but possibly singular semi-algebraic sets.

In this paper we describe a method for path-finding in the general case which has a running time of  $(k^n \log^2 k) d^{O(n^2)} c^2$  randomized, which is within  $\log k$  of the original roadmap algorithm. Making the algorithm deterministic increases the complexity to  $(k^n \log k) d^{O(n^4)} c^2$ . Note that the combinatorial complexity in terms of  $k$  in both cases is very close to the worst-case lower bound of  $\Omega(k^n)$  on the number of connected components. The new method performs a direct reduction from the general case to the compact, regularly stratified case, so that the algorithm of [Can88a] can be applied. This reduction increases the size of the input by a log factor. Each polynomial in the formula defining the original semi-algebraic set is replaced by polynomials which differ in their constant coefficients by various infinitesimals. The new collection of polynomials define a semi-algebraic set which is compact and regularly stratified, and has the same connected components as the original.

## 2 Preliminaries

The input to the algorithm is a semi-algebraic set  $S$  defined by polynomials with rational coefficients. For a set defined in  $n$ -dimensional space, the polynomials lie in the ring  $\mathbb{Q}[x] = \mathbb{Q}[x_1, \dots, x_n]$ . Formally, a semi-algebraic set is defined as:

**Definition** Let  $F_1, \dots, F_k \in \mathbb{Q}[x]$  be a collection of polynomials with rational coefficients. A *semi-algebraic set*  $S \subseteq \mathbb{R}^n$  is a set derived from the sets

$$S_i = \{x \in \mathbb{R}^n \mid F_i(x) > 0\} \quad (1)$$

by finite union, intersection and complement.

A semi-algebraic set can be defined as the set of points in  $\mathbb{R}^n$  satisfying a certain predicate of the form  $B(A_1, \dots, A_k)$  where  $B : \{0, 1\}^k \rightarrow \{0, 1\}$  is a boolean function and each  $A_i$  is an atomic formula of one of the following types:

$$(F_i = 0) \quad (F_i \neq 0) \quad (F_i > 0) \quad (F_i < 0) \quad (F_i \geq 0) \quad (F_i \leq 0) \quad (2)$$

with  $F_i \in \mathbb{Q}[x]$ . In the analysis that follows it will be helpful to assume a certain form for the defining predicate.

**Definition** A formula  $B(A_1, \dots, A_k)$  is said to be in *monotone standard form* if the boolean function  $B$  is monotone, and all atomic formulae  $A_i$  are either  $(F_i = 0)$  or  $(F_i > 0)$ .

An arbitrary formula can be converted to monotone standard form with a constant factor increase in size. Assume we are given a boolean circuit  $C$  to represent the function  $B$ . This circuit can be converted to a negation-free, and therefore monotone circuit  $C_M$  as follows. For each node  $v \in C$  there are two in  $C_M$ , one of which represents  $v$  and the other  $\neg v$ . Now all primitive logical operations between nodes in  $C_M$  can be implemented with  $\wedge$  and  $\vee$ . e.g. if  $C_7 = C_3 \vee \neg C_5$  in the circuit  $C$ , in  $C_M$ , the node representing  $C_7$  is the  $\vee$  of the node representing  $C_3$  and the node representing  $\neg C_5$ . We also need to compute  $\neg C_7$  for later use, and this node is the  $\wedge$  of the node representing  $\neg C_3$  and the node representing  $C_5$ .

The circuit  $C_M$  defines a monotone boolean function of the original atomic formulae and their negations. The negations can be pushed into the atomic formulae by replacing  $\neg(F_i > 0)$  with  $(F_i \leq 0)$  etc. This formula can be converted to standard form by substituting for the inequalities  $\leq, \neq, \geq$  with a union of a pair of inequalities using  $>$  and  $=$ . Overall the number of atomic formulae increases by a factor of at most four compared to the original predicate.

### Predicate complexity

We measure the complexity of a predicate with four quantities, the number of polynomials  $k$ , the number of variables  $n$ , the maximum degree of the polynomials  $d$ , and the maximum coefficient length  $c$  of the coefficients of the polynomials.

There remains one sticky point with regard to the boolean formula  $B$ . There is a certain cost associated with evaluating  $B$ , given the signs of the  $F_i$ 's. This time is clearly linear for the first evaluation. The algorithm of [Can88a] requires frequent re-evaluation of  $B$  when a single  $F_i$  changes sign. Our complexity bounds will be valid if the time to re-evaluate  $B$  when a single  $F_i$  changes sign is  $O(\log k)$ . In [Can90] we show that this will be true if the function  $B$  is defined by a formula. Basically, we show there that an arbitrary boolean formula can be converted to an equivalent log-depth *ternary* formula (in polynomial time). The predicate  $B$  has been assumed to be a formula in most previous work on semi-algebraic sets. Our algorithm will still work if the function  $B$  is represented by a general circuit, but if the time to re-evaluate  $B$  when a single input changes is greater than  $\log k$ , then we must substitute this larger time for  $\log k$  in the complexity bounds above.

## 2.1 Stratifications

**Definition** A stratification  $\underline{S}$  of a set  $S \subset \mathbf{R}^n$  is a partition of  $S$  into a finite number of disjoint subsets  $S_i$  called *strata* such that each  $S_i$  is a manifold.

A regular stratification satisfies some additional conditions which are well described in [GWDL76]. There are several basic ways to construct regular stratifications. We will only need two:

- **Taking products.** If  $\underline{C}$  and  $\underline{D}$  are regular stratifications of the spaces  $C$  and  $D$  respectively, then the product  $\underline{C} \times \underline{D}$  is a regular stratification of  $C \times D$ .
- **Preimage of a transversal map.** If  $F : M \rightarrow N$  is transversal to  $\underline{D}$  for a regular stratification  $\underline{D}$  of a subset  $D \subseteq N$ , then  $F^{-1}(\underline{D})$  is a regular stratification of  $F^{-1}(D)$ .

where  $F \bar{\cap} \underline{D}$  means  $F$  is transversal to all the strata of  $\underline{D}$ , and  $F^{-1}(\underline{D})$  is the set  $\{F^{-1}(\sigma) \mid \sigma \in \underline{D}\}$ . If we now define

**Definition** Let  $F_i \in \mathbb{Q}[x]$ ,  $i = 1, \dots, k$  be a collection of polynomials that define a map  $F : \mathbf{R}^n \rightarrow \mathbf{R}^k$ . A *sign sequence*  $\sigma$  is an element of  $\{\mathbf{R}^-, \{0\}, \mathbf{R}^+\}^k$ . The sets  $F^{-1}(\sigma)$  are called *sign-invariant sets* of  $F$ .

Then we can view a semi-algebraic set  $S$  as a finite union of sign-invariant sets of some polynomial map  $F$ . The sign partition  $(\underline{\mathbf{R}})^k$  of  $\mathbf{R}^k$  is a regular stratification of  $\mathbf{R}^k$ . So if a map  $F : \mathbf{R}^n \rightarrow \mathbf{R}^k$  is transversal to  $(\underline{\mathbf{R}})^k$ , then the preimage  $F^{-1}((\underline{\mathbf{R}})^k)$ , which is the collection of sign-invariant sets of  $F$ , is a regular stratification.

## 2.2 Infinitesimals

We will make extensive use of extensions of real fields by infinitesimals. This process is simple to implement computationally, and has been well formalized in [BCR87] using the real spectrum. An elementary description of the use of infinitesimal elements is given in [Can88c] in an algorithm for the existential theory of the reals.

**Definition** For a given real field  $R$ , we say that an element  $\epsilon$  is *infinitesimal* with respect to  $R$  if the extension  $R(\epsilon)$  is ordered such that  $\epsilon$  is positive, but smaller than any positive element of  $R$ .

We will have cause to make use of towers of such field extensions. We will use the suggestive notation  $\delta \gg \epsilon$  for two infinitesimals to mean that  $\epsilon$  is infinitesimal with respect to the real closure of the field  $R(\delta)$ .

Strictly speaking, in what follows we do not need true infinitesimals. Any result that we prove for an infinitesimal will hold for all sufficiently small real values. This follows because the “bad sets” or critical sets for the various calculations we perform are semi-algebraic, and there is a smallest bad real value. Even when we use towers of infinitesimals, as long as we choose a good real value for each variable, there will be a good real value for the next variable that is “small enough” with respect to it, and which avoids the bad set. Some of our proofs will be phrased as though the infinitesimals were real numbers. This

saves us having to use awkward definitions of compactness, connectivity and regularity for arbitrary real closed fields. This idea may be seen as a special case of the “Transfer Principle” [Tar48].

### 3 Reduction from an unbounded to a bounded set

This is a rather standard reduction which is used in a number of places, see for example [Can88c]. First we show that the unbounded set  $S \subset \mathbf{R}^n$  is homotopy equivalent to a bounded set. For this purpose, let  $\rho(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$  be a polynomial radius function. Consider the set  $S \cap D_r$  where  $D_r = \rho^{-1}([0, r])$ .

**Lemma 3.1** *There exists a positive  $r_0$  such that for all  $r > r_0$ ,  $S \cap D_r$  is a deformation retract of  $S$ .*

**Proof** Let  $F = (F_1, \dots, F_k)$  be a set of polynomials which define  $S$ . Let  $\underline{\mathbf{R}}^n$  be a Whitney regular stratification of  $\mathbf{R}^n$  which is compatible with the sign-invariant sets of  $F$ . All sign-invariant sets are unions of strata. Such a stratification is described in [BCR87]. By the semi-algebraic Sard theorem [BCR87], the map  $\rho$  has only finitely many critical values when restricted to any of these strata. Let  $r_0$  be the largest critical value, then for all  $r > r_0$ ,  $S \cap D_r$  has the same homotopy type (hence number of connected components) as  $S$ . To see this, use  $\rho$  to lift a vector field on  $[r, \infty)$  to one on  $\mathbf{R}^n - \text{Int}(D_r)$  which is compatible with the stratification of  $\mathbf{R}^n$ . The flow along this vector field defines a retraction of  $S$  onto  $S \cap D_r$ .  $\square$

So to find connected components of  $S$  it suffices to find components of  $S \cap D_r$  for sufficiently large  $r$ . In practice this is done by treating  $r$  as an indeterminate element of the base field. When it comes time to determine the sign of a base field element, which will be a polynomial in  $r$ , we use the sign of the highest degree term in  $r$ . This correctly gives the sign for sufficiently large  $r$ .

### 4 Reduction to a regular stratification

As was shown in [Can88a] one can obtain a regular stratification by taking the sign-invariant sets of a system of polynomials in sufficiently general position. In the present case, the given  $F_i$  will not be in general position. In [Can88a] a fixed perturbation was applied to their constant coefficients to achieve this. Now consider the following symbolic perturbation of the  $F = (F_1, \dots, F_k) : \mathbf{R}^n \rightarrow \mathbf{R}^k$ . Define

$$F_{\epsilon a} = F + \epsilon a \tag{3}$$

where  $a \in \mathbf{R}^k$  is supposed constant for the time being, and  $\epsilon$  is a single infinitesimal. As shown in [Can88a] for almost all choices of the constant coefficients  $\epsilon a$ , the map  $F_{\epsilon a}$  is transversal to the natural stratification by sign  $(\underline{\mathbf{R}})^k$  of  $\mathbf{R}^k$ , where  $\underline{\mathbf{R}} = \{\mathbf{R}^-, \{0\}, \mathbf{R}^+\}$ . In particular, if we fix  $a = (a_1, \dots, a_k)$ , then for almost all such choices, the map  $F_{\epsilon a}$  is transversal to  $(\underline{\mathbf{R}})^k$  for almost all  $\epsilon$ . This implies that the sign-invariant sets of  $F_{\epsilon a}$  form a Whitney regular stratification, a very important property for us later. We assume for

now that such  $a_i$ 's have been found. Later we will show how they can be determined either deterministically or probabilistically.

Assume the  $a_i$ 's were chosen to be positive, the sign-invariant sets of (3) are the same as for

$$\begin{aligned} F_1/a_1 + \epsilon \\ F_2/a_2 + \epsilon \\ \vdots \\ F_k/a_k + \epsilon \end{aligned} \tag{4}$$

Define  $G_i = F_i/a_i$ . Then another way to describe the sign-invariant stratification of the last paragraph is as the preimage under  $G$  of the stratification  $(\underline{\mathbf{R}})^k$ .

Now suppose instead of a single polynomial, we replace each  $F_i$  by four polynomials as follows:

$$F_i/a_i + \delta, \quad F_i/a_i + \epsilon, \quad F_i/a_i - \epsilon, \quad F_i/a_i - \delta \tag{5}$$

this gives us a system of  $4k$  polynomials  $H_j$ , and we claim that for almost all  $a$  and for almost all pairs  $(\epsilon, \delta)$ , the map  $H$  is transversal to  $(\underline{\mathbf{R}})^{4k}$ .

We can say this another way as follows. Let  $\underline{\mathbf{R}}_{\delta\epsilon}$  be the stratification of the real line into the points  $-\delta, -\epsilon, \epsilon, \delta$  and the open intervals in between and to  $\pm\infty$ . Then we have the following general position lemma:

**Lemma 4.1** *For almost all  $a \in \mathbf{R}^k$ , the map  $G = (F_1/a_1, \dots, F_k/a_k)$  is transversal to the stratification  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$  of  $\mathbf{R}^k$ , for almost all  $(\delta, \epsilon)$ .*

**Proof** We say that a value of  $a$  is *regular* if the set of  $(\delta, \epsilon)$ -values such that  $G$  is *not* transversal to  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$  has measure zero in  $\mathbf{R}^2$ . The map  $G$  is transversal to  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$  if it is transversal to all the strata of  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$ . If we can find a value of  $a$  which is regular with respect to every stratum, we are done because the set of values of  $(\delta, \epsilon)$  for which  $G$  is not transversal to  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$  is the union of the sets of values for which it is not transversal to the individual strata. This is a finite union of measure zero sets, and so has measure zero.

We show below that the set of  $a$  values which are not regular with respect to a given stratum has measure zero in  $\mathbf{R}^k$ . From this it follows that the set of  $a$  values which are not regular with respect to the stratification  $(\underline{\mathbf{R}}_{\delta\epsilon})^k$  is measure zero in  $\mathbf{R}^k$ . Again this is because it is the union of measure zero sets corresponding to non-regular values for the individual strata. This will complete the proof.

So it remains to show that the set of non-regular values  $a$  of  $G$  with respect to a given stratum  $\sigma_{\delta\epsilon}$  in  $\underline{\mathbf{R}}_{\delta\epsilon}$  has measure zero in  $\mathbf{R}^k$ .

Now  $\sigma_{\delta\epsilon}$  is a product of points and open intervals in  $\mathbf{R}$ . From the definition of transversality given earlier, it should be clear that  $G$  will be transversal to  $\sigma_{\delta\epsilon}$  if  $dG$  is surjective in the subspace  $V$  of  $\mathbf{R}^k$  comprising those coordinates where  $\sigma_{\delta\epsilon}$  is a point. Write  $G|_V$  for the map  $G$  restricted to those coordinates, and  $\sigma_{\delta\epsilon}|_V$  for the restriction of  $\sigma_{\delta\epsilon}$ . By definition,  $p = \sigma_{\delta\epsilon}|_V$  is a single point in  $V$ . The condition that  $G$  be transversal to  $\sigma_{\delta\epsilon}$  is equivalent to the condition that  $p$  be a regular value of  $G|_V$ .

Now define  $q \in V$  by  $q_i = a_i p_i$ . The condition that  $p$  is a regular value of  $G|_V$  is the same as the condition that  $q$  is a regular value of  $F|_V$ . By Sard's theorem, we know that the set of non-regular  $q$  values has measure zero in  $V$ . In fact the semi-algebraic version of

Sard's theorem [BCR87] tells us that the set of bad values is semi-algebraic. Since it is a measure zero set, it must be contained in a algebraic proper subset  $Z \subset V$ .

The  $i^{\text{th}}$  coordinate of  $q$  is either  $\pm a_i \epsilon$  or  $\pm a_i \delta$ . Choose a  $q$  that avoids  $Z$  and temporarily suppose  $\epsilon = \delta = 1$ . This fixes the corresponding values  $a_i$  to  $\pm q_i$ . Let  $L$  denote the map taking  $q$  to  $a$  when  $\epsilon = \delta = 1$ , and suppose henceforth that  $a$  is fixed at some value  $L(q)$  where  $q \notin Z$ . If  $\epsilon$  and  $\delta$  are supposed variable again, the plane in  $V$  parametrized by  $\epsilon$  and  $\delta$  intersects  $Z$  in an algebraic set. Since this set does not contain the image of the point  $(1, 1)$ , it must be a codimension one subset. Hence for almost all pairs  $(\delta, \epsilon)$ , the point  $q$  is a regular value of  $F|_V$ .

Now notice that any value of  $a$  which is not in the codimension one set  $L(Z)$  is a regular value in the sense defined above, with respect to a given stratum. Taking the union of the  $L(Z)$  for all strata, we obtain a codimension one set of  $a$  values for which  $G$  is not transversal to  $\sigma_{\delta\epsilon}$  for almost all  $\delta$  and  $\epsilon$ .  $\square$

Once we have this regular stratification, we can use a certain subset of the strata to approximate an arbitrary semi-algebraic set. Define  $\mathbf{R}_{\text{sep}} = (-\infty, -\delta] \cup [-\epsilon, \epsilon] \cup [\delta, \infty)$ , then  $\mathbf{R}_{\text{sep}}$  is a closed set. Intuitively, this is a partition of the real line separating values that are nearly zero from those that are definitely non-zero. We will show later that the connected components of the preimage  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$  are in one-to-one correspondence with the connected components of sign-invariant sets of  $F$  (if  $\delta \gg \epsilon > 0$  are both sufficiently small).

For each sign-sequence  $\sigma \in (\underline{\mathbf{R}})^k$ , there is also "separated" sign-sequence  $\sigma_{\text{sep}}$  defined as

$$(\sigma_{\text{sep}})_i = \begin{cases} [\delta, \infty) & \text{if } \sigma_i = \mathbf{R}^+ \\ [-\epsilon, \epsilon] & \text{if } \sigma_i = \{0\} \\ (-\infty, -\delta] & \text{if } \sigma_i = \mathbf{R}^- \end{cases} \quad (6)$$

and the  $\sigma_{\text{sep}}$  are exactly the connected components of  $(\mathbf{R}_{\text{sep}})^k$ .

The set  $\mathbf{R}_{\text{sep}}$  has a regular stratification, denoted  $\underline{\mathbf{R}}_{\text{sep}}$ :

$$\underline{\mathbf{R}}_{\text{sep}} = \{(-\infty, -\delta), \{-\delta\}, \{-\epsilon\}, (-\epsilon, \epsilon), \{\epsilon\}, \{\delta\}, (\delta, \infty)\} \quad (7)$$

but because we chose  $a$  carefully, the preimage  $G^{-1}((\underline{\mathbf{R}}_{\text{sep}})^k)$  is a Whitney regular stratification of  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$ . Since the preimage is also a closed, bounded set, and therefore compact, the algorithm of [Can88a] can be applied directly to compute its connected components. This takes us a long way toward computing the connected components of a given semi-algebraic set, and leaves us only with the task of determining adjacencies between connected components of sign-invariant sets. For now we must show

**Theorem 4.2** *Let  $G = (F_1/a_1, \dots, F_k/a_k)$  and  $\mathbf{R}_{\text{sep}}$  be defined as above. Then the connected components of sign-invariant sets of  $F$  are in one-to-one correspondence with the connected components of  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$  for almost all  $a \in \mathbf{R}^k$  and for all sufficiently small  $\delta \gg \epsilon > 0$*

The result follows from the next two lemmas. There is a natural correspondence between each non-empty sign-invariant set  $F_\sigma = F^{-1}(\sigma)$  and the set  $G^{-1}(\sigma_{\text{sep}})$ . The lemmas show

that these two sets have the same homotopy type, by showing that they can be retracted onto a common subset.

Consider a particular sign-invariant set  $F_\sigma$  of  $F$ , and number the  $F_i$ 's such that  $F_1, \dots, F_m$  are all zero on  $F_\sigma$ , and the remaining polynomials are non-zero. Assume also for simplicity that all non-zero  $F_i$ 's are positive on  $F_\sigma$ . We replace each inequality  $F_i > 0$  for  $i > m$  with a new inequality  $F_i \geq \delta$ . Together with the inequality  $\rho \leq r$  ( $\rho$  is the radius function defined earlier), this defines a closed set  $F_\sigma^-(\delta)$  which is a subset of  $F_\sigma$  for  $\delta > 0$ . Since it is also a subset of the compact ball of radius  $r$ , it is compact.

We first show that the homotopy type of  $F_\sigma^-(\delta)$  is the same as that of  $F_\sigma$  for small enough  $\delta$ . In fact we have

**Lemma 4.3** *There exists a positive  $\delta_0$  such that for all positive  $\delta < \delta_0$ , the set  $F_\sigma^-(\delta)$  is a deformation retract of  $F_\sigma$ .*

**Proof** Consider the set  $D$  in  $\mathbb{R}^{n+1}$  defined as  $D = \{(x_1, \dots, x_n, \delta) \mid x \in F_\sigma^-(\delta)\}$ . Think of  $D$  as the “graph” of  $F_\sigma^-(\delta)$ .  $D$  is certainly semi-algebraic, and so has a Whitney regular stratification compatible with the signs of the polynomials  $F_i$  and  $(F_i - \delta)$ . The projection  $\pi_\delta : (x, \delta) \mapsto \delta$  has a finite number of critical values when restricted to these strata. Choose  $\delta_0 > 0$  to be the smallest positive critical value, and let  $\delta$  be any positive number less than  $\delta_0$ . Then  $\pi_\delta$  is regular on all strata for values in the range  $(0, \delta]$ . We can use  $\pi_\delta$  to lift a vector field on  $(0, \delta]$  to a vector field on  $D$  which is compatible with its strata. Since  $D$  is compact when restricted to  $[0, \delta]$ , this gives us a deformation retraction of the set  $D|_{(0, \delta]} = \pi_\delta^{-1}(0, \delta]$  onto the compact set  $D_\delta = \pi_\delta^{-1}(\delta)$ .

But if we define  $\pi_x : (x, \delta) \mapsto x$ , then the projection  $\pi_x(G|_{(0, \delta]})$  is just  $F_\sigma$ , and  $\pi_x(D|_\delta)$  is just  $F_\sigma^-(\delta)$ . Furthermore, composing  $\pi_x$  with the deformation retraction of the last paragraph gives us a deformation retraction of  $F_\sigma$  onto  $F_\sigma^-(\delta)$ .  $\square$

To guarantee that  $\delta$  is small enough, we leave  $\delta$  as an indeterminate element of the base field (like  $r$ ), and when it comes time to evaluate the sign of a base field element, which is a polynomial in  $r$  and  $\delta$ , we first find the term of lowest degree in  $\delta$ , then among all terms with this degree in  $\delta$  we take the sign of the highest degree term in  $r$ . This is equivalent to preceding all evaluations with the quantification  $\exists r_0 \forall r > r_0 \exists \delta_0 \forall \delta < \delta_0 \dots$

Operationally, this is also equivalent to working in a real field extension  $\mathbb{R}(r, \delta)$  where  $r$  is larger than any element of  $\mathbb{R}$  and  $\delta$  is smaller than any element of the real closure of  $\mathbb{R}(r)$ . However, in the algorithms that follow, all the numerical calculations we make will involve polynomials from  $\mathbb{R}[r, \delta]$ . We do not need  $r$  and  $\delta$  to be values that do not lie in  $\mathbb{R}$ , but only “sufficiently large” or “sufficiently small” real values that the signs of all polynomials are correctly computed. If we work only over the reals, life is much easier, since the usual notions of compactness and connectivity apply.

In the last lemma we defined compact sets  $F_\sigma^-(\delta)$  with the useful properties that they are compact and are deformation retracts of sign-invariant sets. This means that each *connected component* of a sign-invariant set  $F_\sigma$  contains a single component of  $F_\sigma^-(\delta)$ . Next we define a set  $F_\sigma^+(\delta, \epsilon)$  which is a “neighborhood” of  $F_\sigma^-(\delta)$  and can be retracted onto it. To define  $F_\sigma^+(\delta, \epsilon)$ , we take each inequality in an  $F_i$  and replace it with one or two inequalities:

$$\begin{aligned} (F_i + \epsilon a_i \geq 0) \wedge (F_i - \epsilon a_i \leq 0) & \text{ if } \sigma_i = \{0\} \\ (F_i - \delta a_i \geq 0) & \text{ if } \sigma_i = \mathbb{R}^+ \end{aligned} \tag{8}$$



and notice that  $F_\sigma^+(\delta, \epsilon)$  is now compact. By the previous general position lemma, it is also regularly stratified by the signs of the polynomials that define it. Notice also that  $F_\sigma^-(\delta)$  is a subset of  $F_\sigma^+(\delta, \epsilon)$ .

**Lemma 4.4** *Assume  $\delta$  is chosen to satisfy lemma 4.3. There exists a positive  $\epsilon_0$  such that for all  $0 < \epsilon < \epsilon_0$ , the set  $F_\sigma^-(\delta)$  is a deformation retract of  $F_\sigma^+(\delta, \epsilon)$ .*

**Proof** We only sketch this proof since it is almost identical to the proof of lemma 4.3. Let  $D$  in  $\mathbb{R}^{n+1}$  defined as  $D = \{(x_1, \dots, x_n, \epsilon) \mid x \in F_\sigma^+(\delta, \epsilon)\}$ , treating  $\delta$  as a constant. For all sufficiently small  $\epsilon > 0$ , the set  $D$  has a Whitney regular stratification  $\underline{D}$  into sign-invariant sets of the polynomials that define it. There is some  $\epsilon_0$  which is the smallest positive critical value of the projection  $\pi : (x, \epsilon) \mapsto \epsilon$  restricted to  $\underline{D}$ . For  $\epsilon < \epsilon_0$ , We can use  $\pi$  to lift a vector field on  $(0, \epsilon]$  and thereby define the desired retraction.  $\square$

Notice that  $F_\sigma^+(\delta, \epsilon)$  is exactly the set  $G^{-1}(\sigma_{\text{sep}})$ , the non-singular approximation of  $F^{-1}(\sigma)$ . Since  $F_\sigma$  and  $F_\sigma^+(\delta, \epsilon)$  have a common retract  $F_\sigma^-(\delta)$ , they have the same homotopy type and hence the same number of connected components. This completes the proof of theorem 4.2.

Finally, we observe that the sets  $F_\sigma^+(\delta, \epsilon)$  and  $F_{\sigma'}^+(\delta, \epsilon)$  are disjoint for  $\sigma$  and  $\sigma'$  distinct. This follows because if the union of a component of  $F_\sigma^+(\delta, \epsilon)$  and a component of  $F_{\sigma'}^+(\delta, \epsilon)$  were connected, then its image under  $G$  would have to be connected also. But that image must lie in the union of the disjoint sets  $\sigma_{\text{sep}}$  and  $\sigma'_{\text{sep}}$ , and it must intersect both, which is impossible.

So the connected components of  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$ , which are the union of connected components of  $F_\sigma^+(\delta, \epsilon)$ 's, correspond exactly to the connected components of sign-invariant sets of  $F$ . Since  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$  is a compact set which is regularly stratified as  $G^{-1}((\mathbf{R}_{\text{sep}})^k)$ , we can apply the algorithm of [Can88a], modified to work over arbitrary real coefficient fields as described in [Can88c].

All that remains to determine connectivity of a given semi-algebraic set is to determine the adjacencies between connected components of sign-invariant sets. This we deal with in the next section.

## 5 Determining adjacencies between sign components

In the last section, we modeled connected components of sign-invariant sets  $F_\sigma$  with regularly stratified compact “neighborhood” sets  $F_\sigma^+(\delta, \epsilon)$ . In this section, we show that these neighborhood sets can also be used to determine adjacencies between components. We will need to make use of a “big” neighborhood of one set and a “small” neighborhood of the other.

First we remark that if disjoint sets  $A$  and  $B$  have a connected union, then either  $A \cap \overline{B} \neq \emptyset$ , or  $B \cap \overline{A} \neq \emptyset$ . For if this were not true, both  $A$  and  $B$  would be open in the union  $A \cup B$ , and therefore both closed in  $A \cup B$ . So to check whether connected  $A$  and  $B$  have a connected union, it suffices to check whether either  $A \cap \overline{B} \neq \emptyset$ , or  $B \cap \overline{A} \neq \emptyset$ .

Let  $A$  be a connected component of  $F_\sigma$ , and let  $A^+(\delta_1, \epsilon_1)$  be the corresponding connected component of  $F_\sigma^+(\delta_1, \epsilon_1)$ . Let  $B$  be a connected component of a second sign-invariant set  $F_{\sigma'}$ , and  $B^+(\delta_0, \epsilon_0)$  be the corresponding connected component of  $F_{\sigma'}^+(\delta_0, \epsilon_0)$ . Note the

use of two different sets of infinitesimals. The next lemma shows that for suitable choices of the infinitesimals, adjacency of  $A$  and  $B$  can be checked by testing overlap of  $A^+(\delta_1, \epsilon_1)$  and  $B^+(\delta_0, \epsilon_0)$ :

**Lemma 5.1** *With  $A^+(\delta_1, \epsilon_1)$  and  $B^+(\delta_0, \epsilon_0)$  as defined above, and for all sufficiently small  $\delta_1 \gg \epsilon_1 \gg \delta_0 \gg \epsilon_0 > 0$ ,*

$$(A \cap \overline{B} \neq \emptyset) \iff A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0) \neq \emptyset$$

**Proof** First we clarify what we mean by “sufficiently small”. Let  $\overset{\circ}{\forall} \alpha$  mean that there is some positive  $\alpha'$  such that the formula that follows the quantifier is true for all positive  $\alpha$  less than  $\alpha'$ . The lemma states that  $A \cap \overline{B} \neq \emptyset$  is equivalent to

$$\overset{\circ}{\forall} \delta_1 \overset{\circ}{\forall} \epsilon_1 \overset{\circ}{\forall} \delta_0 \overset{\circ}{\forall} \epsilon_0 (A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0) \neq \emptyset) \quad (9)$$

We can simplify this formula by doing quantifier elimination from the inside out. Let  $B^-(\delta_0)$  be the component of  $F_{\sigma'}^-(\delta_0)$  that is a retract of  $B^+(\delta_0, \epsilon_0)$ . We claim that

$$\overset{\circ}{\forall} \epsilon_0 (A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0) \neq \emptyset) \iff (A^+(\delta_1, \epsilon_1) \cap B^-(\delta_0) \neq \emptyset) \quad (10)$$

To see this, notice that the set  $(A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0))$  in  $(x, \epsilon_0)$ -space is compact when restricted to  $\epsilon_0 \in [0, \epsilon'_0]$ , and so achieves a minimum non-negative value of  $\epsilon_0$ . If this value is greater than zero, both formulae are false, if it is zero, both formulae are true. Next we need to show that

$$\overset{\circ}{\forall} \delta_0 (A^+(\delta_1, \epsilon_1) \cap B^-(\delta_0) \neq \emptyset) \iff (A^+(\delta_1, \epsilon_1) \cap B \neq \emptyset) \quad (11)$$

Let  $p$  be a point in  $(A^+(\delta_1, \epsilon_1) \cap B)$ , and let  $\delta'_0$  be the minimum of the values of the  $F_i$ 's at  $p$ , excluding those  $F_i$ 's which are zero at  $p$  (we assume wlog that non-zero  $F_i$ 's are positive). The first formula will be true for all  $\delta_0$  less than  $\delta'_0$ . Conversely, if the first formula is true for any  $\delta_0$ , then there is a point  $p$  in  $(A^+(\delta_1, \epsilon_1) \cap B^-(\delta_0))$ . This point is also in  $(A^+(\delta_1, \epsilon_1) \cap B)$  since  $B^-(\delta_0)$  is a subset of  $B$ , so the second formula is true also. It remains to show that

$$\overset{\circ}{\forall} \delta_1 \overset{\circ}{\forall} \epsilon_1 (A^+(\delta_1, \epsilon_1) \cap B \neq \emptyset) \iff (A \cap \overline{B} \neq \emptyset) \quad (12)$$

Suppose first that  $A \cap \overline{B} \neq \emptyset$ , and let  $p$  be a point in  $A \cap \overline{B}$ . Choose  $\delta_1$  small enough so that  $p$  is in the (relative) interior of  $A^+(\delta_1, \epsilon_1)$ , i.e. none of the inequalities  $F_i(p) \geq a_i \delta_1$  has equal arguments. We know that all neighborhoods of  $p$  intersect  $B$ , and the rest of the formula checks this. Specifically, for every  $\epsilon_1 > 0$ ,  $A^+(\delta_1, \epsilon_1)$  contains an absolute neighborhood  $U$  of  $p$ . This follows because all the polynomials that define  $A^+(\delta_1, \epsilon_1)$  are non-zero at  $p$  (some  $F_i$ 's may be zero of course, but  $A^+(\delta_1, \epsilon_1)$  is defined by these polynomials  $\pm a_i \epsilon_1$ ). Since  $U$  intersects  $B$ , so does  $A^+(\delta_1, \epsilon_1)$ .

Conversely, suppose  $A \cap \overline{B} = \emptyset$ . We show that  $\overset{\circ}{\forall} \epsilon_1 A^+(\delta_1, \epsilon_1) \cap B \neq \emptyset$  is false for any  $\delta_1 > 0$ . Pick a  $\delta_1 > 0$ . Now choose any monotonically decreasing sequence  $v_i \rightarrow 0$  of positive  $\epsilon_1$  values. Then for at least one of these values  $A^+(\delta_1, \epsilon_1) \cap B = \emptyset$ .

Why? Suppose the intersection were non-empty for all  $v_i$ . Let  $p_i$  be a point in  $A^+(\delta_1, v_i) \cap B$ . The sequence  $(p_i)$  lies in the compact set  $A^+(\delta_1, v_1) \cap \overline{B}$ , and so has a

convergent subsequence. But let  $p$  be a limit point, we must have  $p \in A^-(\delta_1) \subset A$  by continuity of the polynomials defining  $A^+$ . So we have a subsequence of  $(p_i) \in B$  which converges to  $p \in A$ . This shows  $A \cap \overline{B} \neq \emptyset$ , contrary to our assumption of the last paragraph. So the above assumption was false, and  $A^+(\delta_1, \epsilon_1) \cap B$  must be empty for some (in fact almost all)  $v_i > 0$ .  $\square$

To compute with this quantification, we once again treat  $\delta_1, \epsilon_1, \delta_0$  and  $\epsilon_0$  as real elements of the ground field. This is equivalent to a real field extension by infinitesimals  $\mathbb{R}(\delta_1, \epsilon_1, \delta_0, \epsilon_0)$  where  $\delta_1, \epsilon_1, \delta_0$  and  $\epsilon_0$  are adjoined in that order, and each is taken to be smaller than any positive element of the previous extension field. However, the correct semantics is that they are real numbers that are small enough (relative to previously quantified values) that they give the same sign for the ground field elements computed by our algorithm.

An important corollary of the above theorem is that we can compute the connected components of an arbitrary semi-algebraic set by computing the connected components of a certain compact, regularly stratified set. First let us define a fine stratification of the real line  $\underline{\mathbb{R}}_{\text{fin}}$  which consists of the eight points  $\pm\delta_1, \pm\epsilon_1, \pm\delta_0, \pm\epsilon_0$  and the open intervals in between and to  $\pm\infty$ .

Let  $S$  be the original semi-algebraic set defined by polynomials  $F_1, \dots, F_k$ . Assume without loss of generality or efficiency, that the formula defining  $S$  is a monotone boolean function of inequalities of the form  $F_i = 0$  or  $F_i > 0$ . Suppose a suitable  $a \in \mathbb{R}^k$  has been chosen so that  $G = (F_1/a_1, \dots, F_k/a_k)$  is transversal to  $(\underline{\mathbb{R}}_{\text{fin}})^k$ . Replace each  $(F_i = 0)$  in the formula for  $S$  with the conjunction

$$(F_i \leq \epsilon_0 a_i) \wedge (F_i \geq -\epsilon_0 a_i) \tag{13}$$

and each  $(F_i > 0)$  with  $(F_i \geq \delta_0 a_i)$ . Call this new set  $S_{\text{sep0}}$ . We need the defining boolean formula to be monotone so that the new set  $S_{\text{sep0}}$  is a union of separated sign-invariant sets  $G^{-1}(\sigma_{\delta_0 \epsilon_0})$ .

Similarly, we can define a set  $S_{\text{sep1}}$  as above using  $\epsilon_1$  and  $\delta_1$ . We can now state the result relating the connected components of  $S$  and the regularly stratified sets  $S_{\text{sep0}}$  and  $S_{\text{sep1}}$ :

**Corollary 5.2** *There is a one-to-one correspondence between connected components of  $S$  and connected components of  $S_{\text{sep0}} \cup S_{\text{sep1}}$ .*

**Proof** Let  $A_1, \dots, A_q$  be the connected components of sign-invariant sets of the map  $F$  that are contained in  $S$ . Note that this means there may be more than one  $A_i$  within the same sign-invariant set (in fact if one component of a given sign-invariant is in  $S$ , all the others must be).

We saw in the last section that each  $A_i$  corresponds in a simple way with a certain connected component  $A_i^+(\delta_0, \epsilon_0)$  of  $S_{\text{sep0}}$ , and to a certain component  $A_i^+(\delta_1, \epsilon_1)$  of  $S_{\text{sep1}}$ . Now consider the union  $A_i^{++} = A_i^+(\delta_0, \epsilon_0) \cup A_i^+(\delta_1, \epsilon_1)$ . This is a connected set, and it intersects  $A_i$ . We complete the proof by showing that  $A_i \cup A_j$  is connected for distinct  $A_i$  and  $A_j$  if and only if  $A_i^{++} \cup A_j^{++}$  is connected.

If  $A_i \cup A_j$  is connected, then one set intersects the closure of the other. Assume for instance that  $A_i \cap \overline{A_j} \neq \emptyset$ . We know from the theorem of this section that this implies  $A_i^+(\delta_1, \epsilon_1) \cap A_j^+(\delta_0, \epsilon_0) \neq \emptyset$ , so that  $A_i^{++} \cup A_j^{++}$  must be connected.

Conversely, suppose  $A_i^{++} \cup A_j^{++}$  is connected. We know from the last section that  $A_i^+(\delta_0, \epsilon_0)$  and  $A_j^+(\delta_0, \epsilon_0)$  are disjoint, and similarly for  $A_i^+(\delta_1, \epsilon_1)$  and  $A_j^+(\delta_1, \epsilon_1)$ . So the two remaining possibilities are  $A_i^+(\delta_1, \epsilon_1) \cap A_j^+(\delta_0, \epsilon_0) \neq \emptyset$  which implies  $A_i \cap \overline{A_j} \neq \emptyset$ , or  $A_i^+(\delta_0, \epsilon_0) \cap A_j^+(\delta_1, \epsilon_1) \neq \emptyset$  which implies  $\overline{A_i} \cap A_j \neq \emptyset$ . In either case  $A_i \cup A_j$  is connected.  $\square$

## 5.1 Transformation Algorithm

To summarize, the following algorithm reduces calculation of connected components of a general semi-algebraic set to calculation of connected components of a compact, regularly stratified set.

- Convert the input formula to monotone standard form, and if necessary, collapse the defining formula to a trinary formula of logarithmic depth as described in [Can90].
- Add to the formula a conjunction with the polynomial inequality  $\sum_{j=1}^n x_j^2 \leq r^2$  (converted to standard form), where  $r$  is an infinite positive value (larger than any real). Let the resulting formula be  $B(A_1, \dots, A_k)$ , it defines a bounded set in the extension field  $\mathbf{R}(r)$ .
- Choose an  $a \in (\mathbf{R} - \{0\})^k$  at random, or let  $a_1 \gg a_2 \gg \dots \gg a_k > 0$  be a series of infinitesimals.
- Construct a formula  $B_0$  from the input formula  $B(A_1, \dots, A_k)$  as follows. For each atomic predicate  $A_i$ , replace  $A_i$  with

$$\begin{aligned} \text{if } A_i \text{ is } F_i = 0 \text{ then } (F_i + \epsilon_0 a_i \geq 0) \wedge (F_i - \epsilon_0 a_i \leq 0) \\ \text{if } A_i \text{ is } F_i > 0 \text{ then } (F_i - \delta_0 a_i \geq 0) \end{aligned} \tag{14}$$

where  $\delta_0 \gg \epsilon_0$  are infinitesimals. Then the set  $S_{\text{sep}0}$  defined by this formula is closed and bounded, therefore compact (or semi-algebraic compact, if we think in terms of true infinitesimals rather than sufficiently small reals). By the results of section 4 the connected components of the sign-invariant sets of  $B_0$  are in one-to-one correspondence with those of  $B$ .

- Now define a new formula  $B_1$  from  $B(A_1, \dots, A_k)$  as in the previous step but with  $\epsilon_1$  and  $\delta_1$  replacing  $\epsilon_0$  and  $\delta_0$ . Set  $\delta_1 \gg \epsilon_1 \gg \delta_0$ . The formula  $B_1$  defines a set  $S_{\text{sep}1}$ , and by the results of section 5, the connected components of  $S_{\text{sep}0} \cup S_{\text{sep}1}$  are in one-to-one correspondence with the connected components of  $S$ .
- Return  $B_0 \vee B_1$ , which defines a regularly stratified, compact semi-algebraic set whose connected components correspond one-to-one with those of  $S$ .

## 6 Complexity

A full analysis of the algorithm is given in [Can90], but the main ideas are simple enough to sketch here. The main facts we need are the following:

1. The algorithm of [Can88a], modified to use the BKR lemma as described in [Can88c] makes all its branching decisions based on the signs of query polynomials whose degree in the coefficients of the input polynomials is  $d^{O(n^2)}$ . This bound is obtained by inspecting the resultant matrices used in calculating projections, and the 2-d point ordering algorithm.
2. The number of such polynomials that might ever occur in the calculation is  $(kd)^{O(n^2)}$ , obtained by considering all the possible slices that might be taken recursively.
3. Each query polynomial contains at most  $O(n^2)$  infinitesimals, even if all of  $a_1, \dots, a_k$  are infinitesimal. This seems surprising at first, but one must remember that by using infinitesimal  $a_i$ 's, we have guaranteed that all the algebraic surfaces defined by the input polynomials meet transversally. In particular, any collection of more than  $n$  surfaces will not have a common intersection point. If the query polynomials were only generated by intersection points, there could be at most  $O(n)$  infinitesimals in each one. But the roadmap algorithm also generates hyperplanes which are defined by  $n$  input surfaces. These also meet the input surfaces transversally (except at one point each), and so an intersection point can actually depend on  $O(n^2)$  input surfaces.

Making these observations allows us to determine the running time for the deterministic version of the algorithm, which uses infinitesimal  $a_i$ 's. The bounds in [Can88a] and [Can88c] show that the roadmap algorithm for a particular input requires  $O(k^n \log k) d^{O(n^2)}$  evaluations of query polynomials. The above arguments indicate that each polynomial has degree  $d^{O(n^2)}$  in  $O(n^2)$  variables, or  $d^{O(n^4)}$  coefficients, and they require  $d^{O(n^4)}$  time to evaluate (coefficient length bounds are much better than this, and don't affect the complexity bounds). The running time is the product of the number of query polynomial evaluations and the time for each which is  $(k^n \log k) d^{O(n^4)}$ .

For the randomized algorithm, we need to figure out the number of random bits required in the choice of  $a_i$ 's. We could try to figure out explicitly the conditions for a particular  $a$  to be a good choice, but there is a simpler argument we can use, which takes advantage of the fact that our calculation can be expressed as an algebraic decision tree. A particular  $a$  must be a good choice if *all the query polynomials in the decision tree are non-zero* at that  $a$  (excepting query polynomials which are identically zero, which can be ignored). This follows because for such an  $a$ , there is an open, connected neighborhood  $N(a)$  such that all the query polynomials have the same sign over all of  $N(a)$  as they do at  $a$ . Thus the algorithm's output is that same for all these choices. But almost all of the points in  $N(a)$  must be good choices, since good points are dense. The algorithm must produce the correct output at these points, hence it produces the correct output at  $a$ .

So it suffices to choose  $a$  to avoid the zero sets of all the query polynomials. The query polynomials have degree  $d^{O(n^2)}$  and there are potentially  $(kd)^{O(n^2)}$ . The union of the zero sets gives us a bad set which is an algebraic set in the space  $\mathbb{R}^k$  of possible  $a$ -values which has degree  $(kd)^{O(n^2)}$ . By Schwartz's lemma, we will have probability  $p^n$  of hitting

the bad set if we choose the  $a_i$ 's randomly with  $\log(p^{-1}(kd)^{O(n^2)})$  bits. Fixing  $p$ , we see that  $O(n^2 \log(kd))$  bits suffice for the  $a_i$ 's. This contributes the extra factor of  $\log k$  to the running time of the original roadmap algorithm, and gives a randomized running time of  $(k^n \log^2 k)d^{O(n^2)}c^2$ .

## References

- [BCR87] J. Bochnak, M. Coste, and M-F. Roy. *Géométrie algébrique réelle*. Number 12 in *Ergebnisse der mathematik 3*. Springer-Verlag, Berlin, 1987.
- [BOKR86] M. Ben-Or, D. Kozen, and J. Reif. The complexity of elementary algebra and geometry. *J. Comp. and Sys. Sci.*, 32:251–264, 1986.
- [Can88a] J.F. Canny. *The Complexity of Robot Motion Planning*. M.I.T. Press, Cambridge, 1988.
- [Can88b] J.F. Canny. Constructing roadmaps of semi-algebraic sets I: Completeness. *Artificial Intelligence*, 37:203–222, 1988.
- [Can88c] J.F. Canny. Some algebraic and geometric computations in PSPACE. In *ACM Symposium on Theory of Computing*, pages 460–467, 1988.
- [Can90] J.F. Canny. Computing roadmaps of semi-algebraic sets. Technical report, University of California, Berkeley, 1990.
- [Can91] J.F. Canny. An improved sign determination algorithm. In *AAECC-91*, 1991. New Orleans.
- [CGV91] J.F. Canny, D.Y. Grigor'ev, and N.N. Vorobjov. Finding connected components of a semialgebraic set in subexponential time. *App. Algebra in Eng. Comm. and Comp. (accepted)*, 1991.
- [FGM87] N. Fitchas, A. Galligo, and J. Morgenstern. Algorithmes rapides en séquentiel et en parallèle pour l'élimination des quantificateur en géométrie élémentaire. *Sém. Structures Algébriques Ordonnées*, 1987.
- [Gou91] L. Gournay. Construction of roadmaps in semialgebraic sets. In *AAECC-91 (this volume)*, 1991.
- [Gri88] D.Y. Grigor'ev. Complexity of deciding Tarski algebra. *Journal of Symbolic Computation*, 5:65–108, 1988.
- [GV88] D.Y. Grigor'ev and N.N. Vorobjov. Solving systems of polynomial equations in subexponential time. *Journal of Symbolic Computation*, 5:37–64, 1988.
- [GV89] D.Y. Grigor'ev and N.N. Vorobjov. Counting connected components of a semi-algebraic set in subexponential time. *Computational Complexity (submitted)*, 1989.

- [GWDL76] C.G. Gibson, K. Wirthmüller, A.A. Du Plessis, and E.J.N. Looijenga. *Topological Stability of Smooth Mappings*. Number 552 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1976.
- [HRS90a] J. Heintz, M.F. Roy, and P. Solerno. Complexité du principe de Tarski-Seidenberg. *Bull. Soc. Math. France*, 118:101–126, 1990.
- [HRS90b] J. Heintz, M.F. Roy, and P. Solerno. Single-exponential path finding in semi-algebraic sets I: The case of a smooth bounded hypersurface. In *AAECC-90, Springer LNCS (to appear)*, 1990.
- [HRS90c] J. Heintz, M.F. Roy, and P. Solerno. Single-exponential path finding in semi-algebraic sets II: The general case. In *To appear Proc. 60th Birthday Conf. for S. Abhyankar*, 1990.
- [KY85] D. Kozen and C. Yap. Algebraic cell decomposition in NC. In *IEEE Conference on Foundations of Computer Science*, pages 515–521, 1985.
- [Ren89] J. Renegar. On the computational complexity and geometry of the first-order theory of the reals, parts I, II and III. Technical Report 852,855,856. Cornell University, Operations Research Dept., 1989. *Journal of Symbolic Computation* 1992.
- [SS83] J.T. Schwartz and M. Sharir. On the piano movers' problem, II: General techniques for computing topological properties of real algebraic manifolds. *Advances in Applied Mathematics*, 4:298–351, 1983.
- [Tar48] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, Berkeley, 1948.
- [Tro89] D. Trotman. On Canny's roadmap algorithm: Orienteering in semialgebraic sets. Technical report, Univ. Aix-Marseille, 1989.

