On Discrete Morse Functions and Combinatorial Decompositions

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Abstract

This paper relates the recent theory of discrete Morse functions due to Forman [11] and combinatorial decompositions such as shellability and partitionability, which are known to have many useful applications within combinatorics. First, we present the basic aspects of discrete Morse theory for regular cell complexes in terms of the combinatorial structure of their face posets. We introduce the notion of a *generalized shelling* of a regular cell complex and describe how to construct a discrete Morse function associated with such a decomposition. An application of Forman's theory gives us generalizations of known results about the homotopy properties of shellable complexes. We also discuss the relation of our work to a variety of other decomposition results that have been shown for special classes of complexes.

1 Introduction

This paper will focus on a recent development in topology – namely a discrete version of Morse theory developed by Forman [11], and relate it to combinatorial decompositions such as shellings and interval partitions which have been studied extensively in combinatorics. We refer the reader to Björner's chapter [2] on "Topological Aspects" in the Handbook of Combinatorics for a comprehensive survey of some important topological techniques that have been useful in combinatorics. The primary purpose of this paper is to show that discrete Morse theory provides a unifying framework for many interesting and important problems of topological combinatorics. One of the principal ideas of discrete Morse theory is to construct for a given finite cell complex (which we will assume to be *regular*), a "more efficient" cell-complex (which will not, in general, be regular), while retaining topological properties of the original space as much as possible. The construction of the more efficient complex depends on the existence of *discrete Morse functions* on the original regular cell complex. Forman's work is a discrete analog of "smooth" Morse theory of Milnor [15], [11]. In the original theory, there are canonical choices for Morse functions for smooth manifolds while in the discrete version this does not appear to be the case, at least from Forman's paper. We will show that for a given *generalized* shelling of a regular cell complex there is a canonical discrete Morse function. A simple application of Forman's theory gives us a generalization of known results about the homotopy properties of shellable cell complexes. We should point out that Forman also constructs a differential complex associated with a discrete Morse function which has the same integer homology as the original cell complex – however, we will not cover this aspect of discrete Morse theory here. We will present those definitions and results of Forman which are readily accessible given some familiarity with the tools of combinatorial topology discussed in Part II of Björner's Chapter [2]. Further topological details and applications of this elegant and powerful theory of discrete Morse functions can be obtained from [11]. We will end the paper with a discussion of a variety of applications and interesting open problems that arise when one views combinatorial decompositions from the perspective of discrete Morse theory.

2 Preliminaries

We will assume familiarity with the notion of cell complexes [2], which are traditionally called CW-complexes in standard algebraic topology texts such as Munkres [16] and Massey[13], [14]. Throughout the paper we will assume all such complexes to be finite. In a combinatorial context, it is most natural to consider *regular* cell complexes since with this additional property, the topology of the associated space is completely determined by the *face poset* of closed cells ordered with respect to containment. We refer the reader to Björner's survey [2] or the appendix of Chapter 4 of [6] for further details and terminology. Hence forth, without change of notation we will also regard a regular cell-complex Σ as a poset, whose order and cover relation are denoted by $\leq$ and $\prec$, respectively, with $\geq$, $<$, $\succ$ etc. having the obvious interpretations. For $\sigma \in \Sigma$, let $\delta\sigma$ be the boundary subcomplex of σ and let $\bar{\sigma} = \sigma \cup \delta\sigma$. Recall that if Σ is a regular cell complex , $\bar{\sigma}$ is (homeomorphic to) the $dim\sigma$-ball while $\delta\sigma$ is a $(dim\sigma - 1)$-sphere. The dimension of Σ is the number $max\{dim\sigma : \sigma \in \Sigma\}$, and we will say that Σ is *pure* if all its maximal cells have the same dimension. When the regular cell complex is a simplicial complex, we will refer to its cells as its *faces* and its maximal cells as *facets*.

The property of shellability has been classically been studied only in the context of pure cell complexes and pure simplicial complexes. It leads to very interesting algebraic, enumerative and topological consequences ([21], [1], [3]) for many complexes that arise naturally in combinatorics. Recently, Björner and Wachs [4], [5] have undertaken a systematic study of shellability for general (non-pure) cell complexes and its applications. We now present this definition of shellable complexes.

Definition: An ordering $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the maximal cells of a d-dimensional regular cell complex Σ is a *shelling* if either $d = 0$ or it satisfies the following conditions:

- (S1) There is a ordering of the maximal cells of $\delta\sigma_1$ which is a shelling.

- (S2) For $2 \leq j \leq m$, $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k)$ is pure and $(dim\sigma_j - 1)$-dimensional
- (S3) For $2 \leq j \leq m$, there is an ordering of the maximal cells of $\delta\sigma_j$ which is a shelling and further, the maximal cells of $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k)$ appear first in this ordering.

A regular cell complex is said to be *shellable* if it admits a shelling. In the general non-pure context, the following result is due to Björner and Wachs [4], [5] and it describes the primary topological consequence of shellability.

Theorem 1 : *If a regular cell complex Σ is shellable then it is homotopy equivalent to a wedge of spheres.* ■

Next, we define an even more general decomposition property for regular cell-complexes which has a natural relation to the discrete Morse theory of Forman.

Definition: An ordering $\sigma_1, \sigma_2, \ldots, \sigma_m$ of distinct cells of a regular cell complex Σ is a *generalized shelling* if satisfies the following two conditions *and* (S1), (S2) and (S3):

- (G1) $\Sigma = \cup_{i=1}^{m} \bar{\sigma}_m$.
- (G2) If $\sigma_i \in \delta\sigma_j$ then $i < j$.

Hence if $\sigma_1, \sigma_2, \ldots, \sigma_m$ are maximal cells of Σ, then we get the definition of Björner and Wachs [4]. Clearly, every regular cell complex admits a trivial generalized shelling – a total order on all of its cells which is consistent with its partial order. We will show later that there are examples of complexes that are not shellable but admit non-trivial generalized shellings that are in, some sense, canonical. Figure 1 shows an example of a regular cell complex with 3 maximal cells which is not shellable but admits a generalized shelling sequence with 4 cells as shown.

The next proposition relates the existence of generalized shellings in simplicial complexes to interval-partitions and provides a non-recursive definition for generalized shellings in this context. We omit the proof, which is quite routine. Note that, as is traditional in combinatorial literature, the empty set is also considered to be a face when considering interval partitions of a simplicial complex .

Proposition 1 *Let Σ be a simplicial complex. Then for an ordered subset $F_1, F_2, \ldots F_m$ of faces of Σ the following are equivalent:*

(i) $F_1, F_2, \ldots F_m$ is a generalized shelling for Σ.

(ii) There exist faces $G_1, G_2, \ldots G_m$ with $G_i \subseteq F_i$ such that the sequence $\{[G_i, F_i], i = 1, \ldots, m\}$ of intervals, partitions Σ and further $\cup_{i=1}^{k}[G_i, F_i]$ is a simplicial complex for $k = 1, 2, \ldots, m$. ■

Following [7], we will refer to the ordered sequence of intervals $\{[G_i, F_i], i = 1, \ldots, m\}$ as an *S-partition* of Σ.

3 Elements of discrete Morse theory

We will derive results for the homotopy type of complexes which admit non-trivial generalised shellings by applying the theory of discrete Morse functions developed by Forman [11]. We begin with the definition of these functions.

Definition: Given a (finite) regular cell complex Σ, a *(discrete) Morse function* on Σ is a function $f : \Sigma \to R$ satisfying the following two conditions for every cell σ of Σ :

- $|\{\tau \prec \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$. (M1)
- $|\{\omega \succ \sigma : f(\omega) \leq f(\sigma)\}| \leq 1$. (M2)

A convenient way to think about a Morse function is to regard it as being "almost increasing" with respect to dimension. Clearly, any function which is increasing with respect to dimension would be an (uninteresting!) example of a discrete Morse function. Figure 2 shows an example of a discrete Morse function on the cell complex of Figure 1.

Definition: A p-dimensional cell σ of Σ is *critical* (with respect to a fixed Morse function f) if it satisfies each of the following conditions:

- $|\{\tau \prec \sigma : f(\tau) \geq f(\sigma)\}| = 0$. (C1)

- $|\{\omega \succ \sigma : f(\omega) \leq f(\sigma)\}| = 0$. (C2)

We will denote by $\mathcal{C}(f)$ - the set of critical cells of Σ with respect to f. In the example of Figure 2, there exactly two critical cells for the given Morse function - the 0-cell a and the 1-cell df.

The following is one of the central theorems of discrete Morse theory.

Theorem 2 :

1. *([11] Corollary 3.5) Suppose Σ is regular cell complex with a discrete Morse function. Then Σ is homotopy equivalent to a cell complex with exactly one cell of dimension p for each critical cell of Σ of dimension p.*

2. *([15], [11] Corollary 3.7) "Weak Morse Equalities": Let β_j be the j'th Betti number of Σ with coefficients in some fixed field and m_j be the number of j-dimensional critical cells, then $\beta_j \leq m_j$ for every j.*

We will indicate later an outline of the elegant proof of statement (1) of this theorem using terminology that is some what different from Forman's. If we take the dimension as a Morse function on any complex, every cell would be critical and hence the above theorem tells us nothing new. Hence, it is important to construct "efficient" Morse functions (that have few critical points), especially in view of (2) of the above theorem. For more details about both weak and *strong* Morse inequalities, we refer the reader to [15] and [11]. Observe that applying the above theorem to the example of Figure 2 shows that the complex is homotopy equivalent to the 1-sphere.

We will now restate the basic concepts of discrete Morse theory in graph-theoretic terms to emphasize the combinatorial nature of the theory for regular cell complexes. The discrete vector fields discussed in Forman's paper [11] and the work of Stanley [20] and Duval [10] on decompositions of simplicial complexes have underlying ideas that are similar in nature to what we will present. We begin with following simple lemma of Forman.

Lemma 1 : *If f is a Morse function on a regular cell complex Σ and σ is any cell of Σ, then conditions (C1) and (C2) cannot both be false for σ.*

Proof: If possible, let $\omega \succ \sigma \succ \tau$ satisfy $f(\omega) \leq f(\sigma) \leq f(\tau)$. Now let α be a cell distinct from σ that also satisfies $\omega \succ \alpha \succ \tau$. The existence of such an α for a regular cell complex follows from the fact that $\delta\omega$ is a sphere. Applying condition (M1) to ω and (M2) to τ, we have $f(\omega) > f(\alpha) > f(\tau)$ which leads to a contradiction. ■

In particular, if a cell is not critical then it violates *exactly one* of (C1) and (C2). In graph theoretic terminology, this implies that there exists a matching $M(f)$ on the Hasse diagram of Σ associated with every discrete Morse function f of Σ such that the set of cells of Σ not incident to any edge of M is exactly $C(f)$. From the definition of a discrete Morse function, it is clear that $M(f)$ is the precisely the set of the cover relations where f is non-increasing with respect to dimension. We can easily perturb f without changing the set of critical cells (or the matching relations) so that it is (strictly) increasing or decreasing across all cover relations. Now we appeal to the elementary combinatorial result that a directed graph is acyclic if and only if there is labelling of the nodes such that each edge is directed from a node to another node with strictly lower label. Hence, by assigning directions on the cover relations of Σ according to increasing value of f, we obtain a unique acyclic orientation on the Hasse diagram of Σ for every discrete Morse function on Σ. In the other direction, we can regard the Hasse diagram of Σ as a directed graph which we call $G(\Sigma)$ with the edges being cover relations directed from higher to lower dimensional cells. Clearly, $G(\Sigma)$ is acyclic in the directed sense. Now we choose a matching M (possibly empty) of edges of this graph such that the directed graph $G_M(\Sigma)$ obtained from $G(\Sigma)$ by *reversing* the direction of edges in M is also acyclic. In particular, we have $G_\emptyset(\Sigma) = G(\Sigma)$. Evidently, by choice of M, any node labelling on the acyclic $G_M(\Sigma)$ which is increasing along the direction of the edges, is a discrete Morse function on Σ with the critical cells being precisely the nodes are not incident to any of the edges in M. We have essentially proved the following proposition.

Proposition 2 : *A subset C of the cells of a regular cell complex Σ is the set of critical cells for some discrete Morse function f if and only if there* *exists a matching M on $G(\Sigma)$ such that $G_M(\Sigma)$ is acyclic and C is the set of nodes of $G(\Sigma)$ not incident to any edge in M.* ∎

Proof of Theorem 2 (an outline): We will now present a rough sketch of Forman's proof of Theorem 2 using the language of the above proposition. Since $G_M(\Sigma)$ corresponding to a given discrete Morse function f is acyclic, it must have a source node, say a p-dimensional cell σ. Now assume that σ is *not* maximal in Σ. Then by the construction of $G_M(\Sigma)$, σ is contained in the boundary of exactly one $(p+1)$-dimensional cell, say τ and further σ and τ must be matched to each other in M. It is well known that the subcomplex of Σ, defined by $\Sigma \setminus \{\sigma, \tau\}$ is homotopy equivalent to, in fact is a deformation retract of, Σ. (See [11], [2] where such a reduction is referred to as *an elementary collapse*). The proof essentially follows by induction. On the other hand, if σ is a maximal cell then σ must be critical. Now we apply the result inductively to the subcomplex $\Sigma \setminus \{\sigma\}$, which completely contains the boundary of the cell σ. Then if we glue the open cell σ back on, along its boundary, the resulting complex has the desired properties. We remark that in this case, the resulting complex need not be regular, as the boundary of σ might be collapsed to a point, as can be seen for the cell *df* in the example of Figure 2. Following conventional terminology [2], we will refer to complexes that admit a Morse function with only one critical point as being *collapsible* since they admit a sequence of elementary collapses that reduce them to a point.

4 Generalized shellings and discrete Morse functions

In this section, we will construct discrete Morse functions for complexes with given generalized shellings. We will first prove these results for shellable *pseudomanifolds*. Recall that a d-pseudomanifold is a pure d-dimensional regular cell complex such that

(i) every $(d-1)$-cell is contained in at most two d-cells,

(ii) Given any two d cells β and τ there exists a sequence of d-cells $\beta = \sigma_1, \sigma_2, \ldots, \sigma_m = \tau$ such that σ_i and σ_{i+1} share a common $(d-1)$-cell for $1 \leq i \leq m-1$.

The *boundary* of a d-pseudomanifold is the subcomplex generated by the set of $(d-1)$-cells which are contained in exactly one d-cell.

We begin with a fundamental result due to Bing, and Danaraj and (Klee [9], [6] Chapter 4).

Lemma 2 : *Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a shelling of a d-pseudomanifold Σ. Then $\cup_{k=1}^{j}\bar{\sigma}_k$ is a d-ball for every $1 \leq j < m$. $\Sigma = \cup_{k=1}^{m}\bar{\sigma}_k$ is a d-sphere if Σ has empty boundary, otherwise Σ is a d-ball.* ■

Thus a purely combinatorial property, namely shellability, enables us to deduce a topological property – of being (homeomorphic to) a ball or sphere. It is easy to show ([11] Corollary 4.4) that a simplex and its boundary admit Morse functions with exactly one and two critical points respectively. We extend this result in the next proposition to shellable balls and spheres. Following Björner [1], we call the cell σ_j, for $j \geq 2$, a *homology cell* with respect to a fixed generalised shelling $\sigma_1, \sigma_2, \ldots, \sigma_m$ of a complex Σ if $\delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k) = \delta\sigma_j$.

Proposition 3 : *Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a shelling of a d-pseudomanifold Σ and let v be any 0-cell in $\bar{\sigma}_1$. Then, Σ admits a Morse function f such that*

(i) If Σ is the d-sphere then v and σ_m are only critical cells, while if Σ is a d-ball then v is the only critical cell.

(ii) When restricted to $\cup_{k=1}^{j}\bar{\sigma}_k$ for $1 \leq j < m$, the only critical cell of f is v.

Proof: We will prove the result by induction on the dimension d. For $d = 0$, the result is obviously true. Now for $d \geq 1$, we will construct the required Morse function f inductively.

First, we note that $\delta\sigma_1$ is a shellable $(d-1)$-sphere and hence by induction admits a Morse function f_1 on $\delta\sigma_1$ in which v and some $(d-1)$ cell ω_1 are the only two critical points. Thus there exists a matching M_1 on $G(\delta\sigma_1)$ as per Proposition 2 such that $G_{M_1}(\delta\sigma_1)$ is acyclic and v and ω_1 are the only unmatched cells. It is easy to see that v and ω_1 are the unique sink and source node, respectively, in $G_{M_1}(\delta\sigma_1)$. Now, extend the matching M_1

to a matching $\bar{M}_1$ of $G(\bar{\sigma}_1)$ by adding the matching relation (ω_1, σ_1). We claim that $G_{\bar{M}_1}(\bar{\sigma}_1)$ is also acyclic. First, we note that ω_1 is also the unique source node in $G_{\bar{M}_1}(\bar{\sigma}_1)$ and hence cannot be in any directed cycle. But any directed cycle of $G_{\bar{M}_1}(\bar{\sigma}_1)$ must contain the node σ_1 as $G_{M_1}(\delta\sigma_1)$ is acyclic. However the only edge of $G_{\bar{M}_1}(\bar{\sigma}_1)$ *into* σ_1 is (ω_1, σ_1), and hence it must be on any directed cycle contradicting the fact that the node ω_1 is a source, thus proving the claim. By Proposition 2, we have constructed a Morse function with the desired properties for $\bar{\sigma}_1$.

Now for $j \geq 2$ let $\alpha_j = \delta\sigma_j \cap (\cup_{k=1}^{j-1} \delta\sigma_k)$ and suppose σ_j is not a homology cell. Now there is a shelling of $\delta\sigma_j$ (which is a $(d-1)$-sphere) such that the maximal cells of the pure and $(d-1)$-dimensional complex α_j appear first. Then by induction, there exists a Morse function f_j on $\delta\sigma_j$ with a 0-cell v_j and a $(d-1)$-cell ω_j being the only critical cells such that v_j is the only critical cell when f_j is restricted to α_j. In particular, if M_j is the matching associated with f_j in $G(\delta\sigma_j)$, then there is no edge of M_j from a cell in α_j to a cell in $\delta\sigma_j \setminus \alpha_j$. As a consequence, the subgraph of the directed graph $G_{M_j}(\delta\sigma_j)$ when *restricted* to the cells of $\delta\sigma_j \setminus \alpha_j$ is also acyclic with ω_j being the only unmatched cell. We can extend this acyclic subgraph with the induced matching to $\bar{\sigma}_j \setminus \alpha_j$ as before by adding the matching relation (ω_j, σ_j) and the other appropriate edges all directed away from σ_j. Now this acyclic graph with no unmatched cells corresponding to $\bar{\sigma}_j \setminus \alpha_j$ is attached to the (inductively constructed) acyclic graph for $\cup_{k=1}^{j-1}\bar{\sigma}_k$ with v being the only unmatched cell. Since the attaching edges contain no matching edges and hence are all directed away from $\bar{\sigma}_j \setminus \alpha_j$, the resulting graph is also acyclic with a well defined matching such that v is the only critical cell.

If σ_m is a homology cell, then we let σ_m be attached unmatched to the inductively constructed acyclic graph. The resulting graph is also clearly acyclic since we are adding a source node and from the construction, v and σ_m are the only critical cells for the resulting Morse function. This completes the proof of the proposition. ■

Thus we have shown that shellable pseudomanifolds are "nice" examples for discrete Morse theory in that we can construct the most efficient Morse functions for them. For boundary complexes of convex polytopes, we can use a strengthening of the famous Brugesser and Mani result about shellability of these complexes [23] to obtain the following result. (compare this result to Theorem 4.6 of Forman [11])

Corollary 1 : *If Σ is the boundary complex of a convex d-polytope then for any vertex v and any facet F of the polytope there exists a discrete Morse function on Σ such that v and F are the only two critical cells.* ■

It would be interesting to find triangulations of balls (spheres) that do not admit discrete Morse functions with exactly one (two) critical cells. Clearly such triangulations cannot be shellable and hence must be of dimension $d \geq 3$ ([9]). The obvious first candidates for these examples would be Rudin's unshellable triangulation of the tetrahedron [9], [23] and an unshellable 3-sphere due to Lickorish [12].

The construction of discrete Morse functions for shellable pseudomanifolds can easily be extended to prove the next theorem which states the precise connection between generalized shellings and Morse functions.

Theorem 3 : *Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a generalized shelling of a regular cell complex Σ and let v be any 0-cell in $\bar{\sigma}_1$. Then there exists a discrete Morse function f of Σ such that v is critical and further any other cell σ is critical if and only if it is a homology cell.* ■

Applying Theorem 2 to the Morse function of the above theorem we get the following result.

Corollary 2 : *For a d-dimensional regular cell complex Σ, let m_j be the number of j-dimensional homology cells in some generalized shelling, $j = 0, 1, \ldots, d$ and suppose they are not all zero. Then we have the following:*

1. *Σ is homotopy equivalent to a cell complex with $m_0 + 1$ points and m_j j-dimensional cells for $j = 1, 2 \ldots, d$.*
2. *If the homology cells appear in non-increasing order of dimension in the generalized shelling then Σ is homotopy equivalent to a wedge of spheres consisting of m_j j-dimensional spheres, $j = 0, 1, 2, \ldots, d$.*

Proof: The proof of (1) follows immediately from Theorem 2 and Theorem 3. We can prove (2) easily by induction on the number of homology cells in the generalized shelling. The key observation is that if a p-cell σ_j is a homology cell, then it is critical and further it is a maximal cell for the subcomplex $\cup_{k=1}^{j}\bar{\sigma}_k$. As per the proof of Theorem 2, it is attached along its boundary which is contained in $\cup_{k=1}^{j-1}\bar{\sigma}_k$. But by induction, $\cup_{k=1}^{j-1}\bar{\sigma}_k$ is homotopy equivalent to a wedge of spheres all of dimension at least p. Hence the boundary of σ_j must have been collapsed into the single 0-cell and therefore $\cup_{k=1}^{j}\bar{\sigma}_k$ is homotopy equivalent to wedge of spheres. ■

Remarks and an example:

(i) The condition of non-increasing dimension for homology cell is inspired by the *rearrangement lemma (2.6)* of Björner and Wachs [4], [5]. They show that the maximal cells of a shellable regular cell complex can be rearranged to give a shelling order in which the maximal cells appear in non-increasing order of dimension. Hence for shellable complexes, statement 2 of the above corollary reduces to Theorem 1.

(ii) The following example further illustrates the significance of the homology cells appearing in non-increasing order of dimension. Consider the (disconnected) simplicial complex on the vertex set {a,b,c,d,e} whose facets are ab,bc,ca,de. This is an example of a pure simplicial complex which is not shellable. Consider the following S-partition: [∅,de], [a,a], [b,ab], [c,ac], [bc,bc]. Now, (1) of the above corollary says that this complex is homotopy equivalent to a cell complex with two 0-cells and one 1-cell, and we also know that there is a subcomplex which has the homotopy type of 0-sphere. Clearly, these two facts by themselves do not rule out a contractible complex! On the other hand, for the S-partition [∅,bc], [c,ac], [bc,bc], [d,d], [e,de] , the corollary gives a wedge of spheres as the only possibility.

5 Applications and some open problems

5.1 Decompositions of surfaces

An interesting class of regular cell complexes which are not shellable, in general, are regular cell decompositions of surfaces (by which we mean compact, connected 2-manifolds without boundary). It is well-known [9] that regular cell decompositions of 2-spheres are shellable - indeed along with Lemma 2, this means that the class of shellable 2-pseudomanifolds with empty boundary is the class of "spherical" 2-pseudomanifolds. Therefore 2-pseudomanifolds which decompose surfaces of non-zero genus cannot be shellable. We now consider two such examples.

1. Consider the triangulation of the projective plane shown in Figure 3 whose facets are $\{123,125, 136, 145, 146, 234, 246, 256, 345, 356\}$. We can construct the following S-partition: [$\emptyset$,123], [5,125], [6,136],[4,145], [46,146],[24,24], [34,234], [26,246], [56,256], [35,345], [356,356]. This S-partition gives a discrete Morse function which results in the most efficient representation of the projective plane as a cell complex namely a complex with one cell each in dimensions $0, 1, 2$.

2. Consider the regular cell complex which decomposes the torus as shown in Figure 4 with 2-cells A,B,C,D,E. Then we have the generalized shelling: A, B, 12, C, 13, D, E. Hence the homology cells are 12, 13 and E and this gives a Morse function with one 0-cell, one 2-cell and two 1-cells being critical.

In general, any surface S of genus g has a (most efficient) representation as a cell complex with one 0-cell, one 2-cell and $p(S)$ 1-cells, where $p(S) = 2g(g)$ if S is orientable (non-orientable) and its Euler characteristic is $\chi(S) = 2 - p(S)$. Then we ask the following question:

Given a 2-pseudomanifold Σ without boundary, of Euler characteristic $\chi(\Sigma)$, is there a combinatorial decomposition property $P(\chi(\Sigma))$, such that Σ is a surface if and only if it satisfies $P(\chi(\Sigma))$?

The motivating special case is the 2-sphere for which the answer is in the affirmative and the decomposition property in question is shellability.

A related question is whether every regular cell complex Σ on a surface S, admits the most efficient Morse function possible for S. This can be also be formulated in terms of the graph G, which is the 1-skeleton of Σ. In the terminology of [17], which is a survey of results on embeddings of graphs on surfaces, Σ is a *closed 2-cell embedding* of the graph G on the surface S and it is shown that such a graph has to be 2-connected. It is well known that any 2-connected graph admits an *ear-decomposition* (see [19] for details). However, we ask the following question: Given such an embedding Σ on S, does G admit an ear-decomposition, which starts with a facial circuit and contains $n - 1$ facial circuits in all, where n is the number of faces of G (2-cells of Σ) in the embedding ? We leave it to the reader to construct a discrete Morse function for Σ which is the most efficient possible for S from such an ear-decomposition.

5.2 Relation to the decompositions of Stanley and Duval

There is a result of Duval [10] which describes a canonical combinatorial decomposition for simplicial complexes with a prescribed set of Betti numbers over some field. This settled a conjecture of Stanley and Kalai made in [20], where a special case of the result was proved by Stanley. Hence these results of Duval and Stanley, and results from discrete Morse theory presented in this paper are, in some sense, partial converses of each other. For instance, Stanley [20] considers a simplicial complex Σ, all of whose Betti numbers are zero (he calls such a complex acyclic but we avoid this term for obvious reasons). In the terminology of Proposition 2, he shows, using techniques from exterior algebra, that the Hasse diagram of such a simplicial complex Σ admits a matching M such that

(a) The only unmatched face in Σ is a vertex.

(b) The set of lower dimensional faces of the matching edges is a subcomplex of Σ.

In the special case when Σ is a cone over some vertex, the result is easy to see. In relation to Proposition 2, we remark that in the general instance of Stanley's theorem, it is clear that $G_M(\Sigma)$ need not be acyclic since this would imply, by Proposition 2, that Σ is collapsible. For instance, any triangulation of the projective plane satisfies the hypothesis of Stanley's theorem. For the triangulation of Example 1, Section 5.2, one can construct a matching M satisfying the above two conditions using the given S-partition. It can also be easily seen that $G_M(\Sigma)$ is not acyclic for any such M, as expected.

In the other direction, since a collapsible simplicial complex has trivial reduced homology, one should be able to modify the matching M on a collapsible simplicial complex Σ constructed by Proposition 2, to obtain a matching on Σ satisfying the above two conditions. We leave it to the interested reader to devise such an algorithm.

5.3 On some complexes related to matroids

In this section we discuss the topology of a set of simplicial complexes related to matroids called *Steiner complexes* which were introduced by Colbourn and Pulleyblank [8], motivated by applications to K-connectedness reliability on graphs. We will not present the original definition but rather a simpler reformulation in terms of matroid ports which is shown to be equivalent to the original by the author [7]. In what follows, we assume familiarity with the basic concepts of matroid theory.

Definition: Given a connected matroid N and an element e of the ground set of N, the port of N at the element e is the set

$\mathcal{P} = \{C - \{e\} : e \in C, C \text{ is a circuit of } N\}$.

A Steiner complex on a ground set E is a simplicial complex $\mathcal{S}$ defined by

$$\mathcal{S} = \{E - A : P \subseteq A \text{ for some } P \in \mathcal{P}\}$$

where $\mathcal{P}$ is the port of some connected matroid N on ground set $E \cup \{e\}$ at the element e. Now consider the set $\mathcal{P}^*$ of inclusion-minimal elements of $2^E - \mathcal{S}$. It follows from elementary matroid theory that $\mathcal{P}^*$ is the port of the matroid N^* at the element e. An important consequence of this fact is that if $\mathcal{S}$ is a Steiner complex defined with respect to the matroid N then $\mathcal{S}^b$ is a Steiner complex associated with N^*, where $\mathcal{S}^b = \{E - F : F \in 2^E - \mathcal{S}\}$. Note that by Alexander duality on the sphere ([1], page 278, exercise 7.4.3) we have that

$$H_i(\mathcal{S}) \cong H_{|E|-i-3}(\mathcal{S}^b)$$

for all i, where $H_i(\Sigma)$ represents the i-th dimensional (reduced) homology with coefficients in some field. Before we discuss the topological properties of Steiner complexes further, it might be appropriate to give some examples of ports and Steiner complexes.

The most important example of ports from the point of view of applications are *Steiner trees* of a graph with respect to fixed subset K of the vertices of a connected graph. The number of faces of the corresponding Steiner complex is of great interest in network reliability applications and we refer to [7] for further details. Of course a matroid complex is also a Steiner complex, however, it is easily seen that Steiner complexes are neither pure nor shellable in general. We will apply results of previous section to the following S-partitioning result for Steiner complexes [7], which was motivated primarily by applications to network reliability. Let $\mathcal{B}$ be the set of bases of N which do not contain e and let $\beta(N^*)$ be the β-invariant of N^* (see [1]).

Theorem 4 : *Let S be a Steiner complex defined with respect to the port of a connected matroid N of rank ρ at an element e of its ground set $E \cup \{e\}$.*

(i) ([7]) There is an ordering $B_1, B_2, \ldots, B_m$ of the bases in $\mathcal{B}$ and sets G_i and F_i satisfying $G_i \subseteq E - B_i \subseteq F_i$ such that the sequence of intervals $\{[G_i, F_i], i = 1, \ldots, m\}$ forms an S-partition of $\mathcal{S}$.

(ii) (implicit in [7]) The number of homology faces of the S-partition is equal to $\beta(N^*)$. ∎

Combining Theorem 4 and Corollary 2, we get

Corollary 3 : *The Steiner complex $\mathcal{S}$ is homotopy equivalent to a wedge of $\beta(N^*)$ $(|E| - \rho)$-dimensional spheres, where ρ is the rank of N.*

One could also have derived the homotopy properties of Steiner complexes by using some of the techniques that are described in Björner's survey [2]. For instance, let L^o be the geometric semi-lattice (see [18] for definitions and [22] for application to affine hyperplane arrangements) obtained by deleting the order filter of flats containing the element e from the geometric lattice defined by the flats of the matroid N^*. Then the maximal elements of L^o correpond precisely to the facets of $\mathcal{S}$ and the matroid closure operator of N^* is an order-preserving map from $\mathcal{S}$ to L^o. Now one could apply the fiber theorem of Quillen ([2], 10.5) to this map from $\mathcal{S}$ to L^o or one could also apply a version of the cross-cut theorem ([2] ,10.8) to L^o and obtain $\mathcal{S}$ as a cross-cut complex. Both approaches would show that the Steiner complex $\mathcal{S}$ is homotopy equivalent to the order complex of L^o. It is known that ([22]) that the order complex L^o is homotopy equivalent to reduced broken-circuit complex of N^*, $RBC(N^*)$, defined with respect to any order in which e is the smallest element. Both these pure complexes can be shown to have the homotopy type of a wedge of $\beta(N^*)$ $(|E| - \rho)$-dimensional spheres by shellability arguments ([1] [18], [22]). We remark that $RBC(N^*)$ is, hence, a proper and pure subcomplex of $\mathcal{S}$ with the same homotopy type as $\mathcal{S}$. In fact, one can show using the S-partition of the Steiner complex of [7] and the canonical shelling of the reduced broken-circuit complex shown in [1] and [22], that this homotopy equivalence can be constructed by a sequence of elementary collapses. We can use this observation to explain a certain topological duality for reduced broken-circuit complexes which was observed by Björner in [1]. For the connected matroid N and its dual N^* on the ground set $E \cup \{e\}$, we have $\beta(N) = \beta(N^*)$. Now for any fixed total order on the ground set in which e is the smallest element, if $RBC(N)$ and $RBC(N^*)$ are the reduced broken circuit complexes associated with N and N^*, then this implies the following for every i. ([1], (7.39))

$$H_i(RBC(N)) \cong H_{|E|-i-3}(RBC(N^*))$$

To quote Björner [1] - "(this is)...a curious topological duality for reduced broken circuit complexes that seems to lack a systematic explanation". We have already shown that $\mathcal{S}$ is homotopy equivalent to $RBC(N^*)$. By matroid port duality, $\mathcal{S}^b$ is homotopy equivalent to $RBC(N)$. Therefore, the topological duality of the reduced broken-circuit complexes observed by Björner is actually inherited from the (Alexander) duality of the pair of Steiner complexes.

5.4 A discrete Morse theory for posets ?

It would be interesting to try and generalize discrete Morse theory from regular cell complexes to posets with the hope that the existence of Morse functions with certain properties would enable us to derive topological consequences for its order complex. Most known results about topological properties of posets derive from lexicographic shellability type arguments due to Björner and Wachs [3] which show that the order complex is shellable. However, there are examples of interesting posets which are not known to be lexicograpically shellable but have nice homotopy properties. For instance P_n, the poset of partitions of the integer n ordered by dominance is not known to be lexicographically shellable. However, it is known through other fairly complicated methods [5] that (the order complex of) every interval of the poset P_n has the homotopy type of a sphere or is contractible. Since discrete Morse theory is a more general tool for deriving homotopy properties than traditional shellability techniques, an extension of the theory to posets, if possible, may indeed be worthwhile.

References

[1] A. Björner, Homology and Shellability of Matroids and Geometric Lattices, in *Matroid Applications*, (N. White, Ed.) (1992), Encyclopaedia of Mathematics and its applications, Cambridge University Press.

[2] A. Björner, Topological Methods, in *Handbook of Combinatorics*, Ed. R. Graham, M. Grötschel and L. Lovaśz (1996) North-Holland, Amsterdam.

[3] A. Björner and M. Wachs, On lexicographically shellable posets, *Trans. Amer. Math. Soc.* 277 (1983), 323-341.

[4] A. Björner and M. Wachs, Shellable and non-pure complexes and posets, I. (preprint dated September 1994) - to appear in *Trans. Amer. Math. Soc.*

[5] A. Björner and M. Wachs, Shellable and non-pure complexes and posets, II. (preprint dated June 1995) - to appear in *Trans. Amer. Math. Soc.*

[6] A. Björner, M. Las Vergnas, B.Sturmfels, N. White, G. Ziegler, *Oriented Matroids*, Encyclopaedia of Mathematics and it Application, 46 (1993), Cambridge University Press, Cambridge.

[7] M.K. Chari, Steiner complexes, matroid ports, and shellability, *J. Combin. Theory Ser B.* 59 (1993), 41-68.

[8] C.J. Colbourn and W.R. Pulleyblank, Matroid Steiner problems, the Tutte polynomial and network reliability, *J. Combin. Theory Ser. B* 41 (1989), 20-31.

[9] G. Danaraj and V. Klee, Which spheres are shellable, in :*Algorithmic Aspects of Combinatorics* (B. Alspach et. al eds) *Annals of Discrete Mathematics.*2 (1978), pp. 33-52.

[10] A.M. Duval A combinatorial decomposition for simplicial complexes, *Israel J. of Math.* 87 (1994), 77-87.

[11] R. Forman, A Morse theory for cell-complexes - preprint (1995), Department of Mathematics, Rice University. - to appear in *Advances in Math.*

[12] W.B.R. Lickorish, Unshellable triangulations of spheres. *Europ. J. Combinatorics* 12 (1991), 527-530.

[13] W.S. Massey, *Algebraic Topology: An Introduction* (1977), Springer Verlag, New York.

[14] W.S. Massey, *Singular Homology Theory* (1980), Springer Verlag, New York.

[15] J. Milnor, *Morse Theory*, Annals of Mathematics Study No. 51 (1962) Princeton University Press.

[16] J. R. Munkres, *Elements of Algebraic Topology* (1984) Addison-Wesley, Menlo Park.

[17] N. Robertson and R. Vitray, Representivity of surface embeddings, in *Paths, Flows and VLSI-Layout* (B. Korte. Ed), Algorithms and Combinatorics. 9, (1990), Springer-Verlag, Berlin Heidelberg.

[18] M. Wachs and J.W. Walker, On Geometric Semilattices, *Order* 2 (1986), 367-385.

[19] D.B. West, *Introduction to Graph Theory* (1996), Prentice Hall Inc.

[20] R.P. Stanley, A combinatorial decomposition of acyclic simplicial complexes, *Discrete Math.* 118 (1993), 175-182.

[21] R.P. Stanley, *Combinatorics and Commutative Algebra*, (2nd. Edition) (1996), Birkhäuser, Boston.

[22] G.M.Ziegler, Matroid shellability, β-systems, and affine hyperplane arrangements, *Journal of Algebraic Combinatorics* 1 (1992), 283-300.

[23] G.M.Ziegler , *Lectures on Polytopes* (1995) Springer Verlag, New York.

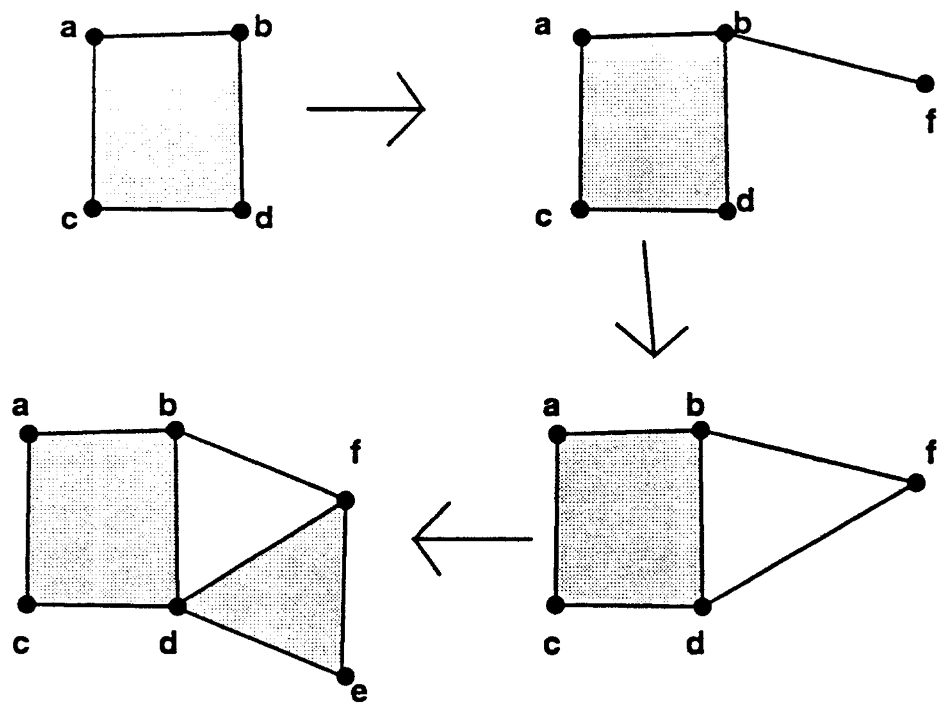

Figure 1. A generalized shelling of a complex which is not shellable

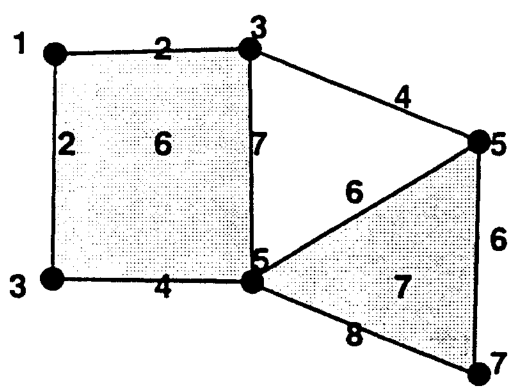

Figure 2. Example of a discrete Morse function

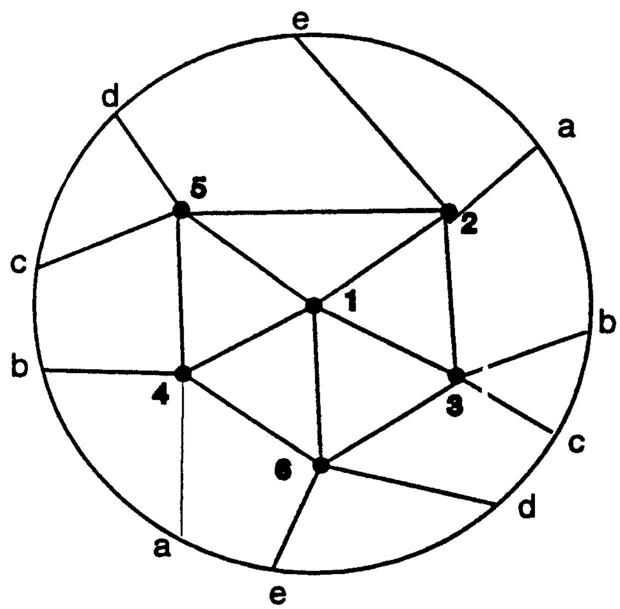

Figure 3.

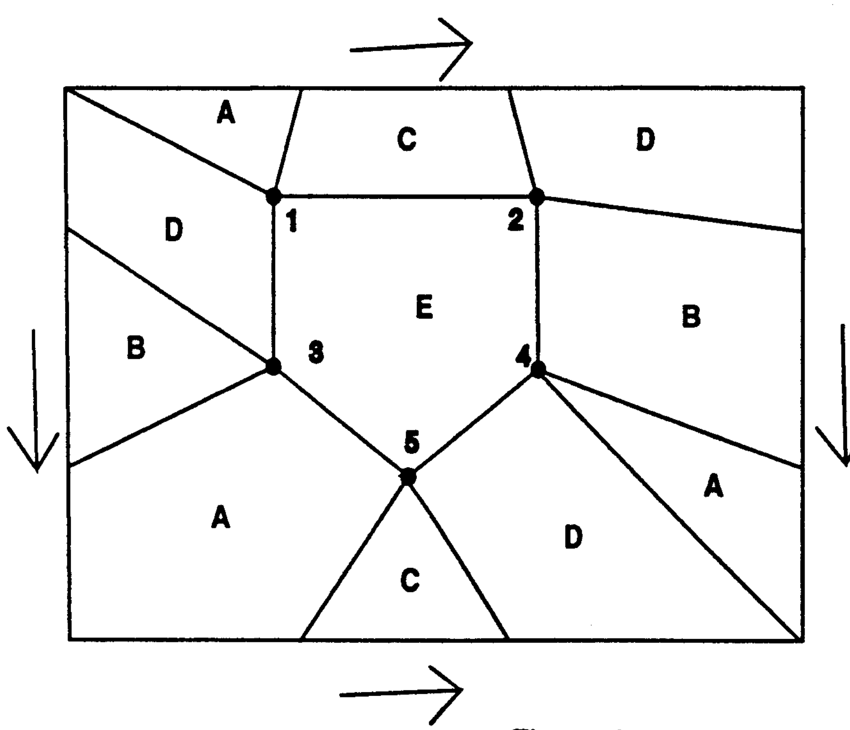

Figure 4.