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Oriented Matroid Pairs, Theory and an Electric Application

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ABSTRACT. The property that a pair of oriented matroids $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ on E have free union and no common (non-zero) covector generalizes oriented matroid duality. This property characterizes when certain systems of equations whose only nonlinearities occur as real monotone bijections have a unique solution for all values of additive parameters. Instances include sign non-singularity of square matrices and generalizations of positive definiteness given by Fiedler and Pták. Other instances of this property include various kinds of characterizations of when an electric network problem is well-posed. Such characterizations have been given in terms of matrix pairs by Sandberg and Willson and in terms of electrical network graphs by Duffin, Minty, Hasler and Neirnyck, and by Nishi and Chua.

Cases of the general common covector problem are classified. Natural matroid rank conditions are sufficient for a common covector to exist. An algorithm to construct a common covector by composing certain fundamental cocircuits is given. If $\mathcal{M}_L$ and $\mathcal{M}_R$ have two common bases with opposite relative orientation (chirotope value) then $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ have a common covector. This abstracts the realizable case of a determinant expansion having terms of opposite sign. An open problem is whether $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ having a common covector implies that $\mathcal{M}_L$ and $\mathcal{M}_R$ have two common bases with opposite relative orientation, when the latter have one common basis and are not realizable. A weaker conjecture is $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ have a common covector if and only if $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ have a common vector, when $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis.

The computational complexity of the problem "Do $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ have a common covector?" when $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis is at least as high as telling if a square matrix is not sign solvable or if a digraph has an even directed circuit. When $\text{rank}(\mathcal{M}_L^{\perp}) + \text{rank}(\mathcal{M}_R) < |E|$ the problem is strongly NP-complete and it generalizes non L-matrix sign pattern detection.

Introduction

The theory begins with the definition of an elementary property of a pair of real linear subspaces, say the row spaces of two real matrices. The signature function $\sigma : \mathbf{R}^E \to \{0, +, -\}^E$ maps each real tuple into the pattern of its signs.

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DEFINITION 0.1. Two linear subspaces L_1 and L_2 of $\mathbf{R}^E$ for finite set E have a common covector if there exist $x \in L_1$ and $y \in L_2$ for which $\sigma(x) = \sigma(y) \neq 0$.

This definition of common covectors for realized oriented matroid pairs naturally generalizes. The covector set of an oriented matroid is denoted by $\mathcal{L}$. The dual of an oriented matroid $\mathcal{M}$ is denoted by $\mathcal{M}^{\perp}$. See [**1**] for an exposition of oriented matroids that emphasizes how the matroid dual abstracts the orthogonal complement of a real linear space.

DEFINITION 0.2. A pair of oriented matroids $\mathcal{M}_L^{\perp}, \mathcal{M}_R$ on the same ground set E have a common covector X if $X \in \mathcal{L}(\mathcal{M}_L^{\perp}) \cap \mathcal{L}(\mathcal{M}_R)$ and $X \neq 0$.

Orthogonal pairs of subspaces, and, more generally, dual pairs of oriented matroids, never have common covectors.

The Electric Network Model. This section distills material from [**11, 12, 18, 21, 26, 29, 30, 34**]. A finite, lumped analog DC electric network model is a set of devices and a network graph which represents their interconnection. The graph nodes model maximally connected electrically conducting regions typically comprised of physically connected metal wires. Some graph edges correspond to idealized two terminal electrical devices such as voltage sources (batteries), resistors, diodes, etc. Each terminal is identified with a node. Every two terminal device will be identified with its edge. Other devices such as transistors, ideal operational amplifiers, and other kinds of controlled sources have three or more terminals. For each device, the model has some edges between some pairs of that device's terminals. See [**12**, Ch. 13].

The usual schematic diagram of such a network uses solid lines for the wires, dots for wire junctions and standard symbols for the devices. See parts (a-c) of Figure 1 and Figure 4 for examples. The edges for devices with three or more terminals are usually omitted. One node is often distinguished as the "ground." The ground node is understood to be connected by wires between multiple ground symbols in addition to the explicit wire lines.

Let E be the set of network graph edges. The matroids that motivate our subject all have ground sets that are either subsets of E or subsets of disjoint unions of copies of E. Many are graphic or cographic.

The electric network model determines a set of real equations on $2|E|$ variables: Variable v_e for $e \in E$ represents the potential difference or *voltage* between the endpoints of e, and i_e represents the rate of charge flow or *current* through edge e. Flow is conserved at nodes. The equations fall into two classes: the structural laws (Kirchhoff's laws) and the constitutive laws (the device characteristics). Kirchhoff's voltage law (KVL) says $v_E = (v_e, e \in E)$ is in the cocycle space of the network graph. Kirchhoff's current law (KCL) says $(i_e, e \in E)$ belongs to the cycle space. See [**21, 29, 30**]. The fact that these spaces are orthogonal is known in the electric circuit theory literature as Tellegen's theorem.

The constitutive law for a voltage source edge e (i.e., an ideal battery) is v_e equals a constant. For current source edge e, i_e equals a constant. These constants are considered independent "input signals" to the system. They will generally be parameters. Notice that when, say, e is a voltage source, the current i_e is an unknown variable.

The constitutive law for a positive, linear resistor edge e is called Ohm's law: $v_e = r_e i_e$, where constant $r_e > 0$ is called the resistance (of e). The reciprocal

$g_e = r_e^{-1}$ is called the conductance. For a diode the law is $i_e = D(v_e) - D(0)$ where $D : \mathbf{R} \to \mathbf{R}^+$ is exponential. For a more realistic model with reverse breakdown, this current function would be onto $\mathbf{R}$ but still monotonic. An ideal operational amplifier device has 4 terminals and two disjoint edges, say e and f. The output edge f is incident to the ground. The constitutive law is $v_e = 0$ and $i_e = 0$. This law is the limit, as A goes to infinity and v_f is bounded, of the more realistic (DC) law $i_e = 0$ and $v_f = Av_e$. The constant A here is called the open loop gain, which is typically at least 10^5 and is over 10^7 in some modern commercial units [**23**]. Either model is a good approximation when the non-ideal operational amplifier has sufficiently large gain, the system is stable (as a dynamic system stabilized by feedback), and the voltages and currents of the amplifier are within the ranges for "active operation." See [**12**, Ch. 9 and 11].

DEFINITION 0.3. The network model is called *well-posed* when for all real values for the input signal parameters, the equations (in the voltage and current variables) have a unique solution. Otherwise it is *ill-posed.*

The linear or non-linear constitutive laws for many devices other than the (constant) sources are generally known only approximately. The central motivating question for this paper is what combinatorial properties of the network graph can distinguish three possibilities: (1) the network model is well-posed for every choice of continuous, monotone increasing constitutive law functions; (2) the model is well-posed for some and ill-posed for other choices of such constitutive laws; (3) the model is ill-posed for all such choices. In this paper, we will relate the answer to this question given by [**18, 19, 20, 12**, Ch. 31] and work cited below to results about the common covector problem for oriented matroid pairs. For example, the uniqueness proofs given when the constitutive laws for two terminal devices are monotone cite Tellegen's theorem. However, they only use the its consequence that the network graph's cycle and cocycle spaces over $\mathbf{R}^E$ have no common covector.

We analyze the voltage divider in part (a) of Figure 1 for an example. Let us eliminate the current and voltage variables for the voltage source V_0. Kirchhoff's laws constrain the rest of the voltages $v = (v_e, v_f)$ and currents $i = (i_e, i_f)$ to affine lines in $\mathbf{R}^2$. The equations below show representative homogeneous coordinates in $\mathbf{R}^3$ of the points on these lines as s and t range over $\mathbf{R}$. The corresponding $\mathbf{R}^2$ coordinates v and i are also shown. The cocycle and cycles spaces of the 2 edge circuit graph are denoted by $\mathcal{C}^\perp$ and $\mathcal{C}$ respectively. This graph is the contraction by edge V_0 of the original network graph.

$$[s,1]\begin{array}{|ccc|} & e & f \\ \hline 0 & -1 & 1 \\ 1 & V_0 & 0 \\ \hline \end{array} = [1, V_0 - s, s] \qquad [t,1]\begin{array}{|ccc|} & e & f \\ \hline 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline \end{array} = [1, t, t]$$

$\{v = (V_0 - s, s) \;:\; s \in \mathbf{R}\}$ = voltages v feasible under KVL. $\{i = (t,t) \;:\; t \in \mathbf{R}\}$ = currents (flows) i feasible under KCL.

$$v \in v^0 + \mathcal{C}^\perp \qquad\qquad i \in i^0 + \mathcal{C}$$

Ohm's law for this problem is

$$v = iR = i\begin{pmatrix} r_e & 0 \\ 0 & r_f \end{pmatrix}.$$

To prove that a solution is unique, let $\delta v = v - v'$ and $\delta i = i - i'$ for two solutions $(v; i)$, $(v'; i')$. Then $\delta v \in \mathcal{C}$, $\delta i \in \mathcal{C}^\perp$ and $\delta v = i'R - iR$. If δv and δi are not

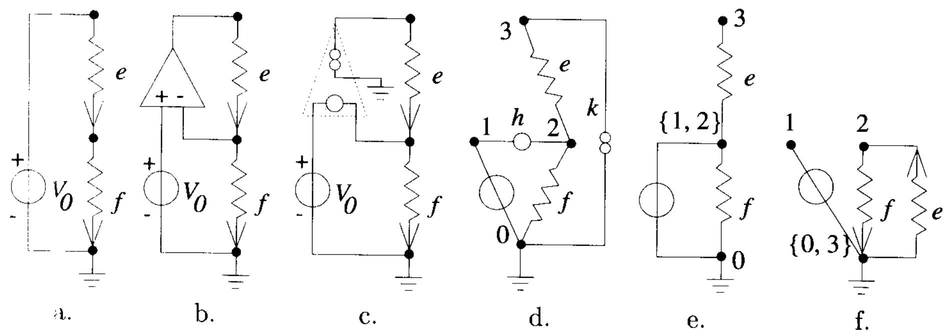

FIGURE 1. Examples of electrical network models with unique solutions because the oriented matroids coding feasible current and voltage sign patterns have no common covector. Edges V_0 are voltage sources with value V_0. Part (a) is a classical voltage divider. Part (b) is the schematic of a feedback system with an ideal operational amplifier. Part (c) shows that device with its edges. The nullator edge $h = (1,2)$ and the norator edge $k = (0,3)$ are ideal two-terminal devices that signify the constitutive law for the amplifier. Part (d) is the network graph drawn as in graph theory books. Suppose the amplifier's constitutive laws are used to eliminate v_n and i_n for both $n = (1,2)$ and $n = (0,3)$. The remaining voltages are constrained by KVL to the cocycle space $\mathcal{C}_V^\perp$ of the "voltage graph" shown in part (e). The remaining currents are constrained to the cycle space $\mathcal{C}_I$ of the "current graph" of part (f). [**11, 26**]

both zero, then $\sigma(\delta v) \neq \sigma(\delta i)$ since $\mathcal{C}$ and $\mathcal{C}^\perp$ have no common covector. However $\sigma(\delta v) = \sigma(\delta i)$ since the resistance values (entries in diagonal matrix R) are positive. The same argument would apply for nonlinear resistance functions $v_e = r_e(i_e)$ that are strictly monotone increasing. The oriented matroids with no common covector are realized by matrices $M_L^\perp = [-1\ 1]$ (whose row space is $\mathcal{C}^\perp$) and $M_R = [1\ 1]$ (whose row space is $\mathcal{C}$.)

A network model as in part (b) of Figure 1 with an ideal operational amplifier can be expressed by a network model as in parts (c-d) of Figure 1 with two kinds of special device edges in addition to resistor, voltage sources and current sources: *nullators* and *norators*. Kirchhoff's laws constrain the voltage drops and currents to the cocycle and cycle spaces of this graph as before. However, a nullator edge h represents the further constraint that its current i_h is 0 and its voltage v_e is 0. Edge k is called a norator to indicate that it conducts current but is not subject to any constitutive law constraint directly.

Let us analyze the network given by part (b) and equivalently by part (c) of Figure 1 as we did for part (a). This time, we eliminate the variables v_h, i_h, v_k and i_k for the nullator and norator in addition to the voltage source.

Here are the equations:

$$[s,1]\begin{array}{c} \begin{matrix} e & f & \end{matrix} \\ \boxed{\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & V_0 \end{matrix}} \end{array} = [1, s, V_0] \qquad [t,1]\begin{array}{c} \begin{matrix} & e & f \end{matrix} \\ \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}} \end{array} = [1, t, t]$$

$\{v = (s, V_0) \ : \ s \in \mathbf{R}\}$ = voltages v feasible under KVL.

$$v \in v^0 + \mathcal{C}_V^\perp$$

$\{i = (t,t) \ : \ t \in \mathbf{R}\}$ = currents (flows) i feasible under KCL.

$$i \in i^0 + \mathcal{C}_I$$

Kirchhoff's voltage law now constrains the remaining voltages $v = (v_e, v_f)$ to (Constant)$v^0 + \mathcal{C}_V^\perp$, where $\mathcal{C}_V^\perp$ is the cocycle space of the *voltage graph* (part (e) of Figure 1). The KCL constraint on $i = (i_e, i_f)$ uses the cycle space $\mathcal{C}_I$ of a *different* graph (part (f) of Figure 1) called the *current graph.*

A solution for the model in parts (b-c) of Figure 1 is unique because the *non-orthogonal* row spaces $\mathcal{C}_V^\perp$ and $\mathcal{C}_I$ of $M_L^\perp = [1\ 0]$ and $M_R = [1\ 1]$ respectively do not have a common covector.

Let $\mathcal{G}$ be a network graph with nullators P and norators Q. After eliminating both variables for each edge in $P \cup U$, the voltages feasible under KVL are the cocycles $\mathcal{C}_V^\perp$ of the voltage graph $\mathcal{G}/P\backslash Q$. The feasible currents are the cycles $\mathcal{C}_I$ of the current graph $\mathcal{G}/Q\backslash P$. Such distinct graphs to represent KVL and KCL constraints for nullator and norator models as well as models with controlled sources are described in [**10, 26, 11, 18, 19, 34**]; see also [**6**]. Realistic (DC) models for multiterminal devices such as transistors can be expressed either by generally nonlinear relations among voltages and currents of the device's edges (called *ports*) or by a network of 2-terminal devices whose edges either have (generally nonlinear) resistance or conductance functions, or are nullators or norators. Thus a multiterminal device would be replaced by a subnetwork composed only of 2-terminal devices (which are called 1-ports). Nullators and norators generalize to matroid pairs $\mathcal{M}_L, \mathcal{M}_R$:

- When e is a nullator the reduction by e is $(\mathcal{M}_L/e), (\mathcal{M}_R\backslash e)$.
- When e is a norator the reduction by e is $(\mathcal{M}_L\backslash e), (\mathcal{M}_R/e)$.

Related Work and Summary. The no common covector condition for uniqueness also can be used to establish the *existence* of solutions when rank conditions are satisfied and the nonlinear real functions are onto as well as monotone. This theme appears in the work of Sandberg and Willson [**36**] (see also the survey [**42**]). We will relate this theory, expressed in terms of $\mathcal{W}_0$ pairs of square matrices [**40, 41**], to oriented matroids in section 4. However, similar results developed with graph theory appear in [**32**] and in [**18, 19, 20**]. Many of the arguments given in [**18, 19**] extend immediately to oriented matroids because they are based on Minty's painting theorem and simple properties of digraphs which together can axiomatize oriented matroids. Earlier work of Duffin [**13**] and Minty [**28**] treated only orthogonal subspaces as sources for sign relationships that imply existence and uniqueness. Also see [**35**].

The role of common bases in telling if an electric network model is well-posed with generic coefficients in linear constitutive laws is apparent in [**18, 19**] and is treated explicitly in [**29, 30, 34**]. Common bases used to address solution properties appears in [**8**]. Common bases and algorithms for cases of the graph theory analysis [**32**] are used in [**27**]. See also the literature on symbolic simulation [**43, 12,** Ch. 52], and the matrix tree theorem [**6, 8, 7**]. Ported matroid Tutte polynomials [**9**] will be extended to oriented matroids and applied to electric problems in a future publication.

Section 1 begins with theorems that show that natural conditions on the ranks of two oriented matroids and their union are sufficient for them to have a common covector. G. Ziegler mentioned [**44**] that such results could be proved using the methods of [**5**]. However, our proofs construct the covector by elementary algorithms.

The rank conditions do not apply to those cases of the linear subspace (i.e., realizable oriented matroid) common covector problem that are formulated to distinguish possibilities (1) from (2) among the three possibilities given after definition (0.3). Instead, in these cases, a common covector exists if and only if there are terms of opposite sign in the Laplace expansions of certain determinants. In section 1.1 we prove that the natural generalization of this term sign condition to general oriented matroid chirotopes implies that common covectors exist. We leave as an open problem the converse. A combinatorial proof of the converse might lead to algorithms that search for "substructures" (i.e., minors) in electrical networks and other nonlinear systems that are necessary and sufficient for non-uniqueness in some instances of systems with a given "structure." In the graph of a network that includes transistors, each transistor appears as a triangle with one distinguished edge[1]. For networks with the (quite accurate) Ebers-Moll model used for transistors, Nielsen and Willson applied the theory of $\mathcal{W}_0$ pairs of matrices to prove [**31**] that all instances of networks with the same structure (i.e., network graph) have a unique solution if and only if the graph does not have a "feedback structure" graph minor, which is a triangle of parallel edge pairs from exactly two transistors with the two distinguished edges in two distinct sides. (See [**38, 12**, Ch. 31].)

Section 2 classifies instances of the common covector problem and summarizes the results. Properties known only for the realizable case are given in section 3. In particular, the existence of a common covector is equivalent to the existence of a common vector when the rank conditions do not imply either. These properties are applied in section 4 to give an oriented matroid interpretation of the class of $\mathcal{W}_0$ matrix pairs [**41**]: A pair (A, B) of square real matrices is in $\mathcal{W}_0$ if and only if the row spaces of $[A\ B]$ and $[I\ -I]$ have full rank and no common covector. We use this to derive a problem dual to the original problem given by Willson (see Theorem 4.3) where matrix pairs of type $\mathcal{W}_0$ were applied.

Section 5 has some rather pessimistic facts about the computational complexity of the common covector problem. First, when $\text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R) < |E|$, the common covector problem (even in the realizable case) is strongly NP-complete. Second, the case relevant to the given applications (complementary rank and free union) includes the (complement of the) sign non-singularity (SNS) question for square matrices of signs [**3**]. This problem is known to be polynomial time equivalent to the even cycle problem for digraphs [**22**]. These problems have been recognized as deep, unsolved combinatorial problems for which it is unknown whether they lie in complexity classes P, NP-complete or in between [**22, 39**].

Rigidity and Elasticity. It would be interesting to know electrical analogs of rigidity [**17**] properties, or if some rigidity properties are equivalent to no common covectors. We mention the basic analogies. Stress (a signed scalar for each bar) in

[1] The distinguished edge represents the emitter and collector terminal pair.

a multidimensional bar framework is an analog of edge conductance in an electrical network: the force vector in a bar is analogous to current; joint position is analogous to absolute node potential. The fact that a non-zero stress must be positive in some edges and negative in others is a manifestation of the fact that the (dual pair of) graphic and cographic oriented matroids of the same graph have no common covector. We therefore note that the electric network analysis problem "given the conductances find the voltages" and the problem applicable to rigidity analysis "determine what stresses a given framework can sustain" are opposite problems.

The electrical analog of an elastic "spring" network with some vertices pinned is a network with fixed positive conductances whose only sources are voltage sources all joined at a common node. The elastic analog of parts (b-c) of Figure 1 is easy to visualize: A robot standing on the ground watches node 2 and pulls up on node 3 just enough to align node 2 with the top of a rod that stands V_0 meters high.

Standard Theory and Terminology. Our use of standard matroid and oriented matroid terminology and results generally follows [**2**]. Matroid union is denoted by $\vee$. The row space of matrix M is denoted by $L(M)$. The sign tuples of members of this space comprise the covectors $\mathcal{L}(M)$ of the oriented matroid $\mathcal{M}(M)$ realized by M. The collection of *vectors* of oriented matroid $\mathcal{M}$ is denoted by $\mathcal{V}(\mathcal{M})$ The vectors of realizable $\mathcal{M}(M)$ are the sign tuples of members of the orthogonal complement of $L(M)$.

Section 1 uses the tableau matrix notation to express fundamental cocircuits and conditions for basis exchanges that is developed in Chapter 10 of [**2**]. Our notation differs slightly as we include the current basis elements in the column set of the tableau. A matrix decomposed horizontally into disjoint blocks A, B, etc. is denoted by $[A\ \ B\ \ \cdots]$.

1. Common Covector Existence Theorems

The first theorem is used for the others. Its proof contains an algorithm to construct a common covector efficiently by composing covectors with cocircuits.

THEOREM 1.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids on the ground set $A \uplus Z \uplus S \uplus R$. Assume the covectors $C_0 \in \mathcal{L}(\mathcal{M}_1)$ and $D_0 \in \mathcal{L}(\mathcal{M}_2)$ satisfy the properties:*

1. *$A \neq \emptyset$ and $C_0(a) = D_0(a) \neq 0$ for all $a \in A$,*
2. *$S \uplus Z$ is independent in $\mathcal{M}_1$ and $C_0(S \uplus Z) = 0$, and*
3. *$R \uplus Z$ is independent in $\mathcal{M}_2$ and $D_0(R \uplus Z) = 0$.*

Then $\mathcal{M}_1, \mathcal{M}_2$ have a common (non-zero) covector $C \in \mathcal{L}(\mathcal{M}_1) \cap \mathcal{L}(\mathcal{M}_2)$ that is compatible with both C_0 and D_0. In other words, $C_0 \preceq C$ and $D_0 \preceq C$.

PROOF. There exist sets of fundamental cocircuits $\{c_e : e \in S\}$ and $\{d_e : e \in R\}$ and one covector C_0 and D_0 in each of $\mathcal{M}_1$ and $\mathcal{M}_2$ as described by the two tableaux in Figure 2. Observe $c_e(S\backslash e) = 0$, $d_e(R\backslash e) = 0$, and $c_e(Z) = d_e(Z) = 0$. The assertions marked "//" in the algorithm can be verified by induction.

Input: Covector C_0 of $\mathcal{M}_1$, a covector D_0 of $\mathcal{M}_2$, a cocircuit c_e of $\mathcal{M}_1$ for each $e \in S$, and a cocircuit d_e of $\mathcal{M}_2$ for each $e \in R$ as described.

Output: Common covector $C = D$ of $\mathcal{M}_1$ and $\mathcal{M}_2$.

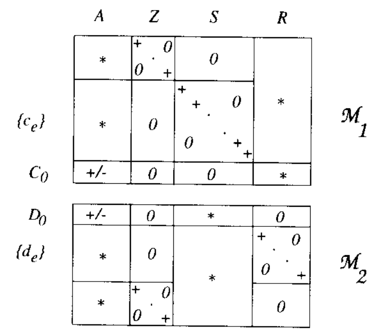

FIGURE 2. The tableau for $\mathcal{M}_1$ shows the cocircuits c_e for $e \in S$, other cocircuits for $e \in Z$, and the covector C_0. These cocircuits are fundamental with respect to some basis that extends $Z \cup S$. The algorithm constructs a covector C by starting with $C \leftarrow C_0$ and composing $C \leftarrow C \circ (\pm c_e)$ to make $C(e) = D(e)$ when necessary, while similar operations are applied to D using cocircuits from $\mathcal{M}_2$.

(1) $C \leftarrow C_0$; $D \leftarrow D_0$;
Repeat
(2) For each $e \in S$ such that $D(e) \neq 0$ and $C(e) = 0$
(3) do $C \leftarrow C \circ (D(e)c_e)$;
//$C(f)$ for $f \in (S \backslash e) \cup A \cup Z$ is unchanged.
//$C \in \mathcal{L}(\mathcal{M}_1)$.
(4) For each $e \in R$ such that $C(e) \neq 0$ and $D(e) = 0$
(5) do $D \leftarrow D \circ (C(e)d_e)$;
//$D(f)$ for $f \in (R \backslash e) \cup A \cup Z$ is unchanged.
//$D \in \mathcal{L}(\mathcal{M}_2)$.
(6) Until $C(S) = D(S)$ & $D(R) = C(R)$;

Throughout the execution, whenever $C(e) \neq D(e)$, such e must satisfy either (i) $e \in S$, $D(e) \neq 0$, and $C(e) = 0$, or (ii) $e \in R$, $C(e) \neq 0$, and $D(e) = 0$. Each execution of step (3) or (5) makes $C(e) = D(e) \neq 0$ for one such e. The step might cause. for some f, one of $C(f)$ or $D(f)$ to change so that $C(f) = D(f)$ is no longer true. However, $C(f) \neq D(f)$ where both $C(f) \neq 0$ and $D(f) \neq 0$ is impossible. A later step will make $C(f) = D(f)$ true again. The common value will then be non-zero so neither $C(f)$ nor $D(f)$ will change again. Therefore the execution must terminate after $|R| + |S|$ or fewer composition steps. □

The next lemma is a special case of the theorem that follows it.

LEMMA 1.2. *If* $|E| <$ *rank*$(\mathcal{M}_L^\perp)+$*rank*$(\mathcal{M}_R)$ and $\mathcal{M}_L^\perp \vee \mathcal{M}_R$ *is a free matroid, that is,* $E = B_L \cup B_R$ *for some* $B_L \in \mathcal{B}(\mathcal{M}_L^\perp)$ *and* $B_R \in \mathcal{B}(\mathcal{M}_R)$, *then* $M_L^\perp, M_R$ *have a common covector. Moreover, every sign tuple* X *over* $B_L \cap B_R$ *with some non-zero entry is the restriction of a common covector.*

PROOF. Since rank$(\mathcal{M}_L^\perp)$+rank$(\mathcal{M}_R) > |E|$, every B_L and B_R as above satisfy $B_L \cap B_R \neq \emptyset$. The pair of tableaux for these bases is shown in part (a) of Figure 3. For each $e \in B_L \cap B_R$, $\{c_e\}$ and $\{d_e\}$ are the fundamental cocircuits in $\mathcal{M}_L^\perp$, M_R, respectively, of e. Given a sign tuple $X \neq 0$, let C_0 be an arbitrary composition of

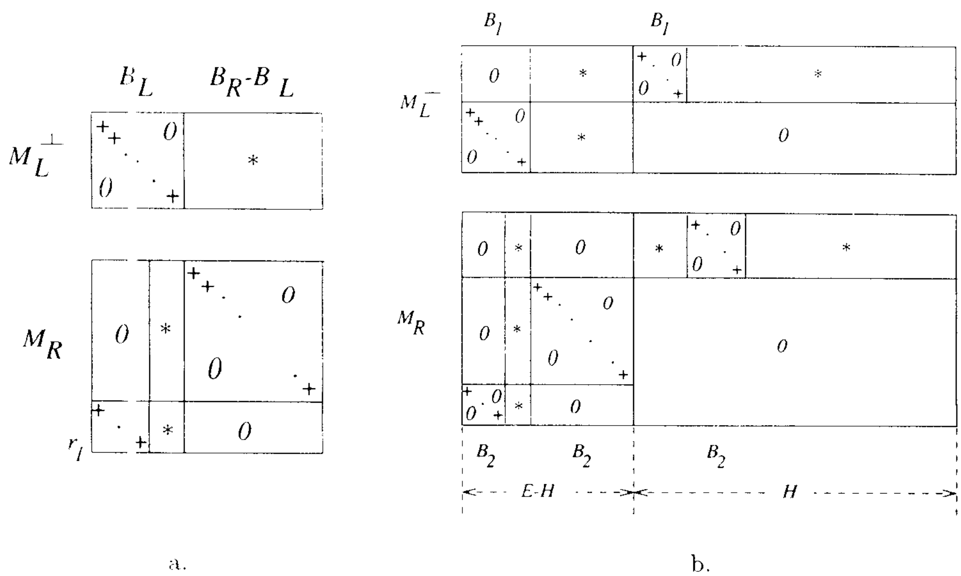

FIGURE 3. Tableau pairs used respectively in the proofs of Lemma 1.2 and Theorem 1.3.

$X(e)c_e$ and let D_0 be an arbitrary composition of $X(e)d_e$. Now C_0 and D_0 satisfy the hypotheses of Theorem 1.1 with $A \cup Z = B_L \cap B_R$, $A \neq \emptyset$, $S = B_L \backslash B_R$ and $R = B_R \backslash B_L$. $\square$

THEOREM 1.3. *If* $\operatorname{rank}(\mathcal{M}_L^\perp \vee \mathcal{M}_R) < \operatorname{rank}(\mathcal{M}_L^\perp) + \operatorname{rank}(\mathcal{M}_R)$, *then* $\mathcal{M}_L^\perp, \mathcal{M}_R$ *have a common covector.*

PROOF. Let $B_1 \in \mathcal{B}(\mathcal{M}_1)$, $B_2 \in \mathcal{B}(\mathcal{M}_2)$ and $H \subseteq E$ realize the extrema in the formula for the rank r of the union matroid $\mathcal{M}_1 \vee \mathcal{M}_2$ below.

$$r := \max_{\substack{B_1' \in \mathcal{B}(\mathcal{M}_1) \\ B_2' \in \mathcal{B}(\mathcal{M}_2)}} |B_1' \cup B_2'| = \min_{H' \subseteq E} (\operatorname{rank}_1(H') + \operatorname{rank}_2(H') + |E \backslash H'|)$$

For these values, the general inequalities

$$\begin{aligned} |B_1 \cup B_2| &= |(B_1 \cup B_2) \cap H| + |(B_1 \cup B_2) \backslash H| \\ &\leq |B_1 \cap H| + |B_2 \cap H| + |E \backslash H| \\ &\leq \operatorname{rank}_1(H) + \operatorname{rank}_2(H) + |E \backslash H| \end{aligned}$$

become equations and so

$$\begin{aligned} &|B_1 \cap H| = \operatorname{rank}_1(H), && |B_2 \cap H| = \operatorname{rank}_2(H), \\ &B_1 \cap B_2 \cap H = \emptyset, && \text{and } |(B_1 \cup B_2) \backslash H| = |E \backslash H|. \end{aligned}$$

Hence there are cocircuits described by the tableau pair shown in part (b) of Figure 3. Lemma 1.2 can now be applied to the cocircuits whose support is contained in $E \backslash H$. $\square$

1.1. An Orientation Condition. Theorem 1.3 does not apply to the case covered by the next theorem. For realizable oriented matroids, the proof and the proof of its converse (Theorem 3.1) are much easier. Whether the converse of Theorem 1.6 is true for non-realizable oriented matroid pairs is an open problem.

DEFINITION 1.4. The oriented matroids $\mathcal{M}_L$ and $\mathcal{M}_R$ with chirotope functions χ_L and χ_R have a pair B_1 and B_2 of common bases with opposite relative orientations if the ordered sets B_1 and B_2 satisfy

$$\chi_L(B_1)\chi_R(B_1)\chi_L(B_2)\chi_R(B_2) = -. \tag{1.1}$$

REMARK 1.5. Matroids $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis if and only if $\text{rank}(\mathcal{M}_L) = \text{rank}(M_R)$ and $\mathcal{M}_L^\perp \cup \mathcal{M}_R$ is a free matroid on E.

THEOREM 1.6. *If $\mathcal{M}_L$ and $\mathcal{M}_R$ have a pair B_1 and B_2 of common bases with opposite relative orientation, then $\mathcal{M}_L^\perp, \mathcal{M}_R$ have a common covector.*

PROOF. Let B_1 and B_2 satisfy (1.1) with minimum separation $|B_1 \backslash B_2| = |B_2 \backslash B_1|$. Consider the tableau of the fundamental cocircuits in $\mathcal{M}_L^\perp$ relative to basis B_1^c of $\mathcal{M}_L^\perp$ and the tableau of the fundamental cocircuits of $\mathcal{M}_R$ relative to basis B_1.

To conveniently denote submatrices of these tableaux, we assume E is ordered with the subset $B_1^c \cap B_2^c$ coming first, followed by $B_2 \backslash B_1$, $B_1 \backslash B_2$ and finally $B_1 \cap B_2$. Let P be the submatrix of the tableau for $\mathcal{M}_L^\perp$ whose rows are indexed by $B_2 \backslash B_1$ and whose columns are indexed by $B_1 \backslash B_2$. When this tableau is restricted to rows $B_2 \backslash B_1$, matrix P appears as a block in $[0 \ \ I \ \ P \ \cdots]$ where I is the identity matrix whose rows and columns are indexed by $B_2 \backslash B_1$.

Similarly, let Q denote the block with columns $B_2 \backslash B_1$ in $[\cdots \ \ Q \ \ I \ \ 0]$; the latter is the submatrix with rows $B_1 \backslash B_2$ in the tableau of $\mathcal{M}_R$.

The minimality assumption implies that B_1 and B_2 are the only common bases of the restrictions of $\mathcal{M}_L$ and $\mathcal{M}_R$ to $B_1 \cup B_2$. For if $B' \subset B_1 \cup B_2$ were a common basis strictly between B_1 and B_2 the minimality subject to (1.1) would be violated for exactly one of the pairs B_1 and B' or B_2 and B' since $\chi_R(B')\chi_L(B') = \pm 1$.

Consider the tableau for any matroid $\mathcal{M}$ relative to the basis $B \in \mathcal{B}(\mathcal{M})$ and let $T(X, Y)$ be the square submatrix with rows corresponding to $X \subseteq B$ and columns $Y \subseteq E \backslash B$. The non-zero entries of $T(X, Y)$ define the bipartite graph G with vertices $X \uplus Y$.

THEOREM 1.7. *(Krogdahl, see* **[24]** *or* **[37**, Ch. 3**]**.*)*

1. *If $B \backslash X \cup Y \in \mathcal{B}(\mathcal{M})$, then G contains a perfect matching.*
2. *If G contains a unique perfect matching, then $B \backslash X \cup Y \in \mathcal{B}(\mathcal{M})$.*

Part 1. of Theorem 1.7 shows that the graphs of P and of Q each contain perfect matchings. Consider these bipartite graphs to be binary relations: their composition is a binary relation R on $B_2 \backslash B_1$ that therefore contains a permutation. Now part 2. of Theorem 1.7 together with the fact that B_1 and B_2 are the only common bases of M_L and M_R in $B_1 \cup B_2$ show that the graphs of P and Q are just perfect matchings and R is a single cycle permutation. For if this were not true, R would contain a minimum length cycle of length less than $|B_1 \backslash B_2|$. This cycle would be the composition of the binary relations of proper square submatrices of P and Q, where these relations are each perfect matchings.

We return to oriented matroid analysis. The product PQ of signed permutation matrices P and Q is now known to represent a single cycle permutation. Let $\mathcal{M}_l$

$= \mathcal{M}_L\backslash(B_1^c \cap B_2^c)/(B_1 \cap B_2)$ and $\mathcal{M}_r = \mathcal{M}_R\backslash(B_1^c \cap B_2^c)/(B_1 \cap B_2)$. Hence $\mathcal{M}_l^\perp = \mathcal{M}_L/(B_1^c \cap B_2^c)\backslash(B_1 \cap B_2)$. The (particularly simple) oriented matroids $\mathcal{M}_l^\perp$ and $\mathcal{M}_r$ are realized by $[I\ P]$ and $[Q\ I]$, respectively, interpreted as real matrices with entries in $\{0, +1, -1\}$. This is because each column of $[I\ P]$ and $[Q\ I]$ has a single non-zero entry, so the circuits, cocircuits, etc., are completely determined by the sign patterns of these matrices.

CLAIM 1.1. $|P|\ |Q| = -(-1)^{|B_1\backslash B_2|}$.

PROOF. The ground set of $\mathcal{M}_l$ and $\mathcal{M}_r$ is $A = (B_1\backslash B_2) \cup (B_2\backslash B_1)$. Each of $\mathcal{M}_R$ and $\mathcal{M}_L$ are spanned by $(B_1^c \cap B_2^c)^c = B_1 \cup B_2$. Therefore (see [**2**, pages 133-135]) chirotope functions for $\mathcal{M}_l$ and $\mathcal{M}_r$ are given by

$$\chi_l(X) = s_{l0}\chi_L(X;F) \quad\text{and}\quad \chi_r(X) = s_r\chi_R(X;F),$$

where $F = B_1 \cap B_2$. Here, ";" denotes concatenation of ordered sets or sequences. The s_i denote constant signs. A chirotope function for $\mathcal{M}_l^\perp$ is given by

$$\chi_l^\perp(X) = s_l\chi_l(\overline{X})\epsilon(X, A).$$

Here, $\overline{X} = A\backslash X$ and $\epsilon(X, A)$ is the parity of the number of inversions when X would be shuffled in A so A is ordered by $X;\overline{X}$. Let the signs s_l and s_r used be those for which $\chi_l^\perp(B_2\backslash B_1) = +1$ and $\chi_r(B_1\backslash B_2) = +1$. Therefore $\chi_l^\perp$ and χ_r are realized by matrices $[I\ P]$ and $[Q\ I]$ respectively. Hence we have

$$\begin{aligned}
&|P|\ |Q|\ |I|\ |I| \\
&= \chi_l^\perp(B_1\backslash B_2)\chi_r(B_2\backslash B_1)\chi_l^\perp(B_2\backslash B_1)\chi_r(B_1\backslash B_2) \\
&= s_l^2\chi_l(B_2\backslash B_1)\epsilon(B_1\backslash B_2, A)\chi_l(B_1\backslash B_2)\epsilon(B_2\backslash B_1, A)\chi_r(B_2\backslash B_1)\chi_r(B_1\backslash B_2) \\
&= (-1)^{|B_1\backslash B_2|}s_{l0}^2\chi_L(B_2\backslash B_1;F)\chi_L(B_1\backslash B_2;F)s_r^2\chi_R(B_2\backslash B_1;F)\chi_R(B_1\backslash B_2;F) \\
&= \chi_L(B_2)\chi_L(B_1)\chi_R(B_2)\chi_R(B_1)(-1)^{|B_1\backslash B_2|} \\
&= -(-1)^{|B_1\backslash B_2|}.
\end{aligned}$$

The calculation uses the fact that $\epsilon(X, A)\epsilon(\overline{X}, A) = (-1)^{|X||\overline{X}|}$ and when $|X| = |\overline{X}|$ this is $(-1)^{|X|}$. ◁

CLAIM 1.2. The number of -1 entries in P and Q together is even.

PROOF. The non-zero entries of PQ represent a permutation of $|B_1\backslash B_2|$ elements with exactly one cycle. Therefore,

$$|PQ| = -(-1)^{|B_1\backslash B_2|}(-1)^{\text{number of } -1 \text{ entries in } P \text{ and } Q}.$$

But by claim 1.1, $|PQ| = -(-1)^{|B_1\backslash B_2|}$. ◁

CLAIM 1.3. There exist covectors $C_0 \in \mathcal{L}(\mathcal{M}_L^\perp)$ and $D_0 \in \mathcal{L}(\mathcal{M}_R)$ such that

- $C_0(a) = D_0(a) \neq 0$ for all $a \in A = (B_1\backslash B_2) \cup (B_2\backslash B_1)$,
- $C_0(B_1^c \cap B_2^c) = 0$ (note $B_1^c \cap B_2^c$ is independent in $\mathcal{M}_L^\perp$), and
- $D_0(B_1 \cap B_2) = 0$ (note $B_1 \cap B_2$ is independent in $\mathcal{M}_R$).

PROOF. For $e \in B_2\backslash B_1$ let $c_e \in \mathcal{L}(\mathcal{M}_L^\perp)$ be the unique cocircuit for which $c_e(e) = +$ and $c_e(f) = 0$ for all $f \in B_1^c\backslash e$. Thus $\{c_e : e \in B_2\backslash B_1\}$ is a subset of the fundamental cocircuits in $\mathcal{M}_L^\perp$ with respect to $B_1^c \in \mathcal{B}(\mathcal{M}_L^\perp)$. It corresponds to the rows indexed by $B_2\backslash B_1$ in the full tableau of which the tableau we showed for $\mathcal{M}_l^\perp$ is a submatrix.

For $e \in B_1 \backslash B_2$ let $d_e \in \mathcal{L}(\mathcal{M}_R)$ be the unique cocircuit for which $d_e(e) = +$ and $d_e(f) = 0$ for all $f \in B_1 \backslash e$. These are some of the fundamental cocircuits in $\mathcal{M}_R$ with respect to B_1.

Consider the bipartite graphs of P and of Q to which we applied Theorem 1.7. Form $\mathcal{N}$ as the union of P and the reverse of Q. We have shown that $\mathcal{N}$ is the cycle $(v_0, v_1, ..., v_{N-1})$ where $N = 2|B_1 \backslash B_2|$. Call an arc in $\mathcal{N}$ negative if the entry in P or Q it corresponds to is -1. Define $s : \{0, \ldots, N-1\} \to \{+1, -1\}$ by

$$s(i) = (-1)^{\text{number of negative arcs in the path from } v_0 \text{ to } v_i \text{ in } \mathcal{N}.}$$

Observe that since claim 1.2 shows that number of negative arcs in $\mathcal{N}$ is even,

$$s(i)s(i+1) = \begin{cases} -1 & \text{if arc } (v_i, v_{i+1}) \text{ is negative,} \\ +1 & \text{otherwise} \end{cases}$$

is true for all subscripts $0 \leq i \leq N-1$ with $i+1$ taken mod N. Let $v_0 \in B_2 \backslash B_1$ be arbitrary. The compositions satisfy the claim:

$$C_0 = (s(0)c_{v_0}) \circ (s(2)c_{v_2}) \circ \cdots \circ (s(N-2)c_{v_{N-2}})$$
$$D_0 = (s(1)d_{v_1}) \circ (s(3)d_{v_3}) \circ \cdots \circ (s(N-1)d_{v_{N-1}}) \quad \triangleleft$$

Claim 1.3 says C_0 and D_0 satisfy the hypotheses of Theorem 1.1 with $\mathcal{M}_1 = \mathcal{M}_L^\perp$, $\mathcal{M}_2 = \mathcal{M}_R$, $A = (B_1 \backslash B_2) \cup (B_2 \backslash B_1)$, $S = B_1^c \cap B_2^c$, $R = B_1 \cap B_2$ and $Z = \emptyset$. Hence $C = D$ from Theorem 1.1 is the common covector. $\square$

2. Common Covector Problem Classification

Since $\text{rank}(\mathcal{M}_L^\perp \vee \mathcal{M}_R) \leq \text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R)$ let us distinguish oriented matroid pairs with $\text{rank}(\mathcal{M}_L^\perp \vee \mathcal{M}_R) < \text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R)$ from pairs with equality here. The latter we will say have "full union rank." Theorem 1.3 says that pairs that do not have full union rank always have a common covector. See [**25, 33**] for polynomial time algorithms to compute $\text{rank}(\mathcal{M}_L^\perp \vee \mathcal{M}_R)$.

We will classify common covector problems into three categories. For the first category, full union rank is impossible. For the other two categories we summarize the properties of pairs with full union rank. Whether or not such pairs have a common covector is the interesting question. The ground set cardinality $|E|$ is denoted by m.

Excess Rank Sum: $\text{rank}(\mathcal{M}_R) > \text{rank}(\mathcal{M}_L)$, in other words, $\text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R) > m$. Since $m \geq \text{rank}(\mathcal{M}_L^\perp \vee \mathcal{M}_R)$ Theorem 1.3 always applies.

Balanced Rank Sum: $\text{rank}(\mathcal{M}_R) = \text{rank}(\mathcal{M}_L)$, in other words, $\text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R) = m$. Assume full union rank. This case includes the given applications to electrical networks and to $\mathcal{W}_0$ matrix pairs. Theorem 5.2 below shows the common covector problem is as hard as deciding if a square matrix is not sign non-singular [**3**] (SNS) and thus it is as hard as the digraph even cycle problem [**22**], even for rather simple classes of oriented matroids. Theorem 1.6, its converse for realizable oriented matroids, and its conjecture for all oriented matroids apply to this case.

Deficient Rank Sum: $\text{rank}(\mathcal{M}_R) < \text{rank}(\mathcal{M}_L)$, in other words, $\text{rank}(\mathcal{M}_L^\perp) + \text{rank}(\mathcal{M}_R) < m$. Assume full union rank. Corollary 5.5 shows the common covector problem is NP-complete.

3. Realizable Oriented Matroid Pairs

THEOREM 3.1. *If realizable $\mathcal{M}_L^\perp$ and $\mathcal{M}_R$ have a common non-zero covector and $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis then $\mathcal{M}_L$ and $\mathcal{M}_R$ have a pair of oppositely directed bases.*

PROOF. The determinant ("bracket") of the square submatrix of M_i with columns B is denoted by $M_i[B]$. A chirotope representation χ_i for $\mathcal{M}(M_i)$ has values $\chi_i(B) = \sigma(M_i[B])$. The product of real variables g_e for $e \in B$ is denoted g_B. Let $G = \mathrm{diag}(g_e)$. Laplace's theorem and Lemma 1.1 in [**4**] show that

(3.1)

$$\Delta = \left|\left[\begin{array}{c} M_R G \\ M_L^\perp \end{array}\right]\right| = \sum_{B\subset E} \epsilon(B,E) M_R[B] M_L^\perp[E\backslash B] g_B = C \sum_{B\subset E} M_R[B] M_L[B] g_B$$

where $C = \pm M_L^\perp[E\backslash B_0]/M_L[B_0]$ for an arbitrary B_0 for which $M_L[B_0] \neq 0$. Thus Δ is not identically 0 since $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis. Since $\Delta = 0$ for some positive values for the g_e, two terms in (3.1) have opposite sign. □

THEOREM 3.2. *Suppose realizable $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis. The following conditions are equivalent.*

- *$\mathcal{M}_L^\perp, \mathcal{M}_R$ have a common covector.*
- *$\mathcal{M}_L, \mathcal{M}_R^\perp$ have a common covector.*
- *$\mathcal{M}_L^\perp, \mathcal{M}_R$ have a common vector.*
- *$\mathcal{M}_L, \mathcal{M}_R^\perp$ have a common vector.*

PROOF. By Theorems 3.1 and 1.6, the first two conditions are each equivalent to $\mathcal{M}_L$ and $\mathcal{M}_R$ having a pair of oppositely directed common bases. The other two conditions follow because $\mathcal{L}(\mathcal{M}^\perp) = \mathcal{V}(\mathcal{M})$ and $\mathcal{L}(\mathcal{M}) = \mathcal{V}(\mathcal{M}^\perp)$. □

4. Sandberg-Willson Theory and its Dual

Consider the problem to solve the equation $AF(x) + Bx = c$ for $x \in \mathbf{R}^n$, where A and B are $n \times n$ matrices, $F : \mathbf{R}^n \to \mathbf{R}^n$ has the form $F(x)_k = f_k(x_k)$ with each f_k being a strictly monotone increasing function from $\mathbf{R}$ *onto* $\mathbf{R}$, and $AF(x)$ denotes the real column vector whose kth entry is $\sum A_{ki} f_i(x_i)$. Suppose the equation has two distinct solutions x and x'. Then $A(F(x')-F(x))+B(x'-x) = 0$. The strict monotonicity assumption for F means $\sigma(F(x') - F(x)) = \sigma(x' - x) = X \neq 0$. Therefore $\mathcal{M}[A\ B]$ and $\mathcal{M}[I\ \ -I]$ have a common non-zero vector $[X\ X]$. Conversely, suppose $\mathcal{M}[A\ B]$ and $\mathcal{M}[I\ \ -I]$ have a common non-zero vector. This means some $x, y \in \mathbf{R}^n$ satisfy $Ay + Bx = 0$ and $\sigma(x) = \sigma(y) \neq 0$. Define $F = (f_e)$ so $f_e(t) = (y_e/x_e)t$ if $x_e \neq 0$ and $f_e(t) = t$ otherwise. With this F, $AF(x)+Bx = 0$ has multiple solutions.

These ideas were observed by Sandberg and Willson who proved that, for given (A, B), the solution x exists and is unique for each choice of functions f_k and $c \in \mathbf{R}^n$ is equivalent to the properties of (A, B) below.

THEOREM 4.1. *(Willson, [**41**].) These properties of a pair of $n \times n$ matrices (A, B) are equivalent.*

1. *$|AD + B| \neq 0$ for every diagonal matrix $D > 0$.*

2. *There exists a matrix*[2] $M \in C(A,B)$ *such that* $|M| \neq 0$ *and for all* $N \in C(A,B)$, $|M| \cdot |N| \geq 0$.
3. *For each* $x \in \mathbf{R}^n$ *with* $x \neq 0$, *there is an index* k *such that* $(xA)_k \neq 0$ *or* $(xB)_k \neq 0$, *and such that* $(xA)_k(xB)_k \geq 0$.
4. *For each* $x \in \mathbf{R}^n$ *with* $x \neq 0$, *there is a diagonal matrix* $D_x \geq 0$ *such that either* $xAD_xA^tx^t > 0$ *or* $xBD_xB^tx^t > 0$ *and such that* $xAD_xB^tx^t \geq 0$.
5. *For each complementary pair* (M,N) *taken from* $C(A,B)$, *(that is,* $M = (A,B)(S' \cup (E\backslash S)'')$ *and* $N = (A,B)((E\backslash S)' \cup S'')$*) we have that each real root* λ *of* $|M - \lambda N|$ *is non-negative.*
6. *There exists a complementary pair* (M,N) *taken from* $C(A,B)$ *such that* $M^{-1}N \in \mathcal{P}_0$, *in the sense of Fiedler and Pták* [**14**].
7. *There exists a non-singular* $M \in C(A,B)$ *and for any complementary pair* (M,N) *taken from* $C(A,B)$ *with* M *non-singular,* $M^{-1}N \in \mathcal{P}_0$.

A pair of matrices (A,B) *that satisfies these properties is called a* $\mathcal{W}_0$ *pair, denoted* $(A,B) \in \mathcal{W}_0$.

This section shows two conditions on the oriented matroid $\mathcal{M} = \mathcal{M}[A\ B]$ realized by the row space of $[A\ B]$ are each equivalent to $(A,B) \in \mathcal{W}_0$. Each condition is a combinatorial property of $\mathcal{M}$ together with the involution $i \leftrightarrow (n+i)$ given on the ground set E where $|E| = 2n$. The two conditions are known to be equivalent for realizable oriented matroids from Theorem 3.2. One condition reflects the argument given above for uniqueness of solutions to $AF(x) + Bx = c$. The other pertains to a problem dual to $AF(x) + Bx = c$ in the sense that the row space of $[A\ B]$ plus a constant is an affine feasible set that is then further constrained by strict monotone diagonal nonlinearities.

REMARK 4.2. A common covector of $\mathcal{M}[A\ B], \mathcal{M}[I\ -I]$ is $[Z\ -Z]$ where for some $t \in \mathbf{R}^n$, $\sigma(tA) = -\sigma(tB) = Z$. A common *vector* of $\mathcal{M}[A\ B], \mathcal{M}[I\ -I]$ is $[W\ W]$ where there exist $x, y \in \mathbf{R}^n$ such that $Ax + By = 0$ and $W = \sigma(x) = \sigma(y)$.

THEOREM 4.3. *For a pair of* $n \times n$ *matrices* (A,B), *the following conditions are equivalent.*

1. $(A,B) \in \mathcal{W}_0$ *in the sense of Theorem 4.1, e.g.,* $|AD + B| \neq 0$ *for all positive diagonal* D, *etc.*
2. $\operatorname{rank} \mathcal{M}[A\ B] = n$ *and* $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ -I] = \{0\}$.
3. $\operatorname{rank} \mathcal{M}[A\ B] = n$ *and* $\mathcal{V}[A\ B] \cap \mathcal{V}[I\ -I] = \{0\}$.
4. *(Fundamental theorem of Sandberg and Willson* [**36, 40**]*) For all functions* $F : \mathbf{R}^n \to \mathbf{R}^n$ *of the form* $F(x)_k = f_k(x_k)$ *where each* f_k *is a strictly monotone increasing function from* $\mathbf{R}$ onto $\mathbf{R}$ *and for all* $c \in \mathbf{R}^n$, *the equation*
$$AF(x) + Bx = c$$
has a unique solution x.
5. *For all functions* $G : \mathbf{R}^n \to \mathbf{R}^n$ *of the form* $G(w)_k = g_k(w_k)$ *where each* g_k *is a strictly monotone increasing function from* $\mathbf{R}$ onto $\mathbf{R}$ *and for all* $d', d'' \in \mathbf{R}^n$, *the equations*
$$u^t = z^tA + d', \quad w^t = z^tB + d'', \quad u = -G(w) \tag{4.1}$$
have a unique solution (u, w, z).

[2]The following notation is used in [**41**]: $C(A,B)$ is the collection of all 2^n matrices of order $n \times n$ that are constructed by juxtaposing for each i in the order 1, 2, ..., n, either column A_i or B_i.

PROOF. The equivalence of 1. and 4. was proved in [**40, 41**]. We will use it to prove 5. below.

The equivalence of 1. and 2. is proved using property 3. of Theorem 4.1. The first part of property 3., $x[A\ B] = [xA\ xB] \neq 0$ for all $x \neq 0$ is equivalent to $\text{rank}\,\mathcal{M}[A\ B] = \text{rank}[A\ B] = n$.

Suppose $\text{rank}\,\mathcal{M}[A\ B] = n$ and $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$. Then for all $x \neq 0$, $\sigma(xA) \neq -\sigma(xB)$; so for at least one k, $\sigma(xA)(k) \neq -\sigma(xB)(k)$. At least one of $(xA)_k$ and $(xB)_k$ is non-zero and $(xA)_k(xB)_k \geq 0$.

For the converse, note that when $\text{rank}[A\ B] = n$, $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$ is equivalent to $\sigma(xA) \neq -\sigma(xB)$ for all $x \neq 0$. Therefore $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$ because $\sigma(xA)(k) \neq -\sigma(xB)(k)$ for the index k that satisfies property 4.1(3). Therefore, 1. and 2. are equivalent.

To use Theorem 3.2 (known for realizable oriented matroids only) to prove that 2. implies 3. one must show $\text{rank}(\mathcal{M}[A\ B] \vee \mathcal{M}[I\ -I]) = 2n$, i.e., this union matroid is free. However, if it were not, Theorem 1.3 would contradict $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$ since $\text{rank}(\mathcal{M}[A\ B]) = n$ is assumed and $\text{rank}(\mathcal{M}[I\ \ -I]) = n$.

Let the rows of matrix $[P\ Q]$ be a basis for the orthogonal complement of the row space $L[A\ B]$. When $\text{rank}[A\ B] = n$, $[P\ Q]$ is a rank n matrix with n rows and $2n$ columns. By oriented matroid duality principles, that $\mathcal{M}[P\ Q]$ and $\mathcal{M}[I\ I] = \mathcal{M}[I\ \ -I]^{\perp}$ have no common covector is equivalent to $\mathcal{V}[A\ B] \cap \mathcal{V}[I\ \ -I] = \{0\}$. Given condition 3., Theorem 1.3 implies as before that $\mathcal{M}[P\ Q] \vee \mathcal{M}[I\ I]$ is free. Hence Theorem 3.2 tells us $\mathcal{M}[P\ Q], \mathcal{M}[I\ I]$ have no common vector, since we are working with realizable oriented matroids. But no common vector for this pair means that their duals $\mathcal{M}[A\ B], \mathcal{M}[I\ \ -I]$ have no common covector. Therefore, condition 3. implies condition 2.

Equations (4.1) are equivalent to

$$Pu - Q(-w) = Pd'^t + Qd''^t \qquad u = -G(w) = G_1(-w)$$

for some vector function G_1 satisfying the same conditions as G.

Suppose (A, B) satisfies 1. and therefore 2. Hence $\text{rank}\,\mathcal{M}[A\ B] = n$ and so $\text{rank}(\mathcal{M}[P\ \ -Q]) = n$. By duality, $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$ is equivalent to $\mathcal{V}[P\ Q] \cap \mathcal{V}[I\ I] = \{0\}$. In general, the existence of common non-zero (co)vectors of oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on ground set E is invariant under reorientation of $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same subset of E. Specifically, $\mathcal{V}[P\ Q] \cap \mathcal{V}[I\ I] = \{0\}$ if and only if $\mathcal{V}[P\ \ -Q] \cap \mathcal{V}[I\ \ -I] = \{0\}$. Condition 3., applied to $(P, -Q)$, is now known to imply condition 4. with $(P, -Q)$ taking the place of (A, B). Therefore, the solution components $(u^t, -w^t)$ exist and are unique. The z^t component is unique since $\text{rank}[A\ B] = n$. Hence condition 5. is proven for (A, B).

Conversely, suppose condition 5. is true so (4.1) has a unique solution (u, w, z) for all G as specified and for all d' and d''. Then $\text{rank}[A\ B] = n$ since z is unique. Therefore $\text{rank}[P\ \ -Q] = n$ because $\text{rank}[P\ Q] = n$. The latter also shows that for all c there exist d' and d'' so $c = Pd' + Qd''$. So, for all c, $Pu^t - Q(-w^t) = c$ and $u = G_1(-w)$ have a unique solution $(u^t, -w^t)$. This satisfies condition 4. applied to the matrix pair $(P, -Q)$. Thus 3. applies to $(P, -Q)$. By reorientation, $\mathcal{V}[P\ Q] \cap \mathcal{V}[I\ I] = \{0\}$. By duality, $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$. However, this and $\text{rank}(\mathcal{M}[A\ B]) = n$ is condition 2. for (A, B). □

COROLLARY 4.4. *Let there be given four real n column matrices in pairs (A, B) and (P, Q), where each pair has full row rank and the row spaces are orthogonal complements, i.e., $L[A\ B] = L[P\ Q]^{\perp}$. The following conditions are equivalent.*

1. $(A, B) \in \mathcal{W}_0$.
2. $(P, -Q) \in \mathcal{W}_0$.
3. $\operatorname{rank}(\mathcal{M}[A\ B]) = n$ *and* $\mathcal{M}[A\ B], \mathcal{M}[I\ \ -I]$ *have no common covector.*
4. $\operatorname{rank}(\mathcal{M}[A\ B]) = n$ *and* $\mathcal{M}[A\ B], \mathcal{M}[I\ \ -I]$ *have no common vector.*
5. $\operatorname{rank}(\mathcal{M}[P\ Q]) = n$ *and* $\mathcal{M}[P\ Q], \mathcal{M}[I\ I]$ *have no common covector.*
6. $\operatorname{rank}(\mathcal{M}[P\ Q]) = n$ *and* $\mathcal{M}[P\ Q], \mathcal{M}[I\ I]$ *have no common vector.*
7. *For all* $x, y \in \mathbf{R}^n$, $Ax + By = 0$ *and* $\sigma(x) = \sigma(y)$ *implies* $x = y = 0$; *i.e., if* $x \neq 0$ *or* $y \neq 0$, *then for some index* k, $x_k \neq 0$ *or* $y_k \neq 0$, *and* $x_k y_k \leq 0$.
8. *For all* $u \in \mathbf{R}^n$, $\sigma(u^t A) = -\sigma(u^t B)$ *implies* $u = 0$ (q.v. *Theorem 4.1(3).)*

REMARK 4.5. Case 3. of Corollary (4.4) is a specialization of $\mathcal{L}(\mathcal{M}_L^{\perp}) \cap \mathcal{L}(\mathcal{M}_R) = \{0\}$ and the matroids $\mathcal{M}_L$ and $\mathcal{M}_R$ have a common basis. However, the latter condition is equivalent to $\mathcal{M}_L^{\perp} \oplus \mathcal{M}'_R$ and $\mathcal{M}[I_2\ \ -I_2]$ having no common covector and $\mathcal{M}_L \oplus \mathcal{M}'^{\perp}_R$ and $\mathcal{M}[I_2\ \ -I_2]$ having a common basis. Here $\mathcal{M}'_R$ is isomorphic to $\mathcal{M}_R$ on a disjoint copy of the ground set of $\mathcal{M}_L$ of size m, and I_2 is the order $2m$ identity matrix.

5. Sign Solvability and Computational Complexity

A sign matrix A is by definition an *L-matrix* if every real matrix with sign pattern A has all linearly independent rows. A square L-matrix is said to be sign non-singular (SNS, see [**3**] for a discussion of these topics.) We show that A is not an L-matrix is equivalent to a pair of rather simple linear subspaces (i.e., realized oriented matroids) having a common covector.

LEMMA 5.1. *Given $m \times n$ sign matrix A with no rows of zeros, let E be the set of positions ij where $A_{ij} \neq 0$ and define an $m \times |E|$ matrix M_L and an $n \times |E|$ matrix M_R by*

$$M_L(i, e) = \begin{cases} A_{ij} & \text{if } e = ij \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad M_R(j, e) = \begin{cases} 1 & \text{if } e = ij \\ 0 & \text{otherwise.} \end{cases}$$

Then A is an L-matrix if and only if $\mathcal{M}(M_L)$ and $\mathcal{M}(M_R)^{\perp}$ have no common covector.

PROOF. Let G be the diagonal matrix of g_{ij} for $ij \in E$. Every real A' with $\sigma(A') = A$ can be written as $A' = M_L G M_R^t$ for some real $g_e > 0$.

Suppose some linear combination of the rows of A' were 0. Some non-zero member of the row space $x \in L(M_L)$ would be in $L(M_R G)^{\perp}$. The signatures from the latter subspace are the covectors of $\mathcal{M}(M_R)^{\perp}$. Hence $\sigma(x)$ is a common covector.

Conversely, suppose $x \in L(M_L)$ and $y \in L(M_R)^{\perp}$ satisfy $\sigma(x) = \sigma(y) \neq 0$. Define $g_e = y(e)/x(e)$ when $x(e) \neq 0$ and $g_e = 1$ otherwise. Hence G is positive diagonal, $xG \in L(M_R)^{\perp}$ and so $x \in L(M_R G)^{\perp}$. □

THEOREM 5.2. *The problem of telling if a square sign matrix matrix A is not SNS is polynomial time reducible to the common covector problem for $\mathcal{M}_L = \mathcal{M}(M_L)$ and $\mathcal{M}_R^{\perp} = \mathcal{M}(M_R)^{\perp}$ where M_L and M_R realize oriented matroids with a common basis and whose underlying matroids are partition matroids.*

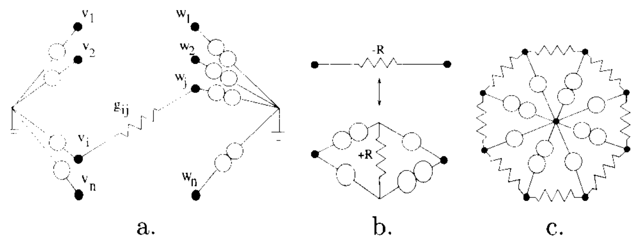

FIGURE 4. a. Network for matrix A' of Theorem 5.2, with positive resistors E, nullator edges P and norator edges Q. Let $\mathcal{M}$ be the graphic oriented matroid. Let $\mathcal{M}_L = \mathcal{M}/P\backslash Q$ and $\mathcal{M}_R = \mathcal{M}\backslash P/Q$. This electric network model is well-posed for all positive resistance values if and only if $\mathcal{M}_L, \mathcal{M}_R^\perp$ have no common covector. The SNS problem for $0, +1$ matrices is reducible to cases of this electrical problem. For $0, \pm 1$ matrices, negative resistors can be simulated by (b.). When E is a cycle with $4k$ edges, the network determinant Δ has exactly two terms and they have opposite sign. The Wheatstone bridge (a K_4 with the nullator and norator as a disjoint pair of edges) is the case of $k = 1$. The case of $k = 2$ is (c.), $\Delta = g_1 g_3 g_5 g_7 - g_2 g_4 g_6 g_8$.

PROOF. To reduce, first test if $\det A'$ is identically 0 by a polynomial time bipartite matching algorithm [**25, 33**]. If so, A is not SNS. Otherwise, the reduction given in Lemma 5.1 satisfies the theorem. □

REMARK 5.3. Matrix A' above is the system matrix for the nullator, norator, and resistor network shown in Figure 4. Therefore the common covector problem cases from electrical applications, to determine if a network model is sometimes but not always ill-posed [**18, 19, 20**], are no easier than the (non) SNS problem.

REMARK 5.4. Generalizing part (c) of Figure 4 gives a family of graph pairs $\{(\mathcal{G}_L^k, \mathcal{G}_R^k)\}$. For each member, $\mathcal{M}(\mathcal{G}_L^k), \mathcal{M}(\mathcal{G}_R^k)^\perp$ have a common covector. But no proper minor pair $(\mathcal{G}_L^k/X\backslash Y, \mathcal{G}_R^k/X\backslash Y)$ satisfies this property. See [**32, 15**].

THEOREM 5.5. *The common covector problem for row spaces of integer matrices is strongly NP-complete.*

PROOF. The NP algorithm is to guess the common covector and verify it by solving two instances of the integer linear programming feasibility problem. Theorem 13.4 on page 320 of [**33**] gives an upper bound on the magnitudes of some solution to a feasible integer linear program in terms of the magnitudes of the coefficients. The upper bound implies that the verification can be done in time polynomial in the number of bits needed to code the matrices.

To prove the NP-hardness, use Lemma 5.1 to reduce the problem of telling if rectangular A is not an L-matrix to the common covector problem. The former problem was shown to be NP-complete by Klee, Ladner and Manber in [**22**]. Since the instances from the reduction are coded with $0, \pm 1$s, the NP-completeness is strong [**16**]. □

REMARK 5.6. The proof presented at the conference used reduction from the "feedback arc set" problem for directed graphs [**16**].

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