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INTERCALATE MATRICES: I. RECOGNITION OF DYADIC TYPE

GILBERTO CALVILLO, ISIDORO GITLER, AND JOSÉ MARTÍNEZ-BERNAL

Abstract

This is the first of a series of papers where we study the characterization problem of which intercalate matrices determine integral formulas of sum of squares. Matrices that can be embedded into the Cayley table of a group of exponent two are intercalate matrices, called dyadic. This paper is devoted to giving combinatorial criteria to recognize dyadic matrices.

1. Introduction

The aim of the present paper is to characterize which intercalate matrices can be embedded into the Cayley table of a group of exponent two, such matrices are called dyadic. To this end we give five criteria: the first two hinge on a simple theorem of alternatives given in Section 2. Lemmas (3.1) and (4.1) found below rely on the fact that an intercalate matrix is completely determined by the position of its intercalations (or co-intercalations); their proofs are not given, the reader should have no problem in proving them, as well as proving Lemmas 5.1 and 5.2. The other criteria are based on connectedness, homogeneity and duality of intercalate matrices, respectively. All of these concepts are defined further below.

In this paper we consider the group of exponent two $(\mathbb{N}, \oplus)$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\oplus$ is the **dyadic sum** defined as follows: consider the binary representation of a and b, then add componentwise mod2 to obtain $a \oplus b$. We regard $\mathbb{N}$ as a vector space over $GF(2)$.

A matrix is **intercalate** if its entries, thenceforward called colors, along any row and along any column are all distinct, and furthermore, each of its 2×2 submatrices has an even number of distinct colors. Those 2×2 submatrices with two (four) distinct colors are called **intercalations** (**co-intercalations**).

The Cayley table of $(\mathbb{N}, \oplus)$ is an infinite intercalate matrix that we denote by $D[\mathbb{N} : \mathbb{N}]$. For (ordered) subsets $R = \{r_1, \dots, r_p\}$ and $C = \{c_1, \dots, c_q\}$ of $\mathbb{N}$, denote by $D[R : C]$ the $p \times q$ submatrix of $D[\mathbb{N} : \mathbb{N}]$ whose (i, j)-color is $r_i \oplus c_j$; R and C are called the set of row and column generators of $D[R : C]$, respectively.

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When $R = \{0, 1, \ldots, m-1\}$ and $C = \{0, 1, \ldots, n-1\}$ we will write $D[m : n]$. Matrices $D[2^n : 2^n]$ (see Example 1.1) are the Cayley tables of all the possible finite groups of exponent two (up to isomorphism). They will play an important role in our subject of study. Throughout this paper all intercalate matrices are assumed to be finite and to have colors in the set $\mathbb{N}$.

Example (1.1).

$$\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 & 13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 & 14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 \\
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 & 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{bmatrix}$$

$D[16 : 16]$

Two intercalate matrices are **isotopic** if one can be brought to the other by row and column permutations and, if necessary, a relabelling of its colors (we always assume that a relabelling associates distinct new labels to distinct colors). In case permutations are restricted to columns, we call them **c-isotopic**. Isotopy is an equivalence relation on the set of intercalate matrices. In these terms, an intercalate matrix is **dyadic** if it is isotopic to the matrix $D[R : C]$ for some sets of generators R and C. Observe that being dyadic is preserved under isotopy.

An intercalate matrix is **consistently signed** if it is possible to associate a *plus* or *minus* sign to each of its coordinates in such a way that every intercalation results with an odd number of *minus* signs. It is well-known ([10], [14], [16], [17], [18], [21], [23]) that there exists a consistently signed $r \times s$ intercalate matrix with n distinct colors if and only if there exists a sum of squares formula

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2,$$

where each z_k is an integral bilinear form in the indeterminates $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$; that is, $z_k = \sum_{i,j} a_{i,j}^k x_i y_j$ with the $a_{i,j}^k$ integers. Determining if a given intercalate matrix is consistently signed will be refered to as the **signability problem**.

The fact that any intercalate matrix containing a non-consistently signable submatrix cannot be consistently signed, suggests searching for minimal obstructions. In forthcoming papers we develop a matroid approach to the signability problem obtaining diverse structures that an intercalate matrix must avoid to be consistently signed. In a subsequent paper we will exhibit the smallest forbidden matrices for the signability problem. In another paper we will show a relationship between such smallest forbidden matrices and the Petersen graphs; this relation depends strongly on the fact that these matrices are dyadic. This is a main reason why we are interested in studying dyadic matrices.

2. A theorem of alternatives

Let F be a field and let $\mathcal{M}_{r\times s}(F)$ denote the set of $r \times s$ matrices with entries in F. Denote by e_i the vector whose ith entry is 1 and all other entries are zero. We write $x \rhd 0$ to express that all the entries of a vector x are distinct from zero.

THEOREM (2.1). *Let E be a vector space over the field F. Let $A \in \mathcal{M}_{p\times n}(F)$ and $B \in \mathcal{M}_{q\times n}(F)$ be matrices distinct from the zero matrix. If $n - \mathrm{rank}(A) \leq \dim_F(E) \leq \infty$, then one and only one of the next two conditions holds.*

(I) $\exists\, x \in E^n$: $Ax = 0$ *and* $Bx \rhd 0$.

(II) $\exists\, i \in \{1, \ldots, q\}$ *and* $\exists\, \alpha \in F^p$: $\alpha A = e_i B$.

Proof. Let r be the rank (over F) of the matrix A. Let I_r denote the identity matrix of order r. By Gauss-Jordan elimination, we can find a matrix $\Lambda \in \mathcal{M}_{r\times p}(F)$ and a permutation matrix Φ such that $\Lambda A \Phi = [I_r\ R]$, for some matrix R of size $r \times n - p$.
Case $r = n$. In this case we have a solution of (II) by taking i such that $e_i B \neq 0$ and $\alpha = e_i B \Phi \Lambda$.
Case $r < n$. Assume that the linear system (I) does not have a solution.

Take a vector z in E^{n-r} so that its entries are linearly independent over F. Now define $y = -Rz \in E^r$ and $x = (y, z) \in E^n$. Since $\Lambda A \Phi x = [I_r\ R](y, z) = y + Rz = 0$, and because of the assumption the system $A\Phi x = 0$ and $B\Phi x \rhd 0$ has no solution, there must exist an i such that $e_i B \Phi x = 0$. Hence we obtain, with $B\Phi = [B^y\ B^z]$, that $0 = e_i B \Phi x = e_i[B^y\ B^z]x = e_i(B^y y + B^z z) = e_i(-B^y R z + B^z z) = e_i(-B^y R + B^z)z$. But the choice of z implies that $e_i(-B^y R + B^z) = 0$, so $e_i B^y R = e_i B^z$. Therefore we have a solution of (II) with $\alpha = e_i B^y \Lambda$. In fact, we have

$$\alpha A \Phi = e_i B^y \Lambda A \Phi = e_i B^y [I_r\ R] = [e_i B^y\ e_i B^y R]$$
$$= [e_i B^y\ e_i B^z] = e_i [B^y\ B^z] = e_i B \Phi.$$

Finally, it can easily be verified that both conditions cannot be simultaneously satisfied.

We will use Theorem (2.1) in the proof of Theorems (3.2) and (4.2) with $E = \mathbb{N}$ and $F = GF(2)$. Note that in this case $\dim_F(E) = \infty$.

3. Co-intercalation criterion

LEMMA (3.1). *An intercalate matrix is dyadic if and only if there is a relabelling of its colors for which every co-intercalation has zero dyadic sum.*

Lemma (3.1) gives a method to verify if a given intercalate matrix M is dyadic. The method consists of finding new labels $x_1, \dots, x_n \in \mathbb{N}$ for the distinct colors $m_1, \dots, m_n$ of M, through the system of equations and inequalities

$$x_i \oplus x_j \oplus x_k \oplus x_l = 0, \quad \text{for every co-intercalation with colors } m_i, m_j, m_k, m_l, \text{ and} \tag{3.1}$$

$$x_i \oplus x_j \neq 0, \quad \text{for } i, j = 1, \dots, n; \ i \neq j. \tag{3.2}$$

For a set X denote its cardinality by $|X|$, and for a matrix N denote its set of distinct colors by $v(N)$. Let Δ denote the symmetric difference operator. The following theorem gives a characterization of dyadic matrices based on the theorem of alternatives of Section 2.

THEOREM (3.2). *Let M be an intercalate matrix. Then either M is dyadic or else there exists a family $\mathscr{C}$ of co-intercalations of M such that $|\Delta_{C\in\mathscr{C}} v(C)| = 2$.*

Proof. Let A and B be the coefficient matrices of the systems (3.1) and (3.2), respectively. A has a row for each co-intercalation and a column for each distinct color of M. The result follows by a direct application of Theorem (2.1) with $E = \mathbb{N}$ and $F = GF(2)$.

Example (3.3). The matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 5 & 6 & 7 & 8 \\ 6 & 5 & 9 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 3 & 2 & 7 & 8 \\ 9 & 10 & 11 & 8 & 7 \\ 10 & 9 & 12 & 13 & 14 \\ 15 & 16 & 13 & 12 & 17 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 3 & 2 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 10 & 9 \\ 15 & 16 & 14 & 13 & 17 & 18 \\ 16 & 15 & 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 21 & 20 & 26 \end{bmatrix}$$

are not dyadic. The first one contains the two co-intercalations $\{1, 3, 5, 7\}$ and $\{1, 3, 5, 0\}$, whose symmetric difference is $\{7, 0\}$; the second one contains the co-intercalations $\{1, 3, 10, 12\}$, $\{3, 7, 10, 8\}$, $\{8, 7, 12, 17\}$, whose symmetric difference is $\{1, 17\}$; and the third one contains the co-intercalations $\{1, 4, 16, 20\}$, $\{2, 4, 16, 13\}$, $\{2, 9, 13, 10\}$, $\{9, 10, 20, 26\}$, whose symmetric difference is $\{1, 26\}$.

4. Intercalation criterion

Let M be an intercalate matrix whose rows and columns have been labelled in some arbitrary way. For every submatrix I of M, let $(R(I), C(I))$ be the pair of its row and column indices. For a family $\mathscr{I}$ of intercalations of M we define its **border difference** as the pair of sets $(R(\mathscr{I}), C(\mathscr{I}))$, where $R(\mathscr{I}) = \Delta_{I\in\mathscr{I}} R(I)$ and $C(\mathscr{I}) = \Delta_{I\in\mathscr{I}} C(I)$. We denote the emptyset by $\emptyset$.

A family of intercalations $\mathscr{I}$ of M is **ill aligned** if its border difference $(R(\mathscr{I}), C(\mathscr{I}))$ satisfies one of the following three conditions.

1. $|R(\mathscr{I})| = |C(\mathscr{I})| = 2$ and $(R(\mathscr{I}), C(\mathscr{I}))$ determine a co-intercalation of M.
2. $|R(\mathscr{I})| = 2$ and $C(\mathscr{I}) = \emptyset$.
3. $R(\mathscr{I}) = \emptyset$ and $|C(\mathscr{I})| = 2$.

LEMMA (4.1). *An intercalate matrix M is dyadic if and only if there exist sets of labels R for the rows and C for the columns with the property that the labels of two rows have the same dyadic sum as the labels of two columns when, and only when, such rows and columns determine an intercalation of M.*

The role of Lemma (4.1) parallels that of Lemma (3.1). But now one must find labels $r_1, \ldots, r_p \in \mathbb{N}$ for the rows, and labels $c_1, \ldots, c_q \in \mathbb{N}$ for the columns, so that

(4.1)	$r_i \oplus r_j \oplus c_k \oplus c_l = 0,$	when r_i, r_j, c_k, c_l	determine an intercalation,
(4.2)	$r_i \oplus r_j \oplus c_k \oplus c_l \neq 0,$	when r_i, r_j, c_k, c_l	determine a co-intercalation.
(4.3)	$r_i \oplus r_j \neq 0,$	for $i, j = 1, \ldots, p;$	$i \neq j,$
(4.4)	$c_i \oplus c_j \neq 0,$	for $i, j = 1, \ldots, q;$	$i \neq j$

THEOREM (4.2). *Let M be an intercalate matrix. Then either M is dyadic or else it has an ill aligned family of intercalations.*

Proof. Let A denote the coefficient matrix of the system (4.1) and let B denote the coefficient matrix of the system of inequalities

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_p \\ c_1 \\ \vdots \\ c_q \end{pmatrix} \rhd 0,$$

where B_1, B_2 and B_3 are the coefficient matrices of the systems (4.2), (4.3) and (4.4), respectively. If the systems (4.1), (4.2), (4.3), and (4.4) have a simultaneous solution, then M is dyadic, as is verified by relabelling the (i, j)-color of M with $r_i \oplus c_j$. If any of these systems has no solution, then, by Theorem (2.1), a row of B is obtained as a $GF(2)$-linear combination of the rows of A. Depending whether such a row is in B_1, B_2 or B_3, we obtain one of the three conditions for an intercalate matrix to be ill aligned.

Example (4.3). If M is an intercalate matrix of size $4m \times 4n$, having a partition into 2×2 submatrices so that all, except one, are intercalations, then M is not dyadic. Note that all the intercalations in the partition are ill aligned.

5. Connectedness criterion

An intercalate matrix is an extension of another intercalate matrix if the latter appears as submatrix of the former.

Following Yiu [18], we will say that an intercalate matrix is:

- **complete,** if the colors in each row are a permutation of the colors in every other row.
- **connected,** if the two submatrices induced by any bi-partition of its set of columns have at least one common color.
- **saturated,** if whenever we extend it to an intercalate matrix by adding a new row, and if in this new row at least one new color appears, then all the other colors appearing in this new row are also new.

LEMMA (5.1). *Any complete and connected intercalate matrix is saturated.*

The columns of an intercalate matrix can be partitioned so that each submatrix in this partition is connected and different parts have disjoint set of colors. These parts are called the connected components (by columns) of the matrix.

LEMMA (5.2). *Every connected component of a complete intercalate matrix is complete.*

LEMMA (5.3). *Any complete and connected intercalate matrix M, with at least two rows and two columns, can be partitioned (up to isotopy) in the form*

$$M = \begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix},$$

where $\left[\begin{smallmatrix} P_1 \\ Q_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} P_2 \\ Q_2 \end{smallmatrix}\right]$ are the same matrices up to relabelling, P_1 is complete and connected, the first row of Q_1 is equal to the first row of P_2, and P_1 and Q_1 have no common colors.

Proof. Consider a maximal set of rows determining a non-connected submatrix P of M. It is not difficult to verify, using Lemmas (5.1) and (5.2), that P has exactly two complete and connected components which can be arranged to obtain matrices P_1 and P_2. The partition of P induces a partition in its complement within M to obtain Q_1 and Q_2. Permuting the columns of $\left[\begin{smallmatrix} P_2 \\ Q_2 \end{smallmatrix}\right]$, one can ensure that the first row of Q_1 is identical to the first row of P_2. Hence, from this and the intercalacy of M, the required condition on the equality of the submatrices follows.

THEOREM (5.4). *A complete and connected intercalate matrix M is dyadic.*

Proof. Proceed by induction on the size of M. The result is trivially true if M has just one row or column. So assume M has at least two rows and two columns. Apply Lemma (5.3) to M and let

$$M' = \begin{bmatrix} P_1 & P_2 \\ Q_1' & Q_2' \end{bmatrix}$$

be the matrix obtained from M by removing the first row $[R_1\ R_2]$ of the submatrix $[Q_1\ Q_2]$. Using the induction hypothesis one can conclude that M' is dyadic. Since $\left[\begin{smallmatrix}P_1\\Q_1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}P_2\\Q_2\end{smallmatrix}\right]$ are equal up to relabelling, one can assume that the generators of the columns of $\left[\begin{smallmatrix}P_2\\Q_2\end{smallmatrix}\right]$ are obtained by adding 2^k to every column generator of $\left[\begin{smallmatrix}P_1\\Q_1\end{smallmatrix}\right]$, where k has been fixed adequately and the addition is the dyadic one. Then the generator of the row $[R_1\ R_2]$ is $r \oplus 2^k$, where r is the generator of the first row of M'.

COROLLARY (5.5). *An intercalate matrix is dyadic if and only if it can be embedded into a complete and connected intercalate matrix.*

Example (5.6). Symmetric (hence connected) intercalate matrices are not necessarily dyadic, as the following example shows.

$$\begin{bmatrix} a & b & c & d & e & 1 & 2 & 3 & 4 & 5 \\ b & a & h & i & j & 6 & 3 & 2 & 7 & 8 \\ c & h & a & f & k & 9 & 10 & 11 & 8 & 7 \\ d & i & f & a & g & 10 & 9 & 12 & 13 & 14 \\ e & j & k & g & a & 15 & 16 & 13 & 12 & 17 \\ 1 & 6 & 9 & 10 & 15 & a & f & l & m & n \\ 2 & 3 & 10 & 9 & 16 & f & a & b & o & p \\ 3 & 2 & 11 & 12 & 13 & l & b & a & g & q \\ 4 & 7 & 8 & 13 & 12 & m & o & g & a & h \\ 5 & 8 & 7 & 14 & 17 & n & p & q & h & a \end{bmatrix}$$

This matrix contains a 5×5 submatrix in its right upper corner which is not dyadic (see Example (3.3)).

6. Homogeneity criterion

Let R be a subset of $\mathbb{N}$ and let $G(R)$ be the subgroup of $(\mathbb{N}, \oplus)$ generated by R. We denote by $E(R)$ ($O(R)$) the subset of $G(R)$ consisting of those elements which are the sum of an even (odd) number of elements of R. $E(R)$ is a subgroup of $G(R)$ of index less than or equal to two. For every $x \in O(R)$ we have $x \oplus E(R) = O(R)$. We denote by $\langle R\rangle$ ($\langle R\rangle^*$) the dyadic matrix $D[R : E(R)]$ ($D[R : O(R)]$). Since each row of $\langle R\rangle$ is a permutation of the elements of $O(R)$, it follows that $\langle R\rangle$ is complete. Furthermore, $\langle R\rangle$ is isotopic to $\langle R\rangle^*$.

LEMMA (6.1). *$\langle R\rangle$ is a connected intercalate matrix.*

Proof. Let P and Q be a bi-partition of the set $E(R)$, and assume that $0 \in P$. Define $A_0 = \{0\}$ and

$$A_i = \{x \in E(R) : x \text{ can be written as the sum of } 2i \text{ elements in } R\}, \text{ for } i \geq 1.$$

Since $E(R)$ is the union of the A_i, there must exist $j \geq 1$ such that $A_{j-1} \subseteq P$ and $A_j \bigcap Q \neq \emptyset$. Let $x \in A_j \cap Q$, say $x = r_1 \oplus \cdots \oplus r_{2j}$. Define $y = x \oplus r_1 \oplus r_2 \in$

$A_{j-1} \subsetneq P$. Because $x \oplus y = r_1 \oplus r_2$, it follows that $r_1 \oplus x = r_2 \oplus y$ is a common color in the partition of the matrix $\langle R \rangle$ induced by P and Q.

An intercalate matrix is **c-embedded** into another intercalate matrix if the former is c-isotopic to a submatrix of of the latter.

LEMMA (6.2). *If the matrix $D[R : C]$ is connected, then $D[R : C]$ is c-embedded into $\langle R \rangle$.*

Proof. Since $D[R : C]$ is connected, it must be c-embedded into exactly one of the connected components of the matrix $D[R : \mathbb{N}]$. But each of these components is c-isotopic to $\langle R \rangle$, so the result follows immediately.

THEOREM (6.3). *Any complete and connected intercalate matrix is isotopic to $\langle R \rangle$ for some subset R of $\mathbb{N}$.*

An intercalate matrix is **homogeneous** if there exists a subset R of $\mathbb{N}$ such that each connected component of M is c-embedded into $\langle R \rangle$.

THEOREM (6.4). *Let M be an intercalate matrix. Then M is dyadic if and only if it is homogeneous.*

Proof. Necessity follows directly by Lemma (6.2). For sufficiency, let R be a subset of $\mathbb{N}$ such that the connected components of M are c-embedded into $\langle R \rangle$. We can assume that each of these components is c-embedded into a different connected component of the matrix $D[R : \mathbb{N}]$. So clearly M is dyadic.

7. Duality criterion

The following definition is given in [18]. A **partial intercalate matrix** is a matrix that may have some undefined colors, and such that (1) the defined colors along each row and along each column are distinct, (2) whenever three colors in a 2×2 submatrix are defined and two of them are identical, then the fourth is also defined and such submatrix is an intercalation.

A partial intercalate matrix is said to **support** an intercalate matrix if it is possible to determine all its undefined colors so that the resulting matrix is intercalate. For example the partial intercalate matrix

$$\begin{bmatrix} a & b & * \\ c & * & b \\ * & c & d \end{bmatrix},$$

does not support an intercalate matrix.

Let $M = (m(r_i, c_j))$ be a partial intercalate matrix with its rows indexed by $r_1, \dots, r_p$, its columns indexed by $c_1, \dots, c_q$, and whose (distinct) colors are $d_1, \dots, d_n$. The **dual** matrix $M^c = (m^c(r_i, d_k))$ of M is the $p \times n$ partial intercalate matrix with its rows indexed by $r_1, \dots, r_p$, its columns indexed by $d_1, \dots, d_n$, and whose defined colors are $c_1, \dots, c_q$: $m^c(r_i, d_k) := c_j$ if and only if $m(r_i, c_j) = d_k$, undefined otherwise. Observe that $(M^c)^c = M$.

THEOREM (7.1). *An intercalate matrix M is dyadic if and only if its dual supports a dyadic intercalate matrix.*

Proof. The result follows by observing that in the group $(\mathbb{N}, \oplus)$ the dyadic sum $\oplus$ has the property $r \oplus c = k$ if and only if $r \oplus k = c$. For the necessity assume that the matrix M is of the form $D[R : C]$ for some subsets R and C of $\mathbb{N}$. Then, by the observation above, the matrix M^c supports the dyadic matrix $D[R : K]$, where K is the set of distinct colors of M. For the sufficency assume that $D[R' : K']$ is the dyadic matrix supported by M^c. Let C' and C'' be the sets of distinct colors of $D[R' : K']$ and M^c, respectively; $C'' \subseteq C'$. Again by the observation above, we have that $M = D[R' : C'']$ (up to isotopy).

Example (7.2). The following process shows that the (bordered) 4×4 matrix M is not dyadic.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ e & g & h & f \\ f & i & j & e \end{pmatrix} \end{array}$$

$$M^c = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccc} a & b & c & d & e & f & g & h & i & j \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . & . & . & . \\ 2 & 1 & 4 & 3 & . & . & . & . & . & . \\ . & . & . & . & 1 & 4 & 2 & 3 & . & . \\ . & . & . & . & 4 & 1 & . & . & 2 & 3 \end{pmatrix} \end{array}$$

we introduce colors 5 and 6;

$$M^c = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccc} a & b & c & d & e & f & g & h & i & j \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & . & . & . & . & 6 \\ 2 & 1 & 4 & 3 & 6 & . & . & . & . & 5 \\ 5 & . & . & . & 1 & 4 & 2 & 3 & . & (2) \\ . & . & 6 & . & 4 & 1 & . & . & 2 & 3 \end{pmatrix} \end{array}.$$

This matrix does not support any intercalate matrix, so by Theorem (7.1), M is not dyadic.

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GILBERTO CALVILLO.
BANCO DE MÉXICO,
5 DE MAYO # 6,
MÉXICO DF 06059,
MÉXICO.
gcalvill@banxico.org.mx

ISIDORO GITLER.
DEPARTAMENTO DE MATEMÁTICAS,
CINVESTAV-IPN,
A.P. 14-740,
MÉXICO D.F. 07000,
MÉXICO.
igitler@math.cinvestav.mx

JOSÉ MARTÍNEZ-BERNAL.
DEPARTAMENTO DE MATEMÁTICAS,
CINVESTAV-IPN,
A.P. 14-740,
MÉXICO D.F. 07000,
MÉXICO.
jmb@math.cinvestav.mx

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