11/23/99

Jesus,

The handwritten notes are from a lecture I gave ages ago. This gives a nice summary of the Wills' Functional and the Wills Conjecture, and Hadwiger's work on it.

Hadwiger's 1975 article gives the basics. His 1974 article with Wills proves the conjecture for certain rotation bodies. His 1979 article gives a counterexample to the conjecture in general. Larman's list of problems talks about this on p. 431.

Klee discusses the problem of maximizing volume in the "Viewer's Manual" (I don't have the film!) Steen mentions it on p. 413 of his article from Science News (copy also enclosed)

Enclosed also are copies of pp. 143-148 from the book Unsolved Problems in Geometry, by Croft, Falconer, Guy, these pp. dealing with lattice point problems,

Also the book by J. Hammer on Unsolved Problems Concerning Lattice Points. But there is a later (and I believe more comprehensive?) book by Erdős, Hammer, and Scott (?) (I think) on the same subject. I don't have the book, but I believe the library has it.

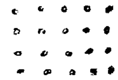
Don

Convex Sets and Lattice Pts

Food for thought: Is there for each n a circle enclosing exactly n lattice pts?

mention such problems in connection with Sierpinski reference ...

1) Lattice pt = (m, n), m, n integers ... similarly in $\mathbb{R}^n$



[illegible] [illegible] between number theory and Geometry [[illegible], A selection of [illegible] in the [illegible] of numbers, Pergamon 1964]

2) Pick's Thm.:

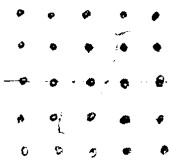
$A = I + \frac{1}{2} B - 1$ [Coxeter Intr. to Geom]

$A(P) = I(P) + \frac{1}{2} B(P) - 1$

$Z(P) = I(P) + B(P)$

$A(P) = Z(P) - \frac{1}{2} B(P) - 1$

3) Minkowski's Thm (Geometry of Numbers) [[illegible] AMM 81 (1974), p. 445]

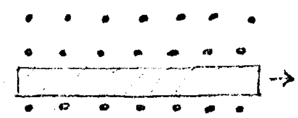


K closed odd convex, centrally symm. with center at $\underset{\sim}{0}$ in $\mathbb{R}^2$

K misses lattice pts $\neq \underset{\sim}{0} \Rightarrow A < 4$ i.e. $Z = 1 \Rightarrow A < 4$

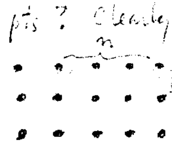
(in $\mathbb{R}^n$, free of lattice pts $\neq \underset{\sim}{0} \Rightarrow V < 2^n$) i.e. $Z = 1 \Rightarrow V < 2^n$

4) Bender (AMM 1962): remove restriction of central symm. centered at $\underset{\sim}{0}$



What can one say if K misses all lattice pts? Clearly no bound on volume

Bender proved in this case $A/L < \frac{1}{2}$

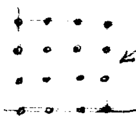


$A = n$ $L = 2n + 2$

$A/L = \frac{n}{2n+2} = \frac{1}{2} \frac{n}{n+1} \to \frac{1}{2}$ so $\frac{1}{2}$ is "best" constant

Let $Z = \#$ lattice pts in K: Then Bender $\Rightarrow$ $\boxed{Z = 0 \Rightarrow A/L < \frac{1}{2}}$

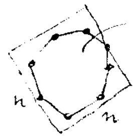
5) Newman E 1954 (Sol in AMM 75 (1968) p. 545)



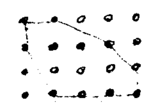
"best" position covers exactly $(n+1)^2$

If Q is an $n \times n$ square, then $Z(Q) \le (n+1)^2$

Prf:



P = convex hull of those lattice pts covered by Q ... clearly $L(P) \ge B(P)$:



$Z(Q) = Z(P) = I(P) + B(P) = I(P) + \frac{1}{2} B(P) + \frac{1}{2} B(P) \le I(P) + \frac{1}{2} B(P) + \frac{1}{2} L(P) = \left(I(P) + \frac{1}{2} B(P) - 1\right) + \frac{1}{2} L(P) + 1 =$

$= A(P) + \frac{1}{2} L(P) + 1 \le A(Q) + \frac{1}{2} L(Q) + 1 = n^2 + 2n + 1 = (n+1)^2$

note proof [illegible] [illegible] with Q replaced by K.

For later: How does Newman's problem generalize to $\mathbb{R}^n$?

6) Both anticipated by Nozarzewska, Colloq. Math 1 (1948), 305-311. For any plane K:

$$\boxed{A(K) - \tfrac{1}{2} L(K) < Z(K) \le A(K) + \tfrac{1}{2} L(K) + 1}$$

[note this generalizes Bender, since $Z = 0 \Rightarrow A - \frac{1}{2} L < 0$]

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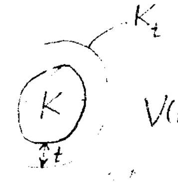
7) Following Leo Moser's injunction "be wise, generalize":

Wills (1970): $Z = 0 \Rightarrow V/S < \frac{1}{2}$ in $\mathbb{R}^n$, $n = 3, 4$ (Generalizing Bender)

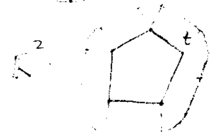
Hadwiger (1970): $Z = 0 \Rightarrow V/S < \frac{1}{2}$ in $\mathbb{R}^n$, $n \ge 2$

Schmidt (1972): $V - \frac{1}{2} S < Z$ in $\mathbb{R}^3$ (generalizing Nozarzewska in one direction)

Bokowski, Hadwiger, Wills (1972): $V - \frac{1}{2} S < Z$ in $\mathbb{R}^n$, all n



$$V(K_t) = \sum_{k=0}^{n} \binom{n}{k} W_k(K) t^k$$



$$A(K_t) = A(K) + tL(K) + \pi t^2$$

$$V(K_t) = V(K) + tS(K) + t^2 M(K) + \frac{4\pi}{3} t^3$$

$$M = \frac{1}{2}\sum \alpha \ell = 2\pi \overline{w}$$
$$= 3W_2$$

$\mathbb{R}^n$: $W_0 = V$, $nW_1 = S$, $W_{n-1} = \frac{1}{2}\omega_n \overline{w}$, $W_n = \omega$

8) The Wills Functional: $W(K) = \sum_{k=0}^{n} \frac{1}{\omega_k} \binom{n}{k} W_k(K)$

$\boxed{\text{Wills Conjecture: } Z(K) \le W(K)}$ with equality for box in standard position

Note case $n=2$: $W(K) = W_0 + \frac{1}{2}\cdot 2W_1 + \frac{1}{\pi} W_2$
$= A + \frac{1}{2}L + 1$

$\therefore$ Wills Conjecture is Nozarzewska's Thm in $\mathbb{R}^2$

$W_0 + \frac{1}{2}\cdot 3W_1 + \frac{1}{\pi}\cdot 3W_2 + \frac{1}{\omega_3}\cdot\omega_3$

Overhagen (1975): Wills Conjecture in $\mathbb{R}^3$, i.e. $Z(K) \le V + \frac{1}{2}S + \frac{1}{\pi}M + 1$

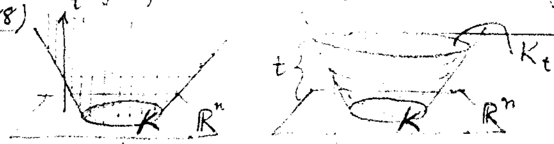
Hadwiger, Wills (1974): Wills Conjecture for rotation bodies in $\mathbb{R}^n$, $n \le 6$

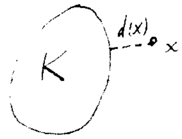
→ According to Larman (1978): counterex. by Hadwiger, simplex of dimen ≥ 441.
(See Hadwiger Math. Ann. 239 (1979), 271-288)

note this generalizes Newman's prob. to $\mathbb{R}^3$: A $n\times n\times n$ cube in $\mathbb{R}^3$ covers at most $(n+1)^3$ lattice pts

Exercise: calculate this...

Hadwiger's representation of Wills functional





$$\int_{\mathbb{R}^n} e^{-\pi d(x)^2} dx = \int_{\mathbb{R}^n} \left\{ \int_{d(x)}^{\infty} 2\pi t\, e^{-\pi t^2} dt \right\} dx = \int_0^{\infty} \left\{ \int_{x \in K_t} dx \right\} 2\pi t\, e^{-\pi t^2} dt = \int_0^{\infty} V(K_t)\, 2\pi t\, e^{-\pi t^2} dt$$

$$= \int_0^{\infty} \left\{ \sum \binom{n}{k} W_k t^k \right\} 2\pi t\, e^{-\pi t^2} dt = 2\pi \sum \binom{n}{k} W_k \underbrace{\int_0^{\infty} t^{k+1} e^{-\pi t^2} dt}_{= \frac{1}{2\pi \omega_k}} = W(K)$$

$$\left\{ \int_0^{\infty} t^{k+1} e^{-\pi t^2} dt \overset{u=\pi t^2}{=} \int_0^{\infty} \left(\frac{u}{\pi}\right)^{\frac{k+1}{2}} e^{-u} \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{u}} du = \frac{1}{2\pi^{1+k/2}} \int_0^{\infty} u^{k/2} e^{-u} du = \frac{1}{2\pi^{1+k/2}} \Gamma(1+\tfrac{k}{2}) = \frac{1}{2\pi} \cdot \frac{1}{\pi^{k/2}/\Gamma(1+k/2)} = \frac{1}{2\pi\omega_k} \right\}$$

Hadwiger's representation can be used to prove following properties of $W(K)$ neatly:

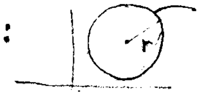
(i) $W(K)$ is "independent of imbedding dimension"

McMullen defines "intrinsic k-volume of K" by $V_k(K) = \frac{1}{\omega_{n-k}} \binom{n}{k} W_{n-k}(K)$

Then Wills functional is $\boxed{W(K) = \sum_{k=0}^{n} V_k(K)}$

Note e.g. ... Then ... $\mathbb{R}^n$, $V_1(K) = \frac{1}{2}L$

ex of indep of imbedding space:



disk D of radius r:

$W(D) = A + \frac{1}{2}L + 1 = \pi r^2 + \pi r + 1$ in $\mathbb{R}^2$

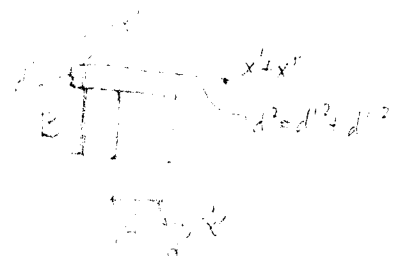
... L for 2-dim ...

$W(D) = V + \frac{1}{2}S + \frac{1}{\pi}M + 1 = 0 + \frac{1}{2}\cdot 2\pi r^2 + \frac{1}{\pi}\cdot\frac{1}{2}\pi\cdot 2\pi r + 1$ in $\mathbb{R}^3$

Same as in $\mathbb{R}^2$

(3

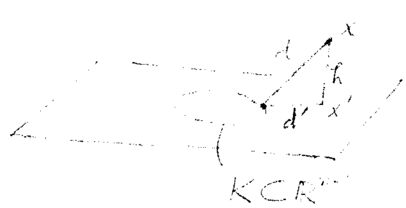
R^{n-1} $B, 0$



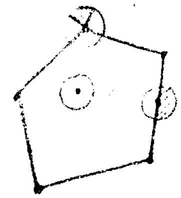
(ii)

$$W(A+B) = \int e^{-\pi d^2} dx' dx'' = \int e^{-\pi d'^2} dx' \int e^{-\pi d''^2} dx'' = W(A)\,W(B)$$

Note: ... using the ... representation



$$W(K) = \int e^{-\pi d^2} dx = \int e^{-\pi(d'^2 + h^2)}\, dh\, dx' = \underbrace{\int_{\mathbb{R}^{n-1}} e^{-\pi d'^2} dx'}_{W(K')} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\pi h^2} dh}_{=1}$$



$W(\text{Polygon}) = A + \frac{1}{2}L + 1$

$W(\text{3-polytope}) = V + \frac{1}{2}S + \frac{1}{4\pi}\sum \alpha_i l_i + \ldots$

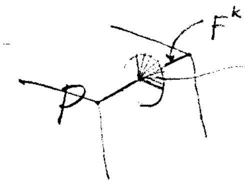
Remarks from McMullen's paper:

For P a polytope,

$$\boxed{W(P) = \sum_F \gamma(F,P)\, V(F)}, \quad F \text{ the faces of } P$$

Follows since for polytope P $\qquad V(F^k) = k\text{-dim vol. of } F^k$

$$V_k(P) = \sum_{F^k} \gamma(F^k, P)\, V(F^k) \quad \text{sum over all } k\text{-faces of } P$$



$\gamma(F^k, P) =$ "external $\measuredangle$ of P at F^k"

To prove Wills Conjecture suffices to prove for lattice polytopes, since $W(K)$ is monotone

"Inversion" of W:
$$V(P) = \sum_F (-1)^{\dim P - \dim F} \underbrace{\beta(F,P)}_{\text{internal } \measuredangle} W(F)$$

Bokowski: $W(K)$ is the unique continuous rigid motion invariant valuation which agrees with $Z(K)$ on lattice boxes

For lattice polytopes:
$$\left.\begin{aligned} Z(mP) &= \sum_{k=0}^{n} Z_k(P)\, m^k \\ W(mP) &= \sum_{k=0}^{n} V_k(P)\, m^k \end{aligned}\right\} \quad m \text{ any positive integer}$$

where $Z_k(P)$ are certain functionals

McMullen Conjecture: P lattice polytope $\Longrightarrow Z_k(P) \le V_k(P)$ all $k \ge 0$

reciprocity law:
$$\underbrace{Z^{\circ}(mP)}_{\#\text{ lattice pts interior to } mP} = (-1)^{\dim P} \sum_{k=0}^{n} Z_k(P)(-m)^k$$

From Hadwiger, Vorlesungen .... p. 216

for a box with edge lengths $l_1, \dots, l_n$ in $\mathbb{R}^n$

$$W_k = \frac{\omega_k}{\binom{n}{k}} \underbrace{\sum l_1 \cdots l_{n-k}}_{n-k^{th} \text{ symm. fctn}}$$

$$\therefore V_k = \underbrace{\sum l_1 \cdots l_k}_{k^{th} \text{ symm. fctn}}$$

$$\therefore W = 1 + \sum l_i + \sum l_i l_j + \dots + l_1 \cdots l_n = (l_1+1)\cdots(l_n+1)$$

$\therefore$ If the box ~~is~~ is $m \times m \times \dots \times m$, then

$$W = (m+1)^n$$

If oriented as standard lattice box, then # lattice pts covered is exactly $(m+1)^n$ as expected

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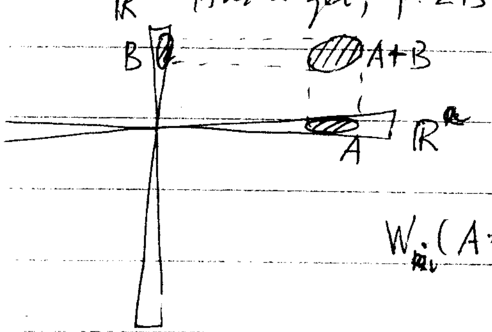
$\mathbb{R}^{n-a}$ Hadwiger, p. 215



A, B in complementary orthogonal subspaces

$$W_i(A+B) = \frac{\omega_i}{\binom{n}{i}} \sum \frac{\binom{a}{\nu}}{\omega_\nu} \frac{\binom{n-a}{i-\nu}}{\omega_{i-\nu}} W'_\nu(A) W''_{i-\nu}(B)$$

or $$\frac{\binom{n}{i}}{\omega_i} W_i(A+B) = \sum \frac{\binom{a}{\nu}}{\omega_\nu} W'_\nu(A) \cdot \frac{\binom{n-a}{i-\nu}}{\omega_{i-\nu}} W''_{i-\nu}(B)$$

$$\frac{\binom{n}{i}}{\omega_{n-i}} W_{n-i}(A+B) = \sum_{\nu=0}^{i} \frac{\binom{a}{\nu}}{\omega_\nu} W'_\nu(A) \frac{\binom{n-a}{n-i-\nu}}{\omega_{n-i-\nu}} W''_{n-i-\nu}(B)$$

$$V_i(A+B) = \sum_{\nu=0}^{i} V_{a-\nu}(A) \, V_{\dots}$$

$$W(A+B) = \sum V_i(A+B) = \quad = W(A) W(B)$$