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G. T. Toussaint, On translating a set of spheres, Tech. Rep. 84.4, McGill University, Montreal, 1984.

E9. Lattice point problems. The next few problems involve properties of sets in relation to lattice points. For our purposes a lattice point will always be taken to mean a point in $\mathbb{R}^d$ with integer Cartesian coordinates. The **square lattice** or **integer lattice** will refer to the set of all such lattice points in $\mathbb{R}^d$ [see Figure E3(a)].

It is possible to pose lattice point problems for lattices other than the square one [see Figure E3(b)]. Sometimes such problems can be shown to be equivalent to that for the square lattice, though often different problems may result.

A large collection of unsolved problems on lattice points may be found in Hammer's book.

J. Hammer [Ham].

E10. Sets covering constant numbers of lattice points. Steinhaus has asked whether there exists a plane set E that covers exactly n lattice points however it is placed on the lattice (rotations being allowed). One might hope to construct such an E by using the axiom of choice or other appropriate axioms of logic, but apparently no one has yet succeeded in doing this. (This is a particular case of the problem in Section G9.)

If we require E to be a Borel or measurable set, the problem takes on a completely different character. Certainly the area of such an E must equal n. The problem is equivalent to seeking a set such that the 2-dimensional Fourier transform of the characteristic function of E is zero on all circles centered on the origin with radii $\sqrt{i^2 + j^2}$, where i and j are positive integers. Croft has used the idea of the density boundary to show that E cannot be

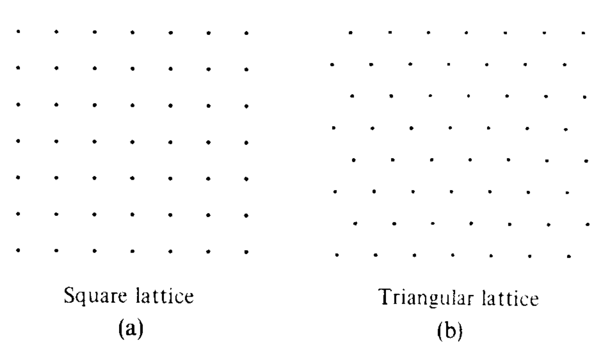

Figure E3. Examples of plane lattices: (a) for square lattice and (b) for triangular lattice. Our problems generally concern the square lattice, consisting of points with integer coordinates.

measurable and bounded, but his argument does not extend to the unbounded case.

Croft also poses the following problem. Put a measure on the set of congruence transformations ρ using some 3-dimensional parameterization (x, y, θ) in the natural way. Let E be plane-measurable. We say that the integer k is **essentially represented** by E if the set of ρ for which $\rho(E)$ contains k points has positive measure (i.e., if such covering positions are non-exceptional). Then is the set of integers essentially represented by E always a block of consecutive integers? This is not true if "essentially" is omitted: the closed disk of radius $\sqrt{5}$ can be moved to cover 17 or 21 lattice points, but not 18, 19, or 20.

H. T. Croft, Three lattice point problems of Steinhaus, *Quart. J. Math. Oxford Ser.* **33** (1982) 71–83; *MR* **85g**:11051.

I. Niven & H. S. Zuckerman, Lattice points in regions, *Proc. Amer. Math. Soc.* **18** (1967) 364–370; *MR* **35** #136.

H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, Oxford, 1969.

E11. Sets that can be moved to cover several lattice points.

It is well-known that every plane (measurable) set of area greater that $\frac{1}{4}\pi$ may be moved (rigidly) to a position so that it covers at least two points of the unit square lattice. What is the critical area for covering at least three, and more generally, at least n points?

A simple argument shows that any set E of infinite area (i.e., infinite inner measure) can be positioned to cover at least n lattice points for each integer n. A question sent to us by Steinhaus asks whether E can always be positioned to cover infinitely many lattice points. If not, what additional hypotheses are required for this to be so? Croft has shown that it is enough for E to have unbounded interior.

One can ask similar questions for convex sets. The following problem from L. Moser's collection is of particular interest. Let $f(A)$ be the largest number such that every plane convex set of area A can be positioned to cover $A + f(A)$ lattice points. Does $f(A) \to \infty$ as $A \to \infty$ and if so, how fast? Recently, Beck has shown that there is always a position covering more than $A + cA^{1/9}$ points and also one covering fewer than $A - cA^{1/9}$. He also points out that, using his method, the exponent $\frac{1}{9}$ can be replaced by $\frac{1}{8} - \varepsilon$ for any $\varepsilon > 0$. Simple examples show that $f(A) = O(A^{1/2})$, and Beck conjectures that $f(A) = O(A^{1/4})$ is the right order or magnitude.

We propose the following strengthening of Moser's problem: Can a convex set of area A be placed so as to intersect the parallel line set $\{(x, y): x \text{ is an integer}\}$ in total length at least $A + f(A)$ for some $f(A)$ which tends to infinity with A?

Of course, by placing further restrictions on the class of sets (e.g., circles, triangles, squares, n-gons, etc.,) we get a plethora of further problems (see Moser & Pach, Problem 53 for a discussion of the case of a square, and, for example, Kendall & Rankin in the case of a disk or sphere).

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H. T. Croft, Three lattice point problems of Steinhaus, *Quart. J. Math. Oxford Ser.* (2) **33** (1982) 71–83; *MR* **85g**:11051.
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R. Schark & J. M. Wills, Translationen, Bewegungen und Gitterpunktanzahl konvexer Bereiche, *Geom. Dedicata* **3** (1974) 251–256; *MR* **50** #4498.

E12. Sets that always cover several lattice points. Santaló asked for the convex set of least area with the property that every congruent copy contains at least one lattice point. Schäffer and Sawyer showed independently that the smallest such set is $K = \{(x, y) : |x| \le \frac{1}{2}, |y| \le \frac{3}{4} - x^2\}$ which has area $\frac{4}{3}$. According to Reich, Schäffer conjectures that K is also the covering set of least perimeter. What is the (measurable) set of minimal area if the convexity condition is dropped—is it still the same set or can area $1 + \varepsilon$ be achieved for any $\varepsilon > 0$?

Describe the sets, convex or otherwise, of least area or of least perimeter with all congruent copies covering at least n lattice points for $n = 2, 3, \ldots$. Presumably for large n they approach a circular shape—are there any n for which they are circular? Mögling gives exact conditions for regular p-gons always to cover a lattice point if p is odd. Characterize such regular polygons for even p, and more generally find conditions for arbitrary p-gons always to cover n lattice points.

It is also useful to have conditions involving other measures of a convex set K that ensure that K contains at least one (more generally, at least n) lattice points however it is situated. The intuitive idea here is that if K can cover few points compared with its area A, then it must be "long and thin." A basic result in this direction is due to Nosarzewska who showed that however K is placed it must cover at least $A - \frac{1}{2}L$ and at most $A + \frac{1}{2}L + 1$ lattice points, where L is the perimeter of K. (See the papers of Bokowski, Hadwiger & Wills, of Schmidt and of Wills for higher dimensional analogs.)

Bender proved the "isoperimetric" result that if $\frac{1}{2}L \le A$ then K must contain a lattice point, and he extended this to higher dimensions. How big must A/L be to guarantee catching n lattice points? Scott showed that if $(w - 1)(D - 1) \ge 1$ then K contains some lattice point, and if $(w - \sqrt{2})(D - \sqrt{2}) \ge 2$ then K contains at least two points. (See McMullen & Wills for higher-dimensional versions of this.) What are the analogs for n points? Scott suggests that the more natural thing to look at here rather than w and D is what he terms the axial diameter. Scott also showed that a convex set satisfying $A > 1.144nD$ must contain n lattice points, and by a simple refinement of his argument, Hammer pointed out that such a set in fact contains n^2 points! Can these results be improved, or generalized to d dimensions for $d \ge 3$?

E. A. Bender, Area–perimeter relations for two dimensional lattices, *Amer. Math. Monthly* **69** (1962) 742–744.

K. Bezdek, The thinnest holding-lattice of a set, *Monatsh. Math.* **103** (1987) 177–185; *MT* **88j**:52031.

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J. M. Wills, Zur Gitterpunktenzahl Konvexer Mengen, *Elem. Math.* **28** (1973) 57–67; *MR* **53** #3898.

E13. Variations on Minkowski's theorem.

Let K be a convex set in $\mathbb{R}^d$ that is centro-symmetric about the origin **o**. If K has d-dimensional volume $V \geq 2^d$, then K contains at least two lattice points other than **o** (see Figure E4). This is Minkowski's theorem, which is of fundamental importance in the geometry

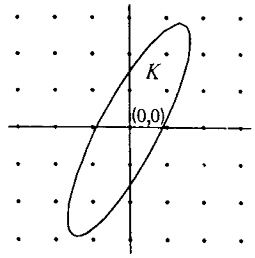

Figure E4. Minkowski's theorem: If K is centro-symmetric about the origin and has an area of at least 4, it contains further lattice points.

of numbers (see Hardy & Wright or Cassels for proofs and consequences of this result). Many problems on convex bodies and lattice points come from trying to generalize Minkowski's theorem or varying the hypotheses. We mention some of these here, others may be found in Hammer's book and in the various papers of Scott.

Ehrhart conjectures that if K is any (not necessarily symmetric) convex body in $\mathbb{R}^d$ with centroid at **o** and containing no further lattice points, then $V \leq (d + 1)^d/d!$, this bound being attained by a certain simplex. He proved this in the plane case. Must sets of any larger area or volume always contain a pair of lattice points symmetric with respect to **o**? Might it also be true that if $V \geq k(d + 1)^d/d!$ then K contains $2k$ lattice points situated symmetrically about **o**?

The extremal convex set for Ehrhart's result in the plane is the triangle with vertices $(-2, -1)$, $(2, 1)$, and $(1, -1)$. Scott asks if this is also the convex set of greatest width with no interior lattice points other than **o**.

Scott also asks for the convex set of largest area with circumcenter at **o** containing no other interior lattice points. He conjectures that the maximum is 4.04 ... with the intersection of the circle center **o** and radius 1.637 ... and a triangle with two lattice points on each side as the extremal.

Scott proposed maximizing A/L for a plane centro-symmetric convex set with center **o** and containing no other lattice points. This problem was solved by Arkinstall & Scott, and by Croft who considered the more general case of A/L^α for $0 < \alpha < 2$. The 3-dimensional analogs promise to be awkward!

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E. Ehrhart, Une généralization probable du théorème fondamental de Minkowski, *C. R. Acad. Sci. Paris* **258** (1964) 4885–4887; *MR* **29** #522.
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P. R. Scott, Convex bodies and lattice points, *Math. Mag.* **48** (1975) 110–112; *MR* **51** #1610.
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P. R. Scott, On the area of planar convex sets containing many lattice points, *Bull. Austral. Math. Soc.* **35** (1987) 441–454; *MR* **88e**:52017.
P. R. Scott, Modifying Minkowski's theorem, *J. Number Theory* **29** (1988) 13–20; *MR* **89f**:11087.

E14. Positioning convex sets relative to discrete sets. Many of the problems of the last few sections concerning lattice points can be asked for more general sets of points scattered across the plane and subject to some density condition.

The following perplexing problem is due to Danzer. Let S be a point set in $\mathbb{R}^2$ of bounded density, that is, with the number of points in $S \cap B_r$ at most cr^2 for some constant c, where B_r is the disk center the origin and radius r. Do there always exist convex sets of arbitrarily large area not containing any points of S? This is so if S is a finite union of lattices, but it is false for certain S with logarithmically greater density, i.e., with (roughly) $cr^2 \ln r$ points of S in B_r.

In the other direction, suppose that $\lim_{r\to\infty}$ (number of points in $S \cap B_r)/\pi r^2 = 1$. Steinhaus asks whether every domain (convex, or more generally, measurable) with an area of at least n, has a congruent copy covering n points of S. Macbeath & Rogers showed that for any bounded domain K of area 1 not including the origin there exists a linear transformation of determinant 1 mapping K to a set disjoint from S.

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