An Optimal Condition for Determining the Exact Number of Roots of a Polynomial System

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Abstract: It was shown in [Ber75] that the number of roots in $(\mathbf{C}^*)^n$ of a polynomial system depends only on the Newton polytopes of the system, for almost all specializations of the coefficients. This result, henceforth referred to as the *BKK bound*, gives an upper bound on the number of roots of a polynomial system. The BKK bound is often much better than the Bezout bound for the same system, but the original theorem gives an exact bound only if all the coefficients corresponding to Newton polytope boundaries are generically chosen. In this paper, we show that the BKK bound is exact under much weaker assumptions: only coefficients corresponding to certain *vertices* of the Newton polytopes need be generic. This result allows application of the BKK bound to many practical problems.

1 Introduction

Various authors [Ber75; [illegible]] [illegible] about the surprising [illegible] of roots of a polynomial system [illegible] $(\mathbf{C}^*)^n$ [illegible] mixed volume [illegible] polytopes. [illegible] These polytopes, commonly referred to as *Newton polytopes*, are higher dimensional analogues of the classical Newton polygon [Art67]. The relationship described in [Ber75; Kus76; Kho78] gives an upper bound (henceforth referred to as the *BKK bound*) on the number of roots of a polynomial system.

D. N. Bernshtein [Ber75], among others, described a sufficient condition under which the BKK bound is exact. Roughly speaking, the condition is that several related Laurent polynomial systems have no roots in $(\mathbf{C}^*)^n$. Thus, given Bernshtein's result, one can derive that the number of roots of a polynomial system is equal to the mixed volume of its Newton polytopes, for a dense constructible set of specializations of the system's coefficients. However, with a bit more work, one can derive the following stronger result:

Proposition *Let $F = (f_1, \ldots, f_n)$ be a polynomial system in n variables with indeterminate boundary coefficients and P its n-tuple of Newton polytopes. Then there exists a "small" set C of vertex coefficients of F with the following property: For any specialization of the remaining boundary coefficients, there exists a dense constructible set Z of specializations of C such that F has exactly $\mathcal{M}(P)$ roots (counting multiplicities) in $(\mathbf{C}^*)^n$. Moreover, Z does not depend on the internal coefficients of F.*

In essence, the proposition states that the number of roots of a polynomial system depends strongly on its vertex coefficients, has almost no dependence on its in-

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ternal coefficients, and depends weakly on its remaining efficients. The notions of "vertex", "boundary", and "internal" arise naturally from the Newton polytopes of the system, and this connection is made clear in the next section. Our main result is a stronger statement of the above proposition: the vertex coefficient theorem. In section 3 we prove the vertex coefficient theorem and discuss its optimality, e.g., what is meant by "small."

The BKK bound therefore has considerable practical significance, considering that root counting methods based on Bezout's theorem typically overestimate the number of roots of a non-homogeneous polynomial system. Examples of this this phenomenon occur in inverse kinematics [Mor90; RR90; WMS90]. Also, root counts for polynomial systems are germane to complexity bounds of many algorithms in computational algebra.

Before proving our result, we first establish some necessary facts about convex bodies and complex algebraic geometry.

2 Preliminaries

Recall the following standard conventions:

$\mathbb{Z}$ = The ring of integers

$\mathbb{Q}$ = The field of rational numbers

$\mathbb{R}$ = The field of real numbers

$\mathbb{C}$ = The field of complex numbers

F^* = The multiplicative group of nonzero elements of the field F

$R[x_1,\ldots,x_n]$ = The ring of polynomials in $x_1,\ldots,x_n$ with coefficients in the ring R

S^c = The set-theoretic complement of the set S

$\#S$ = The cardinality or dimension of S, according as S is a set or a vector

$\langle \cdot , \cdot \rangle$ = The standard inner-product on the vector space $\mathbb{R}^n$

$dist(A,x)$ = The standard minimum Euclidean distance from $A \subset \mathbb{C}^k$ to $x \in \mathbb{C}^k$

By a *Laurent polynomial* we mean an element of the polynomial ring $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Laurent polynomials will be regarded as functions on the manifold $(\mathbb{C}^*)^n$. Let $f(x) = \sum_{q \in \mathbb{Z}^n} c_q x^q$ be a Laurent polynomial. To simplify our notation we freely use multi-indices, e.g., the equality $x^q = x_1^{q_1} \cdots x_n^{q_n}$ is understood. Also, we assume all n-tuples to be ordered and all subsets and subvectors partially ordered by inclusion.

We define the *support* of f, $supp(f)$, to be the set of all q for which $c_q \neq 0$, and the *Newton polytope* of f to be the convex hull of $supp(f)$ in $\mathbb{R}^n$. Also, we define a *Laurent polynomial system* to be an n-tuple $F = (f_1, \ldots, f_n)$ of non-zero Laurent polynomials and $supp(F)$ to be the n-tuple $(supp(f_1), \ldots, supp(f_n))$.

For convenience, we let $\mathbf{0}$ denote the n-tuple $(0, \ldots, 0)$. Let S be a compact subset of $\mathbb{R}^n$ and $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$. Define $m(\alpha, S) = \min_{q \in S}\{\langle \alpha, q \rangle\}$ and $S_\alpha = \{q \in S \mid \langle \alpha, q \rangle = m(\alpha, S)\}$. Thus S_α is the intersection of S with its hyperplane of support in the direction α. If $supp(f) = S$ define $f_\alpha(x) = \sum_{q \in S_\alpha} c_q x^q$. Finally, let $F = (f_1, \ldots, f_n)$ be a Laurent polynomial system and define $F_\alpha = (f_{1\alpha}, \ldots, f_{n\alpha})$. Note that $\{F_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\}$ is finite since F has only n Newton polytopes, each with only finitely many faces.

Let P be the Newton polytope of f. The *boundary* coefficients of f are the c_q with $q \in \partial P$, while the c_q with $q \in P \setminus \partial P$ are referred to as *internal* coefficients. Also, the *vertex* coefficients of f are the c_q with q a vertex of P. Note that a vertex coefficient is a boundary coefficient, but not necessarily vice-versa. A boundary, internal, or vertex coefficient of F is respectively a boundary, internal, or vertex coefficient of f_j for some $j \in \{1, \ldots, n\}$. If some of the coefficients of F are indeterminate, then a *specialization* is an assignment of complex values to these coefficients. A specialization of N indeterminate coefficients is said to be *generic* if its values are chosen from some fixed dense subset of $\mathbf{C}^N$ not of the form $W' \times \mathbf{C}$.

We will assume all polytopes to be bounded and non-empty. Let $V_n(P)$ denote the standard n-dimensional Lebesque measure of a polytope P in $\mathbf{R}^n$. We define an addition on polytopes in $\mathbf{R}^n$ by setting $P_1+P_2 = \{x_1+x_2 | x_1 \in P_1, x_2 \in P_2\}$ for any polytopes $P_1, P_2 \subset \mathbf{R}^n$. Then if $\mathcal{P} = (P_1, \ldots, P_n)$ is an n-tuple of polytopes in $\mathbf{R}^n$, the *mixed volume* of $\mathcal{P}$ is

$$V(\mathcal{P}) = (-1)^{n-1} \sum_i V_n(P_i) + (-1)^{n-2} \sum_{i<j} V_n(P_i+P_j) + \cdots$$

$$\cdots + V_n(P_1 + \cdots + P_n)$$

We remark that $V(\mathcal{P})$ is a non-negative symmetric multilinear function of $P_1, \ldots, P_n$. For a more detailed discussion of these definitions, we refer the reader to [Oda85].

We define $dim\mathcal{P}$ to be the dimension of the smallest flat in $\mathbf{R}^n$ containing $P_1 + \cdots + P_n$ and let $\mathcal{P}_\alpha = (P_{1\alpha}, \ldots, P_{n\alpha})$. The set of all n-tuples of polytopes in $\mathbf{R}^n$ thus has a natural partial ordering: for any two such n-tuples $\mathcal{Q}$ and $\mathcal{Q}'$, define $\mathcal{Q}' \prec \mathcal{Q} \iff \mathcal{Q}' \neq \mathcal{Q}$ and there exists $\gamma \in \mathbf{Q}^n \setminus \{0\}$ with $\mathcal{Q}_\gamma = \mathcal{Q}'$. Using this partial ordering, we introduce a colored tree structure on $\mathcal{P}$ and its $\{\mathcal{P}_\alpha\}$ as follows:

Definition *An* inductive degeneration tree *(ID tree) of $\mathcal{P}$ is a rooted tree of n-tuples of colored polytopes defined as follows:*

(T1) *The root is $\mathcal{P}$ and every P_j is colored red. If $d = dim\mathcal{P} < n$ then re-color one P_j blue and $n-d-1$ other P_j green.*

(T2) *For every non-minimal node $\mathcal{Q}$ define a child $\mathcal{Q}'$ for every maximal $\mathcal{Q}'$ with $\mathcal{Q}' \prec \mathcal{Q}$. Each child $\mathcal{Q}' = (Q'_1, \ldots, Q'_n)$ will have the same coloring as its parent except that any blue Q'_j will be re-colored green and exactly one red Q'_j will be re-colored blue.*

Let T be an ID tree of $\mathcal{P}$ and $\mathcal{Q}$ a height i node of T. Then T has depth $dim\mathcal{P}$, $dim\mathcal{Q} = i$, and $\mathcal{Q}$ has exactly i red polytopes and (iff $i < n$) one blue polytope. Also note that the subtree consisting of $\mathcal{Q}$ and all its descendants is an ID tree for $\mathcal{Q}$. An *ID cover* of $\mathcal{P}$ is a set C of vertices of the P_j with the following property: For any height i node of T ($i < n$), either its blue polytope has a vertex in C or (if $i > 0$) the i-dimensional mixed volume of its red polytopes is 0. Thus any ID cover of $\mathcal{P}$ induces an ID cover of $\mathcal{Q}$ by restricting to a subtree. If $\mathcal{P}$ is an n-tuple of Newton polytopes, we will identify C with its corresponding vector of vertex coefficients.

Let $(S_1, \ldots, S_n) = supp(F)$ and consider F_α. The construction of F_α introduces some dependence into the variables of F and the following lemma makes this notion precise.

Lemma 1 *Suppose $a \in (\mathbf{C}^*)^n$ is a root of F_α. Then at^α is a root of F_α for all $t \in \mathbf{C}^*$.*

Proof: For all $j \in \{1, \ldots, n\}$ we have $f_{j\alpha}(x) = \sum_{q \in S_{j\alpha}} c_q x^q$, $f_{j\alpha}(a) = 0$, and

$$f_{j\alpha}(at^\alpha) = t^{m(\alpha, S_j)} f_{j\alpha}(a) = 0. \quad \square$$

Let $\mathcal{P}$ be the n-tuple of Newton polytopes of F. We paraphrase theorems A and B of [Ber75] as follows:

Bernshtein's Theorem *Counting multiplicities, F has no more than $V(\mathcal{P})$ isolated roots in $(\mathbf{C}^*)^n$. Also, F_α has no roots in $(\mathbf{C}^*)^n$ for any $\alpha \in \mathbf{Q}^n \setminus \{0\} \Longrightarrow F$ has exactly $V(\mathcal{P})$ roots in $(\mathbf{C}^*)^n$, and if $V(\mathcal{P}) > 0$, the converse holds as well.* $\square$

We close this section with some topological considerations: Recall that the *Zariski topology* on $\mathbf{C}^n$ is the topology generated by taking closed sets to be algebraic sets. Henceforth, the only topology we discuss will be the Zariski topology over $\mathbf{C}^n$. An ***affine variety*** is an irreducible closed set and a ***quasi-affine variety*** is an open subset of an affine variety. A subset of $\mathbf{C}^n$ is said to be *constructible* if it is a finite union of quasi-affine varieties. Constructible sets are clearly closed under projection, complementation, finite union, and finite intersection. The following variant of elimination theory will be vital in our proofs:

Coefficient Lemma *Let $G = (g_1, \ldots, g_n)$ be a Laurent polynomial system and C a vector of constants corresponding to supp(G). With this correspondence, regard C as the vector of coefficients of G. Then the set of C such that G has no roots in $(\mathbf{C}^*)^n$ is constructible.*

Proof: [Mum76]. □

From the fact that every non-empty Zariski-open set is dense, we easily obtain the following lemma, whose proof we omit:

Lemma 2 *Any finite intersection of dense constructible sets is a dense constructible set.* □

Let X and Y be sets of coordinates chosen from $\mathbf{C}^m$, and $\Omega \subset \mathbf{C}^m$. By an X *slice* of Ω we mean the intersection of Ω with some translate of the X coordinate subspace of $\mathbf{C}^m$. Thus $\#X = m \Longrightarrow$ any X slice of Ω is Ω. Clearly, constructible sets are closed under taking slices. The X *projection* of Ω is simply the projection of the X coordinates of Ω onto $\mathbf{C}^{\#X}$. If the X projection of an X slice of Ω is dense, we call that slice *dense*. In particular, if $X \subset Y$ and every X slice of Ω is dense, then every Y slice of Ω is dense. Let X' be a set of coordinates chosen from $\mathbf{C}^{m'}$ and $\Omega' \subset \mathbf{C}^{m'}$. Then the following lifting property for slices can easily be verified:

Lemma 3 *Suppose $\varphi : \mathbf{C}^m \longrightarrow \mathbf{C}^{m'}$ is a linear map such that the restriction of φ to the X coordinate subspace of $\mathbf{C}^m$ is a surjection onto a subspace of $\mathbf{C}^{m'}$ containing the X' coordinate subspace. Then every X' slice of Ω' is dense $\Longrightarrow$ every X slice of $\varphi^{-1}(\Omega')$ is dense.* □

For a more detailed discussion of the Zariski topology, we refer the reader to [Har77] or [Mum76].

3 Proof of the Vertex Coefficient Theorem

We will prove the following theorem:

Vertex Coefficient Theorem *Let $\mathcal{P}$ be an n-tuple of polytopes in $\mathbf{R}^n$ with vertices in $\mathbf{Z}^n$, F a Laurent polynomial system with $\mathcal{P}$ as its n-tuple of Newton polytopes, and H, I vectors comprised respectively of the boundary coefficients and the internal coefficients of F. Let C be a vector of vertex coefficients corresponding to some ID cover of $\mathcal{P}$. Then there exists a constructible subset $W \subset \mathbf{C}^{\#H}$ with the following properties:*

(1) *Every C slice of W is dense.*

(2) *$(H, I) \in W \times \mathbf{C}^{\#I} \Longrightarrow F$ has exactly $V(\mathcal{P})$ roots (counting multiplicities) in $(\mathbf{C}^*)^n$.*

Furthermore, if $V(\mathcal{P}) > 0$ or $\#I = 0$, then W can be chosen to satisfy the converse of (2) as well.

Our theorem is optimal in the sense that if $V(\mathcal{P}) > 0$ and W is *any* constructible set satisfying (2), *any* vector of vertex coefficients C satisfying (1) is an ID cover of $\mathcal{P}$. The proof of this fact simply involves applying the vertex coefficient theorem to construct an ID tree of $\mathcal{P}$ from the leaves up. Also, there need not be a W satisfying the converse of (2) if $V(\mathcal{P}) = 0$ and $\#I > 0$. We have constructed a counter-example for this case, but will omit it for brevity's sake.

The existence of a W satisfying (2) (and its converse, if $V(\mathcal{P}) > 0$ or $\#I = 0$) will follow easily from the work of D. N. Bernshtein in [Ber75]. Showing that this W also satisfies (1) is slightly more involved so we will first establish a few facts.

Fix $\alpha \in \mathbf{Q}^n \setminus \{0\}$ and let i be such that $\alpha_i \neq 0$. Let $G_\alpha = (g_{1\alpha}, \ldots, g_{n\alpha})$ be the system in $n-1$ variables obtained by setting $x_i = 1$ in F_α, and $\mathcal{Q} = (Q_1, \ldots, Q_n)$ the n-tuple of Newton polytopes of G_α. Then $dim\mathcal{Q} = dim\mathcal{P}_\alpha \leq n-1$, and we obtain the following lemma:

Lemma 4 *The system G_α has a root in $(\mathbf{C}^*)^{n-1} \iff F_\alpha$ has a root in $(\mathbf{C}^*)^n$.*

Proof: ($\Longrightarrow$) Trivial.
($\Longleftarrow$) Let $a = (a_1, \ldots, a_n)$ be a root of F_α. Then by lemma 1 (using $t = a_i^{-1/\alpha_i}$) we easily obtain a root for G_α. □

Let $j \in \{1, \ldots, n\}$, $S_{j\alpha} = supp(f_{j\alpha})$, and $T_{j\alpha} = supp(g_{j\alpha})$. Also let H_α and I_α be vectors comprised respectively of the boundary coefficients and the internal coefficients of G_α. Considering $\mathcal{P}_\alpha$ and $\mathcal{Q}$, one would naturally expect that setting $x_i = 1$ in F_α induces a bijection between these two n-tuples. More precisely, we have the following lemma:

Lemma 5 *Let $\{c_q \mid q \in S_{j\alpha}\}$ and $\{c'_r \mid r \in T_{j\alpha}\}$ be the support coefficients of $f_{j\alpha}$ and $g_{j\alpha}$, respectively. For any $q = (q_1, \ldots, q_n) \in \mathbf{Z}^n$ define $q(i) = (q_1, \ldots, \widehat{q_i}, \ldots, q_n)$. Then $\{c_q \mid q \in S_{j\alpha}\}$ consists solely of boundary coefficients of F, and the map $\varphi : S_{j\alpha} \longrightarrow T_{j\alpha}$ defined by $q \mapsto q(i)$ is a bijection which preserves ID covers.*

Proof: That $\{c_q \mid q \in S_{j\alpha}\}$ consists solely of boundary coefficients of F is clear, since $S_{j\alpha}$ is the intersection of S_j with a supporting hyperplane. Also, note that any Q_j is the projection of $P_{j\alpha}$ onto the $i^{\underline{\text{th}}}$ coordinate hyperplane in $\mathbf{Z}^n$. Hence φ is clearly a surjection from $S_{j\alpha}$ to $T_{j\alpha}$. Since $\langle \alpha, q \rangle = m(\alpha, S_j)$ for all $q \in S_{j\alpha}$, it easily follows that φ is an injection. Thus, φ is a bijection between $S_{j\alpha}$ and $T_{j\alpha}$. Since φ is also a projection, we obtain that c_q is a vertex coefficient of $f_{j\alpha}$ iff $c'_{q(i)}$ is a vertex coefficient of $g_{j\alpha}$, i.e., φ maps vertices to vertices bijectively. It easily follows that φ maps ID trees of $\mathcal{P}_\alpha$ isomorphically to ID trees of $\mathcal{Q}$. Thus φ preserves ID covers. □

To conclude, define K_α such that $(H_\alpha, I_\alpha) \in K_\alpha \iff G_\alpha$ has no roots in $(\mathbf{C}^*)^{n-1}$. Then by the coefficient lemma, we have the following result:

Lemma 6 *The set K_α is constructible.* □

We now prove our main theorem:

Proof (Vertex Coefficient Theorem): Let K be such that $(H, I) \in K \iff F_\alpha$ has no roots in $(\mathbf{C}^*)^n$ for any $\alpha \in \mathbf{Q}^n \backslash \{\mathbf{0}\}$. Since we need only check finitely many F_α, K is constructible by closure under finite intersection and the coefficient lemma. Since the coefficients of F_α are boundary coefficients of F (lemma 5), it is clear that K is of the form $W \times \mathbf{C}^{\#I}$. Thus, by Bernshtein's theorem, $(H, I) \in W \times \mathbf{C}^{\#I} \Longrightarrow F$ has exactly $V(\mathcal{P})$ roots (counting multiplicities) in $(\mathbf{C}^*)^n$, and if $V(\mathcal{P}) > 0$, the converse holds as well. If $V(\mathcal{P}) = \#I = 0$, then by the coefficient lemma we can easily construct a larger W to again satisfy the converse.

We have thus found a constructible $W \subset \mathbf{C}^{\#H}$ which satisfies (2). If $V(\mathcal{P}) > 0$ or $\#I = 0$ then W also satisfies the converse of (2). To show that this W also satisfies (1), we proceed by induction:

<u>$n = 1$:</u> In this case, we determine W explicitly:

Here, $\mathcal{P} = P_1 = [m, M]$ and $F = f_1(x_1) = \sum_{q=m}^{M} c_q x_1^q$. Then $C = H = \{c_m, c_M\}$, and $I = \{c_{m+1}, \ldots, c_{M-1}\}$. Note that $F_\alpha \in \{F_{-1}, F_1\}$ for any $\alpha \in \mathbf{Q}^*$, $F_1 = c_m x_1^m$, and $F_{-1} = c_M x_1^M$. Letting $W = (\mathbf{C}^*)^{\#H}$, we obtain that $(H, I) \in W \times \mathbf{C}^{\#I} \iff F_1$ and F_{-1} have no roots in $\mathbf{C}^* \iff F$ has exactly $M - m$ roots (counting multiplicities) in $\mathbf{C}^*$ (by the fundamental theorem of algebra). W trivially satisfies property (1), so we are done.

<u>$n > 1$:</u> Assume the result true for $n - 1$. To show that every C slice of W is dense we proceed as follows:

Let W_α be such that $H \in W_\alpha \iff (H_\alpha, I_\alpha) \in K_\alpha$. By lemmas 5 and 6, W_α is well-defined and constructible, and by lemma 4, $H \in W_\alpha \iff F_\alpha$ has no roots in $(\mathbf{C}^*)^n$. Let $J \subset \mathbf{Q}^n \backslash \{\mathbf{0}\}$ be a finite set such that $\{F_\beta \mid \beta \in J\} = \{F_\beta \mid \beta \in \mathbf{Q}^n \backslash \{\mathbf{0}\}\}$. Clearly then,

$$W = \bigcap_{\beta \in J} W_\beta$$

and thus by lemma 2, it suffices to show that every C slice of W_α is dense. Since C restricts to an ID cover C_α of $\mathcal{P}_\alpha$, it suffices to show that every C_α slice of W_α is dense. By lemma 5, C_α induces an ID cover D of $\mathcal{Q}$.

So by lemma 3, it suffices to show that every D slice of K_α is dense.

Fix an ID tree defining D and let $\overline{Q}$ be the $(n-1)$-tuple of red Q_j, $\overline{G}$ the corresponding Laurent polynomial system, and $\overline{H}, \overline{I}$ vectors comprised respectively of the boundary coefficients and the internal coefficients of $\overline{G}$. Let Q_k be the one blue polytope and $g_{k\alpha}$ the corresponding Laurent polynomial. Similarly define $\dot{H}$ and $\dot{I}$ using $g_{k\alpha}$ and let $m = \#(\dot{H}, \dot{I})$. Considering $\overline{Q}$ as an $(n-1)$-tuple of polytopes in $\mathbf{R}^{n-1}$, it is easily seen that D induces an ID cover $\overline{D}$ of $\overline{Q}$. (Simply delete the $k^{\underline{\text{th}}}$ polytope from each node of the ID tree defining D.)

Let $\dot{D}$ be the vector of vertex coefficients of $g_{k\alpha}$ such that $D = (\overline{D}, \dot{D})$. Then by our induction hypothesis, there exists a constructible subset $\overline{W} \subset \mathbf{C}^{\#\overline{H}}$ with the following properties:

(1′) Every $\overline{D}$ slice of $\overline{W}$ is dense.

(2′) $(\overline{H}, \overline{I}) \in \overline{W} \times \mathbf{C}^{\#\overline{I}} \Longrightarrow \overline{G}$ has exactly $V(\overline{Q})$ roots (counting multiplicities) in $(\mathbf{C}^*)^{n-1}$.

For any $y \in \overline{W} \times \mathbf{C}^{\#\overline{I}}$, let $\mathcal{R}(y)$ be the set of roots of $\overline{G}$ in $(\mathbf{C}^*)^{n-1}$, and define $\dot{W}(y) \subset \mathbf{C}^m$ such that $(\dot{H}, \dot{I}) \in \dot{W}(y) \Longleftrightarrow g_{k\alpha}(r) \neq 0$ for all $r \in \mathcal{R}(y)$. Then $\dot{W}(y)$ is a non-empty Zariski-open subset of $\mathbf{C}^m$ since $(\dot{W}(y))^c$ is the union of $V(\overline{Q}) < \infty$ hyperplanes in $\mathbf{C}^m$. Moreover, for any $r \in \mathcal{R}(y)$ and any component $\dot{h}$ of $\dot{H}$, $g_{k\alpha}(r)$ is a non-constant affine function of $\dot{h}$. Thus the $\dot{D}$ projection of any $\dot{D}$ slice of $\dot{W}(y)$ is a non-empty Zariski-open subset of $\mathbf{C}^{\#\dot{D}}$. Define L_α as follows:

$$L_\alpha = \bigcup_{y \in \overline{W} \times \mathbf{C}^{\#\overline{I}}} \{y\} \times \dot{W}(y)$$

Then $(\overline{H}, \overline{I}, \dot{H}, \dot{I}) \in L_\alpha \Longrightarrow G_\alpha$ has no roots in $(\mathbf{C}^*)^{n-1}$. Hence, after a suitable permutation of coordinates, $L_\alpha \subset K_\alpha$. Now note that the D projection of any D slice of L_α is of the following form:

$$A = \bigcup_{\overline{y} \in \overline{\omega}} \{\overline{y}\} \times \dot{\omega}(\overline{y})$$

where $\overline{\omega}$ is the $\overline{D}$ projection of a $\overline{D}$ slice of $\overline{W}$, and $\dot{\omega}(\overline{y})$ is the $\dot{D}$ projection of a $\dot{D}$ slice of the corresponding $\dot{W}(y)$. Since $\overline{\omega}$ and $\dot{\omega}(\overline{y})$ are dense, it is easily seen that $dist(A, x) = 0$ for all $x \in \mathbf{C}^{\#(\overline{D}, \dot{D})} = \mathbf{C}^{\#D}$. Since ϵ-balls form a neighborhood base for the standard topology in $\mathbf{C}^{\#D}$, it easily follows that A is dense. Therefore every D slice of L_α is dense.

Since $L_\alpha \subset K_\alpha$, we obtain that every D slice of K_α is dense. Thus by our previous reductions, every C slice of W is dense and our induction is complete.

Earlier in our proof, we defined a constructible $W \subset \mathbf{C}^{\#H}$ satisfying property (2). We have just shown that this W satisfies property (1), so we are done. □

We remark that for any set of *boundary* coefficients satisfying (1), there exists a set of vertex coefficients with the same or lower cardinality satisfying (1). The proof is based on the following simple argument: If any coefficient satisfying a genericity condition has its corresponding point lying on the interior of a polytope facet, we can just as well replace it by a coefficient whose corresponding point is a vertex of the same facet. Thus we've lost nothing by restricting our attention to vertex coefficients.

4 Conclusion

We have seen that the vertex coefficient theorem gives the weakest possible condition under which a polynomial system achieves its maximum number of isolated roots: a set of coefficients corresponding to some ID cover must be generic. Thus the BKK root bound for a polynomial system is tight "with probability 1," even *after* most of its coefficients have been specialized. This knowledge is particularly useful for polynomial systems occuring in inverse kinematics and surface modelling, where the supports are sparse or the Newton polytopes are identical. Therefore, our result shows that the BKK bound will be tight almost always in practice.

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