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Recent Results in Convexity

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In this talk I shall try to give some of the important results in convexity and related topics which have been obtained since the last congress in 1974. The emphasis will naturally reflect my own interests and due to my own lack of background knowledge, some results will not be adequately treated.

1. Cross sections and volume. It is, perhaps, appropriate to begin with an old problem of H. Busemann and C. M. Petty [8] which was offered as an excercise to the audience by C. A. Rogers [36] at his talk in Vancouver 1974:

Consider two convex bodies K, K' *in* E^n *which are both centrally symmetric about the origin. Suppose, for each* $n-1$ *dimensional linear subspace* L, $V_{n-1}(K\cap L) < V_{n-1}(K'\cap L)$. *Is it true that* $V_n(K) < V_n(K')$?

It is important that the bodies K, K' are convex, see H. Busemann [6] and that both are centrally symmetric about 0. In E^2 the problem is trivial since the condition then ensures that $K\subset K'$. Also, if K, the apparently smaller body, is an ellipsoid then the answer is affirmative. The problem still remains unresolved in E^3. However in E^n, $n\geq 12$, C. A. Rogers and myself [25] were able to give a negative answer in which K' was the unit n ball. The counter example consists of an n ball with small caps removed in a disjoint and a homogeneous manner. Is there still a counter example when $n-1$ dimensional subspaces are replaced by 2-dimensional subspaces? By modifying the methods of [25] it is possible to replace $n-1$ by $[n/2]+5$ dimensional subspaces.

I might also mention the solution to another problem of H. Busemann [7] by Larman–Mani–Rogers [29]. However, it turns out that the problem may have already been solved, within the context of lie groups, by Vinberg [40]. We do not claim, however, to understand exactly what was proved by Vinberg and, even less, his proofs. Anyhow, within convexity the problem was:

Characterise those convex bodies K such that for any two points $\mathbf{x}, \mathbf{y}$ in the interior of K there exists a projective transformation π, permissible for K, such that $\pi K = K$ (set-wise) and $\pi \mathbf{x} = \mathbf{y}$.

In E^2, K must be an ellipse or a triangle. In E^3, K must be an ellipsoid, tetrahedron or a cone on an elliptic base. However, the obvious conjecture, i.e. the convex hulls of disjoint ellipsoids lying in independent subspaces whose union spans E^n, fails in E^4. An example is the convex hull of two touching ellipses in E^4 which lie in two orthogonal 2-dimensional subspaces. The complete description of possible K is algebraic and rather technical.

2. Characterisations of the sphere and ellipsoid. Although it falls outside my time scale, the recent interest in this subject stems from the solution, by P. W. Aitchison, C. M. Petty and C. A. Rogers [1], of the false centre problem: say that a convex body K in E^n has a false centre x in K if x is not the centre of K but every two dimensional section of K through x is centrally symmetric. Clearly every ellipsoid has this property and C. A. Rogers [37] had conjectured that any convex body with a false centre must be an ellipsoid; D. G. Larman [23] extended this result to where x was not necessarily in K. Recently G. R. Burton and P. Mani [5] have proved a more general result which was conjectured by P. Gruber [14]:

Suppose that a convex body K in E^n contains two distinct points $\mathbf{a}$ and $\mathbf{b}$ such that parallel 2-sections of K through $\mathbf{a}$ and $\mathbf{b}$ are directly homothetic. Then K is an ellipsoid.

It is not difficult, assuming that the convex body K has a false centre x to show that K is centrally symmetric, around 0 say. Then x and $-x$ play the roles of a and b in the Gruber property.

It is well known that a convex body in E^n with all its $n-1$ dimensional sections centrally symmetric is an ellipsoid. G. R. Burton [2] has shown that it is enough to assume that all $n-1$ sections of sufficiently small diameter are centrally symmetric.

Suppose that K is a convex body in E^n and that u is a unit vector. Then the scalar product $\langle \cdot, u\rangle$ has a maximum $m(u)$ on K. Suppose also that for each u in S^{n-1} there exists $\lambda(u) > 0$ such that the section

$$K(p; u) = K_n\{x : \langle x, u\rangle = p\}$$

is centrally symmetric whenever $m(u) - \lambda(u) \leq p \leq m(u)$.

Then K is not necessarily an ellipsoid as can be seen from the example of a circular cylinder with hemispherical ends but, see G. R. Burton [3], K must be the sum of an ellipsoid and a finite number of line segments.

Finally in this section I mention the proof by G. R. Burton [4] of a conjecture of Klee:

If the geodesics between any two points of a closed bounded convex surface ∂K in E^n are flat then K is a sphere.

3. Polytopes. Whilst work has declined in the study of convex polytopes in recent years, there have been two outstanding results produced since 1974, both of which involve mathematicians of at least 70 years of age!

The first is due to B. Jessen and A. Thorup [21]:

Say that two polytopes P and Q are equivalent in E^n if Q can be obtained from P by cutting, translating and glueing. A difficult problem has been to give necessary and sufficient conditions for P and Q to be equivalent.

The conditions are (roughly), take $k(x_1, \ldots, x_{n-r})$ any odd real valued function of $x_1, \ldots, x_{n-r}$. Let $u_1, \ldots, u_{n-r}$ be $n-r$ orthogonal vectors in E^n and (with a finite number of exceptions) form the r face $P(u_1, \ldots, u_{n-r})$ by first finding the $n-1$ face $P(u_1)$ of P in direction u_1, then the $n-2$ face of $P(u_1, u_2)$ of P within $P(u_1)$ and so on. Then form

$$\sum k(u_1, \ldots, u_{n-r}) V_r(P(u_1, \ldots, u_{n-r})) \tag{1}$$

where V_r denotes r-dimensional volume. If (1) is equal to a similar sum for Q, for all choices of $u_1, \ldots, u_{n-r}$, $r=0, 1, \ldots, n$, then P is equivalent to Q and conversely.

This was proved previously by Hadwiger and Glur [19] for $n=2$ and by Hadwiger [18] for $n=3$.

The second result is due to Hadwiger [17]: consider the usual square lattice in E_n and a lattice polytope K (i.e. all of the vertices of K are lattice points). Let $G(K)$ denote the number of lattice points inside K.

In 1971 J. M. Wills [41] conjectured that

$$G(K) \leq \int_{E^n} e^{-\pi d^2(x,K)}\, dx$$

where $d(x, K)$ denotes the distance of x from K. It is easy to prove this conjecture for $n=2$ using the well known result $G(K) = (\text{Area of } K) + \frac{1}{2}$ (perimeter of K) $+ 1$. In 1974 T. Overhagen [35] proved it for $n=3$. However, recently Hadwiger has proved that it is false for a certain simplex in E^n, $n \geq 441$.

Finally in this section I shall mention a result of P. McMullen [31] on tiling space with zonotopes (i.e. finite sums of line segments). A zonotope Z in E^d, which is the sum of n line segments, is the orthogonal projection of a cube C in E^n. Of course, there is an associated zonotope $\bar{Z}$ formed by projecting C orthogonally into E^{n-d} (where $E^n = E^d + E^{n-d}$). McMullen proves that if Z tiles E^d by translation, with adjacent zonotopes meeting facet to facet, then $\bar{Z}$ tiles E^{n-d} in the same way.

4. Collision of convex bodies. Suppose two convex bodies K, K' in E^n have parts of their surface painted. Given that the two bodies collide one can ask for the probability that the two bodies collide paint to paint. The forerunner of this type of problem was the solution by P. McMullen [30] of a problem of W. J. Firey: What is the most probable encounter of two unit cubes in E^3?

It is intuitively clear that the only encounters which occur with a positive probability are edge to edge and vertex to face. In fact the proportion is $3\pi:8$ so that the edge to edge collision is more likely.

This work has been extended by W. J. Firey [11] to convex bodies K, K' and a formula derived for the probability of impact paint to paint in terms of the surface area functions of the two bodies. Firey insisted on painting whole faces but this restriction has been removed in the recent work of R. Schneider [38]. The reader is also referred to Schneider's excellent survey article [39].

5. **Helly's theorem.** The famous theorem of E. Helly [20] asserts that if $\mathcal{F}$ is a family of compact convex sets in E^n with any subfamily of $n+1$ sets having a non-empty intersection then the family $\mathcal{F}$ has a non empty intersection.

As an extension of this result B. Grunbaum and T. Motzkin conjectured the following:

Suppose $\mathcal{F}$ is a family of sets, each the union of at most j disjoint compact convex sets in E^n such that the intersection of any k sets in $\mathcal{F}$, $k \leq j$, is also expressible as the union of at most j compact convex sets. Then, if any $j(n+1)$ sets in $\mathcal{F}$ has a nonempty intersection then the intersection of all the sets of $\mathcal{F}$ is nonempty.

They proved this conjecture for $j=2$ and I proved it for $j=3$. It was finally proved in general by H. C. Morris [34].

6. **Distance problems.** Say that a set A in E^n realises the distance d if there are two points of A at a distance d apart. An old result of H. Hadwiger [16] is the following:

If $n+1$ closed sets cover E^n then at least one of the sets realises all distances.

Hadwiger did not believe that $n+1$ was the best possible number but even in E^2 it is still not known whether or not 4 sets will ensure the same result. However, for large n improvements have been made and, to explain these, let us define the concept of a configuration:

A configuration is any finite sequence of points $x_1, \ldots, x_M$, not necessarily distinct. The number of points in the configuration will be M, not necessarily the number of distinct points in the configuration. We say that for a given distance d, the number D is critical (for d and the configuration) if any sub-configuration of $D+1$ points realises d and D is the smallest number with this property.

THEOREM (D. G. LARMAN AND C. A. ROGERS [26]). *Suppose that there is, in E^n, a configuration of M points with critical distance 1 and critical number D. Then, if E^n is covered by less than M/D sets, there is a set of the covering within which all distances are realised.*

Using this result, D. G. Larman and C. A. Rogers [26] extended Hadwiger's result to $\frac{1}{6}n(n-1)$ sets (not necessarily closed). Recently P. Frankl [12] has used this result to prove:

THEOREM. *Given a natural number k there is an integer n_0 such that, provided $n \geqslant n_0$, any n^k sets which cover E^n contain a set which realises all distances.*

CONJECTURE (LARMAN [24]. *If E^n is covered by less than $\frac{1}{3}(\frac{4}{3})^{4n/3}$ sets then there is one set of the covering which realises all distances.*

Finally we mention the work of D. G. Larman, C. A. Rogers and J. J. Seidel [27] which almost resolves the old two distance problem: *Any set in E^n which realises only two distances has cardinality at most $\frac{1}{2}(n+1)(n+4)$.*

7. Tilings in the plane. Let me briefly mention the forthcoming book of B. Grunbaum and G. C. Shephard [15]. This study gives rise to some interesting problems. For example, can any tiling of the plane by bounded tiles be realised as a convex tiling? This, in some sense is a generalisation of the result of Steinitz which asserts that any 3-connected planar graph can be realised as the set of vertices and edges of a convex 3-polytope.

8. Dvoretsky's theorem. It has passed almost unnoticed by geometers that some outstanding work has been done recently, mainly by functional analysists, in shortening and extending the work of A. Dvoretsky [9]. In this work Dvoretsky shows, for $\varepsilon > 0$ and a positive integer k the existence of a positive integer $n(k, \varepsilon)$ such that any centrally symmetric convex body in E^n, $n \geqslant n(k, \varepsilon)$, contains a central k section which is, to within ε, a k-ball. In [28], Larman and Mani show that K need not be centrally symmetric, which although interesting geometrically, is not so useful to a functional analyst. The audience is referred to T. Figiel [10], V. D. Milman [32] and Milman and Wolfson [33].

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