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Vertex-Minimal Simplicial Immersions of the Klein Bottle in Three Space

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Abstract. Although the Klein bottle cannot be embedded in $\mathbf{R}^3$, it can be immersed there, and in more than one way. Smooth examples of these immersions have been studied extensively, but little is known about their simplicial versions. The vertices of a triangulation play a crucial role in understanding immersions, so it is reasonable to ask: How few vertices are required to immerse the Klein bottle in $\mathbf{R}^3$? Several examples that use only nine vertices are given in Section 3, and since any triangulation of the Klein bottle must have at least eight vertices, the question becomes: Can the Klein bottle be immersed in $\mathbf{R}^3$ using only eight vertices? In this paper, we show that, in fact, eight is *not* enough, nine are required. The proof consists of three parts: first exhibiting examples of 9-vertex immersions; second determining all possible 8-vertex triangulations of $\mathbf{K}^2$; and third showing that none of these can be immersed in $\mathbf{R}^3$.

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1. Introduction

Smooth embeddings and immersions of surfaces have been studied extensively, but their simplicial counterparts are not so well understood. Two important questions concerning simplicial surfaces are: What is the minimum number of vertices required to triangulate the surface? and What is the minimum number of vertices needed to produce an embedding or an immersion of the surface into Euclidean space?

For the sphere, the answer to both questions is four: a tetrahedron is a triangulation of the sphere using only four vertices, and it can be embedded in $\mathbf{R}^3$. For a torus, the minimum needed for a triangulation can be shown to be seven (see Section 4), and a somewhat surprising fact is that this triangulation also can be embedded in $\mathbf{R}^3$ using straight edges and planar faces [9]. For a Möbius band, five vertices are needed for a triangulation, and such a triangulation can be embedded easily in $\mathbf{R}^3$. On the other hand, the real projective plane (a Möbius band with a disk attached along its boundary) is a closed, non-orientable surface, so it cannot be embedded in $\mathbf{R}^3$; but it is possible to find immersions (that is, maps that are locally one-to-one) of the projective plane. Banchoff showed [2] that the number of triple points of an immersion is congruent, modulo 2, to the Euler Characteristic χ of the surface, so any immersion of $\mathbf{RP}^2$ must have at least one triple point, since

$\chi(\mathbf{RP}^2) = 1$. In a triangulation, this means that three distinct faces must intersect, and these faces can have no vertices in common by Lemma 2.1; thus at least nine vertices are needed to immerse $\mathbf{RP}^2$. In fact, no additional vertices are needed, and Brehm [7] gives explicit instructions for producing 9-vertex immersions of $\mathbf{RP}^2$. The projective plane can be triangulated using only six vertices [11], so this provides an example where an immersion requires more vertices than the minimal triangulation.

For the Klein bottle, $\mathbf{K}^2$, any triangulation requires at least eight vertices (see Section 4), and we are led to ask: How many vertices are required for an immersion of $\mathbf{K}^2$ into $\mathbf{R}^3$? We cannot use the triple-point argument as we did with the projective plane, since $\chi(\mathbf{K}^2) = 0$; indeed, immersions of $\mathbf{K}^2$ with no triple points are easy to find. On the other hand, 9-vertex immersions of $\mathbf{K}^2$ exist (some are given in Section 3), so we have the following:

THEOREM 1.1. *$\mathbf{K}^2$ can be immersed in $\mathbf{R}^3$ using nine vertices.*

In fact, there are two significantly different types of immersions of $\mathbf{K}^2$ (one cannot be continuously deformed into the other while maintaining an immersion at every step). We give 9-vertex examples of both classes, proving:

THEOREM 1.2. *Both classes of immersions of $\mathbf{K}^2$ can be achieved with nine vertices.*

The question becomes, then: Are there any 8-vertex immersions? To answer this, we first enumerate all possible 8-vertex triangulations of $\mathbf{K}^2$. For the sphere, torus, Möbius band, and real projective plane, the minimal triangulations are unique and equivelar (the vertex stars are the same at every vertex). This is not the case for the Klein bottle: in Section 5 we show that there are exactly six combinatorially distinct 8-vertex triangulations of $\mathbf{K}^2$. Finally, in Section 6 we use combinatorial and geometric means to show that none of these can be immersed in $\mathbf{R}^3$. This, together with our examples from Section 3, proves:

THEOREM 1.3. *Any immersion of a triangulation of $\mathbf{K}^2$ into $\mathbf{R}^3$ has at least nine vertices.*

2. Definitions

By a *simplexwise-linear* map, we mean a map $f\colon M^2 \to \mathbf{R}^3$ from a triangulated surface M^2 to $\mathbf{R}^3$ such that the edges and faces of M^2 are mapped as the convex linear combinations of their vertices. This corresponds directly with our intuitive concept of a triangulated manifold in $\mathbf{R}^3$. The map f is *non-degenerate* if it does not reduce the dimension of any simplex of M^2. That is, no edge of M^2 maps to a single point in $\mathbf{R}^3$, and no triangle in M^2 becomes an edge or a point in $\mathbf{R}^3$. Two simplices of M^2 are said to *intersect* if their images intersect (other than at a common vertex or edge).

The map f is an *embedding* if it is a one-to-one mapping of M^2 into $\mathbf{R}^3$. It is an *immersion* if it is locally one-to-one; that is, for each point $p \in M^2$ there is an open neighborhood U of p such that $f|_U$ is an embedding. In any mapping, a point p with such a neighborhood is said to be *immersed* while one failing to have this property is *non-immersed.*

The *double-set* $D(f)$ is the set of self-intersection for $f(M)$; that is,

$$D(f) = \{p \in M \mid \text{there exists } q \in M,\ q \neq p,\ \text{with } f(p) = f(q)\}.$$

We will refer to the image of the double-set as the *double-curve* of the immersion.

If f is a non-degenerate simplexwise-linear map, then each face is mapped one-to-one by f, hence the interiors of faces are always immersed. Since the map is linear and triangles are convex, if two triangles with a common vertex intersect, they do so in every neighborhood of the vertex, hence a vertex is immersed if and only if none of the triangles containing it intersect. Finally, if the vertices of an edge are immersed, then the two triangles sharing that edge cannot intersect, so their union is mapped one-to-one; since the interior of the edge lies in the interior of this union, the interior of the edge will be immersed whenever the vertices are. Thus we have the following:

LEMMA 2.1. *A non-degenerate simplexwise-linear map $f: M^2 \to \mathbf{R}^3$ is an immersion if and only if no two faces of M^2 with a common vertex intersect in $\mathbf{R}^3$.*

A simplicial map ϕ from a triangulated surface M^2 to itself is a *symmetry* of the triangulation if it is a one-to-one, onto mapping that preserves the dimension of simplicies; that is, ϕ is a one-to-one linear map taking vertices to vertices, edges to edges, and faces to faces. Since edges and faces are determined by their vertices, any such ϕ is completely determined by its action on the vertices of M^2; thus ϕ can be represented as a permutation of the vertices of M^2.

Two maps f, g: $M^2 \to \mathbf{R}^3$ are *equivalent* if there is a symmetry ϕ: $M^2 \to M^2$ of the surface such that $f = g \circ \phi$. Since permutations are invertible, it is clear that this forms an equivalence relation, so when we speak of an immersion f: $M^2 \to \mathbf{R}^3$ we will mean its equivalence class under this relation.

Given a triangulation M of a surface, two immersions f, g: $M \to \mathbf{R}^3$ are *image homotopic* if there exists a continuous map F: $M \times [0, 1] \to \mathbf{R}^3$ such that $F_t(p) = F(p, t)$ is an immersion for each $t \in [0, 1]$, and such that $F_0 = f$ and F_1 is equivalent to g. Two immersions f: $M_1 \to \mathbf{R}^3$ and g: $M_2 \to \mathbf{R}^3$ of different triangulations of a surface are *image homotopic* if there is a common refinement M of M_1 and M_2 such that f and g are image homotopic on this refinement. Image homotopy also defines an equivalence relation on the set of immersions of M^2; Pinkall [12] gives a complete classification of the equivalence classes of immersions under image homotopy.

In particular, for the Klein bottle there are three classes: the familiar 'tube-through-a-tube' (Figure 1), and left- and right-handed versions of the less-familiar 'twisted figure-8 tube' (Figure 1). Since the latter two are identical *via* a reflection,

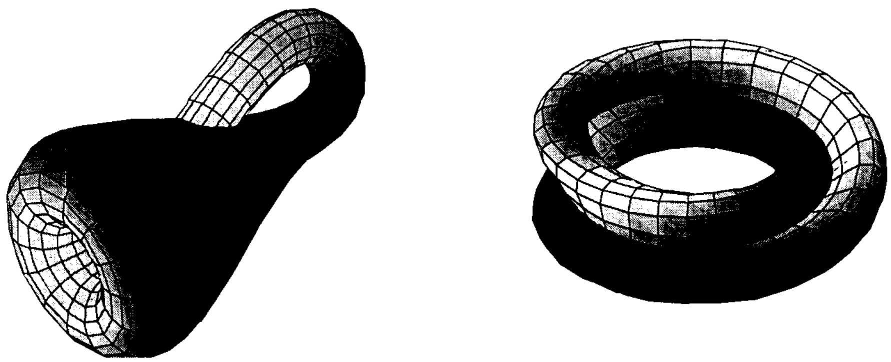


Fig. 1. The two distinct immersions of the Klein bottle: the 'tube-through-a-tube' (left) and the 'twisted figure-8 tube' (right).

any realization of one immediately yields a realization of the other, so we will consider only the two classes shown in the diagram. Note that each class exhibits a symmetry: for the first class it is a reflective symmetry about a plane, and for the other it is a rotational symmetry of order 2.

3. Some 9-Vertex Immersions

We prove in Section 6 that no 8-vertex immersion of $\mathbf{K}^2$ into $\mathbf{R}^3$ exists, but first we produce immersions using nine vertices. There are two basic approaches to creating immersed polyhedral surfaces: either start with an arbitrary (non-immersed) mapping and move or add vertices until it becomes immersed, or start with a many-vertex immersion and remove or identify vertices in such a way that the map remains immersed. Both these approaches are aided by computer-graphic analysis. A three-dimensional modelling program is nearly indispensable for this kind of work; indeed, the models described below were created initially on a graphics workstation.

We can achieve a symmetric version of the usual immersion of the Klein bottle by approaching the problem from a topological viewpoint. The essential feature of this immersion is the self-intersection, which is formed by a tube passing through the side of the surface. Simplicially, we can model this by a triangular tube through a face: the tube requires six vertices while the face has three, so we have a total of nine vertices already.

Placing vertices symmetrically is easy; the trick is to add additional triangles that complete the surface as a Klein bottle without causing additional self-intersections. To do so, we deform the tube by tilting one of the openings so that both openings face the same general direction (Figure 2). It then is possible to add triangles that close up the surface into a Klein bottle, as shown.

Geometrically, this immersion has a reflective symmetry that induces the permutation (AB) (EF) (GH) of the vertices; but this is not a symmetry of the

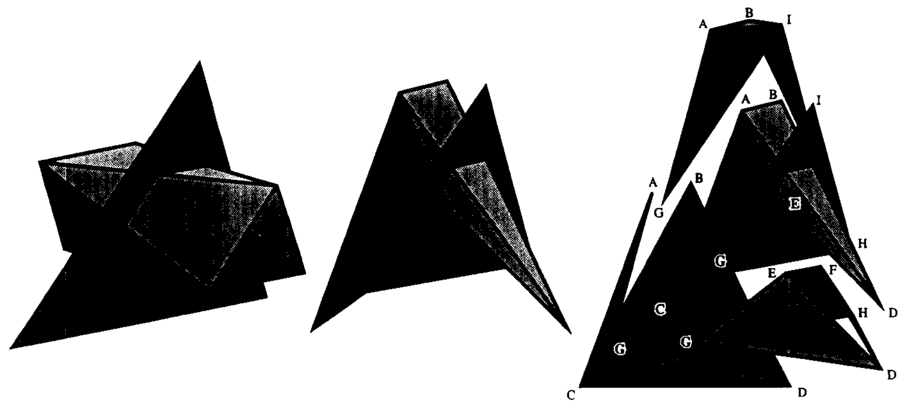


Fig. 2. A tube through a face (left) can be deformed (center) so that the openings no longer point away from each other. Additional triangles (right) can be added to complete a 9-vertex Klein bottle.

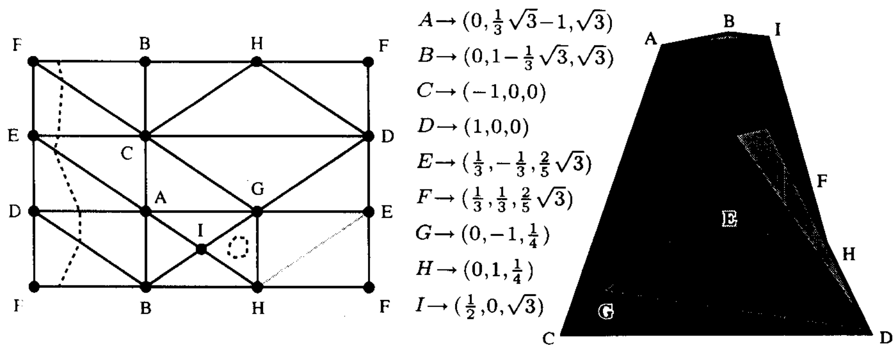


Fig. 3. A symmetric 9-vertex immersion of the Klein bottle. The self-intersection is shown by dotted lines; the shaded line is the diagonal of a symmetric planar quadrilateral in the immersion.

triangulation since it maps the edge EH to the edge FG, which does not appear in the triangulation. Note, however, that the triangles EFH and EGH lie in a plane and form a symmetric convex quadrilateral in $\mathbf{R}^3$, so if we join the two by 'erasing' the edge EH in the triangulation, we have a polyhegdral decomposition of the Klein bottle containing a quadrilateral $EFHG$ instead of the triangles EFH and EGH. To indicate this, the edge EH is shaded in Figure 3. Furthermore, this decomposition exhibits the desired reflective symmetry.

Additional views of this immersion are shown in Figure 4, in which the figure is rotated about its vertical axis. Compare this version to the one shown in Figure 1.

A representative of the second type of immersion can be generated by taking a figure-8 and moving it through space to form a tube, then attaching the ends of the tube so that the top of one end matches the bottom of the other. Another way to view

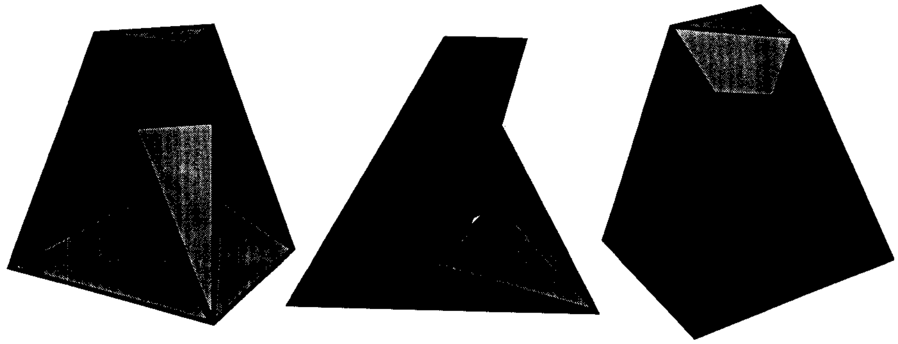


Fig. 4. Three additional views of the immersion shown in Figure 3.

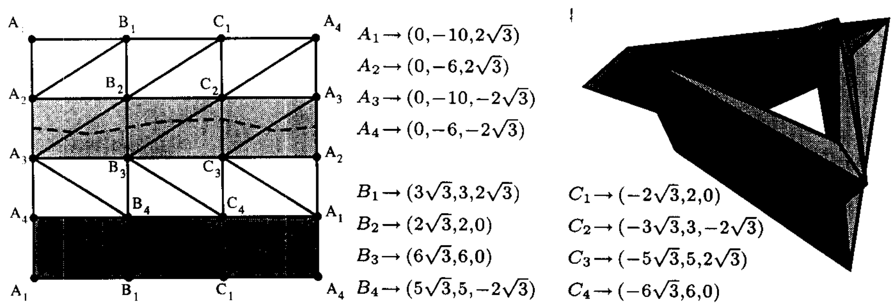


Fig. 5. A 12-vertex example of the 'figure-8 tube' immersion class. The two Möbius bands are shaded in the triangulation.

this is to move the figure-8 around a circle while rotating it by 180°. Modelling this directly by a polyhedral figure-8, which takes four vertices, and placing a copy of this figure-8 at each corner of a triangle, gives a total of 12 vertices. If the figure-8 is rotated by 60° at each corner and we attach corresponding vertices from one figure-8 to the next, a polyhedral model using only 12 vertices can be built (Figure 5).

If we consider only a small neighborhood of the crossing point of the figure-8 as it moves around the circle, each of the two line segments that cross sweeps out a Möbius band, so we get two Möbius bands that intersect along a circle in their interiors. Since a polyhedral Möbius band must have at least five vertices, this would lead us to assume that at least ten vertices should be required to create this figure. The number of vertices can be reduced, however, by allowing the Möbius bands to *share* a vertex, as in Figure 6, giving a total of nine vertices.

Once this figure is produced, it is an easy matter to attach the additional triangles that come from the top and bottom of the figure-8, producing the Klein bottle represented in Figure 7. One would expect that these triangles form a single strip, as they do with the 12-vertex model described above; since two points of the Möbius bands have been identified, however, this 'pinches' the strip, so that it

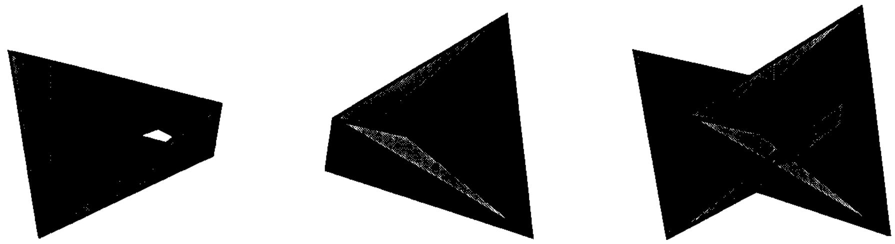


Fig. 6. Two 5-vertex Möbius bands (left) can be combined so that they have a vertex in common, but still are immersed (right).

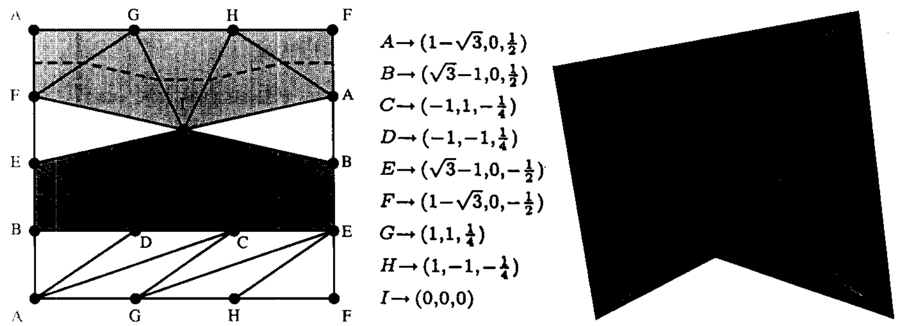


Fig. 7. A 9-vertex example of the 'figure-8 tube' immersion class. The two Möbius bands are shaded in the triangulation.

becomes a disk with two boundary points identified. The two Möbius bands and the pinched strip are clearly visible in the triangulation depicted in the figure.

This final immersion is interesting in a number of ways. First, the mapping does not exhibit the expected 'hole' that is present in the smooth case and in the 12-vertex version: the hole has collapsed completely to form the central vertex I (in fact, if we consider the 12-vertex model and identify the points A_2 with B_2, A_4 with C_1, and then B_2 with C_1, and move the vertices A_1 and A_3 apart somewhat, we achieve (essentially) the 9-vertex immersion of Figure 7. These identifications collapse the hole to a single point).

Second, the star of the central vertex I forms a 'monkey saddle'; moreover, this is *not* the standard monkey saddle, but the polyhedral exotic monkey saddle (Figure 8) that cannot be smoothed without breaking it into two standard saddles [3]. This allows the immersion to have a height function with exactly three critical points rather than the usual four, for example.

Finally, although we were unable to produce a 9-vertex triangulation of the standard immersion with true reflective symmetry, this model does exhibit the expected rotational symmetry (around the y-axis). The triangulation itself has additional symmetries, but none of these can be realized in $\mathbf{R}^3$.

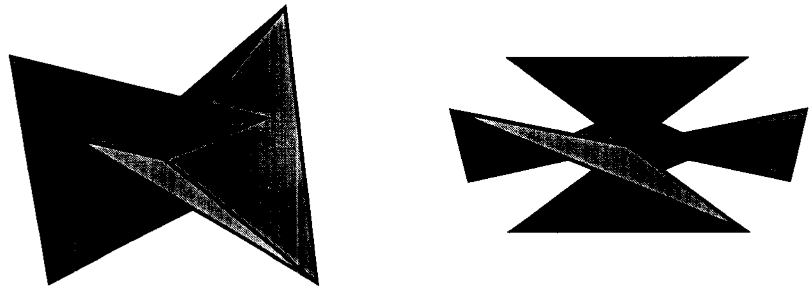


Fig. 8. The polyhedral monkey saddle at the center of the 9-vertex Klein bottle of Figure 7, as viewed from the top (left) and front (right).

4. A Klein Bottle Requires at least Eight Vertices

We have just seen a number of 9-vertex (or more) Klein bottles, and our success in producing them may prompt us to believe that 8-vertex immersions also exist. This is not the case, as we will see in Section 6. In order to prove this, however, we need to know something about what Klein bottles with fewer vertices look like. Our first task is to show that any triangulation of the Klein bottle requires at least eight vertices.

Our main tool will be the Euler formula which relates the number of vertices, edges, and faces of a simplicial complex:

$$\chi(M^2) = V - E + F$$

where V is the number of vertices, E the number of edges, F the number of faces in the triangulation, and $\chi(M^2)$ the Euler characteristic of M^2, a topological invariant. In a triangulation of a closed surface, we also have $2E = 3F$, since each face has exactly three edges, while each edge is on the border of exactly two faces; so $\chi = V - E/3$, or $E = 3(V - \chi)$. On the other hand, if there are V vertices, there can be at most $\binom{V}{2}$ edges in M^2, so $E \leq \binom{V}{2} = V(V-1)/2$. Combining these yields the inequality $V^2 - 7V + 6\chi \geq 0$. Using the quadratic equation and noting that V must be positive and greater than 3, we find

$$V \geq \frac{7 + \sqrt{49 - 24\chi}}{2}.$$

If M is the sphere, for example, $\chi = 2$, so $V \geq 4$, and since a tetrahedron realizes the sphere with only four vertices, we have confirmed our intuition that four is the minimum number of vertices for a triangulation of the sphere. For the torus, $\chi = 0$, so $V \geq 7$, and we need at least seven vertices; but a triangulation of the torus exists using only seven vertices [9], [11], so we see that seven is the minimum triangulation number for the torus.

The projective plane has $\chi = 1$, so $V \geq 6$, and since a 6-vertex triangulation of $\mathbf{RP}^2$ exists [11], this shows that six is the minimum triangulation number for the projective plane.

What about the Klein bottle? In this case, $\chi(\mathbf{K}^2) = 0$, so $V \geq 7$, but there are no 7-vertex triangulations of the Klein bottle. To see this, first note that if there were only seven vertices, then there would have to be $3(7 - 0) = 21$ edges, by the Euler formula; but $\binom{7}{2} = 7 \cdot 6/2 = 21$, so the triangulation would include all possible edges. Thus each vertex would have to be connected to every other vertex. A coloring of the vertices of such a triangulation would require seven colors, but Franklin [10], [8] proved that any graph on the Klein bottle requires no more than six colors, hence a 7-vertex Klein bottle is impossible. (Franklin's result actually deals with map coloring on the Klein bottle, but since this problem is dual to vertex coloring of graphs, the result follows. We do not need the full strength of Franklin's result, as an easy case analysis shows that the only triangulated surface that can be constructed from the complete graph on seven vertices is the torus. Indeed, Franklin proves the dual of this statement as a preliminary to his demonstration of the six-colorability of the Klein bottle, and his proof carries over directly.) Möbius [11] gave an example of an 8-vertex Klein bottle (the one labelled 323-A in Figure 18), so this shows that the minimum triangulation number for the Klein bottle is eight.

5. Determining All 8-Vertex Triangulations

We move now to a consideration of 8-vertex triangulations of the Klein bottle, and begin by constructing all possible such triangulations. We proceed in the following way: first, we describe all vertex-valence combinations that are possible with only eight vertices; for a specific one of these, we consider the possible 1-skeleta (i.e. graphs) that have this valence pattern and attempt to 'fill in' each 1-skeleton with triangles. In doing so, we obtain two different triangulations of the torus and a unique triangulation of the Klein bottle. With this in hand, we turn case-by-case to the other valence combinations, and using an edge-swapping technique, show that each of the others can be converted (in a reversible way) into the triangulation we already have found. Thus the initial triangulation can be used to produce all the others. When the analysis is complete, we find that there are exactly six combinatorially distinct 8-vertex triangulations of the Klein bottle (Figure 18).

5.1. FINDING ALL VALENCE COMBINATIONS

We want a triangulation of $\mathbf{K}^2$ with eight vertices, so $V = 8$, and from the Euler formula we find that $E = 24$ and $F = 16$. Since there are only eight vertices, we know that every vertex has valence less than or equal to seven, and since each vertex is part of a surface, its valence must be at least three. Let v be the vector (v_3, v_4, v_5, v_6, v_7) where v_i is the number of vertices of valence i. Then the following must hold:

$$v_3 + v_4 + v_5 + v_6 + v_7 = V = 8$$

TABLE I. The 10 possible vertex valence combinations

	v_3	v_4	v_5	v_6	v_7
1	0	0	0	8	0
2	0	0	1	6	1
3	0	0	2	4	2
4	0	0	3	2	3
5	0	0	4	0	4
6	0	1	0	5	2
7	0	1	1	3	3
8	0	1	2	1	4
9	0	2	0	2	4
10	1	0	0	4	3

$$3v_3 + 4v_4 + 5v_5 + 6v_6 + 7v_7 = 2E = 48$$

$$v_i = 0 \quad \text{for all } i < v_7$$

The first of these simply counts the total number of vertices; the second counts the number of edges, each edge being counted twice (once for each of its vertices); the third follows from the fact that each valence-7 vertex (called a 7-*vertex*) is adjacent to every other vertex, so any given vertex must be adjacent to every 7-vertex, hence its valence must be at least v_7, and there are no vertices of valence less that v_7.

Given these conditions, it is easy to check that there are ten possible v-vectors (see Table I). To ease notation, we will abbreviate the v-vectors by dropping the parentheses, commas, and initial zeros. For example, the v-vector (0, 1, 1, 3, 3) will be written 1133.

We may rule out the vector 2024 immediately, since each of the 4-vertices must be adjacent to all four 7-vertices, hence to none of the 6-vertices. Thus each 6-vertex is adjacent only to the four 7-vertices and the remaining 6-vertex (only five vertices in all) which contradicts the fact that it is 6-valent.

In the case of the Klein bottle, we also may rule out the vector 10043: the 3-vertex and its star can be removed and replaced by a single triangle, which would leave a 7-vertex triangulation; but any triangulation of the Klein bottle requires at least eight vertices, so case 10 is impossible.

Of the remaining cases, we turn now to the 242 vector which will prove to be instrumental in determining *all* 8-vertex triangulations.

5.2. THE 242 CASE

The 1-skeleton of any surface, in particular a triangulation of the Klein bottle, forms a graph. A complete graph on 8-vertices has $\binom{8}{2} = 8 \cdot 7/2 = 28$ edges, while

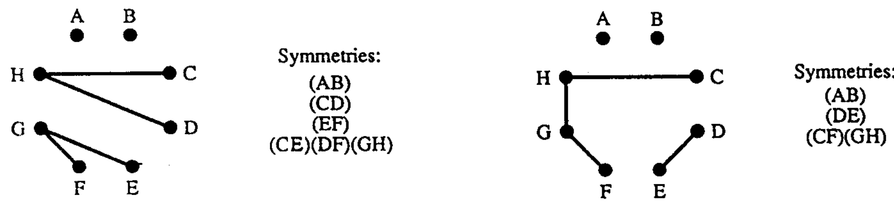


Fig. 9. The two complement graphs for the 242 valence class.

ours will have only 24. Thus exactly four of the possible edges are *not* part of the 1-skeleton. Knowing these four edges tells us which edges are in the graph, and *vice versa*. Thus we can consider the *complement graph* which is the graph made up of the missing edges. The union of a graph and its complement is the complete graph, while their intersection is just a collection of vertices, with no edges at all.

We begin by determining what graphs are possible for the 242 v-vector. In this case, two of the eight vertices are 7-valent and two are 5-valent. Let these be A, B and G, H respectively. The remaining four are 6-valent. Since the 7-vertices A and B are adjacent to every other vertex in the graph, A and B are adjacent to no vertices in the complement. Vertices G and H are 5-valent in the graph so they must be 2-valent in the complement. Similarly, the 6-vertices are 1-valent in the complement.

There are two possibilities: either G and H are adjacent in the complement or they are not. In the latter case, since G and H are not adjacent but are 2-valent, they each must be adjacent to exactly two of the 6-vertices. Furthermore, since the 6-vertices are each 1-valent in the complement, G and H have no 6-vertex in common. This gives the first complement pictured in Figure 9.

On the other hand, if G and H *are* adjacent, then each must have one more edge, necessarily to distinct 6-vertices. The remaining two 6-vertices must be adjacent to each other as the only possible fourth edge. Thus we must have one of the two complements shown in Figure 9.

Generators for the symmetry group of each complement graph are listed in the diagram. Notice that the symmetries of the complement must agree with those of the original graph, so we will use these symmetries freely in our discussions below.

5.2.1. *Case I: The First Possible Complement*

Consider the first of these complements. In it, the 5-vertices are adjacent in the original graph (since they are not adjacent in the complement), so the edge GH appears in our triangulation, and since we have a triangulation of a surface without boundary, there are exactly two triangles that contain the edge GH. Each of the two additional vertices that form these triangles must be adjacent to both G and H. Looking at the complement, we see that G and H are not adjacent to a common 6-vertex, hence the two triangles in question must be formed by the two 7-vertices

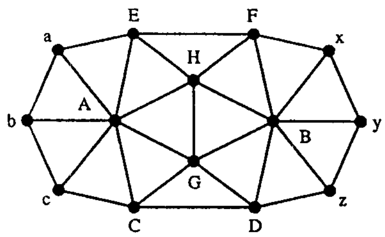


Fig. 10. The initial configuration for the first 242 complement.

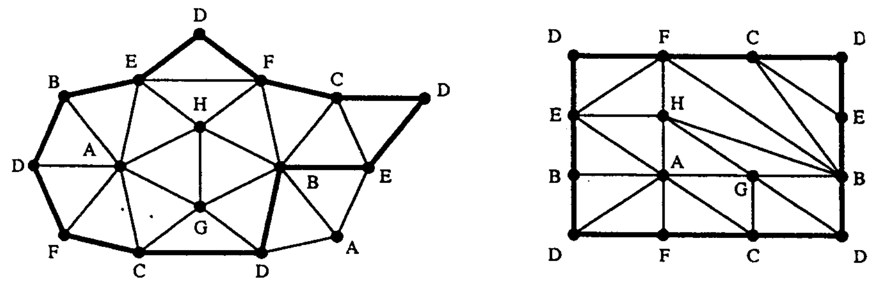


Fig. 11. The completed 242 triangulation – a torus.

(together with the edge GH).

From the complement, we know that the remaining two vertices around H are E and F, while G is adjacent to C and D. By the symmetries (AB), (EF) and (CD) we can assume we have the configuration shown in Figure 10. Now $\{x,\ y,\ z\} = \{A,\ C,\ E\}$, as all three of the latter vertices must appear on the star of B. But no matter how these are placed, A will be adjacent either to C or to E, so we will always have either triangle ACB or AEB. Since we already have triangles ACG and AEH, and since the edges AC and AE each appear in exactly two triangles, we know that either $ACc = ACB$ or $AEa = AEB$; that is, $a = B$ or $c = B$. By the symmetry $(CE)\,(DF)\,(GH)$, we can assume that $a = B$.

Then $\{b,\ c\} = \{D,\ F\}$ since these two vertices must be on the star of A; but $c \neq D$ since D already appears on the C-star (and C is not a 3-vertex), so we must have $b = D$ and $c = F$. This gives us the triangle $Aab = ABD$, but we already have triangle BDG with edge BD, so $BDz = BDA$, i.e. $z = A$. Then $\{x,\ y\} = \{C,\ E\}$, and as above, $x \neq E$ since E already is on the F-star, hence $x = C$ and $y = E$.

At this point, we have enough information to determine that the surface is orientable, hence a torus, as shown in Figure 11 (the additional vertices are determined by combining the partial stars around the two copies of F, and then the partial stars around E). Thus the first of the two possible 242 graphs does not realize a Klein bottle.

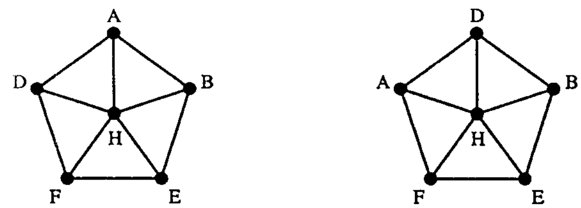


Fig. 12. The two possible H-stars for the second 242 complement.

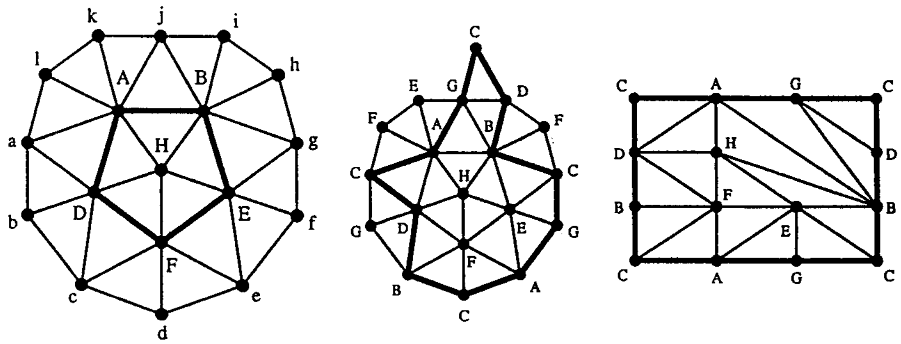


Fig. 13. The first completed H-star – a torus.

5.2.2. *Case II: The Second Complement*

Turning to the second complement graph, consider the 5-vertex H which is adjacent to the 7-vertices A and B, and the 6-vertices D, E and F. Either A and B are adjacent on the star of H or they are not. If they *are* adjacent, then since the edge DE does not exist in the graph, D and E cannot be adjacent on the star of H, hence, up to symmetry, we must have the first configuration of Figure 12. On the other hand, if A and B are *not* adjacent on the star of H, then again since DE does not exist, we must have the second configuration shown in Figure 12 (up to symmetry).

Case II-a: The first H-star. Consider the first of these two possibilities, as in Figure 13. We know that $\{a, b, c\} = \{B, C, G\}$ and $\{c, d, e\} = \{A, B, C\}$ from the complement graph. But $a \neq B$ since B already appears on the A-star, hence $B \in \{b, c\}$. Similarly, $e \neq B$ since B is on the E-star already, so $B \in \{c, d\}$. Now if $c \neq B$, then this would mean $b = B$ and $d = B$. But this is impossible since b and d are both on the star of c, thus we must have $c = B$. Using the star of H and the symmetry $(AB)\,(DE)$ we find $e = A$. This leaves $d = C$ as the only remaining vertex around F.

Since $e = A$, $\{f, g\} = \{C, G\}$, but $f \neq C$ since $d = C$ is on the e-star already, hence $f = G$ and $g = C$. This gives triangle $BgE = BCE$, which (together with triangle $cdF = BCF$) implies $h = F$. This, plus triangle $cDF = BDF$, implies $i = D$ and so $j = G$ as the only remaining vertex around B.

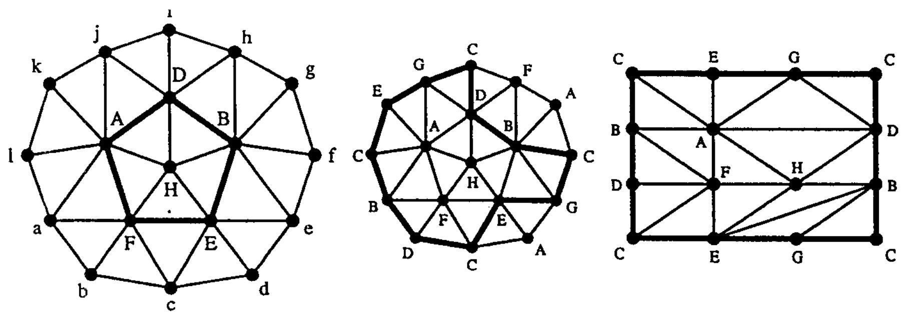


Fig. 14. The second completed H-star – a Klein bottle.

Returning to $\{a, b, c\} = \{B, C, G\}$, we know $\{a, b\} = \{C, G\}$ since $c = B$ already; but $d = C$, so $b \neq C$, hence $b = G$ and $a = C$. Using triangles $deF = CAF$ and $eEF = AEF$ we find that $l = F$ and $k = E$. This is enough to show that the map must be a torus, as indicated. This rules out the first H-star configuration.

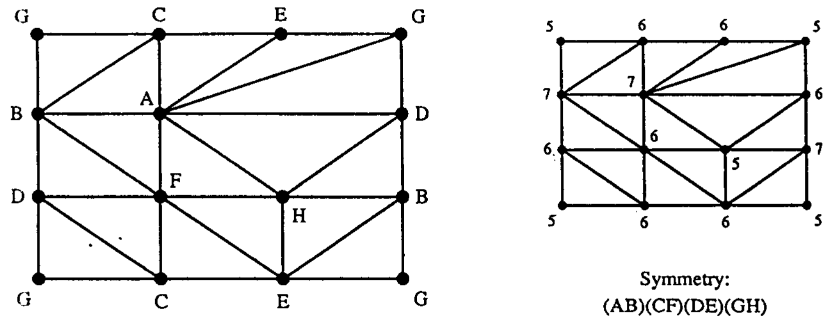
Case II-b: The second H-star. The only remaining possibility is the second H-star configuration, where A and B are not adjacent about H. Labelling this as in Figure 14, we know $\{a, b, c\} = \{B, C, D\}$ and $\{c, d, e\} = \{A, C, G\}$ (from the complement graph) so $c = C$ as the only common member of the two sets; thus $\{a, b\} = \{B, D\}$.

Now $a \neq D$ since D already appears on the A-star, hence $a = B$ and $b = D$. Then $abF = BDF$ together with triangle BDH implies $h = F$. From triangle $DBh = DBF$ and $AaF = ABF$ we find $g = A$. Recall that $\{d, e\} = \{A, G\}$, but $e \neq A$ since $g = A$ is already on the B-star, so $d = A$ and $e = G$, which means $f = C$ as the only remaining vertex about B. From $gBf = ABC$ and $AaF = ABF$ we see that $l = C$. Similarly, $cdE = CAE$ and $alA = BCA$ give $k = E$, and $Ede = EAG$ gives $j = G$. Finally, this implies $i = C$, which is enough to show that the figure is, in fact, a Klein bottle. Since there are no remaining cases for the 242 graphs, this is the unique 242 triangulation of the Klein bottle.

Moving the triangles GCB and GCD from the right side to the left, we get the representation of the unique 242 map shown in Figure 15, which we will use throughout the rest of the paper. The valence of each vertex is shown at the right. Note that the map has the symmetry $(AB)\,(CF)\,(DE)\,(GH)$.

5.3. PRODUCING NEW TRIANGULATIONS FROM KNOWN ONES

Suppose a triangulation of a closed surface contains an edge ab. Then it also must have two vertices c and d which form triangles abc and abd sharing the edge ab. Suppose that the edge cd does *not* exist. Then we can remove the edge ab and



242

Symmetry:
(AB)(CF)(DE)(GH)

Fig. 15. The unique 242 Klein bottle with its valences and symmetries.

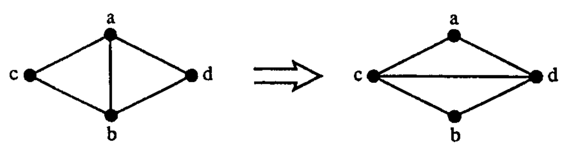


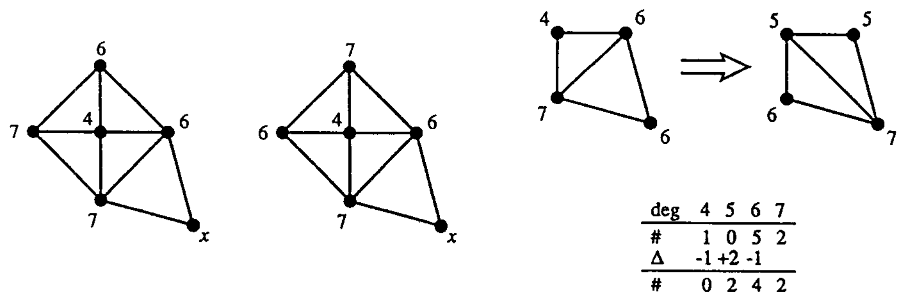
Fig. 16. Switching an edge in order to modify the valences of four vertices.

replace it with the edge cd, transforming the triangles abc and abd into triangles acd and bcd, as in Figure 16. The resulting surface remains a triangulation of the same surface as the original, but the valence of a and b are reduced by 1 each, while those of c and d are increased by 1. We may be able to produce new triangulations from existing triangulations in this way. Such an edge ab will be called a *conversion edge*, and ab together with its associated triangles a *conversion configuration.* Thus we say that ab–cd is a conversion configuration if the edge ab and triangles abc and abd exist, but the edge cd does not.

This conversion procedure is reversible, since the new triangulation produced in this way will not include the edge ab, and hence cd–ab forms a complementary conversion configuration that changes the new triangulation back into the old one.

Since we already have a unique 242 triangulation, we determine the remaining cases by demonstrating that they must each contain conversion configurations that change them into the 242 valence class; we then look for the resulting configurations in the unique 242 triangulation and from them we recover the original triangulations. The uniqueness of these configurations in the (unique) 242 triangulation will guarantee the uniqueness of the remaining cases.

Since there are only four missing edges in an 8-vertex triangulation of the Klein bottle, and since a conversion configuration requires a missing edge, it is easy to find all possible conversion configurations in the 242 triangulation, hence all



deg	4	5	6	7
#	1	0	5	2
Δ	-1	+2	-1	
#	0	2	4	2

Fig. 17. The star of the 4-vertex in a 1052 triangulation must include a 76–46 conversion configuration that changes it into a 242 triangulation.

possible 8-vertex triangulations of the Klein bottle.

5.4. THE REMAINING VECTORS

Consider first the v-vector 1052 and suppose there is a realization of it as a surface. The star of the 4-vertex must contain the two 7-vertices and two of the 6-vertices, hence it must be one of the configurations at the left in Figure 17.

Consider vertex x. In either case, x cannot be one of the vertices already shown, since the stars of the vertices adjacent to x already contain all these vertices; in particular, x is not the 4-vertex, nor is it adjacent to the 4-vertex. Since the only remaining vertices in the triangulation are 6-valent, x must be a 6-vertex. Thus we have a 76–46 conversion configuration, and so we can replace the 76 edge by a 46 edge. The 7- and 6-vertices become 6- and 5-valent, while the 4- and 6-vertices become 5- and 7-valent. Thus there is a net loss of one 4-vertex and one 6-vertex with a corresponding gain of two 5-vertices. This converts the 1052 triangulation into a 242 one containing a 75–56 conversion configuration. But no such configuration appears in our (unique) 242 triangulation of the Klein bottle (the 5-vertices are not adjacent in the 242 map), hence no 1052 triangulation of the Klein bottle exists.

To rule out the 80 v-vector, we first note that the only possible conversion configuration is 66–66; if we assume that no such configuration exists, we arrive at a contradiction using arguments similar to those used to produce the 242 triangulation above. So any 80 triangulation must have a 66–66 conversion configuration. But there is no complementary 77–55 configuration in our 242 triangulation, so no 80 triangulation exists.

Similar arguments show that there are no Klein bottles in the 1214 class, and a unique triangulation in each of the classes 1133 and 404, since their complementary conversion configurations appear exactly once (up to symmetry) in the unique 242 triangulation.

For the 161 v-vector, slightly more complicated arguments are needed, which are helped by constructing the complement graph. Using this to analyze the star of

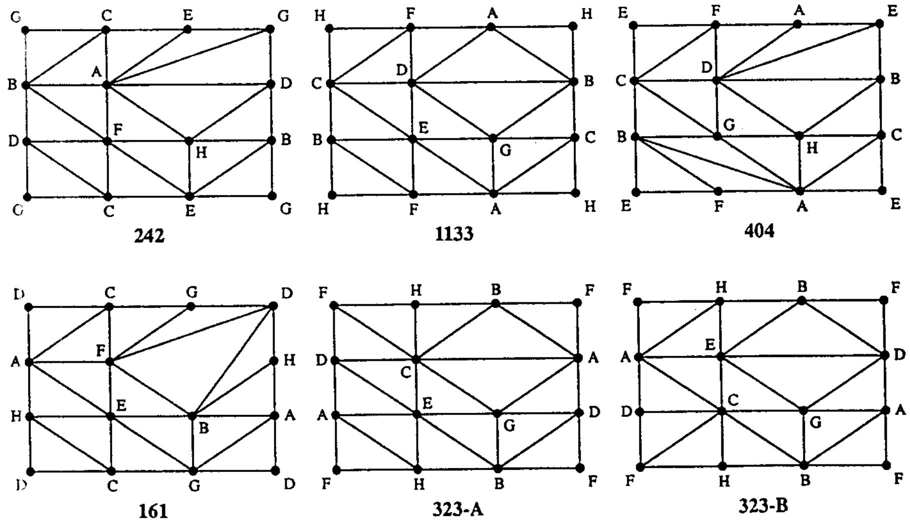


Fig. 18. The six combinatorially distinct 8-vertex triangulations of the Klein bottle.

the 5-vertex, we find that there must be a 66–56 conversion configuration leading to a 76–55 configuration in a 242 triangulation. Two of these appear in our unique 242 triangulation, but they are the same under the symmetry of the 242 triangulation, so there is a unique 161 triangulation of the Klein bottle.

For the 323 v-vector, there are two possible complement graphs: in one the 5-vertices are each connected to the other two, so in the original, the star of any 5-vertex contains a 76–66 conversion configuration transforming it into a 1214 triangulation. Since no 1214 triangulation of the Klein bottle exists, this complement cannot realize a Klein bottle. For the second complement, arguments similar to the ones used for the other cases show that a 75–56 configuration exists which becomes a 76–46 configuration in a 1133 triangulation, a figure that appears twice in our unique 1133 version. The 323 triangulations they generate are distinct since the first of these contains a triangle whose vertices are the three 7-vertices (323-A in Figure 18), while the second does not (323-B). Thus there are exactly two distinct 323 triangulations of the Klein bottle.

This completes the possible v-vectors, so we see that there are exactly six distinct 8-vertex triangulations of the Klein bottle, as pictured in Figure 18.

Generators for the symmetry groups of these triangulations are given in Table II. Note that the symmetry group in the case 404 is isomorphic to the dihedral group D_4 of order 8. We will use these symmetries in Section 6 in order to reduce the number of cases we need to analyze.

TABLE II. Generators for the symmetries of the 8-vertex Klein bottles

Map	Symmetry
242	$(AB)(CF)(DE)(GH)$
1133	No Symmetries
404	$(AD)(B)(C)(EF)(GH)$ $(AC)(BD)(EH)(FG)$
161	$(A)(BE)(CD)(FG)(H)$
323-A	$(AC)(B)(DE)(FH)(G)$
323-B	$(AC)(B)(DE)(FH)(G)$

6. Attempting to Immerse the Klein Bottle

In this section, we show that none of the six triangulations found in the previous section can be immersed in $\mathbf{R}^3$. To do so, we analyze the double-sets that could occur in an immersion buy using combinatorial means to determine which edges can intersect which faces. The homology of these double-sets can be compared to the expected homology of the double-sets of an immersed Klein bottle, and this will rule out all but one of these triangulations.

This last case can be handled through a more careful analysis of the double-set: it is possible to determine that certain edges and faces *must* intersect, while others must *not*. Using this information, a geometric argument shows that the required configuration cannot be achieved in $\mathbf{R}^3$ without forcing additional, disallowed intersections.

6.1. DOUBLE-SETS OF IMMERSED KLEIN BOTTLES

The one-dimensional homology of the Klein bottle (using $\mathbf{Z}/2\mathbf{Z}$ coefficients) has two generators: one whose tubular neighborhood is a cylinder; the other a Möbius band. Let these be $[\alpha]$ and $[\beta]$; then the homology classes are $[0]$, $[\alpha]$, $[\beta]$, and $[\alpha+\beta]$. Banchoff [1] proved that for any immersion of the Klein bottle, the double-set is homologous to $[\alpha]$. The self-intersection of the two different classes of immersions are shown in Figure 19: these are homologous since their union bounds a region.

Suppose we have a closed curve embedded in $\mathbf{K}^2$ that is completely contained within a disk, as in Figure 20. Then the curve must belong to the trivial homology class, since disks are acyclic. Thus the double-set in an immersion cannot be contained entirely within a disk, since the double-set of an immersed Klein bottle must be in class $[\alpha]$, not $[0]$.

Similarly, suppose that a closed curve is contained within a Möbius band, as in Figure 20. Then since the Möbius band supports only two homology classes, the trivial class and $[\beta]$, and neither of these is $[\alpha]$, we can conclude that the double-set of an immersed Klein bottle is not contained completely within a Möbius band.

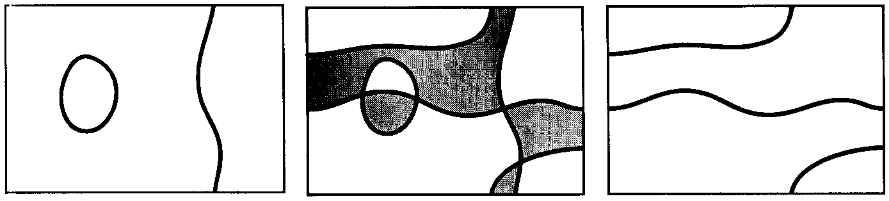


Fig. 19. The homology of the double-sets for the two types of immersions (left, right). They represent the same class since their union bounds a region (center).

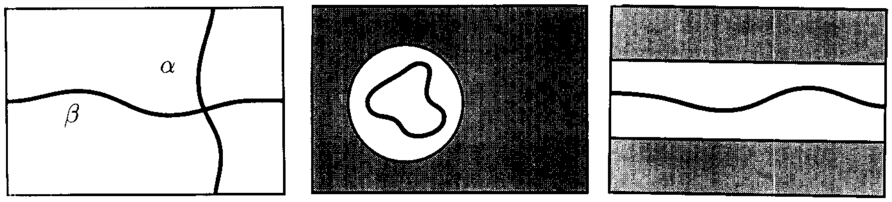


Fig. 20. The two homology classes for the Klein bottle. Curves trapped in a disk or a Möbius band cannot be in class $[\alpha]$.

These observations will be useful in determining which 8-vertex triangulations cannot be immersed: an analysis of their possible double-sets will show that they must lie either in a disk or a Möbius band, which means that the double-sets have the wrong homology.

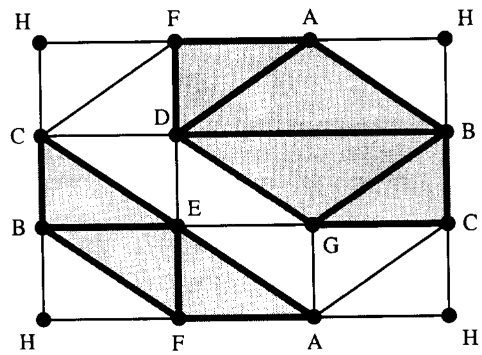
For an immersion of $\mathbf{K}^2$ into $\mathbf{R}^3$, we may assume that no two edges intersect non-trivially, and that no vertex meets a face of which it is not a member. This is reasonable, since if a vertex intersects a face, then moving it slightly to one side of the face will remove the vertex from the double-set without changing whether the map is immersed or not. Similarly, if two edges intersect, then moving one of their end-points slightly will eliminate this intersection without destroying the immersion. The effect of this assumption is to guarantee that if an edge and a face intersect, then they intersect in their interiors. Combining this with Lemma 2.1, we see that for such an immersion, no vertex lies on the double-set.

With this in mind, we can calculate which edges can be involved in the self-intersection of $\mathbf{K}^2_8$. We know that each edge ab is a member of exactly two triangles, abc and abd. Suppose the double-set passes through the edge ab; then ab intersects the interior of some face A not having a or b as a vertex. This face must also intersect the triangles abc and abd, by our assumption that edges and faces meet in their interiors. If $\mathbf{K}^2_8$ is immersed, then, by Lemma 2.1, the face A cannot contain any vertex in the set $\{a, b, c, d\}$. Thus the vertices of A must be three of the four remaining vertices of $\mathbf{K}^2_8$.

Given a specific triangulation and a specific edge ab, we can compute which faces A can intersect ab by checking the four possible triples of vertices that are not in $\{a, b, c, d\}$ to see if any of them form triangles in the given triangulation. For, example, in the 1133 triangulation of Figure 18, we use this technique to calculate

	ABD	ABH	ACG	ACH	ADF	AEF	AEG	BCE	BCG	BDG	BEF	BFH	CDE	CDF	CFH	DEG
AB-DH	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
AC-GH	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.	.
AD-BF	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
AE-FG	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
AF-DE	.	.	.	.	.	.	.	.	.	O	.	.	.	.	.	.
AG-CE	.	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.
AH-BC	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	O
BC-EG	.	.	.	.	O	.	.	.	.	.	.	.	.	.	.	.
BD-AG	.	.	.	.	.	.	.	.	.	.	.	.	.	.	O	.
BE-CF	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
BF-EH	.	.	O	.	.	.	.	.	.	.	.	.	.	.	.	.
BG-CD	.	.	.	.	.	O	.	.	.	.	.	.	.	.	.	.
BH-AF	.	.	.	.	.	.	.	.	.	.	.	.	O	.	.	O
CD-EF	.	O	.	.	.	.	.	.	.	.	.	.	.	.	.	.
CE-BD	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
CF-DH	.	.	.	.	.	.	O	.	.	.	.	.	.	.	.	.
CG-AB	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
CH-AF	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.	O
DE-CG	.	O	.	.	.	.	.	.	.	.	.	O	.	.	.	.
DF-AC	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
DG-BE	.	.	.	O	.	.	.	.	.	.	.	.	.	.	O	.
EF-AB	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
EG-AD	.	.	.	.	.	.	.	.	.	.	.	O	.	.	O	.
FH-BC	.	.	.	.	.	.	O	.	.	.	.	.	.	.	.	O

O = Face and edge can intersect
. = Face and edge can not intersect



1133

Fig. 21. Edge-cut analysis of the 1133 triangulation. Bold edges cannot intersect the double-set, while shaded edges mark triangles that can only be entered from one side. The double-set is homotopic to a curve that remains entirely in the non-shaded region, which is a Möbius band, hence the triangulation cannot be immersed.

which faces can intersect each of the edges, as summarized in the table in Figure 21. Note that eight edges cannot intersect any face in the triangulation; this implies that the double-set cannot pass through these edges (they are marked in bold in the figure).

Shaded lines are marked in those triangles that have only one remaining edge that can be involved in self-intersection. If the double-set passes through one of these edges, it must also return through this same edge in such a way that the double-set is homotopic to a curve that does not cross the shaded edge at all; that is, the double-set is homotopic (hence homologous) to one that remains entirely in the non-shaded portion of the triangulation. Thus the unshaded region can be used to visualize the homology classes to which the double-sets belong.

In the case of the 1133 triangulation, the unshaded region forms a Möbius band (see Figure 23), and since we know that, in an immersion, the double-set must be homologous to a curve in this region, the double-set cannot be in the class $[\alpha]$. But the double-set of an immersed Klein bottle must be in class $[\alpha]$, a contradiction. From this we conclude that 1133 cannot be immersed in $\mathbf{R}^3$.

Performing this same type of calculation for the other five triangulations, we obtain the regions shown in Figure 22. As before, the double-set must be homologous to a curve that lies entirely in the unshaded region of each diagram, from which we can conclude immediately that the 404 triangulation admits only double-sets in the trivial homology class, hence 404 cannot be immersed in $\mathbf{R}^3$.

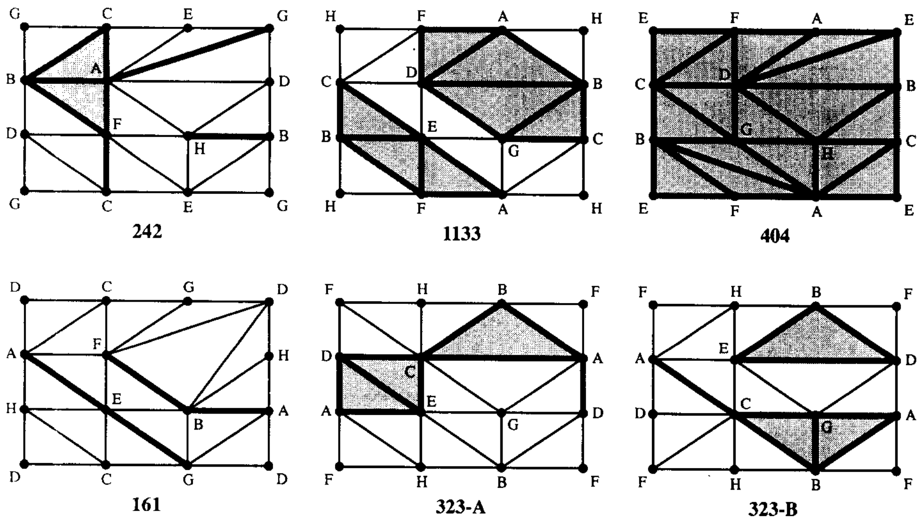


Fig. 22. Edge-cut analysis of all six triangulations.

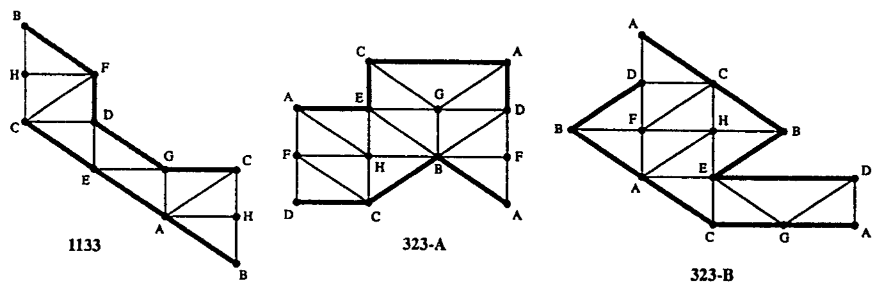


Fig. 23. The non-shaded regions form Möbius bands for these three triangulations, so their double-sets cannot be in the required homology class.

The unshaded area of the 323-A triangulation forms a Möbius band (see Figure 23). We have seen that such a region cannot support curves in the class $[\alpha]$, so the 323-A triangulation cannot be immersed in $\mathbf{R}^3$. Similarly, the unshaded region in 323-B also is a Möbius band, so this triangulation, too, cannot be immersed in $\mathbf{R}^3$.

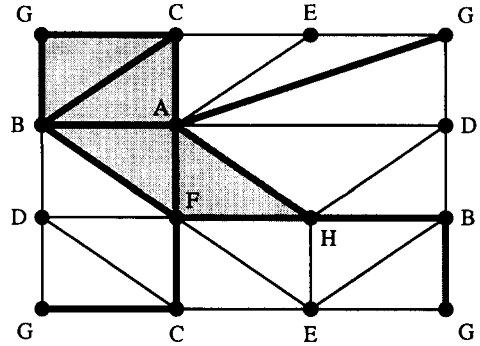
For the remaining two triangulations, we need to look at specific edges more closely, and will increase the shaded areas to the point where the unshaded regions become disks or Möbius bands, thus ruling out an immersion in these cases as well.

6.2. THE 242 CASE

Consider the 242 triangulation. From the table in Figure 24 we see that the edge AH can intersect only faces BCG and BEG, while the edges AD and DH can intersect just face CEF. These three faces are the only ones that can intersect

```
      |AAAAAAABBBBBCCCE
      |BBCDDEFCDDEEDDEF
      |CFEGHGHGFHGHFGFH
------+----------------
AB-CF |................
AC-BE |................
AD-GH |..............O.
AE-CG |........OO......
AF-BH |.............O..
AG-DE |................
AH-DF |.......O..O.....
BC-AG |...............O
BD-FH |..O..O..........
BE-GH |............O...
BF-AD |................
BG-CE |....O.O.........
BH-DE |................
CD-FG |...........O....
CE-AF |.........O......
CF-DE |................
CG-BD |......O........O
DF-BC |.....O..........
DG-AC |...........O...O
DH-AB |..............O.
EF-CH |...O............
EG-AB |............O...
EH-BF |...O.........O..
FH-AE |.......O.....O..

O = Face and edge can intersect
. = Face and edge can not intersect
```



242

Fig. 24. A more careful edge-cut analysis of the 242 triangulation.

ADH (all others contain one of the vertices A, D or H), so if the double-set crosses AH, it cannot leave ADH by AD or DH, hence it must return through AH; therefore, we can revise the diagram for 242 by shading the edge AH and, by symmetry, the edge BG.

This separates the graph into two disks (the stars of D and E). But disks can support only curves in the trivial homology class, so by our remarks above we see that 242 cannot be immersed in $\mathbf{R}^3$.

6.3. THE 161 CASE

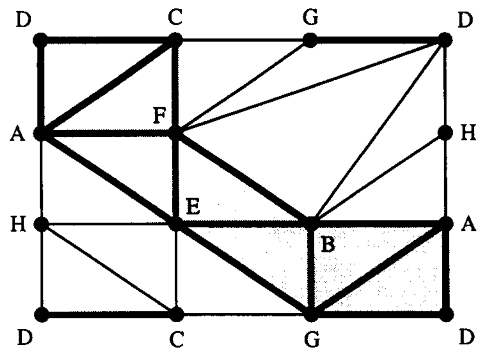
The only remaining case is the 161 triangulation. Using the intersection table in Figure 25, we note that if the double-set passes through the edge AC, the face that cuts it must be BEG, and in this case BEG must intersect ACF; but no edge can intersect ACF, hence BEG must also intersect AF or CF. The face BEG can intersect only the edges AC and CD, however, which is a contradiction; thus the double-set does not cross AC. By symmetry, it also does not intersect AD so we can show these edges in bold, as in Figure 25.

Suppose for the moment that the double-set does not pass through the edge DG. Then the revised triangulation is as shown in Figure 25. The unshaded region becomes a Möbius band, which we have seen cannot support the double-set of an immersion.

We can assume, then, that the double-set *does* pass through DG. The only face that can intersect DG is CEH, so CEH must intersect ADG. From the table, we see that no edge can intersect ADG and AD cannot intersect CEH, so CEH

	A	A	A	A	A	A	A	B	B	B	B	C	C	C	C	D
	B	B	C	C	D	E	E	D	D	E	E	D	E	E	F	F
	G	H	D	F	G	F	H	F	H	F	G	H	G	H	G	G
AB-GH	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
AC-DF	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.	.
AD-CG	.	.	.	.	.	.	.	.	.	O	.	.	.	.	.	.
AE-FH	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
AF-CE	.	.	.	.	.	.	.	.	O	.	.	.	.	.	.	.
AG-BD	.	.	.	.	.	.	.	.	.	.	.	.	.	O	.	.
AH-BE	.	.	.	.	.	.	.	.	.	.	.	.	.	.	O	O
BD-FH	.	.	.	.	.	.	.	.	.	.	.	.	O	.	.	.
BE-FG	.	.	O	.	.	.	.	.	.	.	.	O	.	.	.	.
BF-DE	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
BG-AE	.	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.
BH-AD	.	.	.	.	.	.	.	.	.	.	.	.	O	.	O	.
CD-AH	.	.	.	.	.	.	.	.	.	O	O	.	.	.	.	.
CE-GH	.	.	.	.	.	.	.	O	.	.	.	.	.	.	.	.
CF-AG	.	.	.	.	.	.	.	.	O	.	.	.	.	.	.	.
CG-EF	.	O	.	.	.	.	.	.	O	.	.	.	.	.	.	.
CH-DE	O	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
DF-BG	.	.	.	.	.	.	O	.	.	.	.	.	.	O	.	.
DG-AF	.	.	.	.	.	.	.	.	.	.	.	.	.	O	.	.
DH-BC	.	.	.	.	.	O	.	.	.	.	.	.	.	.	.	.
EF-AB	.	.	.	.	.	.	.	.	.	.	.	O	.	.	.	.
EG-BC	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
EH-AC	.	.	.	.	.	.	.	O	.	.	.	.	.	.	.	O
FG-CD	.	O	.	.	.	.	O	.	.	.	.	.	.	.	.	.

O = Face and edge can intersect
. = Face and edge can not intersect



161

Fig. 25. A more careful edge-cut analysis of the 161 triangulation.

must also intersect AG. Furthermore, CEH cannot intersect AB or BG, so an edge of CEH must intersect ABG. From the table, we see this can only be CH, hence CDH must also intersect ABG, and since CDH cannot intersect AB or AG, CDH must intersect BG.

Thus we have DG intersecting CEH, and BG intersecting CDH but not CEH.

Now CEH determines a plane in $\mathbf{R}^3$, and since DG intersects it transversely, D and G are on opposite sides of this plane. But this means the entire triangle CDH lies on the opposite side of the plane from G (Figure 26). Since BG intersects CDH, this implies that B and D are on the same side of CEH, the one opposite G.

Consider the stereographic projection of $\mathbf{R}^3$ onto the plane determined by CEH using G as the projection point. Since DG intersects CEH, the projection of the point D onto the plane lies in the interior of the triangle CEH. Since CDH shares the edge CH with CEH and the projection of D lies interior to CEH, the projection of the entire triangle CDH lies interior to the triangle CEH (see Figure 26). In particular, the point where the edge BG intersects the face CDH is projected into the interior of CEH. But since B lies on the opposite side of the plane from G, this implies that the edge BG must intersect the interior of the triangle CEH. This is a contradiction, since we know that BG does not intersect CEH.

Thus the 161 triangulation cannot be immersed in $\mathbf{R}^3$. Since this is the last triangulation to consider, we see that no 8-vertex triangulation of the Klein bottle can be immersed in $\mathbf{R}^3$.

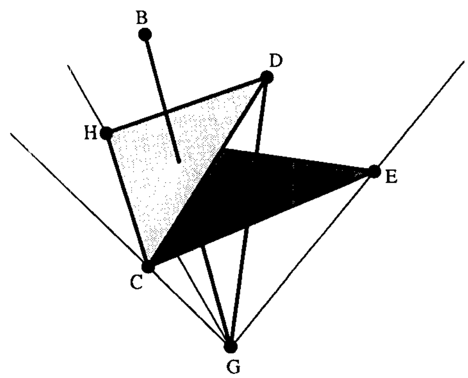


Fig. 26. The geometry of the required self-intersection forces BG to intersect CEH, which is not allowed.

7. Conclusion

In this paper we have determined how many vertices are needed to form an immersion of the Klein bottle, and have done so for each of the different classes of immersion. We may ask the same question for other surfaces: In each of the immersion classes, how many vertices are needed? We have a partial answer to this already, as we have minimal immersions (or embeddings) for the sphere, torus, and real projective plane. Furthermore, Brehm [5], [6] and Bokowski and Brehm [4] give embeddings of minimal triangulations for the orientable surfaces of genus 2, 3 and 4, so we know the answer for at least one class for each of these surfaces.

Pinkall [12] has computed the different classes of immersions for all surfaces in $\mathbf{R}^3$, and for the projective plane there are two, these being left- and right-handed version of the Boy's Surface. Brehm's models [7] can be built with either handedness, so both classes can be achieved minimally with nine vertices. The sphere has only one class, which requires four vertices. The torus, however, has two classes: one an embedding and one an immersion. The 7-vertex torus is an example of the embedded class, but we do not have a vertex-minimal example of the immersed class. This class can be represented by a figure-8 tube with a full twist. Since a figure-8 tube with only a *half*-twist forms a Klein bottle, as discussed in Section 3, which requires nine vertices, it is unlikely that a representation of the twisted figure-8 torus with fewer than nine vertices can be found.

The techniques developed in this paper should prove adequate to find the vertex-minimal immersion in the case of the torus, but they become less readily applicable as the genus of the surface is increased. The edge-cut analysis relies on the fact that there are few vertices in the triangulation, and the case-by-case analysis is aided by the fact that there are few triangulations to check. Neither of these is true for surfaces of higher genus. Nevertheless, these approaches may be valuable in determining upper and lower bounds for the number of vertices needed for immersions of such surfaces.

The process of edge-swapping used in Section 5 shows considerable promise as a technique of determining all triangulations of a surface with a given number of vertices, provided there are sufficiently many 'missing edges'.

Finally, the idea of an immersion generalizes to manifolds of higher dimension, so we may consider the related question: How few vertices are required to immerse a triangulated M^m into $\mathbf{R}^n$ in each of its possible immersion classes? Although this paper completes the study of the vertex-minimal Klein bottles in $\mathbf{R}^3$, it raises a considerable number of related questions that should be a rich source of problems for some time to come.

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