

11/23/99

Jesus,

The handwritten notes are from a lecture I gave ages ago. This gives a nice summary of the Wills' Functional and the Wills Conjecture, and Hadwiger's work on it.

Hadwiger's 1975 article gives the basics. His 1974 article with Wills proves the conjecture for certain rotation bodies. His 1979 article gives a counterexample to the conjecture in general. Larman's list of problems talks about this on p.431.

Klee discusses the problem of maximizing volume in the "Viewer's Manual" (I don't have the film!) Steen mentions it on p.413 of his article from Science News (copy also enclosed)

Enclosed also are copies of pp.143-148 from the book Unsolved Problems in Geometry, by Croft, Falconer, Guy, these pp. dealing with lattice point problems,

Also the book by J. Hammer on Unsolved Problems Concerning Lattice Points. But there is a later (and I believe more comprehensive?) book by Erdős, Hammer, and Scott (?) (I think) on the same subject. I don't have the book, but I believe the library has it.

Don

mention such problems in
connection with Sierpinski
reference ...

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- The figure consists of a 4x5 grid of 20 small, square images. Each image shows a different stage of a face being constructed from a sparse set of black dots. The sequence starts with a single dot in the center of the first image and gradually adds more dots to form the features of a face, including the eyes, nose, mouth, and finally the full outline and details of a human face by the last image in the grid.

of numbers, Pergamon 1964]

- $$A(P) = Z_2(P) - \frac{1}{2} B_1^2 - 1$$

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K misses lattice pts $\neq 0 \Rightarrow A < 4$

$$\therefore Z=1 \Rightarrow A < 4$$

∴ $Z = 1 \Rightarrow Y < 2^m$

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Borden, p. 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917,

$$\frac{A}{L} < \frac{1}{2}$$


 $A = n$ $L = 2n+2$
 $A/L = \frac{n}{2n+2} = \dots$

$$A_1 = \frac{n}{2+1} = \frac{1}{2} \frac{n}{n+1} \rightarrow \frac{1}{2} \text{ so } \frac{1}{2} \text{ is "best" constant}$$
$$\mathcal{Z} = 0 \Rightarrow A/L < \frac{1}{2}$$

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- "best" position error exactly $(n+1)^2$

Prof :

clearly $L(P) \geq B(P)$:

$$\frac{Z(Q)}{Z(P)} = \frac{Z(P)}{Z(P)} = I(P) + B(P) = I(P) + \frac{1}{2} B(P) + \frac{1}{2} B(P) \leq I(P) + \frac{1}{2} B(P) + \frac{1}{2} L(P) = (I(P) + \frac{1}{2} B(P) - 1) + \frac{1}{2} L(P) + 1 =$$

$$= A(p) + \frac{1}{2}L(p) + 1 \leq A(Q) + \frac{1}{2}L(Q) + 1 = n^2 + 2n + 1 = (n+1)^2$$

For later: How does
Newman's problem
generalize to $\mathbb{R}^n$?

- 6) Both anticipated in Nozarzevska, Colloq. Math 1 (1948), 305-311. For any plane K :

$$A(k) - \frac{1}{2}L(k) \leq Z(k) \leq A(k) + \frac{1}{2}L(k) + 1$$

[note this generalizes Poincaré, since $z=0 \Rightarrow A - \frac{1}{2}L < 0$]

- 7) Following Leo Wloss's injunction "be wise, generalize":

Wills (1970): $Z=0 \Rightarrow V/\xi < \frac{1}{2}$ in $\mathbb{R}^n$, $n=3,4$ (Generalizing Bender)

Hadwiger (1970): $Z=0 \Rightarrow V/S' < \frac{1}{2}$ in $\mathbb{R}^n$, $n \geq 2$

Schmidt (1972): $V - \frac{1}{2} S' < \mathbb{Z}$ in $\mathbb{R}^3$ (generalizing Nozarszewska in one direction)

Pokowski, Hradwiger, Wills (1972): $V - \frac{1}{2}S < Z$ in $\mathbb{R}^n$, all n

8) The Wills Functional: $W(K) = \sum_{k=0}^n \frac{1}{\omega_k} \binom{n}{k} W_k(K)$

Wills Conjecture: $Z(K) \leq W(K)$

with equality for box in standard position

Note case $n=2$: $W(K) = W_0 + \frac{1}{2} \cdot 2W_1 + \frac{1}{\pi} W_2$
 $= A + \frac{1}{2} L + 1$

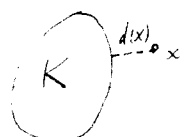
Wills Conjecture is Nozarzewski's Thm in $\mathbb{R}^2$

Overhagen (1975): Wills Conjecture in $\mathbb{R}^3$, i.e. $Z(K) \leq V + \frac{1}{2} S + \frac{1}{\pi} M + 1$

Hadwiger, Wills (1974): Wills Conjecture for rotation bodies in $\mathbb{R}^n$, $n \leq 6$

→ According to Larmann (1978): counterex. by Hadwiger, simplex of dimension ≥ 441 .
 (See Hadwiger Math. Ann. 239 (1979), 271-288)

Hadwiger's representation of Wills functional



$$\int_{\mathbb{R}^n} e^{-\pi |dx|^2} dx = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} 2\pi t e^{-\pi t^2} dt \right\} dx = \int_0^\infty \left\{ \int_{x \in K_t} dx \right\} 2\pi t e^{-\pi t^2} dt = \int_0^\infty V(K_t) 2\pi t e^{-\pi t^2} dt$$

$$= \int_0^\infty \left\{ \sum \binom{n}{k} W_k t^k \right\} 2\pi t e^{-\pi t^2} dt = 2\pi \sum \binom{n}{k} W_k \int_0^\infty t^{k+1} e^{-\pi t^2} dt = W(K)$$

$$\left\{ \int_0^\infty t^{k+1} e^{-\pi t^2} dt = \int_0^{\pi t^2} \frac{t^{k+1}}{\pi} e^{-u} \frac{1}{2\pi t} \frac{1}{\pi} du = \frac{1}{2\pi^{1+k/2}} \int_0^\infty u^{k/2} e^{-u} du = \frac{1}{2\pi^{1+k/2}} \Gamma(1 + \frac{k}{2}) = \frac{1}{2\pi} \cdot \frac{1}{\pi^{k/2}} \frac{1}{\Gamma(1 + k/2)} = \frac{1}{2\pi \omega_k} \right\}$$

Hadwiger's representation can be used to prove following properties of $W(K)$ easily:

(i) $W(K)$ is "independent of embedding dimension"

$W(K)$ defines "intrinsic k-volume of K " by $V_k(K) = \frac{1}{\omega_{n-k}} \binom{n}{k} W_{n-k}(K)$

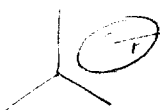
Then Wills functional is $W(K) = \sum_{k=0}^n V_k(K)$

ex of indep of embedding space:

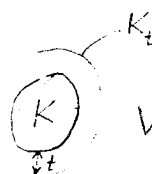


disk D of radius r :

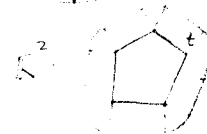
$W(D) = A + \frac{1}{2} L + 1 = \pi r^2 + \pi r + 1$ in $\mathbb{R}^2$



$W(D) = V + \frac{1}{2} S + \frac{1}{\pi} M + 1 = 0 + \frac{1}{2} \cdot 2\pi r^2 + \frac{1}{\pi} \cdot \frac{1}{2} \pi \cdot 2\pi r + 1$ in $\mathbb{R}^3$
 Same as in $\mathbb{R}^2$



$V(K_t) = \sum_{k=0}^n \binom{n}{k} W_k(K) t^k$



$A(K_t) = A(K) + tL(K) + \pi t^2$



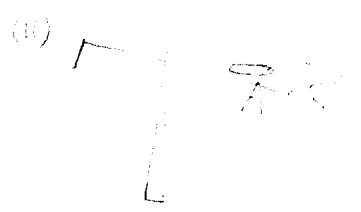
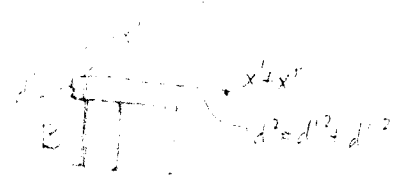
$V(K_t) = V(K) + tS(K) + t^2 M(K) + \frac{\pi}{3} t^3$

$M = \frac{1}{2} \sum \alpha L = 2\pi \overline{w}$
 $= 3W_2$

$\mathbb{R}^n: V_0 = V, nW_1 = S, W_{n-1} = \frac{1}{2} \omega_n \overline{w}, V_{n-1} = \omega$

Note this generalizes Newman's prob. to $\mathbb{R}^3$: A $n \times n \times n$ cube in $\mathbb{R}^3$ covers at most $(n+1)^3$ lattice pts
 Exercise: calculate this...

R^{n-1}
 $B(0)$



$$W(A+B) = \int e^{-\pi d^2} dx' dx'' = \int e^{-\pi d'^2} dx' \int e^{-\pi d''^2} dx'' = W(A)W(B)$$

Note: ... using the wigner representation



$$W(K) = \int e^{-\pi d^2} dx' dx'' = \int e^{-\pi(d'^2 + d''^2)} dx' dx'' = \underbrace{\int_{\mathbb{R}^{n-1}} e^{-\pi d'^2} dx'}_{W(K')} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\pi d''^2} dd''}_{=1}$$

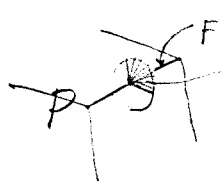
Remarks from McMullen's paper:

For P a polytope,

$$W(P) = \sum_F \gamma(F, P) V(F), \quad F \text{ th. faces of } P$$

Follows since for polytope P

$$V_k(P) = \sum_{F^k} \gamma(F^k, P) V(F^k) \quad \text{sum over all } k\text{-faces of } P$$



$$\gamma(F^k, P) = \text{"external" of } P \text{ at } F^k$$

$$W(\text{Poly}_{\text{pin}}) = A + \frac{1}{2}L + 1$$



$$W(3\text{-poly}) = V + \frac{1}{2}S + \frac{1}{6}\sum_{i=1}^3 \sum_{j=1}^3 \gamma_{ij} + 1$$

To prove Wills Conjecture suffices to prove for lattice polytopes, since $W(K)$ is monotone

"Inversion" of W :

$$V(P) = \sum_F (-1)^{\dim P - \dim F} \underbrace{\beta(F, P)}_{\text{internal } \gamma} W(F)$$

Bokowski: $W(K)$ is the unique continuous rigid motion invariant valuation which agrees with $Z(K)$ on lattice boxes

For lattice polytopes:

$$\left. \begin{aligned} Z(mP) &= \sum_{k=0}^n Z_k(P) m^k \\ W(mP) &= \sum_{k=0}^n V_k(P) m^k \end{aligned} \right\} m \text{ any positive integer}$$

where $Z_k(P)$ are certain functionals

McMullen Conjecture: P lattice polytope $\Rightarrow Z_k(P) \leq V_k(P)$ all $k \geq 0$

reciprocity law:

$$Z_k^{\leftarrow}(mP) = (-1)^{\dim P - k} \sum_{k=0}^n Z_k(P) (m)^k$$

$\leftarrow$ # lattice points interior to nP

From Hadwiger, Vorlesungen... p. 216

for a box with edge lengths $l_1, \dots, l_n$ in $\mathbb{R}^n$

$$W_k = \frac{\omega_k}{\binom{n}{k}} \sum \underbrace{l_1 \cdots l_{n-k}}_{n-k \text{th symm. factor}}$$

$$\therefore V_k = \sum \underbrace{l_1 \cdots l_k}_{k \text{th symm. factor}}$$

$$\therefore W = 1 + \sum l_i + \sum l_i l_j + \cdots + l_1 \cdots l_n = (l_1 + 1) \cdots (l_n + 1)$$

$\therefore$ If the box ~~has~~ is $m \times m \times \cdots \times m$, then

$$W = (m+1)^n$$

If oriented as standard lattice box, then # lattice pts covered is exactly $(m+1)^n$ is expected

$\mathbb{R}^{n-a}$ Hadwiger, p. 215

$B \oplus A+B$

A, B in complementary orthogonal subspaces

$A \subset \mathbb{R}^a$

$$W_{n-i}^i(A+B) = \frac{\omega_k}{\binom{n}{k}} \sum \frac{\binom{a}{v}}{\omega_v} \frac{\binom{n-a}{i-v}}{\omega_{i-v}} W_v'(A) W_{i-v}''(B)$$

$$\approx \frac{\binom{n}{i}}{\omega_i} W_i(A+B) = \sum \frac{\binom{a}{v}}{\omega_v} W_v'(A) \cdot \frac{\binom{n-a}{i-v}}{\omega_{i-v}} W_{i-v}''(B)$$

$$\frac{\binom{n}{i}}{\omega_{n-i}} W_{n-i}(A+B) = \sum_{v=0}^i \frac{\binom{a}{v}}{\omega_v} W_v'(A) \frac{\binom{n-a}{n-i-v}}{\omega_{n-i-v}} W_{n-i-v}''(B)$$

$$V_i(A+B) = \sum_{v=0}^i V_{a-v}(A) V_{i-v}$$

$$W(A+B) = \sum W(A) W(B)$$