This manual consists of the film script, interspersed with paragraphs supplying additional information about the problems discussed in the film and mentioning some additional problems. There is a list of references at the end.

What is the nicest way of placing N points on a circle? Intuitively, it is to place them at the vertices of an inscribed regular polygon. But is there some precise, quantitative sense in which this is the best arrangement? Yes, there are several.

For example, the area of a regular inscribed N-gon is greater than that of any other inscribed N-gon. Thus, for any N greater than 2, the unique way of maximizing the area is to place the points at the vertices of a regular inscribed N-gon. Now, rather than maximizing the area, we might require the points to be dispersed as much as possible. The smallest distance determined by the points should be as large as possible. In other words, we want to maximize the minimum distance between points of the set. Note that it suffices to consider the distances between adjacent points on the circle. Here again, the regular N-gon provides the unique solution. That is, the unique way of maximizing the minimum distance is to place the points at the vertices of an inscribed regular N-gon. The result is the same as when we wanted to maximize the area.

The results just stated are not hard to prove.[1] *But what happens when the points are on a sphere? How should they be arranged to maximize the volume or the minimum distance? These problems are unsolved or, at best, only partially solved. I'm going to discuss them along with some other geometric problems that are unsolved at the time of filming in 1969. Perhaps they will eventually be solved by the discovery of new geometric figures or configurations--that is, by shapes of the future.*

[1] There are many published proofs of the fact that, for a given N and a given circle, only the regular N-gons are of maximum area among those inscribed in the circle. See, for example, Fejes Tóth(1953). Obviously the regular arrangement of N points on the circle is the only one maximizing the minimum distance between the points.

Let's look first at the volume problem. We want to place N points on ...ere so as to maximize the volume of the polyhedron with those N ver-...[2] *This corresponds to the problem for area in two dimensions. ... N is 4 the answer is as expected. A regular tetrahedron maximizes ...volume. Recall that the volume of a tetrahedron is 1/3 hA, where A ...e area of one of the triangular faces and h is the distance from the ... of that face to the fourth vertex. When the vertices lie on a ...e, the base plane's intersection with the sphere is a circle through ... of the vertices. If the base triangle is not equilateral, replacing ... an equilateral triangle will increase the area of that face and hence ...olume of the tetrahedron. Applying this argument to each of the faces of an inscribed tetrahedron, we see that the inscribed tetra-... of maximum volume have all of their faces equilateral and hence ...recisely the regular ones.*

(2) Upon examining his understanding of the phrase, "polyhedron with those N vertices", the reader may find it to be based more on intuition than on mathematics. Since the film's problems are all set in Euclidean 3-space E^3, they can be understood fairly well on an intuitive basis. However, we want also to discuss the higher-dimensional analogues of some of the problems and for those, surely, some precise definitions are required.

A subset C of Euclidean d-space E^d is said to be <u>convex</u> provided that it contains all line segments whose endpoints are in the set; that is, $\lambda x + (1-\lambda)y \in C$ whenever $x \in C$, $y \in C$, and $0 \leq \lambda \leq 1$. Intuitively, a convex set is one that has neither dents nor holes. Several basic notions concerning convex sets are relevant to the film's problems, and they are discussed in this and subsequent footnotes. See Klee (1971b) for a short general survey of convexity theory, including references to the standard texts on various aspects of the subject.

It is plain that the intersection of any family of convex sets is itself convex. Hence we may intersect all convex sets containing a given set X to form the smallest convex set containing X; it is called the <u>convex hull</u> of X. It is known that the convex hull of X is the set of all points expressible in the form $\lambda_1 x_1 + \cdots + \lambda_k x_k$, where k is a positive integer, the λ_i's are nonnegative real numbers whose sum is 1, and the x_i's are points of X. And by a theorem of Carathéodory (1907) (see also Danzer, Grünbaum and Klee (1963)), we may set $k = d + 1$ when $X \subset E^d$. Thus, for example, if X is a subset of E^3 not contained in any plane, the convex hull of X is the union of all tetrahedra whose vertices belong to X.

As the term is used here, a <u>polyhedron</u> is a 3-dimensional subset of E^3 which is the convex hull of a finite set of points; equivalently, it is a convex solid whose boundary is formed by a finite number of

convex polygons. The problem of placing N points on a sphere so to "maximize the volume of the polyhedron with those N vertices" requires that the volume of the convex hull of the set X formed by the N points should be as large as possible.

The volume is maximized when N is 6 by placing the points at the vertices of a regular octahedron, and when N is 12 at the vertices of a regular icosahedron.[3] *That might be expected, by analogy with the regularity of the solution on the circle. However, the analogy is misleading when N is 8, for the inscribed cube does not give the maximum volume in this case. When the sphere is of radius r, the cube's volume is about $1.5r^3$ while that of a double pyramid based on a regular hexagon is about $1.7r^3$; but that isn't the maximum either. Here is the best arrangement of eight points, which yields a volume of about $1.8r^3$. It was discovered in the 1960's when it was approximated by a graduate student with the aid of a computer and later proved by two other graduate students to be optimum.*[4] *The complicated nature of the exact expression for the volume hints at the complexity of the proof. Here's a model of the configuration, mounted so that you can see it can be inscribed in a sphere. Plainly it's far from being regular--for example, its edges are of three different lengths.*

(3) See Fejes Tóth (1953, 1965) for proofs of these statements.

(4) This optimum arrangement of eight points was approximated by Grace (1963) with the aid of a computer search which identified it as providing a local maximum for the volume of the convex hull of eight points on the unit sphere of E^3. Berman and Hanes (1970) described the arrangement precisely and proved that it provides, up to rotation, the unique global maximum for the volume. See references

There are reasonable conjectures for some other values of N. Here is the conjectured optimum for nine vertices, obtained by adding a pyramidal cap over each of the square faces of a triangular prism. The volume problem has actually been solved for all N<9 and for N = 12. You might try to find the known best configurations for 5, 6, and 7 points yourself, in order to get a feeling for the problem.[5][6][7][8]

[5] A famous theorem of Euler(1752) (see also Grünbaum(1967)) asserts that if a polyhedron P has v vertices, e edges, and f faces, then $v - e + f = 2$. If P is of maximum volume among all polyhedra formed as the convex hull of a given number v of points (P's vertices) on the unit sphere of E^3, then all of P's faces are triangles (Fejes Tóth (1953)), whence it follows easily that $3f = 2e$ and hence $2e = 6v - 12$. Defining the valence of a vertex as the number of edges incident to it, let v_n denote the number of n-valent vertices of P. Then the average valence of P's v vertices is $\frac{\Sigma n v_n}{v} = \frac{2e}{v} = 6 - \frac{12}{v}$. Call the polyhedron P medial provided that all of its faces are triangles and the valence of each of its vertices differs by less than 1 from the average $6 - 12/v$. The following conjecture, dual in a sense to one of Goldberg(1935) (see [7] below), was formulated by Grace(1963) and used also by Berman and Hanes(1970): If P is a polyhedron whose v vertices lie on the unit sphere and whose volume is a maximum subject to this condition, then P is medial if a medial polyhedron exists for the v in question. For $v \leq 8$ the validity of this conjecture follows from the work of Berman and Hanes(1970), who state some additional unsolved problems concerning volumes of polyhedra.

[6] For points $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$ of E^d, the inner product $\langle x,y \rangle$ is given by $\langle x,y \rangle = x_1 y_1 + \cdots + x_d y_d$. The norm of a point x is $\| x \| = \langle x,x \rangle^{1/2}$ and the distance between two points x and y is $\| x - y \|$. When x is not the origin 0, the set $H_x = \{y \varepsilon E^d : \langle x,y \rangle = 1\}$ is a hyperplane (a line when d = 2, an ordinary plane when d = 3, $\cdots$, a (d - 1)-dimensional flat in the general case) orthogonal to the ray from 0 through x. In particular, when x belongs to the unit sphere $S = \{s \varepsilon E^d : \| s \| = 1\}$, the hyperplane H_x is tangent to S at x. For any point x of E^d, the polar $\{x\}°$ is defined by

$$\{x\}° = \{y \varepsilon E^d : \langle x,y \rangle \leq 1\},$$

which for $x \neq 0$ is a closed halfspace that contains the origin and is bounded by the hyperplane H_x. The polar of any set X E^d is defined as the intersection of the polars of the members of X; that is,

$$X° = \{y \varepsilon E^d : \langle x,y \rangle \leq 1 \text{ for all } x \varepsilon X\},$$

The problem discussed above ---- of placing a given number N of points on S so as to maximize the d-measure (area when d = 2, volume when d = 3, etc.) of their convex hull ---- is closely related to the problem of placing N points on S so as to minimize the d-measure of their polar. When d = 2, the latter asks for the convex polygons of minimum area circumscribed about S and having N edges; the unique solution is provided by the regular N-gons (see, for example, Fejes Tóth(1953)). When d = 3, it asks for the convex polyhedra of minimum volume circumscribed about S and having N faces. This problem has been extensively studied because of its connections with the isoperimetric problem for polyhedra. See [7] below.

For d > 3, very little is known about either the maximum or the minimum problem stated above, beyond the fact that the regular d-simplex (d-dimensional analogue of equilateral triangles and regular tetrahedra) is the largest simplex contained in a given sphere and the smallest simplex containing a given sphere (Slepian(1969), Ali (1970)). In other words the maximum and minimum problems have been solved when N = d + 1.

[7] The 2-dimensional isoperimetric theorem asserts that, among all plane convex bodies of given perimeter, the circular disks are of maximum area. In fact, if L and A are respectively the perimeter and the area of a 2-dimensional body, then $L^2/A \geq 4\pi$, with equality only for circular disks. For convex polygons with a given number N of vertices or edges, the inequality is sharpened to $L^2/A \geq 4N\tan(\pi/N)$, with equality only for regular N-gons (see Fejes Tóth(1953)).

The 3-dimensional isoperimetric inequality is $A^3/V^2 \geq 36\pi$, where A and V are respectively the surface area and the volume of a 3-dimensional body; equality holds only for spherical balls. The inequality can be sharpened in various ways for polyhedra. For example, with $\omega_n = \frac{n}{n-2}\frac{\pi}{6}$ it is known (Goldberg(1935), Fejes Tóth (1953)) that

$$A^3/V^2 \geq 54(f - 2)\tan\omega_f(4\sin^2\omega_f - 1)$$

whenever A and V come from a polyhedron of f faces; equality holds only for regular tetrahedra, cubes, and regular dodecahedra. Fejes Tóth (1953) gives other conjectured inequalities which have been proved only in special cases.

The "isoperimetric problem for polyhedra" asks for those polyhedra which, for a given surface area and given number k of faces, are of maximum volume; equivalently, it asks for those which minimize the quotient A^3/V^2. But it is known that any such minimizing polyhedron is circumscribed about a sphere, and that $A^3/V^2 = 27V$ for any polyhedron circumscribed about the unit sphere S. Hence the problem reduces to the one, mentioned above, of finding the polyhedra of minimum volume which are circumscribed about S and have k faces. The problem has been rigorously solved for $k \leq 6$, perhaps for k=7, and for k=12, but appears otherwise to be open though it has been studied by distinguished mathematicians since a 1782 paper of Lhuilier. See Steinitz(1927,1928), Goldberg(1935), and Fejes Tóth(1953,1965) for references and a more detailed account.

Goldberg(1935) called a polyhedron with f faces medial provided that each of its vertices is 3-valent and the number of edges of each face differs by less than 1 from the average $6 - 12/f$. For any polyhedron P having the origin in its interior, it is true that P is inscribed in the unit sphere S if and only if the polar P° is circumscribed about S, and that P is medial in Goldberg's sense if and only if P° is medial in the sense of Grace(1963) and Berman and Hanes (1970) (see [5] above). Goldberg's conjecture was that if P is a polyhedron whose f faces are tangent to the unit sphere and whose volume is a minimum subject to that condition, then P is medial (in his sense) if a medial polyhedron exists for the f in question.

The following attractive conjecture seems to be consistent with the few known facts. For each k > d and each set X of k points on the unit sphere S of E^d, the following two statements are equivalent: (a) there is no set of k points on S whose convex hull has greater d-measure than that of X; (b) there is no set of k points on S whose polar has smaller d-measure than that of X.

[8] The term polytope is used here to mean a set (in a finite-dimensional Euclidean space) which is the convex hull of a finite set of points; equivalently a polytope is a bounded set which is

the intersection of a finite number of closed halfspaces. The standard reference on polytopes is Grünbaum(1967). A polytope of dimension d is called a d-polytope; thus 2-polytopes are convex polygons and 3-polytopes are convex polyhedra. When P is a d-polytope in E^d, a face of P is either P itself or the intersection of P with a hyperplane which misses the interior of P. Each face of a polytope is itself a polytope. The 0-faces are vertices and the 1-faces are edges. The 2-faces are often simply called "faces" when P is 3-dimensional.

Let $\Sigma_i(P)$ denote the sum of the i-measures of the various i-faces of P; for example, $\Sigma_1(P)$ is the sum of the lengths of P's edges. For distinct integers i and j between 1 and d, let $\rho_{ij}(P)$ denote the isoperimetric ratio $\Sigma_i(P)^{1/i}/\Sigma_j(P)^{1/j}$. Surprisingly, the following problem is open: For which triples (d,i,j) is $\rho_{ij}(P)$ bounded above as P ranges over all d-polytopes? And even when $\rho_{ij}(P)$ is known to be bounded above, the precise value $\beta(d,i,j)$ of its supremum has been determined only when i = d and j = d − 1.

Eggleston, Grünbaum and Klee(1964) show $\beta(d,i,j)$ is finite if i = d or i = d − 1 > j or i is a multiple of j, and Klee(1970) shows $\beta(d,i,j)$ is infinite whenever i < j. However, the finiteness of $\beta(d,i,j)$ is unsettled whenever $d - 2 \geq i > j \geq 2$ and i is not a multiple of j. From the d-dimensional version of the classical isoperimetric inequality it follows that $\beta(d,d,d-1) = (dW_d^{1/d})^{-1/(d-1)}$, where W_d is the d-measure of a d-dimensional spherical ball of unit radius. The supremum $\beta(d,d-1,d-1)$ is not attained by any d-polytope, but E. Grünbaum conjectures that all other finite suprema are attained. Aberth(1963) shows $\beta(3,2,1) \leq (6\pi)^{-1/2}$. Melzak(1965) conjectures $\beta(3,3,1) \leq 2^{-2/3}3^{-11/6}$, with equality only for a right prism based on an equilateral triangle whose edge-length is equal to the height of the prism. For other results related to the determination of $\beta(d,i,j)$ see Kömhoff(1968,1970) and Larman and Mani(1970b).

Now let's turn to the dispersal problem. We want to place N points on a sphere so as to maximize the minimum distance between the points. Just as in the analogous problem on the circle, we need consider only the distances between points whose line segment is an edge of the polyhedron formed by taking the points as vertices. Quite a simplification, visually [illegible]. To [illegible] on this problem, you might think of [illegible] as the [illegible] N misanthropes, each of whom so hates the others [illegible] wants to get as far away from them as possible. For this, as for [illegible] is a good guide when N is 4, 6, or 12, but [illegible] not when N = 8. The most dispersed arrangement for eight points [illegible] at the vertices of a cube but rather at those of a square antiprism, which yields a larger minimum distance. [illegible] that the [illegible] and dispersal [illegible] answers when N = 8, while on the

circle the area and dispersal problems always have the same answer. The dispersal problem for N points has been solved for N < 10, N = 12, and N = 24. A solution has been announced for N = 11, and there are conjectured solutions for N = 10 and for about a dozen other values of N.[9]

(9) Solutions of the dispersal problem for N ≤ 6 were discovered by Tammes(1930), a botanist. Rigorous solutions were given for N ≤ 6 and N = 12 by Fejes Tóth(1943, 1949, 1953), for N ≤ 9 by Schütte and van der Waerden(1951), and for N = 24 by Robinson(1961). Robinson's result established a conjecture of Schütte and van der Waerden(1951), as did the solutions announced by Ludwig Danzer for N = 10 and N = 11. However, Danzer's arguments, presented at conferences in 1962 and 1958 respectively, have never been published. There are published conjectures of Schütte and van der Waerden(1951), van der Waerden(1952), Jucovic(1959) Strohmajer(1963), Goldberg(1965, 1967a-c, 1969a), and Robinson(1969) which cover all values of N ≤ 42 with the exception of 23, 28, 29, 34, 38, and 39, and cover also the values 44, 48, 52, 60, 80, 110, 120, and 122. The known results and conjectures are summarized by Goldberg(1967a, 1969a) Among the expository accounts of the dispersal problem are those of Fejes Tóth(1953, 1965), Meschkowski(1966), van der Waerden(1961), Coxeter(1962), and Klee(1971a).

As was remarked by Robinson(1961), a general method of Tarski(1951) provides in theory a solution of the dispersal problem for any given number N of points. Indeed, for each N there is a finite number of computational steps leading to an algebraic equation satisfied by the maximum, overall arrangements of N points on the unit sphere, of the minimum distance. However, because of the length of the required computation, the method does not seem to be applicable in practice. For very large values of N one can only seek lower and upper bounds on the maximum of the minimum distance, or, alternatively, lower and upper bounds on the maximum number $C(\varphi)$ of spherical caps of given angular radius φ that can be placed on the sphere without overlapping. It is known that $\frac{2\pi}{\sqrt{3}}\{\csc^2\varphi - (\frac{3}{2}\csc^2\varphi)^{2/3}\} \leq C(\varphi) \leq \frac{2\alpha}{\alpha-30}$, where α is such that $\sec 2\alpha = 1 + \sec 2\varphi$. Here the lower bound is due to van der Waerden(1952) and the upper bound to Fejes Tóth(1949); see also Coxeter(1962) for the latter. For $\varphi < 22°$ a sharper but more complicated upper bound follows from the work of Robinson(1961).

The first systematic study of the dispersal problem was made by a botanist.[9] *This picture shows why. He wanted to explain the distribution on spherical pollen grains of the places through which a pollen tube can emerge in the process of fertilization. As each pollen tube takes up a certain amount of space, the exit places can't be too close together. However, there should be as many exit places as possible in order to maximize the chances of fertilization. We might say that the pollen grain wants to maximize the number of points--exit places--for a fixed minimum* *distance, while the misanthropes want to maximize the minimum distance for a fixed number of points--themselves. However, a complete solution of either problem entails a complete solution of the other.*[10]

[10]For another connection of the dispersal problem with biology, see the mathematical model of cell-nuclei formulated by Şerban and Stroilă(1966). The shapes of virus particles appear not to be related to the solutions of the dispersal problem (see Goldberg(1967d)) but rather to another geometrical problem considered by Goldberg (1937) (see Caspar and Klug(1963) and Wrigley(1969)).

In his studies of molecular geometry, Gillespie(1960,1970) (see also Levine(1970)) notes that the arrangements of electron-pairs in a given valency shell are a consequence of the mutual interactions of electrons due to (a)electrostatic repulsions, and (b)the operation of the Pauli exclusion principle, according to which electrons of the same spin tend to stay as far apart as possible, while electrons of opposite spin tend to be drawn together. He then concludes that, in most cases, (b) is so dominant that (a) can be virtually ignored and the dispersal problem provides the correct model for studying the arrangement. However, Gillespie(1970) appears to claim incorrectly that the most dispersed arrangement of 10 points is at the vertices of a bicapped square antiprism.

In connection with an earlier model of the atom, Thomson(1904) sought to determine the stable equilibrium patterns of N classical electrons constrained to lie on the surface of a sphere while repelling each other according to the inverse square law. (One such pattern would be that of minimum potential energy, but there might be others; there might even be inequivalent patterns having the same minimum potential energy.) The problem, which is no easier than the dispersal problem, was later considered by Föppl(1912), Whyte(1952), Cohn(1956), and Goldberg(1969b), but the total amount of progress was not great. However, Cohn(1960) was able to provide a rather complete treatment of the corresponding problem for the circle, where N not necessarily equal point charges are acted upon by a fairly general repulsion law. For each ordering of the charges on the circle there is (up to rotation) a unique stable configuration, and when all charges are equal it is the regular one.

An interesting aspect of the misanthrope problem is brought out by noting that the best minimum distance for five points is the same as for six. That is, if D(N) denotes the best distance for N points, then D(N-1) = D(N) for N = 6. Thus the five misanthropes are surprised to find they are no better off than six! The same relationship seems to hold between eleven misanthropes and twelve. The regular icosahedron provides the

unique best arrangement for twelve points, and it has been announced that discarding one of those points provides the unique best arrangement for eleven. Similar conjectures have been made for N = 24, 48, 60, and 120. Thus the misanthrope problem leads to the following sub-problem: Is D(N-1) equal to D(N) when N is 6, 12, 24, 48, 60, and 120, and otherwise strictly greater than D(N)? Of course D(N-1) is never less than D(N).[11]

[11]The result for 11 points is Danzer's, as indicated in [9], while the conjecture about 24, 48, 60, and 120 is due to Robinson(1969). The special property of these numbers actually established by Robinson, and related to the conjecture, is as follows: If N points are placed on a sphere in such a way that each point is as near to five others as any two points are to each other, then N is 12, 24, 48, 60, or 120, and for each of these values of N the configuration is unique up to rotation and reflection. This result has been extended in a certain direction by Fejes Tóth(1969b).

A related problem involving points on a sphere was the source of a disagreement in 1694 between Isaac Newton and David Gregory. They wondered how many spheres could be arranged so as to touch a central sphere without any overlapping, all the spheres being congruent. The points of tangency of the outer spheres with the central sphere provide a distribution of points on that sphere. In the corresponding plane problem--arranging congruent circles to touch a central circle--the maximum is easily seen to be 6. For the 3-dimensional maximum, Newton conjectured 12 and Gregory 13. Note that if congruent spheres are tangent, one subtends on the other a cap of angular radius 30°. Hence the Newton-Gregory problem amounts to asking: How many caps of angular radius 30° can be placed on a sphere without overlapping?

One way of arranging twelve such caps is to maximuze the minimum distance between their centers. In this arrangement, which places the centers at the vertices of an inscribed regular icosahedron, the 30° caps can even be enlarged slightly without overlapping. Newton knew a sphere arrangement involving twelve contacts, was unable to produce any involving thirteen, and probably assumed no one else could do it if he couldn't. Thus he conjectured the maximum was 12. Gregory's guess of 13 was also plausible, as the surface area of a sphere is more than thirteen times *that of a cap of angular radius 30°. Well, Newton was right, but that wasn't proved until 180 years later.*(12) *I hope it will be less than 180 years before someone solves the next stage of Newton's problem, in which the single central sphere is replaced by two tangent spheres. That is, how many spheres can be made to contact the figure formed by two tangent spheres, all spheres being congruent and no overlapping permitted?*(12)

(12)See Coxeter(1962,1963) for an account of the Newton-Gregory controversy and for references to solutions. The conjecture for the two-sphere problem is due to R. Robinson(written communication), who observes that it implies a conjecture of L. Fejes Tóth to the effect that any packing of congruent spheres in E^3 in which each sphere touches 12 others must be formed from layers with the usual hexagonal arrangement.

As a step toward a reasonable conjecture for the two-sphere problem, let us consider another arrangement of twelve 30° caps on a sphere, this one associated with a cuboctahedron rather than an icosahedron. In contrast with the icosahedral arrangement, this one does not permit the caps to be enlarged. However, it does lead to a packing of congruent spheres in space, with each contacting twelve others. To see this, simply divide 3-space into cubes in the natural way and place spheres concentric with alternate cubes so that each sphere is tangent to all twelve edges of its cube. This is the "cubic close-packing" of Johannes Kepler. It is of special interest to crystallographers, as it is conjectured to provide the densest packing of spheres in the space. However, that's another unsolved problem, as the conjecture has been proved only under certain regularity assumptions.(13) *If we examine a particular sphere in this configuration, we see that the twelve points of tangency with the other spheres are at the vertices of a cuboctahedron. And if we examine two tangent spheres, we see that there are 18 others touching one or both of them. The conjectured maximum for the two-sphere problem is 18.*(12)

(13)A sphere-packing in E^d is called a <u>lattice-packing</u> provided that, for each pair of sphere-centers x and y, the point 2y - x is also the center of a sphere in the packing. The cubic close-packing

of congruent spheres in E^3 is a lattice-packing and is known to be the densest such. However, there is in E^3 a non-lattice packing of the same density, and it is therefore natural to ask whether some other non-lattice packing may have even greater density. According to Rogers(1958), "many mathematicians believe, and all physicists know" that this is not the case. However, for certain values of d greater than 3 there are non-lattice packings in E^d which are denser than the densest known lattice-packings (Leech and Sloane(1970)).

There are many other unsolved problems concerning arrangements of points on spheres. For example, how should a fixed number of unit charges be placed on a sphere so as to minimize the potential energy of the configuration?(14)(15)(16) *However, I'd like to pass on to a group of unsolved problems related to the famous four-color conjecture. These will require some definitions.*

(14)The problem of finding the most dispersed arrangement of N points in a set X, and closely related problems concerning the packing of circles or spheres in X, have been studied for choices of X not mentioned above. See(15) for higher-dimensional spheres. For the cases of circular disks, rectangles, and cubes see Goldberg(1970, 1971a,1971b), Kravitz(1967,1969), Pvil(1969), Ruda(1970), Schaer (1965), and Schaer and Meir(1965).

(15)To conclude the discussion of points on spheres, we will describe some of the higher-dimensional results and problems.

The special relationship between 5 points and 6 in the case of E^3 was extended to E^d by Rankin(1955), who showed that for $d + 2 \le N \le 2d$, the most dispersed arrangement of N points on the unit sphere of E^d involves a minimum angular distance of $\pi/2$.

Coxeter(1963) was concerned with the maximum number N_d of balls that can touch a ball in E^d, all of the (spherical) balls being congruent and no overlapping permitted; equivalently, N_d is the maximum number of points that can be placed on a sphere in E^d so that the minimum angular distance is at least $\pi/3$. He obtained the following bounds: $24 \le N_4 \le 26$, $40 \le N_5 \le 48$, $72 \le N_6 \le 85$, $126 \le N_7 \le 146$, $240 \le N_8 \le 244$. (See also Leech and Sloane(1971).) His lower bounds are firm, as they result from specific constructions, but his upper bounds result from a formula based on a conjecture which is unproved for all $d \ge 5$. (For d = 4 it was proved by Böröczky and Florian(1964).) Coxeter's formula leads to the asymptotic expression $2^{(d-1)/2}\pi^{1/2}d^{3/2}e^{-1}$ for the upper bound. Fejes Tóth and Heppes(1967) considered the number T_d of congruent balls in E^d that can be arranged so that each ball in the family either touches a certain ball B or touches a member of the family that touches B; they showed $T_2 = 18$, $56 \le T_3 \le 63$, and $168 \le T_4 \le 232$. A further extension of this idea was considered in F^2 by Fejes Tóth(1969a).

As was explained in the expository article of van der Waerden (1961), the dispersal problem on a high-dimensional sphere is of interest in information theory. In a communication system in which all signals are of the same energy and are made up of a limited number of frequencies, each signal may be represented by a point on a sphere centered at the origin in E^d, where the radius of the sphere is determined by the energy, the dimension by the number of frequencies, and the coordinates of a point by the Fourier coefficients of the associated wave-form. The communication is inevitably subject to some "noise", so that when a point p is sent, it is certain only that the received point (signal) is at distance < δ from p, where the value of δ depends on the energy of the noise. In order to avoid ambiguity in communication, any two signals (points) that are used should be at distance ≥ 2δ. Subject to this minimum distance requirement, there should be as many available signals as possible. Alternatively, if the number of signals is fixed, they should be as dispersed as possible in order to maximize the amount of noise that can be tolerated. See Leech and Sloane(1971).

In another form of the communication problem, one seeks to minimize the probability of error. This leads, as Balakrishnan(1961,1965) has shown, to the problem of placing N points on the unit sphere S of E^d so as to maximize the mean width of their convex hull. (For a convex body $B \subset E^d$ and a unit vector $u \in S$, the <u>width</u> $w_u(B)$ in the direction u is defined as the length of the interval $\{\langle u,b\rangle : b \in B\}$. The <u>mean width</u> $w(B)$ is then obtained by averaging $w_u(B)$ over S; that is,

$$w(B) = \frac{\int_{u \in S} w_u(B)}{(d-1)\text{-measure of } S}.$$

The <u>simplex conjecture</u> of information theory asserts that when $N = d + 1$ the mean width is maximized by placing the N points at the vertices of a regular simplex inscribed in S. Balakrishnan showed that the regular arrangement provides a local maximum, but the claimed proof of global optimality of Landau and Slepian(1966) was shown by Farber(1968) to be invalid. Thus the problem is open for all d > 3. Tanner(1970) has provided a survey of the problem and a well-organized exposition of the relevant geometrical knowledge.

(16)The intention here has been to stay rather close to the specific unsolved problems mentioned in the film. Thus, despite the considerable number of references cited, we have barely scratched the surface of the vast literature devoted to packing problems and have not even mentioned the closely related covering problems. The standard references on packing and covering are Fejes Tóth(1953) (and many subsequent papers by him) for two and three dimensions, and Rogers(1964) for higher-dimensional spaces.

Let us say that two 2-dimensional regions are <u>neighbors</u> if their intersection is 1-dimensional, and that two 3-dimensional regions are

<u>neighbors</u> if their intersection is 2-dimensional. A <u>neighborly family</u> of regions is one in which every region is a neighbor of every other region. The four regions of this planar map form a neighborly family, so of course the map cannot be colored in less than four colors. However, a plane cannot contain any neighborly family consisting of more than four regions. Now, what is the situation in 3-space? That is, what is the maximum number of 3-dimensional regions in a neighborly family? It turns out there is no maximum, that for each N there exists a neighborly family consisting of N 3-dimensional regions. Here's an indication of how such a family could be constructed.(17) *But what if the regions are very restricted in shape--for example, if they are all tetrahedra?*

(17)It can even be required that all of the regions are convex polyhedra. Different proofs of this have been given by Tietze(1905), Besicovitch(1947), and Danzer, Grünbaum, and Klee(1963). See Tietze(19 for the history of the problem. The construction of Danzer, Grünbaum and Klee is based upon the startling fact that for each N > 4 there exists a 4-polytope with N vertices such that each pair of vertices is joined by an edge of the polytope. This fact and its higher-dimensional analogues, first established by Carathéodory(1911) and later rediscovered by Gale(1956,1963), play an important role in the study of polytopes (Grünbaum(1967)).

Now we have an unsolved problem. Specifically, what is the maximum number N of members for a neighborly family of tetrahedra? To see that N is at least 8, we begin with two bases. Each consists of four neighborly triangles. We then construct two neighborly families of four tetrahedra each by forming two pyramids. Finally we place the two pyramids base to base and give a slight twist. That yields the configuration shown here in the plane of the bases. Thus each tetrahedron in one pyramid family has a two-dimensional intersection with each in the other, and the eight tetrahedra form a neighborly family. Hence N is at least 8. In the other direction, it has been proved that no neighborly family includes more than nine tetrahedra, though no one knows whether nine are actually possible. The problem, then, is to decide whether the maximum number N is 8 or 9, and the answer is generally believed to be eight. In attempting to prove this you might want to look for a new method rather than refining the old one, for the existing proof that N is at most 9 takes about two hundred pages! (18)

(18)The problem of neighborly tetrahedra is due to Bagemihl (1956), who proved $8 \le N \le 17$. Baston(1965) showed $N \le 9$. An exposition of the problem was provided by Klee(1969a). For the analogous d-dimensional problem, concerning neighborly families of d-dimensional simplices in E^d, the literature does not even seem to contain any good bounds, though Baston(1965) conjectures the maximum is 2^d.

The four-color problem deals with very general maps on a plane or sphere, in which the shapes of the countries may be very complicated. However, the problem can be reduced in several ways. In preparation for some reductions, I'll say that a set is convex if it has no dents or holes in it or, more formally, if it contains all line segments whose endpoints are in the set. The definition applies, of course, to all dimensions. A convex polyhedron is a 3-dimensional convex region whose surface is made up to a finite number of convex polygons. These surface polygons are called the faces of the polyhedron and we're also interested in its edges and vertices. For example, the tetrahedron, cube, and octahedron are all convex polyhedra. Their numbers of vertices, edges, and faces are as shown.

According to one reduction theorem, which we won't prove, the four-color conjecture is equivalent to the conjecture that the faces of any convex polyhedron can be colored in four colors so that neighboring faces never receive the same color.[19] *This plainly is possible for each of the three examples, but there are infinitely many other convex polyhedra to worry about. In order to reduce the four-color problem to a special class of convex polyhedra, we introduce the notion of truncation. That means slicing a polyhedron with a plane that passes between one vertex and the remaining vertices. This replaces the vertex by a small face, and it's easy to see the original polyhedron can be colored in four colors if the truncated version can. Just note that any two faces which would be neighbors with the vertex restored are also neighbors in the truncated version. By truncating any convex polyhedron at each of its original vertices, we obtain one that is 3-valent, meaning that each vertex is on exactly three edges. For example, this is the result of truncating the octahedron at each of its six vertices. If the resulting 3-valent polyhedron can be colored in four colors, then so can the original polyhedron. Thus the four-color conjecture would be proved if we could establish it for all 3-valent convex polyhedra.*

(19)The most complete exposition of the four-color problem appears in the book by Ore(1967), and some important aspects omitted by Ore are discussed by Heesch(1969). May(1969) discusses the problem's origin and Ore and Stemple(1970) show that any map of less than forty countries can be colored in four colors.

The reduction of the four-color problem to the problem of coloring the faces of a (convex) polyhedron follows from reductions given in the literature together with an important theorem of F. Steinitz which characterizes the graphs (combinatorial structures formed by the vertices and edges) of polyhedra in purely combinatorial terms. The theorem of Steinitz first appeared in Steinitz and Rademacher (1934), and simpler proofs were provided by Grünbaum(1967) and Barnette and Grünbaum(1969). The graphs of d-polytopes for $d > 3$ have still not been characterized in purely combinatorial terms, though necessary conditions have been given by Balinski(1961), Barnette(1967), Grünbaum and Motzkin(1963) (see also Grünbaum(1965)), Klee(1964), and Larman and Mani(1970a). See Grünbaum(1970a) for some higher-dimensional analogues of the four-color problem.

Now let's look at a different aspect of truncation. If you truncate the cube four times in the manner shown, you obtain a convex polyhedron in which each face has three edges or six edges. It has been conjectured that every convex polyhedron admits a finite sequence of truncations leading to a polyhedron in which every face has a number of edges that is a multiple of 3. The required truncation sequences may be more complicated than the one already shown for the cube. For example, this one involves truncating a vertex which had itself been introduced by an earlier truncation. In any case, this innocent-sounding conjecture is known to be equivalent to the four-color conjecture![20]

(20)The truncation conjecture is discussed by Hadwiger(1957).

There have been several claimed proofs of the four-color conjecture, all of which turned out to be incorrect.[21] *However, one of them has led to another interesting unsolved problem. In order to introduce that problem and its connection with the four-color conjecture, let's consider any 3-valent convex polyhedron that admits a Hamiltonian circuit. By this I mean a way of traveling along the polyhedron's edges so as to visit each vertex exactly once and return to the starting point. For example,* *that was a Hamiltonian circuit on the cube and here is one on the regular dodecahedron. Any such circuit divides the surface of the polyhedron into two halves in a natural way, and it can be proved that the faces in each half can be colored with only two colors so that neighboring faces receive different colors. That leads to an acceptable coloring of all the faces with just four colors.*

(21) While writing the final version of this viewer's manual (October 1971), I heard of another claimed proof of the four-color conjecture, based on work of Heesch(1969), which sounded much more promising than the many earlier claims. However, I have not yet seen the details of the argument.

Thus the four-color conjecture could be proved by showing that any 3-valent convex polyhedron admits a Hamiltonian circuit. The existence of such circuits stood as a conjecture for more than sixty years until, finally in 1946, a counterexample was produced. However, there remains the problem of finding the smallest counterexample. That is: What is the minimum number M of vertices for a 3-valent convex polyhedron not admitting a Hamiltonian circuit?[22] *It is known that M is between 20 and 38. The proof that M is at least 20 involves an examination of the many different types of 3-valent convex polyhedra with fewer than 20 vertices, and the discovery of a Hamiltonian circuit in each. In the other direction, there is a clever proof that a certain polyhedron of 38 vertices does not admit any Hamiltonian circuit.* [illegible]

(22) A result equivalent to the conjecture on Hamiltonian circuits was stated without proof by Tait(1880), and proofs were claimed by Chuard(1932) and Schönlik(1930). The counterexample of Tutte(1946) has 46 vertices, but an example with 38 vertices was constructed independently by D. Barnette(see Klee(196)), Bosak(1967) and Lederberg (1967). With the aid of a method of generating 3-valent polyhedra due in polar form to Bowen and Fisk(1967) it has been proved by Mrs. Jean Butler that M [illegible] 24. Thus 24 ≤ M ≤ 38, but further progress may be quite difficult. Also unknown is the smallest number of vertices for a 3-valent convex polyhedron not admitting a <u>Hamiltonian path</u>, where this is defined like a Hamiltonian circuit except that it need not return to its starting point. The smallest examples known at present are those of [illegible] A. Pros [illegible] vertices) and T. Zamfirescu (88 vertices), which are described in the expository accounts of Klee(1967) and

Grünbaum(1970b) respectively. Several related unsolved problems are discussed in these accounts. In particular, the following attractive conjectures of D. Barnette are stated by Grünbaum(1970b):

<u>A 3-valent polyhedron admits a Hamiltonian circuit if each of its faces has an even number of edges.</u>

<u>Each 4-valent 4-polytope admits a Hamiltonian circuit.</u>

The second conjecture is unsettled even for the simplest 4-valent 4-polytopes---namely, those formed as prisms over 3-valent polyhedra. The existence of Hamiltonian circuits for such 4-polytopes has been observed by D. Barnette to follow from the four-color conjecture.

A graph is said to be k-connected provided that it has at least k + 1 vertices and is not separated by the removal of any k vertices; equivalently, each pair of its vertices can be joined by k paths that are pairwise disjoint except for their common endpoints. By the theorem of Steinitz(1934) mentioned in (19), as reformulated by Grünbaum and Motzkin(1963), a graph G is isomorphic with the graph of a polyhedron if and only if G is planar and 3-connected. In connection with the above problems on Hamiltonian circuits, it should be mentioned that Tutte(1956) (see also Ore(1967)) proved that every 4-connected planar graph admits a Hamiltonian circuit. And Fleischner(1971) has recently proved that the <u>square</u> of every 2-connected graph admits a Hamiltonian circuit. (The square G^2 of a graph G has the same vertices as G. However, two vertices of G^2 are neighbors (joined by an edge) in G^2 if and only if they are either neighbors in G or have a common neighbor in G.)

For results concerning the existence of Hamiltonian circuits on polyhedra of unrestricted valency, and on certain special classes of 3-valent polyhedra, see Klee(1967) and Grünbaum(1970b) and their references. See especially Grinberg(1968), Sachs(1968), and Barnette and Jucovic(1970).

For additional unsolved problems on various aspects of the geometry of polytopes, see Grünbaum(1967,1970b), Grünbaum and Shephard (1969), Klee(1966), and Shephard(1968).

Though the problem of finding M's true value is a special and difficult one, it is of interest in connection with an important problem from organic chemistry---that of codifying the descriptions of organic compounds in a manner that is amenable to modern methods of electronic data processing.[23] *Having mentioned some slight relationships of convex bodies to botany, crystallography, and organic chemistry, I'll close with three short unsolved problems concerning physical properties of convex bodies.*

(23)See Lederberg(1965,1969) for the use of Hamiltonian circuits in the codification of organic compounds.

A spherical ball has the property that for any two planes the area of the ball's largest cross-section parallel to one plane is the same as that of its largest cross-section parallel to the other. It is unknown whether any non-spherical convex body has that property. The analogous 2-dimensional question has an affirmative answer. In fact, there is a noncircular convex body whose maximum cross-sectional length in any direction is the same as that in any other direction.(24)

(24)The plane convex bodies "whose maximum cross-sectional length in any direction is the same as that in any other direction" are precisely the bodies of constant width, usually defined by the condition that the distance between any pair of parallel tangent lines is the same as the distance between any other such pair. See Klee(1971a) for an expository account of some of the surprising properties of these bodies, and for references to the literature.

The unsolved 3-dimensional problem mentioned above, dealing with areas of plane cross-sections, is related to the problem of determining the Fermi surface of a metal by means of the de Haas-van Alphen effect. See Klee(1969b,1971a) for accounts of this, and Mackintosh (1963) for a readable discussion of Fermi surfaces. Zaks(1971) has constructed some rather well-behaved bodies which are not spherical and yet have the property of "constant maximum cross-sectional area"; however, his examples are not convex.

To introduce the next problem, we note that the non-circular plane convex body shown here is such that for ***any*** *line cutting the body into two halves of equal area, the segment joining the centers of gravity of the two halves is perpendicular to the line. The question is whether any nonspherical 3-dimensional convex body has the analogous property for all planes cutting it into two halves of equal volume. That is, does there exist a nonspherical 3-dimensional convex body such that, for any plane cutting it into two halves of equal volume, the line joining the centers of gravity of the two halves is perpendicular to the plane? This amounts to asking whether there is a nonspherical homogeneous convex solid of density 1/2 which, when floating in water, will rest in equilibrium regardless of its orientation.*(25)

(25)The problem about floating bodies was posed by S. Ulam in the 1930's, and is repeated in his book (Ulam(1960)) on unsolved problems. The 2-dimensional example is due to Auerbach(1938).

The final problem also involves centers of gravity. A homogeneous solid is called ***unistable*** *provided that it is in the shape of a convex polyhedron and has a special face such that, however the solid is placed on a flat surface, it will roll over until it rests on that special face. Here is a unistable solid with 19 faces; the special face is colored red. No matter how I put it down, it will turn until it rests on the red face. However, it is unknown what is the smallest possible number of faces for such a solid.*(26)

(26)For the construction and a number of related unsolved problems, see Guy(1969).

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